An example in the gradient theory of phase transitions

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Abstract: We prove by giving an example that when $n \geq 3$ the asymptotic behavior of functionals $\Omega[|2u|^2+(1-|u|^2)^2/\cdot]$ is quite different with respect to the planar case. In particular we show that the one-dimensional ansatz due to Aviles and Giga in the planar case (see Aviles and Giga 1987) is no longer true in higher dimensions.

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AN EXAMPLE IN THE GRADIENT THEORY OF PHASE TRANSITIONS

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Abstract. We prove by giving an example that when \( n \geq 3 \) the asymptotic behavior of functionals
\[
\int_{\Omega} \left( \frac{\varepsilon |\nabla u|^2 + (1 - |\nabla u|^2)^2}{\varepsilon} \right) \Omega \subset \mathbb{R}^n
\]
as \( \varepsilon \downarrow 0 \), where \( u \) maps \( \Omega \) into \( \mathbb{R} \). This problem was raised by Aviles and Giga in [2] in connection with the
mathematical theory of liquid crystals and more recently by Gioia and Ortiz in [9] for modeling the behavior
of thin film blisters. Recently many authors have studied the planar case giving strong evidences that, as
conjectured by Aviles and Giga in [2], the sequence \( \{F_{\varepsilon}\} \) \( \Gamma \)-converge (in the strong topology of \( W^{1,1} \); see [1] for
a discussion of such a choice and a rigorous setting) to the functional
\[
F^0_{\infty}(u) := \begin{cases} 
\frac{1}{3} \int_{J_{\nabla u}^+} |\nabla u^+ - \nabla u^-|^3dH^{n-1} & \text{if } |\nabla u| = 1, u \in W^{1,\infty} \\
\text{otherwise.} & 
\end{cases}
\]
Here \( J_{\nabla u} \) denotes the set of points where \( \nabla u \) has a jump and \( |\nabla u^+ - \nabla u^-| \) is the amount of this jump. Of
course the first line of the previous definition makes sense only for particular choices of \( u \), such as piecewise \( C^1 \).

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In their first work Aviles and Giga based their conjecture on the following ansatz (which they made in the case $n = 2$):

**Conjecture 1.1.** Let us choose a map $w : \Omega \to \mathbb{R}$ (with $\Omega \subset \mathbb{R}^n$ bounded open set containing 0) such that:

(a) $w$ is Lipschitz and satisfies the eikonal equation $|\nabla w| = 1$;

(b) $\nabla w$ is constant in $\{x_1 < 0\}$ and in $\{x_1 > 0\}$.

Let us define $E := \inf \{\liminf r_F^{\Omega}(w) : \|u_r - w\|_{W^{1,3}} \to 0\}$. Then there exists a family of functions $w_\varepsilon$ such that:

(i) the component of $\nabla w_\varepsilon$ perpendicular to $(1,0,\ldots,0)$ is constant;

(ii) $w_\varepsilon \to w$ in $W^{1,3}$;

(iii) $\lim F^{\Omega}_r(w_\varepsilon) = E$.

This ansatz has been proved by Jin and Kohn in [8] for $n = 2$. It reduces the problem of finding $E$ to a one dimensional problem in the calculus of variations which can be explicitly solved. This analysis leads to the result $E = F^{\Omega}_r(w)$, which means that at $w$ the $\Gamma$-limit of $F^{\Omega}_r$ exists and coincides with $F^{\Omega}_r(w)$. With a standard cut and paste argument (see [4]) it can be proved that the same happens for every $w$ which is piecewise affine.

In the next section we will prove the following theorem:

**Theorem 1.2.** Let $u$ be the function $u(x_1, x_2, x_3) = |x_3|$ and $C$ the cylinder $\{|x_1|^2 + |x_2|^2 < 1\}$. Then there exists $(u_k)$ such that:

(a) every $u_k$ is piecewise affine (being the union of a finite number of affine pieces) and satisfies the eikonal equation;

(b) $\lim_k F^{\infty}_r(u_k) < F^{\infty}_r(u)$;

(c) $u_k \to u$ strongly in $W^{1,p}$ for every $p < \infty$.

The proof can be easily generalized to every $n \geq 3$. As an easy corollary we get that the one-dimensional ansatz fails for $n \geq 3$. Moreover this failure means that $F$ cannot be the $\Gamma$-limit of $F^{\Omega}_r$ for $n \geq 3$.

**Corollary 1.3.** The one-dimensional ansatz is not true for $n \geq 3$.

**Proof.** As already observed, being every $u_k$ piecewise affine, there is a family of functions $u_{k,\varepsilon}$ such that $u_{k,\varepsilon}$ converge to $u_k$ in $W^{1,p}$ (for every $p < \infty$) and $\lim \inf_k F^{\infty}_r(u_{k,\varepsilon}) = F^{\infty}_r(u_k)$. A standard diagonal argument gives a sequence $(u_{k,\varepsilon(k)})$ strongly converging to $u$ in $W^{1,p}$ such that $\lim \inf_k F^{\infty}_r(u_{k,\varepsilon(k)}) < F^{\infty}_r(u)$.

2. The Example

In this section we prove Theorem 1.2. First of all we recall the following fact:

(Curl) If $v : \mathbb{R}^n \to \mathbb{R}^n$ is a piecewise constant vector field, then $v$ is a gradient if and only if for every hyperplane of discontinuity $\pi$ the right trace and the left trace of $v$ have same component parallel to $\pi$.

The building block of the construction of Theorem 1.2 is the following vector field, depending on a parameter $\phi \in (0, \pi/2)$. First of all we fix in $\mathbb{R}^3$ a system of cylindrical coordinates $(r, \theta, z)$ and then we call $A$ the cone given by $\{z > 0, r < 1, (1 - r) > z\tan\phi\}$ and $A'$ the reflection of $A$ with respect to the plane $\{z = 0\}$. Hence we put

\[
\begin{align*}
v(r, \theta, z) &= (0, 0, 1) &\text{if } z > 0 \text{ and } (r, \theta, z) \notin A \\
v(r, \theta, z) &= (\sin(2\phi), \theta + \pi, \cos(2\phi)) &\text{if } z > 0 \text{ and } z \in A \\
v(r, \theta, z) &= (0, 0, -1) &\text{if } z < 0 \text{ and } (r, \theta, z) \notin A' \\
v(r, \theta, z) &= (\sin(2\phi), \theta + \pi, -\cos(2\phi)) &\text{if } z < 0 \text{ and } z \in A'.
\end{align*}
\]
It is easy to see that \( v \) maps every plane \( \{ \theta = \alpha \} \cup \{ \theta = \alpha + \pi \} \) into itself. Moreover the restrictions of \( v \) to these planes all look like as in the following picture.

![Diagram](image)

**Lemma 2.1.** The vector field \( v \) is the gradient of a function \( w \). Moreover there is a sequence of piecewise affine functions \( w_k \) such that:

(a) \( w_k \to w \) strongly in \( W^{1,p} \) for every \( p \);
(b) \( F^w_{\Omega}(w_k) \to F^w_{\Omega}(w) \) for every open set \( \Omega \subset \mathbb{R}^3 \).

**Proof.** We consider the restriction of \( v \) to the plane \( P := \{ \theta = 0 \} \cup \{ \theta = \pi \} \). As already noticed \( v \) maps this plane into itself. Moreover its restriction to it satisfies condition (Curl), hence on \( P \) \( v \) is the gradient of a scalar function \( w \). Moreover we can find such a \( w \) so that it is identically zero on the line \( \{ z = 0 \} \cap P \). Hence \( w \) is symmetric with respect to the \( z \) axis and so we can extend \( w \) to the whole three-dimensional space so to build a cylindrically symmetric function. It is easy to check that the gradient of such a function is equal to \( v \).

We call this function \( w \) as well and we will prove that it satisfies conditions (a) and (b) written above.

(a) Our goal is approximating \( v \) with piecewise constant gradient fields. First of all we do it in the upper half-space \( \{ z > 0 \} \). For every \( n \) we take a regular \( n \)-agon \( B_n \) which is inscribed to the circle of radius 1 and lies on the plane \( \{ z = 0 \} \). The vertices of this \( n \)-agon are given by \( V_i := (1, 2i\pi/n, 0) \).

Hence we construct the pyramid \( A^n \) with vertex \( V := (0, 0, \cot \phi) \) and base \( B_n \). In the pyramid we identify \( n \) different regions \( A^n_1, \ldots, A^n_n \), where every \( A^n_i \) is given by the tetrahedron with vertices \((0,0,0), V, V_i, V_{i+1}\). After this we put \( v_n \) equal to \((0,0,1)\) outside \( A^n \) and in every \( A^n_i \) we put

\[
v_n(r, \theta, z) = (\sin 2\phi, \pi + (2i + 1)\pi/n, \cos 2\phi).
\]

It is easy to see that \( v_n \) satisfies condition (Curl), hence it is the gradient of some function \( w_n \). Moreover we can choose \( w_n \) in such a way that it is identically zero on \( \{ z = 0 \} \). Then we extend \( w_n \) to the lower half space \( \{ z < 0 \} \) just by imposing \( w_n(r, \theta, -z) = w_n(r, \theta, z) \). It is not difficult to see that \( \nabla w_n \) converges strongly to \( \nabla w \) in \( L^{\infty}_{\text{loc}} \) for every \( p \).

(b) Now we check that the previous construction satisfies also the second condition of the lemma. We fix an open set \( \Omega \subset \mathbb{R}^3 \) and we observe that both \( w_k \) and \( w \) satisfy the eikonal equation in \( \Omega \). Moreover we call \( L^n \) the triangle with vertices \( V, V_i, V_{i+1} \) and \( L^n \) the union of \( L^n_i \) (so \( L^n \) is the “lateral surface” of the pyramid \( A^n \)). Finally we denote by \( L \) the lateral surface of the cone \( A \), i.e. the set \((1-r) = z \tan \phi\).

(i) The amount of jump of \( v_n \) (i.e. \(|v^n_+ - v^n_-|\)) on \( L^n \) is constant and equal to the value of \(|v^+ - v^-|\) on \( L \). Moreover the area of \( L^n \) is converging to the area of \( L \). The same happens on the symmetric sets in the lower half-space \( \{ z < 0 \} \).
(ii) Let us call $B$ the base of the cone. The right and left traces of $v_n$ coincide with those of $v$ on $B_n \cup \{z = 0\} \setminus B$). Moreover the area of $B \setminus B_n$ is converging to zero.

(iii) The vector fields $v_n$ are discontinuous also on the triangles $T^n_{i}$ joining $V_i$, $(0,0,0)$ and $V_i$ (and on the symmetric triangles lying on $\{z < 0\}$). The amount of jump of $v_n$ on each of these triangles is given by

$$|v_n^+ - v_n^-| = 2 \sin(\pi/n).$$

Moreover the area of everyone is given by $(\cot \phi)/2$. Hence

$$\int_{\cup_i T_n} |v_n^+ - v_n^-|^3 dH = 4n \cot \phi \sin^3 \pi/n.$$ 

The right hand side goes to zero as $n \to \infty$ and this completes the proof.

\[\square\]

Proof of Theorem 1.2. First of all we pass from the cartesian coordinates of the statement to the cylindrical coordinates $(r, \theta, z)$ given by $x_3 = z$, $x_1 = r \cos \theta$, $x_2 = r \sin \theta$ (and sometimes we will denote the elements of $\mathbb{R}^3$ with $(y, z)$, where $y \in \mathbb{R}^2$ and $z \in \mathbb{R}$).

We take $w$ as in the previous lemma. First of all let us compute $F^c_\infty(w)$ where $C$ is the cylinder $\{r < 1\}$. As in the previous proof we call $L$ the lateral surface of the cone, that is the set $\{r - 1 = z \tan \phi\}$. The value of $|\nabla w^+ - \nabla w^-|$ on the surface $L$ is given by $2 \sin \phi$ and the area of $L$ is given by $\pi/\sin \phi$: the same happens for the symmetric of $L$ lying on the half-space $\{z < 0\}$. On the base of the cylinder we have $|\nabla w^+ - \nabla w^-| = 2 \cdot \cos 2\phi$. Hence

$$a(\phi) := F^c_\infty(u) - F^c_\infty(w) = \frac{\pi}{3} [8 - 8 \cos 3 \phi - 16 \sin^2 \phi]$$

and it can be easily checked that for $\phi$ close enough to zero, $a(\phi)$ is positive.

Therefore let us fix an $\alpha$ for which $a(\alpha) > 0$ and let us agree that $w$ is constructed as in the previous lemma by choosing $\phi = \alpha$. Given $\rho > 0$ and $x \in \mathbb{R}^2$ we define $w_{x,\rho}$ in the cylinder $C_{x,\rho} := \{(y, z) : |y - x| \leq \rho\} \subset \mathbb{R}^3$ as $w_{x,\rho}(y, z) = \rho w((y - x)/\rho, z/\rho)$. It is easy to see that

$$F^c_{\infty,\rho}(u) - F^c_{\infty,\rho}(w_{x,\rho}) = a(\alpha) \rho^2.$$  

Let us fix $\varepsilon$ and take $\rho$ such that $\rho \cot \alpha < \varepsilon$. Thanks to Besicovitch Covering lemma we can cover $H^2$ almost all $D := \{z = 0, r < 1\}$ with a disjoint countable family of closed discs $D_i$ such that every $D_i$ has radius $r_i < \rho$, center $x_i$ and is contained in $D$. We construct $u_{x_i}$ by putting $u_{x_i} \equiv w_{x_i,\rho_i}$ in the cylinder $C_{x_i,\rho_i}$. Since $\nabla u_{x_i}$ coincides with $\nabla u$ in $\{z \geq \varepsilon\}$ and satisfies the eikonal equation, it is easy to see that $u_{x_i} \to u$ locally in the strong topology of $W^{1,p}$. Moreover equation (2) implies that

$$F^c_\infty(u) - F^c_\infty(u_{x_i}) = \sum_i a(\alpha) r_i^2 = a(\alpha).$$

At this point, using the previous lemma we can approximate the function $u_{x}$ in the cylinders $C_{x,\rho_i}$ with piecewise affine functions in such a way that their traces coincide with the trace of $u_{x_i}$ on the boundary of $C_{x_i,\rho_i}$. Using standard diagonal arguments for every $\varepsilon$ we can find a sequence of piecewise affine functions $u^\varepsilon_k$ which converge in $W^{1,p}$ to $u_{x_i}$ and such that $F^c_{\infty}(u^\varepsilon_k) \to F^c_{\infty}(u_{x_i})$. Moreover, again using diagonal arguments, we can construct the sequence $u^\varepsilon_k$ so that each one is a finite union of affine pieces. Finally, one last diagonal argument, gives a sequence $\tilde{u}_k$ such that:

(a) $\tilde{u}_k$ is a finite union of affine pieces;

(b) $\lim_k F^c_{\infty}(\tilde{u}_k) < F^c_{\infty}(u)$;

(c) $\tilde{u}_k \to u$ strongly in $W^{1,p}$ for every $p < \infty$.

\[\square\]
REFERENCES