HILBERT-SAMUEL COEFFICIENTS AND POSTULATION NUMBERS OF GRADED COMPONENTS OF CERTAIN LOCAL COHOMOLOGY MODULES

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Abstract. Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian homogeneous ring with one-dimensional local base ring $(R_0, m_0)$. Let $q_0 \subseteq R_0$ be an $m_0$-primary ideal, let $M$ be a finitely generated graded $R$-module and let $i \in \mathbb{N}_0$. Let $H_{R_+}^i(M)$ denote the $i$-th local cohomology module of $M$ with respect to the irrelevant ideal $R_+ := \bigoplus_{n > 0} R_n$ of $R$. We show that the first Hilbert-Samuel coefficient $e_1(q_0, H_{R_+}^i(M)_n)$ of the $n$-th graded component of $H_{R_+}^i(M)$ with respect to $q_0$ is antipolynomial of degree $< i$ in $n$. In addition, we prove that the postulation numbers of the components $H_{R_+}^i(M)_n$ with respect to $q_0$ have a common upper bound.

1. Introduction

Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian homogeneous ring, so that $R$ is $\mathbb{N}_0$-graded with Noetherian base ring $R_0$ and of the form $R = R_0[\ell_0, \cdots, \ell_r]$ with finitely many elements $\ell_0, \cdots, \ell_r \in R_1$. Let $R_+ := \bigoplus_{n > 0} R_n$ denote the irrelevant ideal of $R$. Moreover let $M$ denote a finitely generated graded $R$-module. For $i \in \mathbb{N}_0$ let $H_{R_+}^i(M) = \bigoplus_{n \in \mathbb{Z}} H_{R_+}^i(M)_n$ denote the $i$-th local cohomology module of $M$ with respect to $R_+$. Keep in mind that the $n$-th graded component $H_{R_+}^i(M)_n$ of $H_{R_+}^i(M)$ is a finitely generated $R_0$-module for all $n \in \mathbb{Z}$ and vanishes for all $n \gg 0$.

Motivated by the “tameness-problem” for coherent sheaves over projective schemes (cf. [B-H]) one is led to ask for the “asymptotic behaviour” of the graded components $H_{R_+}^i(M)_n$ for $n \to -\infty$. Namely, the mentioned tameness-problem is equivalent to the question of whether for each $i \in \mathbb{N}_0$ either $H_{R_+}^i(M)_n = 0$ for all $n \ll 0$ or $H_{R_+}^i(M)_n \neq 0$ for all $n \ll 0$. We do not know any example in which this tameness property is not satisfied. On the other hand, tameness has been established until now in fairly special cases only (cf. [I], [B-F-L]).

A more specific question is, whether for each $i \in \mathbb{N}_0$, the set $\text{Ass}_{R_0}(H_{R_+}^i(M)_n)$ of associated primes of the $R_0$-module $H_{R_+}^i(M)_n$ is asymptotically stable for $n \to -\infty$. This is true, for example, if $R_0$ is semilocal and either of dimension one or of dimension two and essentially of finite type over a field (cf. [B-F-L]). On the other
hand this “asymptotic stability of associated primes” need not hold in general, as can be deduced from examples of Singh \[S\] (cf. \[B-K-S\]) or Katzman \[K\] (cf. \[B-F-T\]). It may fail, for example, if \(R_0\) is regular local and of dimension four.

Finally, to be even more specific, one could ask whether certain numerical invariants of the \(R_0\)-modules \(H^i_{R_+}(M)_n\) “behave well” if \(i\) is fixed and \(n\) goes to \(\infty\). To be more precise we say that a numerical function \(f : \mathbb{Z} \to \mathbb{Z}\) is antipolynomial (of degree \(\leq i\)) if there is some polynomial \(p \in \mathbb{Q}[x]\) (of degree \(\leq i\)) such that \(f(n) = p(n)\) for all \(n < 0\). Moreover, if \((R_0, \mathfrak{m}_0)\) is local and \(q_0 \subseteq R_0\) is an \(\mathfrak{m}_0\)-primary ideal, let \(e_0(q_0, T)\) denote the Hilbert-Samuel multiplicity of the finitely generated \(R_0\)-module \(T\) with respect to \(q_0\). Then, in this terminology we can say (cf. \[B-F-T\]):

1.1 If \((R_0, \mathfrak{m}_0)\) is local of dimension one, \(q_0 \subseteq R_0\) is \(\mathfrak{m}_0\)-primary and \(i \in \mathbb{N}_0\), the functions given by

\[
\begin{align*}
    n \mapsto \text{length}_{R_0} \left( \Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n) \right), & \quad n \mapsto \text{length}_{R_0}(0 :_{H^i_{R_+}(M)_n} q_0), \\
    n \mapsto \text{length}_{R_0} \left( H^i_{R_+}(M)_n/q_0H^i_{R_+}(M)_n \right) & \quad \text{and} \quad n \mapsto e_0(q_0, H^i_{R_+}(M)_n)
\end{align*}
\]

are all antipolynomial of degree \(\leq i\).

The first aim of this note is to show that under the above hypotheses a further numerical invariant of the \(R_0\)-modules \(H^i_{R_+}(M)_n\) is antipolynomial of degree \(\leq i\). Namely, assume that \((R_0, \mathfrak{m}_0)\) is local of dimension one, let \(q_0 \subseteq R_0\) be an \(\mathfrak{m}_0\)-primary ideal and fix \(i \in \mathbb{N}_0\). We then have “asymptotic stability of associated primes” and so \(\dim_{R_0} \left( H^i_{R_+}(M)_n \right) \) takes a constant value \(\leq 1\) if \(n \to \infty\). Assume that this constant value is 1. Then, for all \(n < 0\) the Hilbert-Samuel polynomial of \(H^i_{R_+}(M)_n\) with respect to \(q_0\) can be written as

\[
(1.2) \quad P_{H^i_{R_+}(M)_n, q_0}(x) = e_0 \left( q_0, H^i_{R_+}(M)_n \right) (x + 1) - e_1 \left( q_0, H^i_{R_+}(M)_n \right),
\]

where \(e_1 \left( q_0, H^i_{R_+}(M)_n \right) = -P_{H^i_{R_+}(M)_n, q_0}(-1) \in \mathbb{Z}\) is the so-called first Hilbert-Samuel coefficient of \(H^i_{R_+}(M)_n\) with respect to \(q_0\).

We shall prove that the function given by \(n \mapsto e_1 \left( q_0, H^i_{R_+}(M)_n \right)\) is antipolynomial of degree \(\leq i\) (cf. Theorem 3.3).

In addition we show that the postulation numbers of the \(R_0\)-modules \(H^i_{R_+}(M)_n\) with respect to \(q_0\), hence the numbers

\[
(1.3) \quad \mu \left( q_0, H^i_{R_+}(M)_n \right) := \inf \left\{ r \in \mathbb{N}_0 \mid \text{length}_{R_0} \left( H^i_{R_+}(M)_n/q_0^{r+1}H^i_{R_+}(M)_n \right) = P_{H^i_{R_+}(M)_n, q_0}(t), \forall t \geq r \right\}
\]

have a common upper bound for all \(n \in \mathbb{Z}\) (cf. Theorem 3.3).

As for the unexplained terminology we refer to \[B-S\] and to \[E\].

2. Hilbert-Samuel coefficients in dimension 1

Throughout this section let \((A, \mathfrak{m})\) be a local Noetherian ring of dimension one, let \(q \subseteq A\) be an \(\mathfrak{m}\)-primary ideal and let \(T\) be a finitely generated \(A\)-module of dimension one.
2.1. Notation and Remark. A) Let $P_{T,q}(x) \in \mathbb{Q}[x]$ denote the Hilbert-Samuel polynomial of $T$ with respect to $q$, so that

$$P_{T,q}(n) = \text{length}_{A}(T/q^{n+1}T) \text{ for all } n \gg 0.$$ 

As $\dim(T) = 1$, $P_{T,q}(x)$ is of degree 1 and we may write

$$P_{T,q}(x) = e_{0}(q,T)(x + 1) - e_{1}(q,T)$$

where $e_{0}(q,T) \in \mathbb{N}$ is the (Hilbert-Samuel) multiplicity of $T$ with respect to $q$ and $e_{1}(q,T) \in \mathbb{Z}$ is the first Hilbert-Samuel coefficient of $T$ with respect to $q$.

B) We set

$$\overline{T} := T/\Gamma_{m}(T),$$

where $\Gamma_{m}(T) = \bigcup_{n \in \mathbb{N}} (0 : T : m^{n})$ denotes the $m$-torsion of $T$. Keep in mind that $\text{Ass}_{A}(\overline{T}) = \text{Ass}_{A}(T) \setminus \{m\}$, so that $\dim(\overline{T}) = \text{depth}_{A}(\overline{T}) = 1$.

C) Keep the above hypotheses and notation. Then, for each $n \in \mathbb{N}_{0}$, there is a short exact sequence

$$0 \rightarrow \Gamma_{m}(T)/(\Gamma_{m}(T) \cap q^{n}T) \rightarrow T/q^{n}T \rightarrow \overline{T}/q^{n}\overline{T} \rightarrow 0.$$ 

In particular,

$$\text{length}_{A}(T/q^{n}T) = \text{length}_{A}(\overline{T}/q^{n}\overline{T}) + \text{length}_{A}(\Gamma_{m}(A)) - \text{length}_{A}(\Gamma_{m}(T) \cap q^{n}T).$$

By Artin-Rees there is some $n_{0} \in \mathbb{N}$ such that $\Gamma_{m}(T) \cap q^{n}T = 0$ for all $n \geq n_{0}$. This gives the relations

$$\text{length}_{A}(T/q^{n}T) = \text{length}_{A}(\overline{T}/q^{n}\overline{T}) + \text{length}_{A}(\Gamma_{m}(T)) \text{ for all } n \geq n_{0};$$

$$P_{T,q}(x) = P_{\overline{T},q}(x) + \text{length}_{A}(\Gamma_{m}(T));$$

$$e_{0}(q,T) = e_{0}(q,\overline{T});$$

$$e_{1}(q,T) = e_{1}(q,\overline{T}) - \text{length}_{A}(\Gamma_{m}(T)).$$

2.2. Notation. A) Let

$$\mathcal{R}(q) := \bigoplus_{n \geq 0} q^{n}, \quad \mathcal{R}(q,T) := \bigoplus_{n \geq 0} q^{n}T$$

denote the Rees ring of $q$ and the Rees module of $T$ with respect to $q$, respectively.

B) In addition, let

$$\text{Gr}(q) := \mathcal{R}(q)/q\mathcal{R}(q) = \bigoplus_{n \geq 0} q^{n}/q^{n+1},$$

$$\text{Gr}(q,T) := \mathcal{R}(q,T)/q\mathcal{R}(q,T) = \bigoplus_{n \geq 0} q^{n}T/q^{n+1}T$$

denote the associated graded ring of $q$ and the associated graded module of $T$ with respect to $q$, respectively.

2.3. Notation. For a finitely generated graded module $M$ over the Noetherian homogeneous ring $R = \bigoplus_{n \geq 0} R_{n}$ let $\text{reg}(M)$ denote the (Castelnuovo-Mumford) regularity of $M$, thus

$$\text{reg}(M) := \inf\{r \in \mathbb{Z} \mid H_{R_{+}}^{i}(M)_{n-i+1} = 0, \forall i \in \mathbb{N}_{0}, \forall n \geq r\}.$$

2.4. Lemma. Let $\text{depth}_{A}(T) \neq 0$. Then $\text{reg}(\text{Gr}(q,T)) \leq \text{reg}(\text{Gr}(q))$. 
Proof. Let $x$ be an indeterminate. Then $B := A[x]_{m_{A[x]}}$ is a Noetherian local flat one-dimensional extension ring of $A$ with maximal ideal $n := m_{B}$ and infinite residue field $B/n$. Moreover $qB \subset B$ is an $n$-primary ideal and $T \otimes_{A} B$ is a finitely generated $B$-module of dimension 1 and depth $\neq 0$.

In addition, there is an isomorphism of graded rings $\text{Gr}(B) \cong \text{Gr}(q) \otimes_{A} B$ and an isomorphism of graded $\text{Gr}(qB)$-modules $\text{Gr}(B, T \otimes_{A} B) \cong \text{Gr}(q, T) \otimes_{A} B$. In view of the graded flat base-change property of local cohomology (cf. [B-S]), these latter isomorphisms induce

$$\text{reg} (\text{Gr}(qB)) = \text{reg} (\text{Gr}(q))$$

and

$$\text{reg} (\text{Gr}(qB, T \otimes_{A} B)) = \text{reg} (\text{Gr}(q, T)).$$

Altogether, we thus may replace $(A, m), q$ and $T$ respectively by $(B, n), qB$ and $T \otimes_{A} B$ and hence assume that $A/m$ is infinite.

As $q$ is $m$-primary, its analytic spread equals $\text{dim}(A) = 1$. So there is some $x \in q$ such that $Ax$ is a minimal reduction of $q$. As $A/m$ is infinite the reduction number of $q$ with respect to $Ax$ is $\leq \text{reg} (\text{Gr}(q))$ (cf. [T] Proposition 3.2) so that $xq^{n} = q^{n+1}$ for all $n \geq \text{reg} (\text{Gr}(q))$. Let $x^{*} = (0, x, 0, \cdots) \in R(q)_{1}$ be the element $x \in q$ considered as a one-form in $R(q)$. It follows that $\sqrt{R(q)} = \sqrt{x^{*}R(q)}$ so that $H^{1}_{R(q)+} (R(q, T)) = H^{1}_{x^{*}} (R(q, T))$ for all $i \geq 0$. Therefore $H^{0}_{R(q)+} (R(q, T)) = 0$ for all $i > 1$. As $x \in A$ is a parameter, $\text{depth}_{A}(T) = 1 = \text{dim}(A)$ implies that $x$ is $T$-regular. So $x^{*}$ is $R(q, T)$-regular and hence $H^{0}_{R(q)+} (R(q, T)) = 0$. Moreover we have the following short exact sequence of graded $R(q)$-modules (cf. [B-S] 2.2.17):

$$0 \to R(q, T) \xrightarrow{\eta} R(q, T)_{x^{*}} \to H^{1}_{x^{*}} (R(q, T)) \to 0.$$

As $x^{*}R(q, T)_{n} = xq^{n}T = q^{n+1}T = R(q, T)_{n+1}$ for all $n \geq \text{reg} (\text{Gr}(q))$, the natural map $\eta$ becomes an isomorphism in all degrees $\geq \text{reg} (\text{Gr}(q))$ so that

$$H^{1}_{R(q)+} (R(q, T))_{n} = H^{1}_{x^{*}} (R(q, T))_{n} = 0$$

for all $n \geq \text{reg} (\text{Gr}(q))$. Thus, finally we get $\text{reg} (R(q, T)) \leq \text{reg} (\text{Gr}(q))$.

Now, in view of the well-known behaviour of regularities in short exact sequences (cf. [B-S] 15.2.15) and as $\text{reg} (R(q, T)) \geq 0$ (cf. [B-S] 15.3.1)), we get our claim by the graded exact sequences

$$0 \to qR(q, T) \to R(q, T)(1) \to [T]^{-1} \to 0,$n \geq \text{reg} (\text{Gr}(q))$.

2.5. Lemma. Let $r \geq \text{reg} (\text{Gr}(q))$, let $n \geq r$ and assume that $\text{depth}_{A}(T) = 1$. Then

$$\text{length}_{A}(T/q^{n+1}T) = e_{0}(q, T)n - re_{0}(q, T) + \text{length}_{A}(T/q^{r+1}T).$$

Proof. For all $n \in \mathbb{N}_{0}$ we have

$$\text{length}_{A/q} (\text{Gr}(q, T)_{n}) = \text{length}_{A}(T/q^{n+1}T) - \text{length}_{A}(T/q^{n}T).$$

For all $n \gg 0$, the right-hand side of this equality takes the value $P_{r,q}(n) - P_{r,q}(n-1) = e_{0}(q, T)$, so that the characteristic function of the graded $\text{Gr}(q)$-module $\text{Gr}(q, T)$ takes the constant value $e_{0}(q, T)$. As $\text{reg} (\text{Gr}(q, T)) \leq r$ it follows
that $\text{length}_{A_q} (\text{Gr}(q, T)_n) = e_0(q, T)$ for all $n > r$ and hence $\text{length}_{A}(T/q^{n+1}T) - \text{length}_{A}(T/q^nT) = e_0(q, T)$ for all $n \geq r + 1$. So, for all $n \geq r$ we get

$$\text{length}_{A}(T/q^{n+1}T) = \sum_{m=r+1}^{n} (\text{length}_{A}(T/q^{m+1}T) - \text{length}_{A}(T/q^mT))$$

$$+ \text{length}_{A}(T/q^{r+1}T) = e_0(q, T)(n - r) + \text{length}_{A}(T/q^{r+1}T)$$

$$= e_0(q, T)n - re_0(q, T) + \text{length}_{A}(T/q^{r+1}T).$$

\[ \square \]

2.6. Proposition. Let $r \geq \text{reg} (\text{Gr}(q))$. Then

$$e_1(q, T) = (r + 1)e_0(q, T) - \text{length}_{A}(T/q^{r+1}T) - \text{length}_{A}(q^{r+1}T \cap \Gamma_m(T)).$$

Proof. As $T$ has dimension and depth 1, Lemma 2.5 gives $e_1(q, T) = (r+1)e_0(q, T) - \text{length}_{A}(T/q^{r+1}T)$. As

$$e_0(q, T) = e_0(q, T), \quad e_1(q, T) = e_1(q, T) - \text{length}_{A}(\Gamma_m(T))$$

and

$$\text{length}_{A}(T/q^{r+1}T) = \text{length}_{A}(T/q^{r+1}T) - \text{length}_{A}(\Gamma_m(T))$$

$$+ \text{length}_{A}(\Gamma_m(T) \cap q^{r+1}T),$$

(\text{cf. 2.7 C}), we get our claim. \[ \square \]

3. Antipolynomial growth of first Hilbert-Samuel coefficients and boundedness of postulation numbers

Now, we are ready to formulate and to prove our first main result.

3.1. Theorem. Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian homogeneous ring such that $(R_0, m_0)$ is local and of dimension 1. Let $q_0 \subseteq R_0$ be an $m_0$-primary ideal, let $M$ be a finitely generated graded $R$-module and let $i \in \mathbb{N}_0$. Assume that $\dim_{R_0} \left( H^i_{R^+}(M)_n \right) = 1$ for all $n \ll 0$. Then there is a polynomial $S(x) \in \mathbb{Q}[x]$ of degree $< i$ such that

$$e_1 \left( q_0, H^i_{R^+}(M)_n \right) = S(n) \text{ for all } n \ll 0.$$

Proof. By our hypotheses there is some $r \geq \text{reg} (\text{Gr}(q_0))$ such that $\dim_{R_0} \left( H^i_{R^+}(M)_n \right) = 1$ for all $n \leq -r$. So by Proposition 2.6 we obtain for all $n \leq -r$:

$$e_1 \left( q_0, H^i_{R^+}(M)_n \right) = (r + 1)e_0 \left( q_0, H^i_{R^+}(M)_n \right)$$

$$- \text{length}_{R_0} \left( H^i_{R^+}(M)_n/q_0^{r+1}H^i_{R^+}(M)_n \right)$$

$$- \text{length}_{R_0} \left( q_0^{r+1}H^i_{R^+}(M)_n \cap \Gamma_{m_0} \left( H^i_{R^+}(M)_n \right) \right).$$

According to (1.1) there are polynomials $P, Q \in \mathbb{Q}[x]$ of degree $< i$ such that

$$e_0 \left( q_0, H^i_{R^+}(M)_n \right) = Q(n), \quad \text{length}_{R_0} \left( H^i_{R^+}(M)_n/q_0^{r+1}H^i_{R^+}(M)_n \right) = P(n)$$

for all $n \ll 0$.\[ \square \]
By [3-F-T] Theorem 2.5], the graded $R$-module $\Gamma_{m_0}(H^i_{R^+}(M))$ is Artinian, and hence so is its graded submodule

$$U := q_0^{r+1}H^i_{R^+}(M) \cap \Gamma_{m_0}(H^i_{R^+}(M)).$$

So, there is a polynomial $F \in \mathbb{Q}[x]$ such that $\text{length}_{R_0}(U_n) = F(n)$ for all $n \ll 0$ (cf. [K]). According to (1.1) there is a polynomial $\mathcal{Q} \in \mathbb{Q}[x]$ of degree $< i$ such that $\text{length}_{R_0}(\Gamma_{m_0}(H^i_{R^+}(M)_n)) = \mathcal{Q}(n)$ for all $n \ll 0$. As $U_n \subseteq \Gamma_{m_0}(H^i_{R^+}(M)_n)$ for all $n$, it follows that $\deg(F) \leq \deg(\mathcal{Q}) < i$.

As $U_n = q_0^{r+1}H^i_{R^+}(M)_n \cap \Gamma_{m_0}(H^i_{R^+}(M)_n)$, the equality given at the beginning of this proof yields

$$e_1(q_0, H^i_{R^+}(M)_n) = (r+1)Q(n) - P(n) - F(n) \quad \text{for all} \ n \ll 0.$$ 

This proves our claim. \hfill \Box

Now we prove the announced boundedness of postulation numbers (cf. (1.3)). We begin with a preliminary remark.

3.2. Remark. Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a graded ring, let $W = \bigoplus_{n \in \mathbb{Z}} W_n$ be a graded and Artinian $R$-module and let $(U^{(m)})_{m \in \mathbb{N}_0}$ be a descending sequence of graded submodules of $W$. Assume that for each $n \in \mathbb{Z}$ there is some $m_n \in \mathbb{N}_0$ such that $U_n^{(m_n)} = 0$.

As $W$ is Artinian, there is some $t \in \mathbb{N}_0$ such that $U^{(m)} = U^{(t)}$ for all $m \geq t$. So, for each $n \in \mathbb{Z}$ and for each $m \geq \max\{t, m_n\}$, we obtain $U_n^{(t)} = U_n^{(m)} \subseteq U_n^{(m_n)} = 0$. Therefore $U^{(t)} = 0$.

3.3. Theorem. Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian homogeneous ring such that $(R_0, m_0)$ is local and of dimension $\leq 1$. Let $q_0 \subseteq R_0$ be an $m_0$-primary ideal, let $M$ be a finitely generated graded $R$-module and let $i \in \mathbb{N}_0$. Then, there is some $c \in \mathbb{N}_0$ such that $\mu(q_0, H^i_{R^+}(M)_n) \leq c$ for all $n \in \mathbb{Z}$.

Proof. By [3-F-T] Theorem 3.5 e)] there is some $\delta \in \{0, \pm 1\}$ such that

$$\dim_{R_0}(H^i_{R^+}(M)_n) = \delta \quad \text{for all} \ n \ll 0.$$ 

If $\delta = -1$ we have $H^i_{R^+}(M)_n = 0$ for all but finitely many values of $n$ and our claim is clear.

If $\delta \geq 0$ we shall apply Remark 3.2 to the graded Artinian $R$-module

$$W := \Gamma_{m_0,R}(H^i_{R^+}(M))$$

(cf. [3-F-T] Theorem 2.5 b)].) First, let $\delta = 0$, so that $W_n = H^i_{R^+}(M)_n$ for all $n \ll 0$. If we apply Remark 3.2 with $U^{(m)} := q_0^{m}W$, we find some $t \in \mathbb{N}_0$ with $q_0^tW = 0$ so that $q_0^tH^i_{R^+}(M)_n = 0$ for all $n \ll 0$. As $H^i_{R^+}(M)_n = 0$ for all $n \gg 0$ we get our claim if $\delta = 0$.

So, let $\delta = 1$. If we apply Remark 3.2 with $U^{(m)} := W \cap q_0^{m}H^i_{R^+}(M)$ and use the lemma of Artin-Rees we find some $t \in \mathbb{N}$ with $\Gamma_{m_0,R}(H^i_{R^+}(M)) \cap q_0^{t}H^i_{R^+}(M) = 0$, so that $\Gamma_{m_0,R}(H^i_{R^+}(M)_n) \cap q_0^{t}H^i_{R^+}(M)_n = 0$ for all $n \in \mathbb{Z}$. Now, let $n_0$ be such that $\dim_{R_0}(H^i_{R^+}(M)_n) = 1$ for all $n \leq n_0$. Then, for each $r \geq \max\{\text{reg}(\text{Gr}(q_0)), t-1\}$
and each $n \leq n_0$, Proposition 2.6 — applied with $A = R_0, q = q_0$ and $T = H^i_{R_+}(M)_n$ — gives

$$\text{length}_{R_0} \left( H^i_{R_+}(M)_n / q_0^{r+1} H^i_{R_+}(M)_n \right) = (r + 1)e_0 \left( q_0, H^i_{R_+}(M)_n \right) - e_1 \left( q_0, H^i_{R_+}(M)_n \right) = P_{H^i_{R_+}(M)_n, q_0}(r).$$

As $H^i_{R_+}(M)_n = 0$ for all $n > 0$, this proves our claim. □

3.4. Corollary. Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian homogeneous ring such that $(R_0, m_0)$ is local and of dimension $\leq 1$. Let $q_0 \subseteq R_0$ be an $m_0$-primary ideal, let $M$ be a finitely generated graded $R$-module and let $i \in \mathbb{N}_0$. Then, there are integers $n_0 \in \mathbb{Z}, c \in \mathbb{N}_0$ and polynomials $S, Q \in \mathbb{Q}[x]$ of degree $< i$ such that for each $n \leq n_0$ and each $r \geq c$ we have

$$\text{length}_{R_0} \left( H^i_{R_+}(M)_n / q_0^{r+1} H^i_{R_+}(M)_n \right) = Q(n)(r+1) - S(n).$$

Proof. Immediate by Theorem 3.1, Theorem 3.3 and [B-F-T, Theorem 3.5]. □

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