ON THE DETERMINANT
OF ELLIPTIC BOUNDARY VALUE PROBLEMS
ON A LINE SEGMENT

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Abstract. In this paper we present a formula for the determinant of a matrix-valued elliptic differential operator of even order on a line segment [0, T] with boundary conditions.

1. Introduction and summary of the results

In this paper we present a formula for the determinant of a matrix-valued elliptic differential operator of even order on a line segment [0, T] with boundary conditions. In order to state our results we introduce the following notation:

1. Denote by \( \mathcal{A} = \sum_{k=0}^{2n} a_k(x)D^k \) a differential operator, \( D = D_x = -i \frac{d}{dx} \), where the coefficients are complex-valued \( r \times r \) matrices depending smoothly on \( x, 0 \leq x \leq T \). The leading coefficient \( a_{2n}(x) \) is assumed to be nonsingular and to have \( \theta \) as a principal angle, i.e. \( \text{Re} \cap \text{Spec} a_{2n}(x) = \phi \) for \( 0 \leq x \leq T \), where \( \mathbb{R}_\theta := \{ \rho e^{i\theta} \in \mathbb{C} \mid 0 < \rho < \infty \} \).

2. We impose boundary conditions of the form

\[
\ell_j u(T) = 0, \quad m_j u(0) = 0 \quad (1 \leq j \leq n)
\]

where \( u \in C^\infty([0, T] ; \mathbb{C}^r) \) and \( \ell_j, m_j \) are differential operators of the form

\[
\ell_j := \sum_{k=0}^{\alpha_j} b_{jk} d_x^k, \quad m_j := \sum_{k=0}^{\beta_j} c_{jk} d_x^k \quad \left( d_x = \frac{d}{dx} \right)
\]

such that \( b_{jk}, c_{jk} \) are constant \( r \times r \) matrices with \( b_{j\alpha_j} = c_{j\beta_j} = \text{Id} \) and such that the integers \( \alpha_j, \beta_j \) satisfy

\[
0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq 2n - 1, \quad 0 \leq \beta_1 < \beta_2 < \cdots < \beta_n \leq 2n - 1.
\]

Example 1. Dirichlet boundary conditions: \( \alpha_D = \beta_D = (0, 1, \ldots, n-1) \)

\[
b_{D,jk} = c_{D,jk} := \begin{cases} 
\text{Id} & \text{if } 1 \leq j \leq n, k = j - 1, \\
0 & \text{otherwise.}
\end{cases}
\]
Example 2. Neumann boundary conditions: \( \alpha_N = \beta_N = (n, n+1, \ldots, 2n-1) \)

\[
\begin{cases}
  \mathrm{Id} & \text{if } 1 \leq j \leq n, \ k = n + j - 1,
  \\
  0 & \text{otherwise}.
\end{cases}
\]

For convenience we write \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( |\alpha| = \sum_{j=1}^{n} \alpha_j \) and similarly \( \beta \) and \( |\beta| \). Boundary conditions of the above form are usually called separated. Let \( B = (B_{jk}) \) and \( C = (C_{jk}) \), \( 1 \leq j \leq 2n \), \( 0 \leq k \leq 2n - 1 \), be \( 2n \times 2n \) matrices whose entries are the following \( r \times r \) matrices

\[
\begin{align*}
  b_{jk} & := b_{j-n,k} \quad \text{if } n + 1 \leq j \leq 2n \text{ and } 0 \leq k \leq \beta_j - n, \\
  0 & \quad \text{otherwise};
\end{align*}
\]

\[
\begin{align*}
  c_{jk} & := c_{\alpha_j - n,k} \quad \text{if } 1 \leq j \leq n \text{ and } 0 \leq k \leq \alpha_j, \\
  0 & \quad \text{otherwise}.
\end{align*}
\]

We denote by \( A = A_{B,C} \) the operator \( \mathcal{A} \) restricted to the space of smooth functions \( u : [0, T] \to \mathbb{C}^r \) satisfying the boundary conditions (1.1).

(3) \( \zeta \)-regularized determinant \( \text{Det}_{\theta} A \). In the case where \( A \) is not \( 1-1 \), define \( \text{Det}_{\theta} A = 0 \). In the case \( A \) is \( 1-1 \), one proceeds as follows. As the coefficient \( a_{2n}(x) \) has \( \theta \) as a principal angle, there exists \( \varepsilon > 0 \) so that \( L(\theta - \varepsilon, \theta + \varepsilon) \cap \text{Spec} a_{2n}(x) = \emptyset \), \( 0 \leq x \leq T \), where \( L(\alpha, \beta) := \{ z \in \mathbb{C} | \alpha \leq \arg z \leq \beta \} \). Then the spectrum of \( A \), \( \text{Spec} A \), is discrete, \( \text{Spec} A = \{ \lambda_j, j \in \mathbb{N} \} \), \( |\lambda_j| \to \infty \), and \( \text{Spec} A \cap L(\theta - \varepsilon', \theta + \varepsilon') \) for any \( 0 < \varepsilon < \varepsilon' \) is finite.

If \( R_{\theta} \cap \text{Spec} A = \emptyset \), we define \( \zeta_{A,\theta}(s) = \sum_{j \geq 1} \lambda_j^{-s} = Tr A^{-s} \) where \( s \in \mathbb{C} \), \( \text{Re} s > 1/2n \) and where the complex powers are defined with respect to the angle \( \theta \). It is a well-known fact that \( \zeta_{A,\theta}(s) \) admits a meromorphic extension to \( \mathbb{C} \) with \( s = 0 \) being a regular point. According to Ray and Singer [RS] one defines \( \log \text{Det}_{\theta} A := -\frac{d}{ds}|_{s=0} \zeta_{A,\theta}(s) \). If \( R_{\theta} \cap \text{Spec} A \neq \emptyset \), then choose \( \theta' \in (\theta - \varepsilon, \theta + \varepsilon) \) so that \( R_{\theta'} \cap \text{Spec} A = \emptyset \), and define \( \text{Det}_{\theta} A := \text{Det}_{\theta'}(A) \).

It can be easily checked (cf. [BFK1]) that the definition is independent of the choice of \( \theta' \) in \( (\theta - \varepsilon, \theta + \varepsilon) \).

(4) The fundamental matrix \( Y(x) = Y(x, \mathcal{A}) \). Denote by \( Y(x) = (y_{k\ell}(x)) \) \( (x \in \mathbb{R}) \) the fundamental matrix for \( \mathcal{A} \). Note that \( Y(x) \) is a \( 2n \times 2n \) matrix whose entries \( y_{k\ell}(x) \) \( (0 \leq k, \ell \leq 2n - 1) \) are \( r \times r \) matrices defined by

\[
y_{k\ell}(x) := d_{x}^{k} y_{\ell}(x)
\]

where \( y_{\ell}(x) \) denotes the solution of the Cauchy problem \( \mathcal{A} y_{\ell}(x) = 0 \), \( y_{k\ell}(0) = \delta_{k\ell} \mathrm{Id} \). Of particular interest is the \( 2n \times 2n \) matrix \( Y(T) \), the evaluation of the fundamental matrix at \( x = T \).

(5) Introduce the quantities

\[
g_\alpha := \frac{1}{2} \left( \frac{|\alpha|}{n} - n + \frac{1}{2} \right), \quad h_\alpha = \det \begin{pmatrix}
  w_{1}^{\alpha_1} & \cdots & w_{n}^{\alpha_1} \\
  \vdots & \ddots & \vdots \\
  w_{1}^{\alpha_n} & \cdots & w_{n}^{\alpha_n}
\end{pmatrix}
\]

where \( w_1, \ldots, w_n \) denote the \( 2n \)th roots of \( (-1)^{n+1} \) with \( \text{Re} w > 0 \) given by \( w_k = \exp \left\{ \frac{2k-n-1}{2n} \pi i \right\} \). For a \( r \times r \) matrix \( a \) with principal angle \( \theta \) and eigenvalues \( \lambda_1, \ldots, \lambda_r \), denote \( (\det a)^{\theta}_\alpha = \prod_{j=1}^{r} |\lambda_j|^{g_\alpha} \exp \{ i g_\alpha \arg \lambda_j \} \) where \( \theta - 2\pi < \arg \lambda_j < \theta \).
Example 1. Dirichlet boundary conditions:
\[ g_{\alpha D} = -n/4, \quad h_{\alpha D} = h_n := \prod_{i<j}(w_i - w_j). \]

Example 2. Neumann boundary conditions:
\[ g_{\alpha N} = n/4, \quad h_{\alpha N} = (-1)^n h_n. \]

The main result of this paper is

**Theorem.**

\[ \text{Det}_\theta A = K_\theta \exp \left\{ \frac{i}{2} \int_0^T \text{tr}(a_{2n}^{-1}(x)a_{2n-1}(x)) \, dx \right\} \text{det}(BY(T) - C) \]

where \( K_\theta \equiv K_\theta(\alpha, \beta) \) is given by

\[ K_\theta = ((-1)^{1/2} (2n)^n h_{\alpha -1}^{-\beta} (\text{det} a_{2n}(0))^{\frac{\theta}{2}} (\text{det} a_{2n}(T))^{\frac{\theta}{2}}. \]

**Example 1.** Dirichlet boundary conditions: \[ |\alpha_D| = \frac{n(n-1)}{2}, \]

\[ K_\theta = ((-1)^{1/2} (2n)^n h_{\alpha -1}^{-\beta} (\text{det} a_{2n}(0))^{\frac{\theta}{2}} (\text{det} a_{2n}(T))^{\frac{\theta}{2}}. \]

**Example 2.** Neumann boundary conditions: \[ |\alpha_N| = \frac{n(n-1)}{2}, \]

\[ K_\theta = ((-1)^{1/2} (2n)^n h_{\alpha -1}^{-\beta} (\text{det} a_{2n}(0))^{\frac{\theta}{2}} (\text{det} a_{2n}(T))^{\frac{\theta}{2}}. \]

**Corollary.** \( \text{Det}_\theta A \) is a complex number independent of \( \theta \) up to multiplication with a \( 2n \)th root of unity.

**Remark 1.** In the formula above all terms except the matrix \( Y(T) \) are easily computable from the coefficients of \( \mathscr{A}, \ell_i \) and \( m_j \). The matrix \( Y(T) \) requires the knowledge of the fundamental solutions. The matrix \( Y(T) \) and therefore \( \text{det}(BY(T) - C) \) can be calculated numerically within arbitrary accuracy by solving a finite difference equation approximating \( \mathscr{A} \). So the determinant \( \text{Det}_\theta A \) can be calculated numerically within arbitrary accuracy.

**Remark 2.** Theorem is a companion of the corresponding result on the circle instead of the interval \([0, T]\) which was treated in an earlier paper [BFK1]. Again, the proof of Theorem relies on a deformation argument and explicit computations for certain special operators and special boundary conditions.

**Remark 3.** Introduce a spectral parameter \( \lambda \), and denote the fundamental matrix of \( \mathscr{A} + \lambda \) by \( Y(x, \lambda) = Y(x, \mathscr{A} + \lambda) \). One then verifies \( \text{det}(BY(T; \lambda) - C) = 0 \) iff \( \text{Det}_\theta (A + \lambda) = 0 \), i.e. iff \( -\lambda \) is an eigenvalue of \( A = A_B, C \).

**Remark 4.** First results of the type described in Theorem are due to Dreyfus and Dym [DD] and to Forman [Fo1] (cf. also [Fo2]). Forman proved by different methods that the quotient \( \text{Det}_\theta A / \text{Det}(BY(T) - C) \) only depends on the principal and subprincipal symbols of \( \mathscr{A} \), and the principal symbol of the boundary operators \( \ell_j, m_j \quad (1 \leq j \leq n) \). Our Theorem provides a formula for this quotient.
Remark 5. Analogous to results obtained in \([BFK2]\), Theorem can be extended to the case where \(\mathcal{A}\) is a pseudodifferential operator. The determinant \(\text{Det}_\theta \mathcal{A}\) can be written as a product of local invariants with a Fredholm determinant of a pseudodifferential operator of determinant class, canonically associated to \(\mathcal{A}\). The Fredholm determinant corresponds to \(\det(BY(T) - C)\) in the case when \(\mathcal{A}\) is a differential operator.

2. Auxiliary results

In this section we collect some auxiliary results needed for the proof of Theorem. First we introduce some additional notation. Denote by \(EDO_{2n} \equiv EDO_{2n, r}\) the set of all elliptic differential operators \(\mathcal{A}\) of order \(2n\) on \([0, T]\) as introduced in Section 1. We identify \(EDO_{2n}\) with the open set \(\{(a_2, \ldots, a_0) \in C^\infty([0, T], \text{End} C')^{2n+1} : \det(a_2(x)) \neq 0, 0 \leq x \leq T\}\) of the Fréchet space \(C^\infty([0, T], \text{End} C')^{2n+1}\). Further define \(EDO_{2n, \theta} := \{\mathcal{A} \in EDO_{2n} : \theta\) is principal angle for \(a_2n\}\). Clearly \(EDO_{2n, \theta}\) is an open connected subset in \(EDO_{2n}\).

Given \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n\) with \(0 \leq \alpha_1 < \alpha_2 \cdots < \alpha_n \leq 2n - 1\), we introduce the space \(BDO_\alpha\) of operators used to define the boundary conditions:

\[BDO_\alpha := \{B = (b_{jk})_{0 \leq j, k \leq 2n-1} : b_{jk} \in \text{End} C', b_{ja_j} = \text{Id}, b_{jk} = 0 \text{ if } k \geq \alpha_j + 1\}\]

Given \(\alpha, \beta\), we introduce the space

\[EDO_{2n; \alpha; \beta} := \{A, B, C : \mathcal{A} \in EDO_{2n}, B \in BDO_\alpha, C \in BDO_\beta\}\]

where \(A, B, C\) is the restriction of \(\mathcal{A}\) to the subspace of functions \(u \in C^\infty([0, T]; C')\) satisfying the boundary conditions defined by \(B\) and \(C\). Similarly introduce \(EDO_{2n, \theta; \alpha; \beta} := \{A, B, C \in EDO_{2n; \alpha; \beta} : \mathcal{A} \in EDO_{2n, \theta}\}\). Observe that \(\{A, B, C \in EDO_{2n; \theta; \alpha; \beta} : AB = C\) is 1-1\} is open.

Further, denote by \(EDO_{2n; \alpha; \beta}\) the open subset of \(EDO_{2n; \alpha; \beta} \times S^1\) consisting of pairs \((A, B, C, \theta)\) with \(A, B \in EDO_{2n; \theta; \alpha; \beta}\). As in \([BFK1]\) we have the following

**Proposition 2.1.** (1) \(\text{Det}_\theta(A, B, C)\) is a smooth function on \(EDO_{2n; \alpha; \beta}\) and is locally constant in \(\theta\).

(2) \(\text{Det}_\theta(A, B, C)\) is holomorphic when considered as a function on the open subset of injective operators in \(EDO_{2n; \theta; \alpha; \beta}\).

(3) \(\det(BY(T, \mathcal{A}) - C)\) is holomorphic on \(EDO_{2n} \times BDO_\alpha \times BDO_\beta\).

Observe that a necessary and sufficient condition for \(A, B, C\) to have zero as an eigenvalue is that \(\det(BY(T) - C) = 0\), which in view of Proposition 2.1 (3) implies that the subsets of \(EDO_{2n; \theta; \alpha; \beta}\) and \(EDO_{2n; \alpha; \beta}\) consisting of injective operators are open (as we already noticed) and connected, and therefore, \(EDO_{2n; \alpha; \beta}\) is open and connected as well.

Let \(s : [0, T] \to GL(C')\) be a smooth map. Given \(\mathcal{A} \in EDO_{2n}\) and boundary operators \(\ell_j, m_j (1 \leq j \leq n)\) introduce \(\mathcal{A}_1 := s(x)^{-1} \mathcal{A} s(x)\), \(\ell_{ij} := s(T)^{-1} \ell_j s(x) \mid_{x=T}\), and \(m_{ij} := s(0)^{-1} m_j s(x) \mid_{x=0}\). Denote by \((B_{ijk})\) and \((C_{ijk})\) the matrices introduced in Section 1 corresponding to the boundary operators \((\ell_{ij}, m_{ij})\) for \(1 \leq i \leq n\) and write \(Y_1(x) = Y(x, \mathcal{A}_1)\) for short.

**Proposition 2.2.** \(\det(B_{1}Y_1(T) - C_1) = (\det s(0)s(T)^{-1})^n \det(BY(T) - C)\).
Proof. Let \( L = L(x) \) be a \( 2n \times 2n \) matrix with entries \( L_{k\ell} \) which are the following \( r \times r \) matrices (\( 0 \leq k, \ell \leq 2n - 1 \))

\[
L_{k\ell} := \binom{k}{\ell} d_s^{k-\ell} s(x) \quad \text{if} \ k \geq \ell; \quad L_{k\ell} = 0 \quad \text{if} \ k < \ell.
\]

Thus we obtain

\[
B_1 = \text{diag} (s(T)^{-1}, \ldots, s(T)^{-1}) BL(T)
\]

where \( \text{diag} (s(T)^{-1}, \ldots, s(T)^{-1}) \) is a \( 2n \times 2n \) diagonal matrix whose entries on the diagonal are all equal to the \( r \times r \) matrix \( s(T)^{-1} \). Similarly, one obtains

\[
C_1 = \text{diag} (s(0)^{-1}, \ldots, s(0)^{-1}) CL(0).
\]

Further, by a straightforward computation, \( Y_1 \) is given by

\[
Y_1(x) = L(x)^{-1} Y(x)L(0).
\]

Thus

\[
B_1 Y_1(T) - C_1 = \text{diag} (s(T)^{-1}, \ldots, s(T)^{-1}, s(0)^{-1}, \ldots, s(0)^{-1}) \cdot [BY(T) - C] L(0).
\]

Now observe that \( \det L(0) = (\det s(0))^{2n} \) as \( L(0) \) is lower triangular with diagonal entries all equal to the \( r \times r \) matrix \( s(0) \). This implies that

\[
\det (B_1 Y_1(T) - C_1) = (\det s(0)^{-1})^{2n} \det (BY(T) - C).
\]

Next consider for \( A = ABX \) in \( EDChn-e;\alpha;\beta \) and \( \Phi \in C^\infty ([0, T], GL_r(C)) \) the generalized \( \zeta \)-function \( \zeta_{\Phi, A;\theta} (s) := \text{tr} A^{-s} \). Again this is a function which is holomorphic in \( \text{Re} s > \frac{1}{2n} \) and has a meromorphic extension to the whole complex plane. Moreover \( s = 0 \) is a regular point. Recall that we have introduced \( g_\alpha := \frac{1}{2} (|\alpha| + n + \frac{1}{2}), \) and similarly \( g_\beta \).

Proposition 2.3.

\[
\zeta_{\Phi, A;\theta}(0) = g_\beta \text{tr} \Phi(0) + g_\alpha \text{tr} \Phi(T)
\]

As an immediate consequence we obtain

Corollary 2.4. \( \zeta_{A;\theta}(0) = r(g_\alpha + g_\beta) = r\frac{(|\alpha| + |\beta|)}{2n} - n + 1 \).

Proof (Proposition 2.3). We first prove that there are numbers \( \tilde{g}_\alpha, \tilde{g}_\beta \in \mathbb{C} \) which only depend on \( \alpha \) and \( \beta \) respectively such that (2.1) holds. The actual values of \( \tilde{g}_\alpha, \tilde{g}_\beta \) are computed at the end of section 3 by considering the case \( \Phi(x) \equiv K \) with \( K > 1, \ 2^\alpha = D^n + \lambda, \ \theta = \pi \). In the course of the proof we use a number of results due to Seeley [Se1,2]. For the convenience of the reader we partly keep Seeley's notation. For simplicity, we write \( \zeta(s) = \zeta_{\Phi, A;\theta}(s) \). According to [Se2], the value \( \zeta(0) \) consists of a sum of two terms, \( \zeta(0) = I + II \) where \( I \) represents the contribution to \( \zeta(0) \) of the resolvent of \( \mathcal{A} - \lambda \) and \( II \) represents a correction term due to the boundary conditions. According to [BFK1, p. 8],

\[
I = -\frac{e^{i\theta}}{4\pi n} \sum_{\tau = \pm 1} \int_0^T dx \int_0^\infty dr \text{ tr} \{ \Phi(x)c_{-2n-1}(x, \tau, re^{i\theta}) \}
\]

where \( c_{-2n-1}(x, \tau, \lambda) \) comes from the expansion of the symbol

\[
r(x, \tau, \lambda) = c_{-2n}(x, \tau, \lambda) + c_{-2n-1}(x, \tau, \lambda) + \cdots
\]
of the parametrix for $\mathcal{A} - \lambda = (a_{2n}(x)D^{2n} - \lambda) + \sum_{j=0}^{2n-1} a_j(x)D^j$ and is given by

$$c_{-2n-1}(x, \tau, \lambda) = -\tau^{2n-1}c_{-2n}a_{2n-1}c_{-2n} - i2n\tau^{4n-1}c_{-2n}a_{2n}c_{-2n} \left( \frac{d}{dx} a_{2n} \right) c_{-2n},$$

where $c_{-2n} \equiv c_{-2n}(x, \tau, \lambda) = (a_{2n}(x)\tau^{2n} - \lambda)^{-1}$.

As in [BFK1], Proposition 2.8, in view of the fact that $c_{-2n-1}$ is odd in $\tau$, we conclude $\gamma = 0$. From [Se2], p. 968, it follows that $\gamma \gamma$ is of the form

$$\gamma \gamma = \text{tr} \{ \Delta'_0(0)\Phi(0) + \Delta'(0)\Phi(T) \}$$

where $\Delta'_0(s)$ and $\Delta'(s)$ are smooth functions described below. Let us first consider the scalar case, $r = 1$. In first approximation the kernel $r(x, y, \lambda)$ of $(A_B, \mathcal{A} - \lambda)^{-1}$ is given by

$$\frac{1}{2\pi} \int_{\infty}^{\infty} (a_{2n}(x)\tau^{2n} - \lambda)^{-1}e^{i(x-y)\tau} d\tau + r_0(x, y, \lambda) + r_T(x, y, \lambda)$$

where $r_0(x, y, \lambda)$ and $r_T(x, y, \lambda)$ are correction terms so that in first approximation $r(x, y, \lambda)$ satisfies the boundary conditions at $x = 0$ and $x = T$. Let us explain how to obtain $r_0(x, y, \lambda)$; for $r_T(x, y, \lambda)$ one proceeds in a similar fashion. Consider the boundary value problem

$$(2.2) \quad (aD^{2n} - \lambda)u = 0$$

with the boundary condition

$$(2.3) \quad \lim_{x \to \infty} u(x) = 0; \quad D^\beta_i u(0) = -(a\tau^{2n} - \lambda)^{-1}\tau^\beta_i e^{-iy\tau}$$

where $a = a_{2n}(0)$ and $D = \frac{d}{dx}$. The solution $u(x) = u(x, \tau, y, \lambda)$ of the boundary value problem (2.2)-(2.3) is given by $u(x) = \sum_{\nu=1}^{n} u_\nu e^{ix(-\lambda/a)^{1/2n}w_\nu}$ where $w_\nu$ $(1 \leq \nu \leq n)$ are the $2n$th roots of $-1$ with strictly positive imaginary part and where $(-\lambda/a)^{1/2n} = (-\lambda/a)^{1/2n} = (|\lambda/a|)^{1/2n}e^{i(\theta - \pi - \arg a)/2n}$ with $\lambda = |\lambda|e^{i\theta}$ and $\theta - 2\pi < \arg a < \theta$. The coefficients $u_\nu = u_\nu(\tau, y, \lambda)$ are then determined by (2.3)

$$\sum_{\nu=1}^{n} u_\nu \left( -\frac{\lambda}{a} \right)^{\beta_j/2n} w_\nu^{\beta_j} = -\tau^{\beta_j}(a\tau^{2n} - \lambda)^{-1}e^{-iy\tau}.$$

Thus

$$u_\nu = -\sum_{j=1}^{n} H_{\nu j}(-\lambda/a)^{-\beta_j/2n}\tau^{\beta_j}(a\tau^{2n} - \lambda)^{-1}e^{-iy\tau}$$

with $H_{\nu j}$ defined by

$$(2.4) \quad \sum_{j=1}^{n} H_{\nu j} w_k^{\beta_j} = \delta_{\nu k}.$$

The term $r_0(x, y, \lambda)$ is then given by

$$r_0(x, y, \lambda) = \sum_{\nu=1}^{n} e^{ix(-\lambda/a)^{1/2n}w_\nu} \sum_{j=1}^{n} H_{\nu j} \frac{1}{i} (-\lambda/a)^{-\beta_j/2n} \mathcal{F}$$
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where $\mathcal{J}$ is the sum of residues

$$\mathcal{J} = \sum_{k=1}^{n} \text{Res}_{z=\frac{\lambda}{a}+2\pi i w_k} \{ t^\beta_j (a\tau^{2n} - \lambda)^{-1} e^{-iy\tau} \}$$

of $t^\beta_j (a\tau^{2n} - \lambda)^{-1} e^{-iy\tau}$ in the lower half plane. One obtains

$$\mathcal{J} = \sum_{k=1}^{n} ((-\lambda/a)^{1/2n} w_k)^{\beta_j-(2n-1)} \frac{1}{2na} \exp\{-iy(-\lambda/a)^{1/2n} w_k\}.$$ Summarizing one obtains

$$r_0(x, y, \lambda) = \frac{i}{2na} (-\lambda/a)^{-(2n-1)/2n} \sum_{\nu, j, k} \mathcal{H}_{\nu j} w_k^{\beta_j+1} \exp\{i(-\lambda/a)^{1/2n}(xw_\nu - yw_k)\}.$$ Following Seeley, we now define for $\text{Re} s > 0$

$$\Delta'_0(s) := \int_0^{T/2} ds \frac{1}{2\pi i} \int_{\Gamma_0} d\lambda \lambda^{-s} r_0(x, x, \lambda)$$
where $\Gamma_0$ is the contour that goes from $\infty$ to $0$ along the lower side of ray $\{re^{i\theta} : r > 0\}$, goes around the origin and then returns to $\infty$ along the upper side of the ray $\{re^{i\theta} : r > 0\}$. By a standard computation,

$$\frac{1}{2\pi i} \int_{\Gamma_0} d\lambda \lambda^{-s} r_0(x, x, \lambda) = a^{-s} e^{-i\pi s} \frac{\sin \pi s}{\pi} \Gamma(1 - 2ns) \sum_{\nu, j, k} \mathcal{H}_{\nu j} w_k^{\beta_j+1} ((w_\nu - \bar{w}_k)x)^{-1+2ns}$$

and therefore

$$\Delta'_0(0) = \frac{1}{2n} \sum_{\nu, j, k} \mathcal{H}_{\nu j} \frac{w_k^{\beta_j+1}}{w_\nu - \bar{w}_k}.$$ In the case $r \geq 2$, we first treat the case where all eigenvalues of $a_{2n}(0)$ are different which can be easily reduced to scalar case $r = 1$. By a continuity argument we then conclude that

$$\tilde{g}_\beta = \frac{1}{2n} \sum_{\nu, j, k} \mathcal{H}_{\nu j} (\beta) w_k^{\beta_j+1} (w_\nu - \bar{w}_k)^{-1}$$
where $\mathcal{H}_{\nu j} = \mathcal{H}_{\nu j}(\beta)$ are determined by (2.4). Similarly one obtains

$$\tilde{g}_\alpha = \frac{1}{2n} \sum_{\nu, j, k} \mathcal{H}_{\nu j} (\alpha) w_k^{\alpha_j+1} (w_\nu - \bar{w}_k)^{-1}.$$ 3. Proof of Theorem 1

For the proof of Theorem we need two deformation results. The first one is the analogue of Proposition 3.1 in [BFK1] and proved in a similar way (cf. also [DD] and [Fo1]).
Proposition 3.1. Suppose \( \mathcal{A} = \sum_{k=0}^{2n} a_k(x) D^k \) and \( \mathcal{A}' = \sum_{k=0}^{2n} a'_k(x) D^k \) are in \( EDO_{2n}; \theta \) with \( a_{2n} = a'_{2n} \) and \( a_{2n-1} = a'_{2n-1} \). Then, for \( B \in BDO_\alpha \) and \( C \in BDO_\beta \)

\[
\text{Det}_\theta(A_B, C) \text{det}(B Y(T, \mathcal{A}') - C) = \text{Det}_\theta(A'_B, C) \text{det}(B Y(T, \mathcal{A}) - C).
\]

The second result concerns a deformation of the boundary conditions. Consider boundary operators \((1 \leq j \leq n, d_x = \frac{d}{d_x})\)

\[
\ell_j = \sum_{k=0}^{2n} b_{jk} d_x^k, \quad m_j = \sum_{k=0}^{2n} c_{jk} d_x^k; \quad b_{j\alpha_j} = c_{j\beta_j} = \text{Id}
\]

and

\[
\ell_j' := d_x^{\alpha_j}, \quad m_j' := d_x^{\beta_j}.
\]

Form the matrices \( B, C \) and \( B', C' \) as in Section 1.

Proposition 3.2. Fix \( \mathcal{A} \in EDO_{2n}; \theta \). Then

\[
\text{Det}_\theta(A_{B'}, C') \text{det}(B Y(T) - C) = \text{Det}_\theta(A_B, C) \text{det}(B Y(T) - C').
\]

Proof. Without loss of generality we may assume that both \( A_{B_C} \) and \( A_{B'_C} \) are injective. Note that \( \{A_{B_C} : A_{B_C} \text{ is 1-1, } B \in BDO_\alpha, C \in BDO_\beta\} \) is arcwise connected in \( BDO_\alpha \times BDO_\beta \). Define, for \( 0 < t < 1 \),

\[
\ell_{ij} = d_x^{\alpha_{ij}} + t \sum_{k=0}^{2n} b_{jk} d_x^k, \quad c_{ij} = d_x^{\beta_{ij}} + t \sum_{k=0}^{2n} c_{jk} d_x^k
\]

such that, with \( B_t \) and \( C_t \) the corresponding matrices in \( BDO_\alpha \) and \( BDO_\beta \),

\[
\begin{align*}
& (1) \quad A_{B_t, C_t} \text{ is 1-1 for } 0 \leq t \leq 1; \\
& (2) \quad (B_0, C_0) = (B', C'), \quad (B_1, C_1) = (B, C).
\end{align*}
\]

Introduce

\[
w(t) := \frac{d}{dt} \text{Det}_\theta(A_{B_t, C_t}), \quad \delta(t) := \frac{d}{dt} \text{det}(B_t Y(T) - C_t) / \text{det}(B Y(T) - C_t).
\]

The claimed result follows once we show that \( w(t) = \delta(t) \) \((0 \leq t \leq 1)\). Let us first consider \( \delta(t) \). Denote by \( P_t \) the Poisson operator corresponding to the boundary value problem defined by \( (B_t, C_t) \). Then \( P_t \) is given by \( P_t = Y(x)(B_t Y(T) - C_t)^{-1} \) and

\[
\begin{align*}
\delta(t) &= \text{tr} \{(B_t Y(T) - C_t)(B_t Y(T) - C_t)^{-1}\} \\
&= \text{tr}(\ell_{ij} u(T), \hat{m}_{ij} u(0))_{1 \leq j \leq n}
\end{align*}
\]

when \( \ell_{ij}, \hat{m}_{ij} \) is the operator associating to a section \( u \) the boundary values \( (\ell_{ij} u(T), \hat{m}_{ij} u(0))_{1 \leq j \leq n} \).

Next we consider \( w(t) \); with the notation \( A_t = A_{B_t, C_t} \),

\[
w(t) = F.p.s=0 \text{tr}(A_t^{-1} A_{t'}^{-1-s})
\]

where \( F.p.s=0 \) denotes the finite part at \( s = 0 \). In order to evaluate \( A_t^{-1} A_{t'} = -(A_t^{-1})' A_t \), consider for a fixed section \( u : [0, T] \to C' \) the section \( v_t := A_t^{-1} u \), i.e. \( v_t \) satisfies

\[
\mathcal{A} v_t = u, \quad B_t v_t(T) = 0, \quad C_t v_t(0) = 0.
\]
Taking derivatives with respect to $t$ we obtain

$$\mathcal{A} v'_i = 0, \quad \ell_{ij} v'_i(T) = -\ell_{ij} v'_i(T), \quad m_{ij} v'_i(0) = -m_{ij} v'_i(0) \quad (1 \leq j \leq n).$$

Thus $v'_i = -P_i(\ell_{ij} v'_i(T), m_{ij} v'_i(0))_{1 \leq j \leq n}$ where $P_i$ again denotes the Poisson operator. Thus we have proved that $(A^{-1})' = -P_i(\ell_{ij}, m_{ij})_{1 \leq j \leq n} A^{-1}$. Note that $(A^{-1})' A_t = -P_i(\ell_{ij}, m_{ij})_{1 \leq j \leq n}$ is a singular Green's operator of order $\leq -2$ and then of trace class. Thus

$$w(t)' = \text{tr} P_i(\ell_{k_j}', m_{ij}')_{1 \leq j \leq n}. \quad \square$$

**Proof of Theorem.** We have to prove that

$$f_\theta(A_B, c) := \text{Det}_\theta(A_B, c) - K_\theta \exp \left\{ \frac{i}{2} \int_0^T \text{tr} (a_{2n}(x)^{-1} a_{2n-1}(x)) dx \right\} \cdot \det(BY(T) - C)$$

vanishes identically on $\{ A_B, c \in EDO_{2n}; \theta \in \mathbb{R}; A_B, c \text{ is 1-1} \}$. First observe that it suffices to consider the case $\theta = \pi$: For $\mathcal{A}$ in $EDO_{2n}; \theta \in \mathbb{R}$ we have $\log \text{Det}_\pi(e^{i(\pi - \theta)} A_B, c) = \log \text{Det}_\theta(A_B, c) + \zeta_A, \theta(0) \log e^{i(\pi - \theta)}$ and $\log K_\theta(e^{i(\pi - \theta)} \mathcal{A}) = \log K_\pi(t) + r(g_\beta + g_\alpha) i(\pi - \theta)$; thus Corollary 2.4 allows to conclude the result as soon as we check it for $\theta = \pi$.

To make writing easier, let $f = f_\pi$, $K = K_\pi$, $\theta = \pi$.

**Deformation 1.** Consider the factorization $\mathcal{A} = a_{2n}(D^{2n} + \mathcal{H})$ where $\mathcal{H}$ is a differential operator with ord $\mathcal{H} \leq 2n - 1$. Consider the 1-parameter family

$$s_{\epsilon t} := a_{2\epsilon t}(D^{2\epsilon t} + \mathcal{H}), \quad A_t := A_{t, B, c}$$

when $\alpha_t(x) = ta_{2n}(x) + (1-t)$. Clearly $\theta = \pi$ is a principal angle for $\alpha_t$ and $A_t$ is 1-1 for $0 \leq t < 1$.

Moreover $A_t' = (a_{2n}(x) - 1)(ta_{2n}(x) + (1-t))^{-1} A_t$. Thus, with $w(t) = \log \text{Det}_\pi A_t$ and Proposition 2.3

$$w(t)' = F.p.s \cdot \text{tr} [(a_{2n}(x) - 1)(ta_{2n}(x) + (1-t))^{-1} A(t)^{-t}]$$

$$= g_\beta \text{tr} [(a_{2n}(0) - 1)(ta_{2n}(0) + (1-t))^{-1}]$$

$$+ g_\alpha \text{tr} [(a_{2n}(T) - 1)(ta_{2n}(T) + (1-t))^{-1}]$$

$$= \frac{d}{dt} \{ g_\beta \log \det(ta_{2n}(0) + (1-t))$$

$$+ g_\alpha \log \det(ta_{2n}(T) + (1-t)) \}. $$

Thus

$$\log \text{Det}_\pi A_1 - \log \text{Det}_\pi A_0 = \int_0^1 w(t)' dt = g_\beta \log \det(a_{2n}(0)) + g_\alpha \log \det(a_{2n}(T)).$$

Hence we may and will assume that $a_{2n}(x) \equiv \text{Id} \cdot$ 

**Deformation 2.** Define $s \in C^\infty([0, T]; \text{End} \mathbb{C})$ by

$$\frac{d}{dx} s(x) = \frac{i}{2n} a_{2n-1}(x) s(x) \quad (0 \leq x \leq T); \quad s(0) = \text{Id}.$$
Observe that \( \det(s(x)) = \exp\left\{ \frac{1}{2n} \int_0^x \text{tr} (a_{2n-1}(y)) dy \right\} \neq 0 \) for \( 0 \leq x \leq T \) and therefore \( s(x) \in GL_r(\mathbb{C}) \). Now consider \( \mathcal{A}_1 := s(x)^{-1} \mathcal{A} s(x) \) and boundary conditions defined by \( B_1, C_1 \) (cf. Proposition 2.2). Then \( \text{Det}_\pi(A_1) = \text{Det}_\pi(A) \) as the spectrum of \( A \) and the operator \( A_1 \), defined by \( \mathcal{A}_1 \) and boundary conditions \( (B_1, C_1) \) do coincide. By Proposition 2.2,
\[
\det(B_1 Y_1(T) - C_1) = (\det s(T))^{-n} \det(B Y(T) - C).
\]

As we have noted above, \( \det s(T) = \exp\left\{ \frac{1}{2n} \int_0^T \text{tr} (a_{2n-1}(y)) dy \right\} \). Finally note that \( \mathcal{A}_1 \) is of the form
\[
\mathcal{A}_1 = D^{2n} + \sum_{k=0}^{2n-2} a_{1k}(x) D^k
\]
and then we may and will assume that for \( \mathcal{A} \), \( a_{2n}(x) \equiv \text{Id} \) and \( a_{2n-1}(x) \equiv 0 \).

**Deformation 3.** Applying Proposition 3.1 and Proposition 3.2 we conclude that it remains to prove that \( f(A_B, C) = 0 \) for \( \mathcal{A} = D^{2n} + \lambda \) and \( B, C \) given by
\[
\ell_j = d_{2j}^\alpha, \quad m_j = d_{2j}^\beta \quad (1 \leq j \leq n)
\]
where \( \lambda \) is chosen positive and sufficiently large so that \( A_{B,C} \) is \( 1-1 \). This is verified by an explicit computation. To make writing easier we restrict ourselves to that case \( r = 1 \). However, to obtain the explicit formulas for \( g_\alpha \) and \( g_\beta \) we consider \( \mathcal{A} = \rho D^{2n} + \lambda \) with \( \rho > 1 \). Denote by \( Y(x, \lambda) \) the fundamental matrix for \( \rho D^{2n} + \lambda \). For \( \lambda > 0 \), let \( \mu = (\frac{\lambda}{\rho})^{1/2n} \). Then, with \( w_k := \exp(i2k\pi n^{-1} \pi) \), \( Y(x, \lambda) \) is equal to
\[
\begin{pmatrix}
e^{\mu w_1 x} & \cdots & e^{\mu w_n x} \\
\mu w_1 e^{\mu w_1 x} & \cdots & \mu w_n e^{\mu w_2 x} \\
\vdots & \ddots & \vdots \\
(\mu w_1)^{2n-1} e^{\mu w_1 x} & \cdots & (\mu w_2)^{2n-1} e^{\mu w_2 x}
\end{pmatrix}
\begin{pmatrix}
1 & \cdots & 1 \\
1 & \cdots & \mu w_1 \\
\vdots & \ddots & \vdots \\
(\mu w_1)^{n-1} & \cdots & (\mu w_2)^{2n-1}
\end{pmatrix}^{-1}.
\]

Further define \( B = (B_{jk}), C = (C_{jk}) \) by
\[
B_{jk} = \begin{cases} 
1 & \text{if } 1 \leq j \leq n \text{ and } k = \alpha_j, \\
0 & \text{otherwise};
\end{cases}
C_{jk} = \begin{cases} 
1 & \text{if } n+1 \leq j \leq 2n \text{ and } k = \beta_{j-n}, \\
0 & \text{otherwise}.
\end{cases}
\]

We have to show that
\[
(3.4) \quad \text{Det}_\pi((\rho D^{2n} + \lambda)_{B,C}) = (-1)^{|\beta|}(2n)^n (h_\alpha h_\beta)^{-1} \rho^{g_\alpha + g_\beta} \det( B Y(T, \lambda) - C).
\]

For that purpose we introduce
\[
w(\lambda) := \log \text{Det}_\pi((\rho D^{2n} + \lambda)_{B,C}),
\]
\[
\delta(\lambda) := \log \det( B Y(T; \lambda) - C).
\]

As \( n \geq 1 \), we know from Proposition 3.1 that \( \frac{d}{d\lambda} w(\lambda) = \frac{d}{d\lambda} \delta(\lambda) \). Therefore it suffices to consider the asymptotics of \( w(\lambda) \) and \( \delta(\lambda) \) as \( \lambda \to +\infty \).

First recall from [Fr] (cf. also [Vo]) that \( w(\lambda) \) admits an asymptotic expansion of the form \( \sum_{k=0}^\infty p_k \lambda^{-k/n} + \sum_{j=0}^\infty q_j \lambda^{-j} \log \lambda \) with the property that \( p_0 = 0 \). To find the asymptotics of \( \delta(\lambda) \) as \( \lambda \to +\infty \), write \( Y(T, \lambda) \) in the form
\[
Y(T; \lambda) = LWE(LW)^{-1}
\]
where \( L = \text{diag}(1, \mu, \mu^2, \ldots, \mu^{2n-1}) \), \( E := \text{diag}(e^{\mu w_1 T}, \ldots, e^{\mu w_{2n} T}) \) and

\[
W = \begin{pmatrix}
1 & \cdots & 1 \\
w_1 & \cdots & w_2 \\
\vdots & \vdots & \vdots \\
w_{2n-1} & \cdots & w_{2n}
\end{pmatrix}.
\]

Thus \( \delta(\lambda) = \log(\det W^{-1} L^{-1}) + \log \det(BLCE - CLW) \). Observe that the \((j, k)\)th coefficient of the matrix \( BLCE - CLW \) is of the form \( e^{w_j T} f_{jk}(\mu) + g_{jk}(\mu) \) where \( f_{jk}(\mu) \) and \( g_{jk}(\mu) \) are rational functions of \( \mu \). We conclude that, with \( \Omega = \sum_{j=1}^n w_j = \sum_{j=1}^n \text{Re} w_j \),

\[
\log \det(BLCE - CLW) = \mu \Omega T + \log \det(BLW \begin{pmatrix} \text{Id}_n & 0 \\ 0 & 0 \end{pmatrix} - CLW \begin{pmatrix} 0 & 0 \\ 0 & \text{Id}_n \end{pmatrix}) + e(\lambda)
\]

where \( \lim_{\lambda \to \infty} e(\lambda) = 0 \). The matrix \( BLW \begin{pmatrix} \text{Id}_n & 0 \\ 0 & 0 \end{pmatrix} - CLW \begin{pmatrix} 0 & 0 \\ 0 & \text{Id}_n \end{pmatrix} \) is of the form

\[
\begin{pmatrix}
F^{(1)} \\
F^{(2)}
\end{pmatrix}
\]

where \( F^{(i)} \) are \( n \times n \) matrices given by \((1 \leq j, k \leq n)\)

\[
F^{(1)}_{jk} := \mu^{\alpha_j} w_{kj}, \quad F^{(2)}_{jk} := -\mu^{\beta_j} w_{kj} = (-1)^{\beta_j+1} \mu^{\beta_j} w_{kj}
\]

where we used that \( w_{n+k} = -w_k \). Therefore, with \( |\alpha| = \sum_i \alpha_i \), \( |\beta| = \sum_i \beta_i \)

\[
\log \det(BLW \begin{pmatrix} \text{Id}_n & 0 \\ 0 & 0 \end{pmatrix} - CLW \begin{pmatrix} 0 & 0 \\ 0 & \text{Id}_n \end{pmatrix}) = \mu |\alpha| \log \det(w_{k}^{\alpha_j}) \mu |\beta| (-1)^{|\beta|+n} \det(w_{k}^{\beta_j}).
\]

In view of the fact that \( \det L^{-1} = \prod_{j=0}^{2n-1} (\frac{1}{\beta})^{-j/2n} = \rho^{-\frac{2n-1}{2}} \), this implies that the 0th order coefficient of the asymptotic expansion of \( \delta(\lambda) \) for \( \lambda \to \infty \) is of the form

\[
\delta_{+\infty} := \det L^{-1}_{|\alpha|=1} + \log(\det(W^{-1}) \det(w_{k}^{\alpha_j})(-1)^{|\beta|+n} \det(w_{k}^{\beta_j}) \rho^{-(|\alpha|+|\beta|)/2n})
\]

\[
= \log \rho^{-\frac{2n-1}{2}} - \log \rho^{\left(\frac{|\alpha|+|\beta|}{2n}\right)/2n} + \log((-1)^{|\beta|+n} \det(W^{-1})h_{\alpha}h_{\beta}).
\]

where \( h_{\alpha} = \det(w_{k}^{\alpha_j}) \), \( h_{\beta} \equiv \det(w_{k}^{\beta_j}) \).

By a straightforward computation we have \( \det W = (-1)^n (2n)^n \) and therefore

(3.5) \( w(\lambda) = \delta(\lambda) - \delta_{+\infty} = \delta(\lambda) + \log((-1)^{|\beta|}(2n)^n h_{\alpha}^{-1} h_{\beta}^{-1} \rho^{\left(\frac{|\alpha|}{2n}-\frac{n}{4}+\frac{|\beta|}{2n}-\frac{1}{4}\right)}}.
\]

The claim (3.4) then follows from the following.

**Lemma 3.3.** \( \tilde{g}_\alpha = \frac{1}{2} \left( \frac{|\alpha|}{n} - n + \frac{1}{2} \right) \).

**Proof.** In view of Proposition 2.3 we obtain from (3.5) in the case \( \alpha = \beta \)

\[
2\tilde{g}_\alpha = 2 \left( \frac{|\alpha|}{2n} - \frac{n}{2} + \frac{1}{4} \right) \quad \text{or} \quad \tilde{g}_\alpha = \frac{1}{2} \left( \frac{|\alpha|}{n} - n + \frac{1}{2} \right). \quad \square
\]
References


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