The coupled Seiberg-Witten equations, vortices, and moduli spaces of stable pairs

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Abstract: We introduce coupled Seiberg-Witten equations, and we prove, using a generalized vortex equation, that, for Kaehler surfaces, the moduli space of solutions of these equations can be identified with a moduli space of holomorphic stable pairs. In the rank 1 case, one recovers Witten’s result identifying the space of irreducible monopoles with a moduli space of divisors. As application, we give a short proof of the fact that a rational surface cannot be diffeomorphic to a minimal surface of general type.

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The Coupled Seiberg-Witten Equations, Vortices, and Moduli Spaces of Stable Pairs

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0 Introduction

Recently, Seiberg and Witten [W] introduced new invariants of 4-manifolds, which are defined by counting solutions of a certain non-linear differential equation.

The new invariants are expected to be equivalent to Donaldson’s polynomial-invariants—at least for manifolds of simple type [KM 1]—and they have already found important applications, like e.g. in the proof of the Thom conjecture by Kronheimer and Mrowka [KM 2].

Nevertheless, the equations themselves remain somewhat mysterious, especially from a mathematical point of view.

The present paper contains our attempt to understand and to generalize the Seiberg-Witten equations by coupling them to connections in unitary vector bundles, and to relate their solutions to more familiar objects, namely stable pairs.

Fix a Spin$^c$-structure on a Riemannian 4-manifold $X$, and denote by $\Sigma^\pm$ the associated spinor bundles. The equations which we will study are:

\[
\begin{align*}
\mathcal{D}_{A,b} \Psi &= 0 \\
\Gamma(F^+_A) &= 0
\end{align*}
\]

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This is a system of equations for a pair \((A, \Psi)\) consisting of a unitary connection in a unitary bundle \(E\) over \(X\), and a positive spinor \(\Psi \in A^0(\Sigma^+ \otimes E)\). The symbol \(b\) denotes a connection in the determinant line bundle of the spinor bundles \(\Sigma^\pm\) and \(P_{A_b} : \Sigma^+ \otimes E \rightarrow \Sigma^- \otimes E\) is the Dirac operator obtained by coupling the connection in \(\Sigma^+\) defined by \(b\) (and by the Levi-Civita connection in the tangent bundle) with the variable connection \(A\) in \(E\).

These equations specialize to the original Seiberg-Witten equations if \(E\) is a line bundle. We show that the coupled equations can be interpreted as a differential version of the generalized vortex equations [JT].

Vortex equations over Kähler manifolds have been investigated by Bradlow [B1], [B2] and by Garcia-Prada [G1], [G2]: Given a pair \((\mathcal{E}, \varphi)\) consisting of a holomorphic vector bundle with a section, the vortex equations ask for a Hermitian metric \(h\) in \(\mathcal{E}\) with prescribed mean curvature: more precisely, the equations—which depend on a real parameter \(\tau\)—are

\[
i \Lambda F_h = \frac{1}{2} (\tau \text{id}_\mathcal{E} - \varphi \otimes \varphi^*).
\]

A solution exists if and only if the pair \((\mathcal{E}, \varphi)\) satisfies a certain stability condition (\(\tau\)-stability), and the moduli space of vortices can be identified with the moduli space of \(\tau\)-stable pairs. A GIT construction of the latter space has been given by Thaddeus [T] and Bertram [B] if the base manifold is a projective curve, and by Huybrechts and Lehn [HL1], [HL2] in the case of a projective variety. Other constructions have been given by Bradlow and Daskalopoulos [BD1], [BD2] in the case of a Riemann surface, and by Garcia-Prada for compact Kähler manifolds [G2]. In this connection also [BD2] is relevant. In this note we prove the following result:

**Theorem 0.1** Let \((X, g)\) be a Kähler surface of total scalar curvature \(\sigma_g\), and let \(\Sigma\) be the canonical Spin\(^c\)-structure with associated Chern connection \(c\). Fix a unitary vector bundle \(E\) of rank \(r\) over \(X\), and define \(\mu_g(\Sigma^+ \otimes E) := \frac{\deg_g(E)}{r} + \sigma_g\).

Then for \(\mu_g < 0\), the space of solutions of the coupled Seiberg-Witten equation is isomorphic to the moduli space of stable pairs of topological type \(E\), with parameter \(\sigma_g\).

If the constant \(\mu_g(\Sigma^+ \otimes E)\) is positive, one simply replaces the bundle \(E\) with \(E^\vee \otimes K_X\), where \(K_X\) denotes the canonical line bundle of \(X\) (cf. Lemma 3.1).
Note that the above theorem gives a complex geometric interpretation of the moduli space of solutions of the coupled Seiberg-Witten equation associated to all Spin$^c$-structures on $X$: The change of the Spin$^c$-structure is equivalent to tensoring $E$ with a line bundle.

Notice also that in the special case $r = 1$ one recovers Witten’s result identifying the space of irreducible monopoles on a Kähler surface with the set of all divisors associated to line bundles of a fixed topological type; the stability condition which he mentions is the rank-1 version of the stable pair-condition.

Having established this correspondence, we describe some of the basic properties of the moduli spaces, and give a first application: We show that minimal surfaces of general type cannot be diffeomorphic to rational ones. This provides a short proof of one of the essential steps in Friedman and Qin’s proof of the Van de Ven conjecture [FQ]. More detailed investigations and applications will appear in a later paper.

We like to thank A. Van de Ven for very helpful questions and remarks.

1 Spin$^c$-structures and almost canonical classes

The complex spinor group is defined as Spin$^c := \text{Spin} \times_{\mathbb{Z}_2} S^1$, and there are two non-split exact sequences

$$
1 \rightarrow S^1 \rightarrow \text{Spin}^c \rightarrow \text{SO} \rightarrow 1
$$

$$
1 \rightarrow \text{Spin} \rightarrow \text{Spin}^c \rightarrow S^1 \rightarrow 1
$$

In dimension 4, Spin$^c(4)$ can be identified with the subgroup of $U(2) \times U(2)$ consisting of pairs of unitary matrices with the same determinant, and one has two commutative diagrams:

$$
\begin{array}{cccc}
1 & 1 \\
\downarrow & \downarrow \\
1 \rightarrow \mathbb{Z}_2 & \rightarrow \text{Spin}(4) & \rightarrow \text{SO}(4) & \rightarrow 1 \\
\downarrow & \downarrow \\
1 \rightarrow S^1 & \rightarrow \text{Spin}^c(4) & \rightarrow \text{SO}(4) & \rightarrow 1 \\
\downarrow (\cdot)^2 & \text{det} \downarrow \Delta & \uparrow \\
S^1 = & S^1 & \text{U}(2) \\
\downarrow & \downarrow & \downarrow \text{det} \\
1 & 1 & 1
\end{array}
$$
where $\Delta : U(2) \rightarrow \text{Spin}^c(4) \subset U(2) \times U(2)$ acts by $a \mapsto \left( \begin{pmatrix} \text{id} & 0 \\ 0 & \det a \end{pmatrix}, a \right)$.

and

\[
\begin{array}{cccc}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
S^1 & \text{Spin}^c(4) & \text{SO}(4) \\
\downarrow & \downarrow & \downarrow \\
S^1 \times S^1 & U(2) \times U(2) & \text{ad} \text{SO}(3) \times \text{SO}(3)
\end{array}
\]

where $\lambda^{\pm} : \text{SO}(4) \rightarrow \text{SO}(3)$ are induced by the two projections of $\text{Spin}(4) = \text{SU}(2)^+ \times \text{SU}(2)^-$ [HH]. $\lambda^{\pm}$ can be also be seen as the representations of $\text{SO}(4)$ in $\Lambda^2_+ (\mathbb{R}^4) \simeq \mathbb{R}^3$ induced by the canonical representation in $\mathbb{R}^4$.

Let $X$ be a closed, oriented 4-manifold. Given any principal $\text{SO}(4)$-bundle over $X$, we denote by $P^{\pm}$ the induced principal $\text{SO}(3)$-bundles. If $\hat{P}$ is a $\text{Spin}^c(4)$-bundle, we let $\Sigma^{\pm}$ be the associated $U(2)$-vector bundles, and we set (via the vertical determinant-map in (1)) $\det(\hat{P}) = L$, so that $\det(\Sigma^{\pm}) = L$.

Lemma 1.1 Let $P$ be a principal $\text{SO}(4)$-bundle over $X$ with characteristic classes $w_2(P) \in H^2(X, \mathbb{Z}_2)$, and $p_1(P), e(P) \in H^4(X, \mathbb{Z})$. Then

i) $P$ lifts to a principal $\text{Spin}^c(4)$-bundle $\hat{P}$ if $w_2(P)$ lifts to an integral cohomology class.

ii) Given a class $L \in H^2(X, \mathbb{Z})$ with $w_2(P) \equiv \bar{L} (\text{mod } 2)$, the $\text{Spin}^c(4)$-lifts $\hat{P}$ of $P$ with $\det \hat{P} = L$ are in 1-1 correspondence with the 2-torsion elements in $H^2(X, \mathbb{Z})$.

iii) Let $\hat{P}$ be a $\text{Spin}^c(4)$-principal bundle with $P \simeq \hat{P}/S^1$, and let $L = \det \hat{P}$. Then the Chern classes of $\Sigma^{\pm}$ are:

\[
\begin{align*}
\text{c}_1(\Sigma^{\pm}) &= L \\
\text{c}_2(\Sigma^{\pm}) &= \frac{1}{4} (L^2 - p_1(P) \mp 2e(P))
\end{align*}
\]

Proof: [HH] and the diagrams above.\[\blacksquare\]
Consider now a Riemannian metric $g$ on $X$, and let $P$ be the associated principal SO(4)-bundle. In this case the real vector bundles associated to $P^\pm$ via the standard representations are the bundles $\Lambda^2_\pm$ of (anti-) self-dual 2-forms on $X$.

The integral characteristic classes of $P$ are given by $p_1(P) = 3\sigma$ and $\varepsilon(P) = e$, where $\sigma$ and $e$ denote the signature and the Euler number of the oriented manifold $X$. Furthermore, $w_2(P)$ always lifts to an integral class, the lifts are precisely the characteristic elements in $H^2(X, \mathbb{Z})$, i.e. the classes $L$ with $x^2 \equiv x \cdot L$ for every $x \in H^2(X, \mathbb{Z})$ [HH].

Let $T_X$ be the tangent bundle of $X$, and denote by $\Lambda^p$ the bundle of $p$-forms on $X$. The choice of a Spin$^c$(4)-lift $\hat{P}$ of $P$ with associated U(2)-vector bundles $\Sigma^\pm$ defines a vector bundle isomorphism $\gamma : \Lambda^1 \otimes \mathbb{C} \rightarrow \text{Hom}_\mathbb{C}(\Sigma^+, \Sigma^-)$. There is also a $\mathbb{C}$-linear isomorphism $(\cdot)^\# : \text{Hom}_\mathbb{C}(\Sigma^+, \Sigma^-) \rightarrow \text{Hom}_\mathbb{C}(\Sigma^-, \Sigma^+)$ which satisfies the identity:

$$\gamma(u)^\# \gamma(v) + \gamma(v)^\# \gamma(u) = 2g^\mathbb{C}(u, v)\text{id}_{\Sigma^+},$$

and $\gamma(u)^\# = \gamma(u)^* = g(u, u)\gamma(u)^{-1}$ for real non-vanishing cotangent vectors $u$.

It is convenient to extend the homomorphisms $\gamma(u)$ to endomorphisms of the direct sum $\Sigma := \Sigma^+ \oplus \Sigma^-$. Putting $\gamma(u)|_{\Sigma^-} := -\gamma(u)^\#$, we obtain a vector-bundle homomorphism $\gamma : \Lambda^1 \otimes \mathbb{C} \rightarrow \text{End}_0(\Sigma)$, which maps the bundle $\Lambda^1$ of real 1-forms into the bundle of trace-free skew-Hermitian endomorphisms of $\Sigma$. With this convention, we get:

$$\gamma(u) \circ \gamma(v) + \gamma(v) \circ \gamma(u) = -2g^\mathbb{C}(u, v)\text{id}_\Sigma.$$

Consider the induced homomorphism

$$\Gamma : \Lambda^2 \otimes \mathbb{C} \rightarrow \text{End}_0(\Sigma)$$

defined on decomposable elements by

$$\Gamma(u \wedge v) := \frac{1}{2}[\gamma(u), \gamma(v)].$$

The restriction $\Gamma|_{\Lambda^2}$ identifies the bundle $\Lambda^2$ with the bundle $\text{ad}_0(\hat{P}) \simeq \text{ad}(P)$ of skew-symmetric endomorphisms of the tangent bundle of $X$. 

5
\( \Lambda^2 \) splits as an orthogonal sum \( \Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_- \) and \( \Gamma \) maps the bundle \( \Lambda^2_+ \otimes \mathbb{C} \) (respectively \( \Lambda^2_- \)) isomorphically onto the bundle \( \text{End}_0(\Sigma^\pm) \subset \text{End}(\Sigma) \) (\( su(\Sigma^\pm) \subset su(\Sigma) \)) of trace-free (trace free skew-Hermitian) endomorphisms of \( \Sigma^\pm \).

We give now an explicit description of the two spinor bundles \( \Sigma^\pm \) and of the map \( \Gamma \) in the case of a Spin\(^c\)(4)-structure coming from an almost Hermitian structure.

**Definition 1.2** A characteristic element \( K \in H^2(X, \mathbb{Z}) \) is an almost canonical class if \( K^2 = 3\sigma + 2e \).

Such classes exist on \( X \) if and only if \( X \) admits an almost complex structure. More precisely:

**Proposition 1.3** (Wu) \( K \in H^2(X, \mathbb{Z}) \) is an almost canonical class if and only if there exists an almost complex structure \( J \) on \( X \) which is compatible with the orientation, such that \( K = c_1(\Lambda^0_J) \).

**Proof:** [HH]

Here we denote, as usual, by \( \Lambda^p_q^J \) the bundle of \( (p, q) \)-forms defined by the almost complex structure \( J \).

Notice that any almost complex structure \( J \) on \( X \) can be deformed into a \( g \)-orthogonal one, and that \( J \) is \( g \)-orthogonal iff \( g \) is \( J \)-Hermitian. The choice of a \( g \)-orthogonal almost complex structure \( J \) on \( X \) corresponds to a reduction of the \( \text{SO}(4) \)-bundle \( P \) of \( X \) to a \( U(2) \)-bundle via the inclusion \( U(2) \subset \text{SO}(4) \); since the inclusion factors through the embedding \( \Delta : U(2) \longrightarrow \text{Spin}^c(4) \) (see diagram (1)), this reduction defines a unique Spin\(^c\)(4)-bundle \( \hat{P}_J \) over \( X \). By construction we have \( \hat{P}_J/S^1 \simeq P \), and \( \text{det} \hat{P}_J = -K \).

**Proposition 1.4** Let \( J \) be a \( g \)-orthogonal almost complex structure on \( X \), compatible with the orientation.

i) The spinor bundles \( \Sigma^\pm_J \) of \( \hat{P}_J \) are:

\[
\Sigma^+_J \simeq \Lambda^0_0^+ \oplus \Lambda^0_2^+, \quad \Sigma^-_J \simeq \Lambda^0_0^-.
\]

ii) The map \( \Gamma : \Lambda^2_+ \otimes \mathbb{C} \longrightarrow \text{End}_0(\Sigma^+_J) \) is given by

\[
\Lambda^2_+ \oplus \Lambda^2_- \oplus \Lambda^0_0 \omega_g \ni (\lambda^{20}, \lambda^{02}, \omega_g) \xrightarrow{\Gamma} 2 \left[ \begin{array}{cc} -i & -*(\lambda^{20} \wedge \cdot) \\ \lambda^{02} \wedge \cdot & \lambda^{02} \wedge \cdot \end{array} \right] \in \text{End}_0(\Lambda^0_0 \oplus \Lambda^0_2).
\]
Proof: i) \( c_1(\Sigma^+) = c_1(\Sigma^-) = -K,\ c_2(\Sigma^+) = \frac{1}{4}[K^2 - 3\sigma - 2e],\ c_2(\Sigma^-) = \frac{1}{4}[K^2 - 3\sigma + 2e] = c_2(\Sigma^+) + e,\) and \( U(2)\)-bundles on a 4-manifold are classified by their Chern classes.

ii) With respect to a suitable choice of the isomorphisms i), the Clifford map \( \gamma \) acts by

\[
\gamma(\varphi + \alpha) = \sqrt{2} \left( \varphi u^{01} - i\Lambda_g u^{10} \wedge \alpha \right),
\]

\[
\gamma(u)^\#(\theta) = \sqrt{2} \left( i\Lambda_g (u^{10} \wedge \theta) - u^{01} \wedge \theta \right),
\]

where \( \Lambda_g : \Lambda^{p,q}_J \rightarrow \Lambda^{p-1,q-1}_J \) is the adjoint of the map \( \cdot \wedge \omega_g \) [H1].

\[\blacksquare\]

2 The coupled Seiberg-Witten equations

Let \( P \) be the principal SO(4)-bundle associated with the tangent bundle of the oriented, closed Riemannian 4-manifold \( (X, g) \), and fix a Spin\(^c\)(4) structure \( \hat{P} \) over \( P \) with \( L = \det(\hat{P}) \). The choice of a Spin\(^c\)(4)-connection in \( \hat{P} \) projecting onto the Levi-Civita connection in \( P \) is equivalent to the choice of a connection \( b \) in the unitary line bundle \( L \) [H1]. We denote by \( B(b) \) the Spin\(^c\)(4)-connection in \( \hat{P} \) corresponding to \( b \), and also the induced connection in the vector bundle \( \Sigma = \Sigma^+ \oplus \Sigma^- \). The curvature \( F_{B(b)} \) of the connection \( B(b) \) in \( \Sigma \) has the form

\[
F_{B(b)} = \frac{1}{2} F_b id_\Sigma + F_g = \begin{bmatrix}
\frac{1}{2} F_b id_{\Sigma^+} + F_g^+

0

\frac{1}{2} F_b id_{\Sigma^-} + F_g^-
\end{bmatrix},
\]

where \( F_g \) and \( F_g^\pm \) denote the Riemannian curvature operator, and its components with respect to the splitting \( \text{ad}(\hat{P}) = \Lambda^2_+ \oplus \Lambda^2_- \).

Let now \( E \) be an arbitrary Hermitian bundle of rank \( r \) over \( X \), and \( A \) a connection in \( E \). We denote by \( A_b \) the induced connection in the tensor product \( \Sigma \otimes E \), and by \( \mathcal{D}_{A,b} : A^0(\Sigma \otimes E) \rightarrow A^0(\Sigma \otimes E) \) the associated Dirac operator. \( \mathcal{D}_{A,b} \) is defined as the composition:

\[
A^0(\Sigma \otimes E) \xrightarrow{\nabla_{A_b}} A^1(\Sigma \otimes E) \xrightarrow{m} A^0(\Sigma \otimes E)
\]

where \( m \) is the Clifford multiplication \( m(u, \sigma \otimes e) := \gamma(u)(\sigma) \otimes e \). \( \mathcal{D}_{A,b} \) is an elliptic, self-adjoint operator and its square \( \mathcal{D}_{A,b}^2 \) is related to the usual Laplacian \( \nabla_{A_b}^* \nabla_{A_b} \) by the Weitzenböck formula

\[
\mathcal{D}_{A,b}^2 = \nabla_{A_b}^* \nabla_{A_b} + \Gamma(F_A).
\]
Here $\Gamma(F_{A,b}) \in A^0(\text{End}(\Sigma \otimes E))$ is the Hermitian endomorphism defined as the composition

$$A^0(\Sigma \otimes E) \xrightarrow{F_{A,b}} A^0(\Lambda^2 \otimes \Sigma \otimes E) \xrightarrow{\mathcal{L} A} A^0(\Sigma \otimes E).$$

We set $F_{A,b} := F_A + \frac{1}{2} F_b \text{id}_E \in A^0(\Lambda^2 \otimes \text{End}(E))$.

**Proposition 2.1** Let $s$ be the scalar curvature of the Riemannian 4-manifold $(X,g)$. Fix a Spin$^c(4)$-structure on $X$ and choose connections $b$ and $A$ in $L$ and $E$ respectively. Then

$$\nabla^2_{A,b} = \nabla^*_A \nabla_A + \Gamma(F_{A,b}) + \frac{s}{4} \text{id}_{\Sigma \otimes E}.$$  

**Proof:** Since $\Gamma(F_g) = \frac{s}{4} \text{id}_\Sigma$ [H1], and $F_{A,b} = F_{B(b)} \otimes \text{id}_E + \text{id}_\Sigma \otimes F_A = \frac{1}{2} F_b \text{id}_\Sigma \otimes \text{id}_E + F_g \otimes \text{id}_E + \text{id}_\Sigma \otimes F_A = \text{id}_\Sigma \otimes (F_A + \frac{1}{2} F_b \text{id}_E) + F_g \text{id}_E$, we find $\Gamma(F_{A,b}) = \Gamma(F_{A,b}) + \frac{s}{4} \text{id}_{\Sigma \otimes E}$.  

**Remark 2.2** One has a Bochner-type result for spinors $\Psi$ on which $\Gamma(F_{A,b}) + \frac{s}{4} \text{id}_{\Sigma \otimes E}$ is positive: Such a spinor is harmonic if and only if it is parallel [H1].

Let $(\ , \ )$ be the pointwise inner product on $\Sigma \otimes E$, $\| \|$ the associated pointwise norm, and $\|\|\|$ the corresponding $L^2$-norm. For a spinor $\Psi \in A^0(\Sigma^\pm \otimes E)$ we define $(\Psi \bar{\Psi})_0 \in A^0(\text{End}_0(\Sigma^\pm \otimes E))$ as the image of the Hermitian endomorphism $\Psi \otimes \bar{\Psi} \in A^0(\text{End}(\Sigma^\pm \otimes E))$ under the projection $\text{End}(\Sigma^\pm \otimes E) \rightarrow \text{End}_0(\Sigma^\pm) \otimes \text{End}(E)$.

**Corollary 2.3** With the notations above, we have

$$(\nabla^2_{A,b} \Psi, \Psi) = (\nabla_A^* \nabla_A \Psi, \Psi) + (\Gamma(F^+_{A,b}), (\Psi_+ \bar{\Psi}_+)_0) + (\Gamma(F^-_{A,b}), (\Psi_- \bar{\Psi}_-)_0) + \frac{s}{4} |\Psi|^2,$$

where $(F^-_{A,b})$ is the (anti-)self-dual component of $F_{A,b}$.

**Proof:** Indeed, since $\Gamma(F^\pm_{A,b})$ vanishes on $\Sigma^\mp$ and is trace free with respect to $\Sigma^\pm$, the inner product $(\Gamma(F_{A,b}), (\Psi \bar{\Psi}))$ in the Weitzenböck formula simplifies for a spinor $\Psi \in A^0(\Sigma^\pm \otimes E)$:

$$(\Gamma(F_{A,b}), (\Psi \bar{\Psi})) = (\Gamma(F^\pm_{A,b}), (\Psi \bar{\Psi})_0)$$
For a positive spinor $\Psi \in A^0(E \otimes \Sigma^+)$, the following important identity follows immediately:

$$
\begin{align*}
(\mathcal{D}^2_{A,b} \Psi, \Psi) + \frac{1}{2} \left| \Gamma((F^+_{A,b}) - (\Psi \bar{\Psi})_0) \right|^2 & = (\nabla^*_{A_b} \nabla_{A_b} \Psi, \Psi) + \frac{1}{2} \left| F^+_{A,b} \right|^2 + \frac{1}{2} \left| (\Psi \bar{\Psi})_0 \right|^2 + \frac{s}{4} \left| \Psi \right|^2 \\
\end{align*}
$$

(4)

If we integrate both sides of (4) over $X$, we get:

**Proposition 2.4** Let $(X, g)$ be an oriented, closed Riemannian 4-manifold with scalar curvature $s$, $E$ a Hermitian bundle over $X$. Choose a $\text{Spin}^c(4)$-structure on $X$ and a connection $b$ in the determinant line bundle $L = \det(\Sigma^+) = \det(\Sigma^-)$. Let $A$ be a connection in $E$. For any $\Psi \in A^0(\Sigma^+ \otimes E)$ we have:

$$
|| D_{A,b} \Psi ||^2 + \frac{1}{2} || \Gamma((F^+_{A,b}) - (\Psi \bar{\Psi})_0) ||^2 = || \nabla_{A_b} \Psi ||^2 + \frac{1}{2} || F^+_{A,b} ||^2 + \frac{1}{2} || (\Psi \bar{\Psi})_0 ||^2 + \frac{1}{4} \int_X s |\Psi|^2.
$$

We introduce now our coupled variant of the Seiberg-Witten equations. The unknown is a pair $(A, \Psi)$ consisting of a connection in the Hermitian bundle $E$ and a section $\Psi \in A^0(\Sigma^+ \otimes E)$. The equations ask for the vanishing of the left-hand side in the above formula.

$$
\begin{cases}
D_{A,b} \Psi = 0 \\
\Gamma(F^+_{A,b}) = (\Psi \bar{\Psi})_0
\end{cases}
$$

(SW)

Proposition 2.4 and the inequality $|(\Psi \bar{\Psi})_0|^2 \geq \frac{1}{2} |\Psi|^4$ give immediately:

**Remark 2.5** If the scalar curvature $s$ is nonnegative on $X$, then the only solutions of the equations are the pairs $(A, 0)$, with $F^+_{A,b} = 0$.

If $L$ is the square of a line bundle $L^\frac{1}{2}$, and if we choose a connection $b^\frac{1}{2}$ in $L^\frac{1}{2}$ with square $b$, then $F_{A,b}$ is simply the curvature of the connection $A_b^\frac{1}{2}$ in $E \otimes L^\frac{1}{2}$. The solution of the coupled Seiberg-Witten equations on a manifold with $s \geq 0$ are in this case just $U(r)$-instantons on $E \otimes L^\frac{1}{2}$.

In the case of a Kähler surface $(X, g)$, the coupled Seiberg-Witten equation can be reformulated in terms of complex geometry. The point is that if
we consider the canonical Spin$^c$-structure associated to the Kähler structure, the Dirac operator has a very simple form [H1]. The determinant of this Spin$^c$-structure is the anti-canonical bundle $K_X^\vee$ of the surface, which comes with a holomorphic structure and a natural metric inherited from the holomorphic tangent bundle.

Let $c$ be the Chern connection in $K_X^\vee$. With this choice, the induced connection $B(c)$ in $\Sigma = \Lambda^{00} \oplus \Lambda^{02} \oplus \Lambda^{01}$ coincides with the connection defined by the Levi-Civita connection. Recall that on a Kähler manifold, the almost complex structure is parallel with respect to the Levi-Civita connection, so that the splitting of the exterior algebra $\bigoplus_p \Lambda^p \otimes \mathbb{C}$ becomes parallel, too.

**Proposition 2.6** Let $(X, g)$ be a Kähler surface with Chern connection $c$ in $K_X^\vee$. Choose a connection $A$ in a Hermitian vector bundle $E$ over $X$ and a section $\Psi = \phi + \alpha \in A^0(E) + A^0(\Lambda^{02} \otimes E)$.

The pair $(A, \Psi)$ satisfies the Seiberg-Witten equations iff the following identities hold:

\[
\begin{align*}
F_{A,c}^{20} &= -\frac{1}{2} \phi \otimes \bar{\alpha} \\
F_{A,c}^{02} &= \frac{1}{2} \alpha \otimes \bar{\phi} \\
i\Lambda_g F_{A,c} &= -\frac{1}{2} (\phi \otimes \bar{\phi} - * (\alpha \otimes \bar{\alpha})) \\
\bar{\partial}_A \phi &= i \Lambda_g \partial_A \alpha
\end{align*}
\]

**Proof:** The Dirac operator is in this case $D_{A,c} = \sqrt{2}(\bar{\partial}_A - i \Lambda_g \partial_A)$, and the endomorphism $(\Psi \Psi)_0$ has the form:

\[
\begin{pmatrix}
\frac{1}{2} (\phi \otimes \bar{\phi} - * (\alpha \otimes \bar{\alpha})) & * (\phi \otimes \bar{\alpha} \wedge \cdot ) \\
\alpha \otimes \bar{\phi} & -\frac{1}{2} (\phi \otimes \bar{\phi} - * (\alpha \otimes \bar{\alpha}))
\end{pmatrix}.
\]

Since $\Gamma(F_{A,c}^+) = \Gamma(F_{A,c}^{20} + F_{A,c}^{02} + \frac{1}{2} \Lambda_g F_{A,c} \cdot \omega_g)$ equals

\[
2 \begin{pmatrix}
-\frac{1}{2} \Lambda_g F_{A,c} & * (F_{A,c}^{20} \wedge \cdot ) \\
F_{A,c}^{20} \wedge \cdot & \frac{1}{2} \Lambda_g F_{A,c}
\end{pmatrix},
\]

the equivalence of the two systems of equations follows. $\blacksquare$
3 Monopoles on Kähler surfaces and the generalized vortex equation

Let \((X, g)\) be a Kähler surface with canonical \(\text{Spin}^c(4)\)-structure, and Chern connection \(c\) in the anti-canonical bundle \(K_X\).

We fix a unitary vector bundle \(E\) of rank \(r\) over \(X\), and define \(J(E) := \deg_g(\Sigma^+ \otimes E)\), i.e. \(J(E) = 2r(\mu_g(E) - \frac{1}{2}\mu_g(K_X))\), where \(\mu_g\) denotes the slope with respect to \(\omega_g\).

Every spinor \(\Psi \in A^0(\Sigma^+ \otimes E)\) has the form \(\Psi = \varphi + \alpha\) with \(\varphi \in A^0(E)\) and \(\alpha \in A^0(\Lambda^{02} \otimes E)\).

We have seen that the coupled Seiberg-Witten equations are equivalent to the system:

\[
\begin{align*}
F^A_{20} &= -\frac{1}{2} \varphi \otimes \bar{\alpha} \\
F^A_{02} &= \frac{1}{2} \alpha \otimes \bar{\varphi} \\
i \Lambda_g F_{A,c} &= -\frac{1}{2} (\varphi \otimes \bar{\varphi} - \ast (\alpha \otimes \bar{\alpha})) \\
\bar{\partial}_A \varphi &= i \Lambda_g \partial_A \alpha
\end{align*}
\]

\((SW^*)\)

Lemma 3.1

A. Suppose \(J(E) < 0\):
A pair \((A, \varphi + \alpha)\) is a solution of the system \((SW^*)\) if and only if
i) \(F^A_{20} = F^A_{02} = 0\)
ii) \(\alpha = 0, \partial_A \varphi = 0\)
iii) \(i \Lambda_g F_{A,c} + \frac{1}{2} \varphi \otimes \bar{\varphi} + \frac{1}{2} \text{Id}_E = 0\).

B. Suppose \(J(E) > 0\), and put \(a := \bar{\alpha} \in A^0(E^\vee \otimes K_X)\):
A pair \((A, \varphi + \bar{a})\) is a solution of the system \((SW^*)\) if and only if
i) \(F^A_{20} = F^A_{02} = 0\)
ii) \(\varphi = 0, \partial_A a = 0\)
iii) \(i \Lambda_g F_{A,c} - \frac{1}{2} \ast (a \otimes \bar{a}) + \frac{1}{2} \text{Id}_E = 0\).

Proof: (cf. [W]) The splitting \(\Sigma^+ \otimes E = \Lambda^{00} \otimes E \oplus \Lambda^{02} \otimes E\) is parallel with respect to \(\nabla_{A,c}\), so that, by Proposition 2.4

\[
\| \mathcal{D}_{A,c} \Psi \|^2 + \frac{1}{2} \| \Gamma(F^+_{A,c}) - (\Psi \bar{\Psi})_0 \|^2 = \\
= \| \nabla_{A,c} \varphi \|^2 + \| \nabla_{A,c} \alpha \|^2 + \frac{1}{2} \| F^+_{A,c} \|^2 + \frac{1}{2} \| (\Psi \bar{\Psi})_0 \|^2 + \frac{1}{4} \int_X s(|\varphi|^2 + |\alpha|^2).
\]
The right-hand side is invariant under the transformation \((A, \varphi, \alpha) \mapsto (A, \varphi, -\alpha)\), hence any solution \((A, \varphi + \alpha)\) must have \(F_A^{20} = F_A^{02} = 0\) and \(\varphi \otimes \bar{\alpha} = \alpha \otimes \bar{\varphi} = 0\); the latter implies obviously \(\alpha = 0\) or \(\varphi = 0\). Integrating the trace of the equation \(i \Lambda g F_{A,c} = -\frac{1}{2} (\varphi \otimes \bar{\varphi} - * (\alpha \otimes \bar{\alpha}))\), we find:

\[
J(E) = c_1(\Sigma^+ \otimes E) \cup [\omega_g] = (2c_1(E) - rc_1(K_X)) \cup [\omega_g] = 2 \int_X \frac{i}{2\pi} \text{Tr}(F_{A,c}) \wedge \omega_g = \frac{1}{4\pi} \int_X \text{Tr}(i \Lambda F_{A,c}) \omega_g^2 = \frac{1}{8\pi} \int_X \text{Tr}(-\varphi \otimes \bar{\varphi} + *(\alpha \otimes \bar{\alpha})) \omega_g^2
\]

This equation shows that we must have \(\alpha = 0\), if \(J(E) < 0\), and \(\varphi = 0\), if \(J(E) > 0\). Notice that, replacing \(E\) by \(E^\vee \otimes K_X\), the second case reduces to the first one.

The assertion follows now immediately from the identity \(i \Lambda g F_c = s\).

Notice that the last equation

\[
i \Lambda g F_A + \frac{1}{2} \varphi \otimes \bar{\varphi} + \frac{1}{2} \text{id}_E = 0
\]

has the form of a generalized vortex equation as studied by Bradlow [B1], [B2] and by Garcia-Prada [G2]; it is precisely the vortex equation with constant \(\tau = -s\), if \((X, g)\) has constant scalar curvature.

Let \(s_m\) be the mean scalar curvature defined by \(\int_X s \omega_g^2 = s_m \int_X \omega^2 = 2s_m \text{Vol}_g(X)\).

We are going to prove that the system

\[
\begin{align*}
\bar{\partial}_A^2 & = 0 \\
i \Lambda g F_A + \frac{1}{2} \varphi \otimes \bar{\varphi} + \frac{1}{2} \text{id}_E & = 0
\end{align*}
\]

for the pair \((A, \varphi)\) consisting of a unitary connection in \(E\), and a section in \(E\), is always equivalent to the vortex system with parameter \(\tau = -s_m\), i.e. to the system obtained by replacing the third equation with

\[
i \Lambda g F_A + \frac{1}{2} \varphi \otimes \bar{\varphi} + \frac{1}{2} s_m \text{id}_E = 0.
\]

"Equivalent" means here that the corresponding moduli spaces of solutions are naturally isomorphic.
Let generally \( t \) be a smooth real function on \( X \) with mean value \( t_m \), and consider the following system of equations:

\[
\begin{cases}
\mathcal{D}^2 A = 0 \\
\bar{\partial}_A \varphi = 0 \\
i A_g F_A + \frac{1}{2} \varphi \otimes \bar{\varphi} - \frac{1}{2} t \text{id}_E = 0
\end{cases} \quad (V_t)
\]

\( (V_t) \) is defined on the space \( \mathcal{A}(E) \times A^0(E) \), where \( \mathcal{A}(E) \) is the space of unitary connections in \( E \). The product \( \mathcal{A}(E) \times A^0(E) \) (completed with respect to sufficiently large Sobolev indices) carries a natural \( L^2 \) Kähler metric \( \tilde{g} \) and a natural right action of the gauge group \( U(E) \): \( (A, \varphi)^f := (A^f, f^{-1} \varphi) \), where \( d_A^f := f^{-1} \circ d_A \circ f \).

For every real function \( t \) let

\[ m_t : \mathcal{A}(E) \times A^0(E) \to A^0(\text{ad}(E)) \]

be the map given by \( m_t := \Lambda_g F_A - \frac{1}{2} \varphi \otimes \bar{\varphi} + \frac{1}{2} t \text{id}_E \).

**Proposition 3.2** \( m_t \) is a moment map for the action of \( U(E) \) on \( \mathcal{A}(E) \times A^0(E) \).

**Proof:** Let \( a^\# \) be the vector field on \( \mathcal{A}(E) \times A^0(E) \) associated with the infinitesimal transformation \( a \in A^0(\text{ad}(E)) = \text{Lie}(U(E)) \), and define the real function \( m_t^a : \mathcal{A}(E) \times A^0(E) \to \mathbb{R} \) by:

\[ m_t^a(x) = \langle m_t(x), a \rangle_{L^2}. \]

We have to show that \( m_t \) satisfies the identities:

\[ \iota_a \omega_{\tilde{g}} = dm_t^a, \quad m_t^a \circ f = m^{\text{ad}_f(a)} \quad \text{for all} \quad a \in A^0(\text{ad}(E)), \quad f \in U(E). \]

It is well known that, in general, a moment map for a group action in a symplectic manifold is well defined up to a constant central element in the Lie algebra of the group. In our case, the center of the Lie algebra \( A^0(\text{ad}(E)) \) of the gauge group is just \( i A^0 \text{id}_E \), hence it suffices to show that \( m_0 \) is a moment map. This has already been noticed by Garcia-Prada [G1], [G2].

Note also that in our case every moment map has the form \( m_t \) for some function \( t \), which shows that from the point of view of symplectic geometry, the natural equations are the generalized vortex equations \( (V_t) \).
In order to show that Bradlow’s main result \([B2]\) also holds for the generalized system \((V_t)\), we have to recall some definitions.

Let \(E\) be a holomorphic vector bundle of topological type \(E\), and let \(\varphi \in H^0(E)\) be a holomorphic section. The pair \((E, \varphi)\) is \(\lambda\)-stable with respect to a constant \(\lambda \in \mathbb{R}\) iff the following conditions hold:

1. \(\mu_g(E) < \lambda\) and \(\mu_g(F) < \lambda\) for all reflexive subsheaves \(F \subset E\) with \(0 < \text{rk}(F) < r\).
2. \(\mu_g(E/F) > \lambda\) for all reflexive subsheaves \(F \subset E\) with \(0 < \text{rk}(F) < r\) and \(\varphi \in H^0(F)\).

**Theorem 3.3** Let \((X, g)\) be a closed Kähler manifold, \(t \in A^0\) a real function, and \((E, \varphi)\) a holomorphic pair over \(X\). Set \(\lambda := \frac{1}{4\pi} t \text{Vol}_g(X)\). \(E\) admits a Hermitian metric \(h\) such that the associated Chern connection \(A_h\) satisfies the vortex equation

\[
i A_h F_A + \frac{1}{2} \varphi \otimes \bar{\varphi} - \frac{1}{2} t \text{id}_E = 0
\]

iff one of the following conditions holds:

(i) \((E, \varphi)\) is \(\lambda\)-stable

(ii) \(E\) admits a splitting \(E = E' \oplus E''\) with \(\varphi \in H^0(E')\) such that \((E', \varphi)\) is \(\lambda\)-stable, and \(E''\) admits a weak Hermitian-Einstein metric with factor \(\frac{t}{2}\). In particular \(E''\) is polystable of slope \(\lambda\).

**Proof:** In the case of a constant function \(t = \tau\), the theorem was proved by Bradlow \([B2]\), and his arguments work in the general context, too: The fact that the existence of a solution of the vortex equation implies (i) or (ii) follows by replacing the constant \(\tau\) in \([B2]\) everywhere with the function \(t\).

The difficult part consists in showing that every \(\lambda\)-stable pair \((E, \varphi)\) admits a metric \(h\) such that \((A_h, \varphi)\) satisfies the vortex equation \((V_t)\). To this end let \(\text{Met}(E)\) be the space of Hermitian metrics in \(E\), and fix a background metric \(k \in \text{Met}(E)\). Bradlow constructs a functional \(M_{\varphi, \tau}(-, -) : \text{Met}(E) \times \text{Met}(E) \rightarrow \mathbb{R}\), which is convex in the second argument, such that any critical point of \(M_{\varphi, \tau}(k, \cdot)\) is a solution of the vortex equation; the point is then to find an absolute minimum of \(M_{\varphi, \tau}(k, \cdot)\). The existence of an absolute minimum follows from the following basic \(C^0\) estimate:

**Lemma 3.4** (Bradlow) Let \(\text{Met}_2(E, B) := \{ h = ke^a | a \in L^2_p(\text{End}(E)), a^*k = a, \| \mu_t(A_h, \varphi) \|_{L^p} \leq B \}\). If \((E, \varphi)\) is \(\frac{t}{4\pi} \text{Vol}_g(X)\)-stable, then there exist positive...
constants $C_1$, $C_2$ such that

$$\sup |a| \leq C_1 M_{\varphi, \tau}(k, ke^a) + C_2,$$

for all $k$-Hermitian endomorphisms $a \in L^2_p(\text{End}(E))$. Moreover, any absolute minimum of $M_{\varphi, \tau}(k, \cdot)$ on $\text{Met}^2_0(E, B)$ is a critical point of $M_{\varphi, \tau}(k, \cdot)$, and gives a solution of the vortex equation $V_\tau$.

Let now $t$ be a real function on $X$, and choose a solution $v$ of the Laplace equation $i \Lambda g \bar{\partial} \partial v = \frac{1}{2}(t - t_m)$. If we make the substitution $h = h'e^v$, then $h$ solves the vortex equation $(V_t)$ iff $h'$ is a solution of

$$i \Lambda g F_{h'} + \frac{1}{2} e^v \varphi \otimes \bar{\varphi} - \frac{1}{2} t_m \text{id}_E = 0.$$

Define $\mu_{t_m, v}(h') := i \Lambda g F_{h'} + \frac{1}{2} e^v \varphi \otimes \bar{\varphi} - \frac{1}{2} t_m \text{id}_E = 0$, and

$$M_{\varphi, t_m, v}(k, h) := M_D(k, h) + \| e^v \varphi \|^2_h - \| e^v \varphi \|^2_k - t_m \int_X \text{Tr}(\log(k^{-1}h)),$$

where $M_D$ is the Donaldson functional [D]. Then it is not difficult to show that all arguments of Bradlow remain correct after replacing $\mu_{t_m}$ and $M_{\varphi, t_m}$ with $\mu_{t_m, v}$ and $M_{\varphi, t_m, v}$ respectively. Indeed, let $l$ be a positive bound from below for the map $e^v$. Then

$$M_{\varphi, t_m}(k, ke^a + \log l) \leq M_D(k, ke^a) + M_D(ke^a, lke^a) + \| l \varphi \|^2_h - t_m \int_X \text{Tr} \log(lk^{-1}h)$$

$$\leq M_{\varphi, t_m, v}(k, ke^a) + \| e^v \varphi \|^2_k + 2 \log \text{deg}_g(E) - rt_m \log \text{Vol}_g(X)$$

$$\leq M_{\varphi, t_m, v}(k, ke^a) + C'(k, \varphi, v, l).$$

Similarly, we get constants $n > 0$, $C''$ and an inequality

$$M_{\varphi, t_m, v}(k, ke^{a+\log n}) \leq M_{\varphi, t_m}(k, ke^a) + C'',$$

which shows that the basic $C^0$ estimate in the Lemma above holds for $M_{\varphi, t_m, v}$ iff it holds for Bradlow’s functional $M_{\varphi, t_m}$.

\begin{remark}
In the special case of a rank-1 bundle $E$, a much more elementary proof based on [B1] is possible.
\end{remark}
4 Moduli spaces of monopoles, vortices, and stable pairs

Let $(X, g)$ be a closed Kähler manifold of arbitrary dimension, and fix a unitary vector bundle $E$ of rank $r$ over $X$. We denote by $\hat{A}(E)$ the affine space of connection of type $(0, 1)$ in $E$. the complex gauge group $GL(E)$ acts on $\hat{A}(E) \times A^0(E)$ from the right by $(\hat{\partial}_A, \varphi)^\rho := (g^{-1} \circ \hat{\partial}_A \circ g, g^{-1} \varphi)$; this action becomes complex analytic after suitable Sobolev completions. We denote by $\bar{S}(E)$ the set of pairs $(\hat{\partial}_A, \varphi)$ with trivial isotropy group. Notice that $\varphi \neq 0$ when $(\hat{\partial}_A, \varphi) \in \bar{S}(E)$, and that $\bar{S}(E)$ is an open subset of $\hat{A}(E) \times A^0(E)$, by elliptic semi-continuity [K].

The action of $GL(E)$ on $\bar{S}(E)$ is free, by definition, and we denote the Hilbert manifold $\bar{S}(E)/GL(E)$ by $\mathcal{B}^s(E)$. The map $p : \hat{A}(E) \times A^0(E) \longrightarrow A^{02}(\text{End}(E)) \oplus A^{01}(E)$ defined by $p(\hat{\partial}_A, \varphi) = (F^{02}_A, \hat{\partial}_A \varphi)$ is equivariant with respect to the natural actions of $GL(E)$, hence it gives rise to a section $\hat{p}$ in the associated vector bundle $\bar{S}(E) \times_{GL(E)} (A^{02}(\text{End}(E)) \oplus A^{01}(E))$ over $\mathcal{B}^s(E)$. We define the moduli space of simple pairs of type $E$ to be the zero-locus $Z(\hat{p})$ of this section. $Z(\hat{p})$ can be identified with the set of isomorphism classes consisting of a holomorphic bundle $\mathcal{E}$ of differentiable type $E$, and a holomorphic section $\varphi \neq 0$, such that the kernel of the evaluation map $ev(\varphi) : H^0(\text{End}(\mathcal{E})) \longrightarrow H^0(E)$ is trivial.

In a similar way we define the moduli space $\mathcal{Y}_t^q$ of gauge-equivalence classes of irreducible solutions of the generalized vortex equation $V_t$:

Let $B^+$ denote as usual the subbundle $((\Lambda^{02} + \Lambda^{20}) \cap \Lambda^2) \oplus \Lambda^0 \omega$ of the bundle $\Lambda^2$ of real 2-forms on $X$. We denote by $\mathcal{D}^+$ the open subset of the product $\mathcal{D} := \mathcal{A}(E) \times A^0(E) \simeq \hat{A}(E) \times A^0(E)$ consisting of pairs with trivial isotropy group with respect to the action of the gauge group $U(E)$. The quotient $\mathcal{B}^s(E) := \mathcal{D}^+/U(E)$ comes with the structure of a real-analytic manifold.

Let $v : \mathcal{D}(E) \longrightarrow A^0(B^+ \otimes \text{ad}(E)) \oplus A^{01}(E)$ be the map given by:

$$v(A, \varphi) = (F^{20} + F^{02}, m_t(A, \varphi) \omega \circ id_E, \hat{\partial}_A \varphi).$$

Again $v$ is $U(E)$-equivariant, and the moduli space $\mathcal{Y}_t^q$ of $t$-vortices is defined to be the zero-locus $Z(\hat{v})$ of the induced section $\hat{v}$ of $\mathcal{D}^+/U(E) \times A^0(B^+ \otimes \text{ad}(E)) \oplus A^{01}(E)$ over $\mathcal{B}^s(E)$. 

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Notice now that by Proposition 3.2, the second component \(v^2\) of \(v\) is a moment map for the \(U(E)\) action. It is easy to see that (at least in a neighbourhood of \(Z(v) \cap D^*\)) it has the general property of a moment map in the finite dimensional Kähler geometry: Its zero locus \(Z(v^2)\) is smooth and intersects every \(GL(E)\) orbit along a \(U(E)\) orbit, and the intersection is transversal. This means that the natural map \(A \longrightarrow \partial A\) defines in a neighbourhood of \(Z(\hat{v}) \cap B^\ast\) an open embedding of smooth Hilbert manifolds.

Regard now \(V_{g'}\) as the subspace of \(Z(\hat{v}^2) \subset B^\ast\) defined by the equation \((\hat{v}_1, \hat{v}_3) = 0\). On the other hand, the pullback of the equation \(\hat{p} = 0\), cutting out the moduli space \(Z(\hat{p})\) of simple holomorphic pairs, via the open embedding \(i\) is precisely the equation \((\hat{v}_1, \hat{v}_3) = 0\), cutting out \(V_{g'}\). We get therefore an open embedding \(i_0: V_{g'} \longrightarrow Z(\hat{p})\) of real analytic spaces induced by \(i\), and by Theorem 3.3 the image of \(i_0\) consists of the set of \(\lambda\)-stable pairs, with \(\lambda := \frac{1}{4\pi} t_m \text{Vol}_g(X)\).

Finally we denote by \(M_{\lambda}^X(E, \lambda) \subset Z(\hat{p})\) the open subspace of \(\lambda\)-stable pairs, with the induced complex space-structure. Putting everything together, we have:

**Theorem 4.1** Let \((X, g)\) be a closed Kähler manifold, \(t \in A^0\) a real function, and \(\lambda := \frac{1}{4\pi} t_m \text{Vol}_g(X)\). Fix a unitary vector bundle \(E\) of rank \(r\) over \(X\). There are natural real-analytic isomorphisms of moduli spaces

\[
\mathcal{V}_t^\ast(E) \simeq \mathcal{V}_{t_0}^\ast(E) \simeq M_{\lambda}^X(E, \lambda).
\]

Let us come back now to the monopole equation (SW*) on a Kähler surface. In this case the function \(t\) is the negative of the scalar curvature \(s\), so that the corresponding constant \(\lambda\) becomes:

\[
\lambda = \frac{-s_m}{4\pi} \text{Vol}_g(X) = -\frac{1}{8\pi} \int_X s \omega^2 = -\frac{1}{8\pi} \int_X (i\Lambda F_c) \omega^2 = -\frac{1}{4\pi} \int_X iF_c \wedge \omega = \frac{1}{2} \mu_g(K).
\]

This yields our main result:

**Theorem 4.2** Let \((X, g)\) be a Kähler surface with canonical \(\text{Spin}^c(4)\)-structure, and Chern connection \(c\) in \(K_X^\ast\). Fix a unitary vector bundle \(E\) of rank \(r\) over \(X\), and suppose \(J(E) = \deg_g(\Sigma^+ \otimes E) < 0\). The moduli space of solutions of the coupled Seiberg-Witten equations is isomorphic to the moduli space \(M_{\lambda}^X(E, \frac{1}{2} \mu_g(K))\) of \(\frac{1}{2} \mu_g(K)\)-stable pairs of topological type \(E\).
At this point it is natural to study the properties of the moduli spaces $\mathcal{M}_X^g(E, \lambda)$. We do not want to go into details here, and we content ourselves by describing some of the basic steps.

The infinitesimal structure of the moduli space around a point $[(A, \varphi)] = (\mathcal{C}^\omega_{\delta A, \varphi}, d^\omega_{A, \varphi})$ which is the cone over the evaluation map $ev^*: ev^q(\varphi) : A^0q(\text{End}(E)) \to A^0q(E)$. More precisely

$$C^\omega_{\delta A, \varphi} = A^0q(\text{End}(E)) \oplus A^{0,q-1}(E)$$

and the differential $d^\omega_{A, \varphi}$ is given by the matrix

$$d^\omega_{A, \varphi} = \begin{bmatrix} -\overline{\partial}_A & 0 \\ ev(\varphi) & \partial_A \end{bmatrix},$$

where $\overline{\partial}_A$ and $\partial_A$ are the operators of the Dolbeault complexes $(A^0q(E), \overline{\partial}_A)$ and $(A^0q(\text{End}(E)), \partial_A)$ respectively.

Associated to the morphism $ev^*(\varphi)$ is an exact sequence

$$\ldots \to H^q(\text{End}(\mathcal{E}_A)) \xrightarrow{ev^q(\varphi)} H^q(\mathcal{E}_A) \to H^{q+1}_{\overline{\partial}A, \varphi} \to H^{q+1}(\text{End}(\mathcal{E}_A)) \to \ldots$$

with finite dimensional vector spaces

$$H^q_{\overline{\partial}A, \varphi} = \ker(ev^q(\varphi)) \oplus \text{coker}(ev^{q-1}(\varphi)).$$

$H^0_{\overline{\partial}A, \varphi}$ vanishes for a simple pair $(\overline{\partial}A, \varphi)$, and $H^1_{\overline{\partial}A, \varphi}$ is the Zariski tangent space of $\mathcal{M}_X^g(E, \lambda)$ at $[\overline{\partial}A, \varphi]$.

A Kuranishi type argument yields local models of the moduli space, which can be locally described as the zero loci of holomorphic map germs

$$K_{[\overline{\partial}A, \varphi]} : H^1_{\overline{\partial}A, \varphi} \to H^2_{\overline{\partial}A, \varphi}$$

at the origin.

One finds that $H^2_{\overline{\partial}A, \varphi} = 0$ is a sufficient smoothness criterion in the point $[\overline{\partial}A, \varphi]$ of the moduli space, and that the expected dimension is $\chi(E) - \chi(\text{End}(E))$. The necessary arguments are very similar to the ones in [BD1], [BD2].

The moduli spaces $\mathcal{M}^g(E, \lambda)$ will be quasi-projective varieties if the underlying manifold $(X, g)$ is Hodge, i.e. if $X$ admits a projective embedding such that a multiple of the Kähler class is a polarisation [G1].

A GIT construction for projective varieties of any dimension has been given in [HL2]. The spaces $\mathcal{M}_X^g(E, \lambda)$ vary with the parameter $\lambda$, and flip-phenomena occur just like in the case of curves [T].
5 Applications

The equations considered by Seiberg and Witten are associated to a Spin$^c$(4)-structure, and correspond to the case when (in our notations) the unitary bundle $E$ is the trivial line bundle. Alternatively, we can fix a Spin$^c$-(4)-structure $s_0$ on $X$, and regard the Seiberg-Witten equations corresponding to the other Spin$^c$-(4)-structures as coupled Seiberg-Witten equations associated to $s_0$ and to a unitary line bundle $E$. The Spin$^c$-(4)-structure we fix will always be the canonical structure defined by a Kähler metric. In the most interesting case of rank-1 bundles $E$ over Kähler surfaces the central result is:

**Proposition 5.1** Let $(X, g)$ be a Kähler surface with canonical class $K$, and let $L$ be a complex line bundle over $X$ with $L \equiv K \mod 2$. Denote by $W^g_X(L)$ the moduli space of solutions of the Seiberg-Witten equation for all Spin$^c$-(4)-structures with determinant $L$. Then

i) If $\mu(L) < 0$, $W^g_X(L)$ is isomorphic to the space of all linear systems $|D|$, where $D$ is a divisor with $c_1(O_X(2D - K)) = L$.

ii) If $\mu(L) > 0$, $W^g_X(L)$ is isomorphic to the space of all linear systems $|D|$, where $D$ is a divisor with $c_1(O_X(2D - K)) = -L$.

**Proof:** Use Theorem 4.2 and Bradlow’s description of the moduli spaces of stable pairs in the case of line bundles [B1].

We have already noticed (Remark 2.5) that in the case of a Riemannian 4-manifold with nonnegative scalar curvature $s_g$, the Seiberg-Witten equations have only reducible solutions. In the Kähler case, the same result can be obtained under the weaker assumption $\sigma_g \geq 0$ on the total scalar curvature.

**Corollary 5.2** Let $(X, g)$ be a Kähler surface with nonnegative total scalar curvature $\sigma_g$. Then all solutions of the Seiberg-Witten equations in rank 1 are reducible. If moreover the surface has $K^2 > 0$, then for every almost canonical class $L$, the corresponding Seiberg-Witten equations are incompatible.

**Proof:** The first assertion follows directly from the theorem, since the condition $\sigma_g \geq 0$ is equivalent to $K \cup [\omega_g] \leq 0$. For the second assertion, note that if $L$ is an almost canonical class, then $L^2 = K^2 > 0$, hence (regarded as line bundle) it cannot admit anti-selfdual connections.
Remark 5.3 The Seiberg-Witten invariants associated to almost canonical classes are well-defined for oriented, closed 4-manifolds $X$ satisfying $3\sigma + 2e > 0$.

Proof: Recall that if $L$ is an almost canonical class, then the expected dimension of the moduli space of solutions of the perturbed Seiberg-Witten equations $[W, KM]$ corresponding to a Spin$^c(4)$-structure of determinant $L$ is 0. Seiberg and Witten associate to every such class $L$ the number $n_L$ of points (counted with the correct signs $[W]$) of such a moduli space chosen to be smooth and of the expected dimension. In the case $b_+ \geq 2$, using the same cobordism argument as in Donaldson theory, it follows that these numbers are well-defined, i.e. independent of the metric, provided the moduli space has the expected dimension $[KM]$. The point is that the space of $L$-good metrics $[KM]$ (i.e. metrics with the property that the space of harmonic anti-selfdual forms does not contain the harmonic representative of $c_1^2(L)$) is in this case path-connected. On the other hand, under the assumption $3\sigma + 2e > 0$, it follows that $L^2 > 0$ for any almost canonical class $L$, hence all metrics are $L$-good.

Proposition 5.4 Let $(X, H_0)$ be a polarised surface with $K$ nef and big, and choose a Kähler metric $g$ with Kähler class $[\omega_g] = H_0 + nK =: H$ for some $n \geq KH_0$. Then $W^g_X(L)$ is empty for all almost canonical classes, except for $L = \pm K$, when it consists of a simple point.

Proof: Let $L$ be an almost canonical class with $LH < 0$. Suppose $D$ is an effective divisor with $c_1(O_X(2D - K)) = L$, so that $D(D - K) = 0$. Then $D^2 = DK \geq 0$ since $K$ is nef. If $D^2$ were strictly positive, the Hodge index theorem would give $(D - K)^2 \leq 0$, i.e. $K^2 \leq D^2$. But from $LH < 0$ we get $0 > (2D - K)(H_0 + nK) = (2D - K)H_0 + n(2D^2 - K^2) \geq (2D - K)H_0 + n$, which leads to the contradiction $n < (K - 2D)H_0 \leq KH_0$. Therefore $D^2 = DK = 0$, so that, again by the Hodge index theorem, $D$ must be numerically zero. Since $D$ is effective, it must be empty, and $L = -K$.

Replacing $L$ by $-L$ if $L$ is an almost canonical class with $LH > 0$, we find $L = K$ in this case. The corresponding Seiberg-Witten moduli spaces are simple points in both cases, since $H^2_{\partial_A+\varphi} = H^1(O(D)|_D) = 0$. 

\[20\]
Corollary 5.5 There exists no orientation-preserving diffeomorphism between a rational surface and a minimal surface of general type.

Proof: Indeed, any rational surface $X$ admits a Hodge metric with positive total scalar curvature [H2]. If $X$ was orientation-preservingly diffeomorphic to a minimal surface of general type, then $K^2 > 0$, hence the Seiberg-Witten invariants are well defined (Remark 5.3), and vanish by Corollary 5.2. Proposition 5.4 shows, however, that the Seiberg-Witten invariants of a minimal surface of general type are non-trivial for two almost canonical classes. ■

Witten has already proved [W] that for a minimal surface of general type with $p_g > 0$ ($b_+ \geq 2$), the only almost canonical classes which give non-trivial invariants are $K$ and $-K$. Their proof uses the moduli space of solutions of the perturbation of the Seiberg-Witten equation with a holomorphic form. Proposition 5.4 shows that a stronger result can be obtained with the non-perturbed equations by choosing the Hodge metric $H = H_0 + nK$, $n \gg 0$.

For the proof of Corollary 5.5, we need in fact only the mod. 2 version of the Seiberg-Witten invariants [KM2].
Bibliography


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