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THE CASE OF NONNEGATIVE DATA FOR
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RELAXED SHAPE OPTIMIZATION : THE CASE OF NONNEGATIVE DATA FOR THE DIRICHLET PROBLEM

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0. INTRODUCTION

The aim of this paper is to discuss some qualitative properties of the relaxed solutions of a shape optimization problem for the domain of resolution of an elliptic equation with Dirichlet boundary conditions.

Given a bounded open subset $\Omega$ of $\mathbb{R}^n$, $n \geq 2$, and $f \in H^{-1}(\Omega)$, for any open subset $A$ of $\Omega$ let $u_A$ be the solution of the Dirichlet problem

$$u_A \in H^1_0(A), \quad -\Delta u_A = f \text{ in } A$$

extended by 0 outside $A$. Given a Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, we shall consider the optimal design problem

$$\inf_A \int_{\Omega} g(x, u_A(x)) \, dx \quad (0.1)$$

where $A$ runs over all open subsets of $\Omega$.

It can be shown (see [B.DM]) that the infimum in (0.1) is not always achieved. Therefore, in order to study the behaviour of the minimizing sequences of (0.1), a relaxed formulation of this problem has been introduced in [B.DM.], based on a new family of partial differential equations depending on measures. These equations, called “relaxed Dirichlet problems” in [DM.M.1] and [DM.M.2], can be written formally as

$$-\Delta u + \mu u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (0.2)$$
where $\mu$ is a nonnegative Borel measure on $\Omega$ which vanishes on all sets of (harmonic) capacity zero. Let us denote by $\mathcal{M}_0(\Omega)$ the class of all these measures.

We say that $u$ is a solution to (0.2) if $u \in H^1_0(\Omega) \cap L^2(\Omega, \mu)$ and if

$$\int_{\Omega} \nabla u \nabla \varphi \, dx + \int_{\Omega} u \, \varphi \, d\mu = \langle f, \varphi \rangle$$

(0.3)

for every $\varphi \in H^1_0(\Omega) \cap L^2(\Omega, \mu)$, where $\langle , \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. It can be proved (see [DM.M.1]) that for every $\mu \in \mathcal{M}_0(\Omega)$ there exists one and only one solution $u$ of (0.2). We will denote it by $u_\mu$.

The relaxed formulation of (0.1) is the minimum problem

$$\min_{\mu \in \mathcal{M}_0(\Omega)} \int_{\Omega} g(x, u_\mu(x)) \, dx.$$  \hspace{1cm} (0.4)

It can be shown (see [A.], [DM.M.2], [B.DM.]) that, under some natural growth conditions on $g$, we have

$$\inf_{A} \int_{\Omega} g(x, u_A(x)) \, dx = \min_{\mu \in \mathcal{M}_0(\Omega)} \int_{\Omega} g(x, u_\mu(x)) \, dx.$$  \hspace{1cm} (0.5)

Moreover, $\mu \in \mathcal{M}_0(\Omega)$ is a minimizer of (0.4) if and only if there exists a minimizing sequence $(A_h)$ of (0.1) such that $(u_{A_h})$ converges to $u_\mu$ weakly in $H^1_0(\Omega)$. Finally, the original problem (0.1) admits a solution $A$ if and only if the measure $\mu_A \in \mathcal{M}_0(\Omega)$ defined by

$$\mu_A(B) = \begin{cases} 
0 & \text{if } \text{Cap}(B \setminus A) = 0, \\
+\infty & \text{if } \text{Cap}(B \setminus A) > 0.
\end{cases}$$

(0.6)

is a minimizer of the relaxed problem (0.4).

In this paper we study the case where $f \geq 0$ in $\Omega$. Then (0.4) can be rewritten as a minimization problem on some closed convex set of $H^1_0(\Omega)$ for which it is easy to show the existence of at least one minimizer. This will be proved in section 1. In section 2 we will study more general properties of the solutions. In section 3 we will consider the particular case where $g(x, \cdot)$ is differentiable and we will characterize the minimizers of the relaxed problem (0.4) by means of a complementarity system (Theorem 7). In section 4 we will analyse the shape of the solutions $u_\mu$ of (0.4) in the case where $\Omega$ is a ball and the data are nonnegative and radially symmetric. In this case we are able to give a fairly accurate description of the solutions. In particular the nature of the measure $\mu$ which minimizes (0.4) will be investigated.

Our analysis shows that, under very general assumptions (see Theorems 2 and 9), all the minimizers $\mu$ of the relaxed problem (0.4) are Radon measures. This implies that the measure $\mu_A$ defined by (0.6) cannot be a minimizer of (0.4) unless $A = \Omega$. This leads to the surprising conclusion that, under these hypotheses, the shape optimization problem (0.1) cannot have a minimizer $A$ unless $A = \Omega$. 
We will conclude the paper with the discussion of an example where the optimal measure $\mu$ for (0.4) and the corresponding minimizing sequence $A_n$ for (0.1) can be described explicitly.

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1. FORMULATION OF THE PROBLEM

Let $\Omega$ be a bounded, open subset of $\mathbb{R}^n$, $n \geq 2$.

We will denote by $H^1(\Omega)$, $H^1_0(\Omega)$ the usual Sobolev spaces constructed on $L^2(\Omega)$. $H^{-1}(\Omega)$ will be the dual space of $H^1_0(\Omega)$, $\langle \cdot, \cdot \rangle$ the duality bracket between these two spaces (for details, see, for instance, [K.S.], [G.T.]).

On $H^1_0(\Omega)$ we will find convenient to use the norm

$$||\nabla u||_2 = (\int_{\Omega} |\nabla u|^2 dx)^{1/2} \quad (1.1)$$

where $|\nabla u|$ denotes the usual Euclidean norm of the gradient of $u$, and $|| \cdot ||_2$ the usual $L^2$ norm on $L^2(\Omega)$. $|| \cdot ||_{H^{-1}(\Omega)}$ will be the dual norm of (1.1) on $H^{-1}(\Omega)$.

For every compact subset $E$ of $\Omega$ we define the capacity of $E$ by

$$\text{Cap}(E) = \inf \{ \int_{\Omega} |\nabla u|^2 dx : u \in C^1_0(\Omega), u \geq 1 \text{ on } E \}.$$  

This definition is extended to open subsets $O \subset \Omega$ by

$$\text{Cap}(O) = \sup \{ \text{Cap}(K) : K \text{ compact}, K \subset O \}$$

and to arbitrary sets $E \subset \Omega$ by

$$\text{Cap}(E) = \inf \{ \text{Cap}(O) : O \text{ open}, E \subset O \}.$$  

“a.e.” means “up to a set of measure zero”, “q.e.” means “up to a set of capacity zero”.

$\mathcal{B}(\Omega)$ denotes the $\sigma$-algebra of Borel subsets of $\Omega$.

By “measure” we mean a countably additive set function such that $\mu(\emptyset) = 0$ ($\emptyset$ is the empty set). The Lebesgue measure is denoted by $dx$ or alternatively by $\text{meas}$.

$\mathcal{M}_0(\Omega)$ is the set of all measures $\mu : \mathcal{B}(\Omega) \to [0, +\infty]$ such that (see [DM.M.1], [DM.M.2], [B.DM.]):

$$\mu(B) = 0 \text{ for every } B \in \mathcal{B}(\Omega) \text{ with Cap}(B) = 0.$$
If \( u \in H^1(\Omega) \), we always assume that the pointwise value \( u(x) \) of \( u \) at each point \( x \) of \( \Omega \) satisfies

\[
\lim \inf_{\rho \to 0^+} \frac{1}{|B(x, \rho)|} \int_{B(x, \rho)} u(y) \, dy \leq u(x) \leq \lim \sup_{\rho \to 0^+} \frac{1}{|B(x, \rho)|} \int_{B(x, \rho)} u(y) \, dy
\]

(1.2)

where \( B(x, \rho) = \{ y \in \Omega \mid |y - x| < \rho \} \) and \( |B(x, \rho)| \) the Lebesgue measure of \( B(x, \rho) \).

With this convention the pointwise value \( u(x) \) is uniquely determined q.e. in \( \Omega \) and the function \( u \) is quasi continuous in \( \Omega \) (see [F.Z.]). Therefore, the integrals which appear in the definition (0.3) of a solution of (0.2) are defined unambiguously.

If \( \nu \in H^{-1}(\Omega) \), and \( \nu \geq 0 \) in the sense of distributions, then there exists a nonnegative Radon measure on \( \Omega \), still denoted by \( \nu \), such that \( \nu \in \mathcal{M}_0(\Omega) \), and for every \( u \in H^1_0(\Omega) \) we have

\[
u \in L^1(\Omega, \nu), \quad <\nu, u> = \int_{\Omega} u \, dv,
\]

(1.3)

where the pointwise values of \( u \) are determined according to our convention (1.2).

Let \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) be such that

\[
g(x, s) \text{ is a Carathéodory function}
\]

(1.4)

i.e. measurable in \( x \) for every \( s \in \mathbb{R} \) and continuous in \( s \) for almost every \( x \in \Omega \). Moreover, assume that there exist a function \( a_0 \in L^1(\Omega) \) and a constant \( b_0 \) such that

\[
|g(x, s)| \leq a_0(x) + b_0|s|^2 \quad \text{for a.e. } x \in \Omega, \forall s \in \mathbb{R}.
\]

(1.5)

Let \( f \) be in \( H^{-1}(\Omega) \). All along the paper we will assume that

\[
f \geq 0 \quad \text{in} \quad \Omega.
\]

(1.6)

Here the sign "\( \geq \)" is understood either in the sense of distributions, measures, or \( H^{-1}(\Omega) \), since all these meanings are equivalent by (1.3). We recall that for every \( \mu \in \mathcal{M}_0(\Omega) \) the function \( u_\mu \) is the unique solution of the relaxed Dirichlet problem (0.2) according to the definition (0.3).

We will consider the problem of minimizing the integral:

\[
\int_{\Omega} g(x, u_\mu(x)) \, dx
\]

over all the measures \( \mu \in \mathcal{M}_0(\Omega) \). We will denote in short this problem by

\[
\inf_{\mu \in \mathcal{M}_0(\Omega)} \int_{\Omega} g(x, u_\mu(x)) \, dx
\]

(1.7)

The relationship between (1.7) and the optimal design problem (0.1) has been discussed in the introduction.

We now want to give an equivalent formulation of problem (1.7) in terms of a minimization problem on a closed convex subset of \( H^1_0(\Omega) \).
Set
\[ K = \{ u \in H^1_0(\Omega) : \Delta u + f \geq 0 \text{ in } \Omega \} \] (1.8)
\[ K^+ = \{ u \in K : u \geq 0 \text{ q.e. in } \Omega \}. \] (1.9)

Again, in the definition of \( K \) the sign \( \geq \) is understood either in the sense of distributions, measures, or \( H^{-1}(\Omega) \). \( K \) is the set of all subsolutions to the equation
\[ -\Delta u = f \]
which belong to \( H^1_0(\Omega) \). In particular one has (see [K.S.])
\[ u, v \in K \implies u \vee v \in K. \] (1.10)

\((u \vee v \) denotes the function \( \text{Max}(u, v) \), similarly \( u \wedge v \) denotes \( \text{Min}(u, v) \)). It is easy to check that \( K \) is a closed convex subset of \( H^1_0(\Omega) \). Moreover, we have:

**Proposition 1:** Assume that \( f \in H^{-1}(\Omega) \) satisfies (1.6), then \( K^+ \) is a closed, bounded, convex subset of \( H^1_0(\Omega) \) and, if (1.4), (1.5) hold, then the problem
\[ \inf_{u \in K^+} \int_{\Omega} g(x, u(x)) \, dx \] (1.11)

admits at least one minimizer.

**Proof:** The fact that \( K^+ \) is closed and convex is easy to check. To see that \( K^+ \) is bounded, note that, if \( u \in K^+ \), then
\[ < \Delta u + f, u > \geq 0. \]

This reads also
\[ \int_{\Omega} |\nabla u|^2 \, dx \leq < f, u > \leq |f|_{H^{-1}(\Omega)} \|\nabla u\|_2 \]
and the boundedness of \( K^+ \) follows (see (1.1)).

Next, due to (1.4), (1.5),
\[ u \rightarrow \int_{\Omega} g(x, u(x)) \, dx \]
is continuous from \( L^2(\Omega) \) into \( \mathbb{R} \) (see for instance [N]) and thus it is also sequentially continuous from \( H^1_0(\Omega) \)-weak into \( \mathbb{R} \) (this is due to the complete continuity of the canonical embedding of \( H^1_0(\Omega) \) into \( L^2(\Omega) \)). Then, the result follows from the fact that the closed, convex, bounded set \( K^+ \) is sequentially weakly compact.

We have the following characterization for the elements of \( K^+ \):

**Theorem 1:** Let \( u \) be in \( H^1_0(\Omega) \). Then, \( u \in K^+ \) if and only if \( u = u_\mu \) for some \( \mu \in \mathcal{M}_0(\Omega) \).
Proof: First, let $u$ be an element of $K^+$. Then, $\nu = \Delta u + f$ is a nonnegative Radon measure which belongs to $H^{-1}(\Omega)$ and satisfies (see (1.3)):

$$\int_{\Omega} \varphi \, d\nu = <\nu, \varphi> = -\int_{\Omega} \nabla u \nabla \varphi \, dx + <f, \varphi>$$

(1.12)

for every $\varphi \in H^1_0(\Omega)$.

Now, clearly, from (1.3) we see that $u \in L^1(\Omega, d\nu)$ and from (1.12) we get

$$\int_{\Omega} u \, d\nu = -\int_{\Omega} |\nabla u|^2 \, dx + <f, u>.$$  

(1.13)

In particular $u$ is $d\nu$ measurable and since $u \geq 0$ we can define $\mu : B(\Omega) \to [0, +\infty]$ by:

$$\mu(B) = \begin{cases} 
\int_B \frac{1}{u} \, d\nu & \text{if } \text{Cap}(B \cap \{u = 0\}) = 0, \\
+\infty & \text{if } \text{Cap}(B \cap \{u = 0\}) > 0.
\end{cases}$$

(1.14)

${\{u = 0\}}$ denotes the set $\{x \in \Omega : u(x) = 0\}$. Similar notation will be used without any further notice for $\{u > 0\}, \{u \geq 0\}$...

If $B \in B(\Omega)$ and $\text{Cap}(B) = 0$, then it is easy to show that $\nu(B) = 0$ and thus $\mu(B) = 0$.

Since $u \in L^1(\Omega, d\nu)$, $u$ is $\mu$-measurable. Moreover by (1.13) and (1.14) we have

$$\int_{\Omega} u^2 \, d\mu = \int_{\{u > 0\}} u^2 \, d\mu = \int_{\{u > 0\}} u \, d\nu = \int_{\Omega} u \, d\nu = -\int_{\Omega} |\nabla u|^2 \, dx + <f, u> < +\infty,$$

hence $u \in L^2(\Omega, d\mu)$.

Let $\varphi \in H^1_0(\Omega) \cap L^2(\Omega, d\mu)$. By (1.14), we have $\varphi = 0$ q.e. on $\{u = 0\}$. Therefore, (1.12) gives

$$\int_{\Omega} \nabla u \nabla \varphi \, dx + \int_{\Omega} u \varphi \, d\mu = \int_{\Omega} \nabla u \nabla \varphi \, dx + \int_{\{u > 0\}} u \varphi \, d\mu$$

$$= \int_{\Omega} \nabla u \nabla \varphi \, dx + \int_{\{u > 0\}} \varphi \, d\nu = <f, \varphi>.$$  

This ends the proof of the fact that $u = u_\mu$ for some $\mu \in M_0(\Omega)$.

Conversely, assume now that $u = u_\mu$ for some $\mu \in M_0(\Omega)$. It is shown in [DM.M.1], Proposition 2.9, that $u \geq 0$ q.e. in $\Omega$. Moreover, using the approximation procedure of [DM.M.1], Proposition 2.6, we easily show that

$$\int_{\Omega} \nabla u \nabla \varphi \, dx - <f, \varphi> \leq 0 \quad \forall \varphi \in H^1_0(\Omega), \ \varphi \geq 0 \ \text{a.e. in } \Omega.$$  

Thus,

$$\Delta u + f \geq 0$$
in the sense of $H^{-1}(\Omega)$ and $u \in K^+$. 

**Remark 1**: In general the measure $\mu$ that satisfies (0.2) is not uniquely determined by $f$ and $u$. For instance, when $f = 0$ and $u = 0$, then any measure $\mu \in \mathcal{M}_0(\Omega)$ is suitable. However, we have:

**Proposition 2**: Let $u = u_\mu$ for some $\mu \in \mathcal{M}_0(\Omega)$. Then $\nu = \Delta u + f$ is a nonnegative Radon measure in $H^{-1}(\Omega)$. Moreover,

$$
\nu(B) = \int_B u \, d\mu \quad \forall B \in \mathcal{B}(\Omega), \quad \text{Cap}(B \cap \{u = 0\}) = 0.
$$

(1.15)

If $u > 0$ q.e. in $\Omega$, then

$$
\nu = u \mu, \quad \mu = \frac{\nu}{u}
$$

(1.16)

as measures in $\Omega$. So, in this case, the measure $\mu$ such that $u = u_\mu$ is uniquely determined by $f$ and $u$.

**Proof**: The fact that $\nu$ is a Radon measure has already been proved in Theorem 1 and (1.15), (1.16) are clear from (1.14).

**Remark 2**: If $u > 0$ q.e. in $\Omega$ and $1/u \in L^\infty_{loc}(\Omega, d\mu)$, then $\mu$ is a Radon measure. If, in addition, $1/u \in L^\infty(\Omega, d\mu)$, then $\mu$ belongs to $H^{-1}(\Omega)$. To verify this last assertion, denote by $k$ a constant such that

$$
0 \leq \frac{1}{u} \leq k \quad \mu \text{ a.e. in } \Omega.
$$

Then, for every $\varphi \in \mathcal{D}(\Omega)$,

$$
| < \mu, \varphi > | \leq \int_\Omega \frac{|\varphi|}{u} \, d\nu \leq k < \nu, |\varphi| > \leq k \, |\nu|_{H^{-1}(\Omega)} \, ||\nabla \varphi||_2
$$

which yields $\mu \in H^{-1}(\Omega)$.

As a consequence of Theorem 1 we have:

**Corollary 1**: The problems

$$
\inf_{\mu \in \mathcal{M}_0(\Omega)} \int_\Omega g(x, u_\mu(x)) \, dx
$$

(1.17)

$$
\inf_{u \in K^+} \int_\Omega g(x, u(x)) \, dx
$$

(1.18)

are equivalent in the sense that, if $\mu$ is a measure minimizer of (1.17), then $u_\mu \in K^+$ and $u_\mu$ is a minimizer of (1.18); conversely, if $u$ is a minimizer of (1.18) on $K^+$, then there exists $\mu \in \mathcal{M}_0(\Omega)$ such that $u = u_\mu$ and $\mu$ is a minimizer for (1.17).
**Proof**: This is an immediate consequence of Theorem 1 which claims that

\[ \{ u_\mu \mid \mu \in \mathcal{M}_0(\Omega) \} = K^+ \]

**Remark 3**: In the case where \( g(x,.) \) is strictly convex for a.e. \( x \) in \( \Omega \), then the solution \( u \) to (1.18) is of course unique. Moreover, if \( u > 0 \) q.e. on \( \Omega \), then it results from Proposition 2 that the measure \( \mu \) minimizing (1.17) is unique too.

One can also try to minimize

\[ \int_{\Omega} g(x,u(x)) \, dx \]  \hspace{1cm} (1.19)

over \( K \). In this case one has:

**Proposition 3**: If

\[ g(x,s) > g(x,0) \quad \text{for a.e. } x \in \Omega, \; \forall \; s < 0 \]

then (1.18) is equivalent to

\[ \inf_{u \in K} \int_{\Omega} g(x,u(x)) \, dx \]  \hspace{1cm} (1.20)

(i.e. these two problems have the same infimum and the same minimizers).

**Proof**: Since \( f \geq 0 \) implies that \( 0 \in K \), by (1.10) we have that \( u^+ \in K^+ \) for every \( u \in K \). Therefore, we will be done if we can prove that

\[ \int_{\Omega} g(x,u) \, dx > \int_{\Omega} g(x,u^+) \, dx \]

for every \( u \in K \setminus K^+ \). Let us fix \( u \in K \setminus K^+ \). We have \( \text{meas}(\{ u < 0 \}) > 0 \) and \( g(x,u) > g(x,0) \) a.e. on \( \{ u < 0 \} \). Hence,

\[ \int_{\Omega} g(x,u(x)) \, dx = \int_{\{ u \geq 0 \}} g(x,u(x)) \, dx + \int_{\{ u < 0 \}} g(x,u(x)) \, dx > \]

\[ \int_{\{ u \geq 0 \}} g(x,u(x)) \, dx + \int_{\{ u < 0 \}} g(x,0) \, dx = \int_{\Omega} g(x,u^+(x)) \, dx. \]

The proof is complete.
2. SOME PROPERTIES OF THE SOLUTIONS

In this section we collect some properties of the solutions \( u \) to (1.18) or (1.20), under some special but mild assumptions on \( g \). First of all, for \( x \in \Omega \) introduce the function \( \Psi \) defined by
\[ \Psi(x) = \text{Sup}\{s \mid g(x, s) \text{ is decreasing on } (-\infty, s)\} \tag{2.1} \]

**Theorem 2**: Assume that
\[ f \geq 0 \quad , \quad f \neq 0 \quad \text{in any connected component of } \Omega \tag{2.2} \]
\[ \inf_{\Omega'} \Psi = m_{\Omega'} > 0 \quad \forall \quad \Omega' \subset \subset \Omega \]

\( \inf_{\Omega'} \) denotes the essential infimum on \( \Omega' \) with respect to the Lebesgue measure). Let \( u \) be a minimizer of (1.18) or (1.20). Then
\[ \inf_{\Omega'} u = C_{\Omega'} > 0 \quad \forall \quad \Omega' \subset \subset \Omega. \tag{2.3} \]

Therefore, the measure \( \mu \in \mathcal{M}_0(\Omega) \) which satisfies \( u = u_\mu \) is uniquely determined by \( u \) and is a Radon measure.

**Remark 4**: In the case we are considering, \( \Psi > 0 \) a.e. in \( \Omega \), and thus \( g(x, .) \) is decreasing for a.e. \( x \in (-\infty, 0) \) and \( g(x, s) > g(x, 0) \) for a.e. \( x \) in \( \Omega \) and for every \( s < 0 \). So, (1.18) and (1.20) have the same solutions.

**Proof**: Without loss of generality we can assume that \( \Omega \) is connected. Set
\[ K^\Psi = \{v \in H^1_0(\Omega) \mid v \leq \Psi \text{ a.e. in } \Omega\} \tag{2.4} \]
and introduce \( z \) the solution to
\[ z \in K^\Psi, \quad -\Delta z - f, v - z \geq 0 \quad \forall \ v \in K^\Psi \tag{2.5} \]

\( (K^\Psi \) is not empty since \( 0 \in K^\Psi \). Since \( u \in K \), we have \( \Delta u + f \geq 0 \). Moreover, \( \Delta z + f \geq 0 \), so that \( u \vee z \in K \).

Next, compute
\[ \int_\Omega g(x, (u \vee z)(x)) \, dx = \int_{\{u < z\}} g(x, z(x)) \, dx + \int_{\{u \geq z\}} g(x, u(x)) \, dx \]

Since \( z \leq \Psi \), if \( \{u < z\} \) has a positive Lebesgue measure one would get by the definition of \( \Psi \) (see (2.1)): 
\[ \int_{\Omega} g(x, (u \vee z)(x)) \, dx < \int_{\Omega} g(x, u(x)) \, dx \]

and a contradiction with the definition of \( u \). Thus \( u \geq z \) a.e. in \( \Omega \).

We want to prove that \( \inf_{\Omega'} z > 0 \) for every \( \Omega' \subset \subset \Omega \). For this, fix \( \Omega' \subset \subset \Omega \) and choose \( \Omega'' \) connected and smooth such that

\[ \Omega' \subset \subset \Omega'' \subset \subset \Omega \quad \text{and} \quad f \neq 0 \quad \text{in} \quad \Omega''. \]

Let \( w \) be the solution to

\[ w \in H^1_0(\Omega'') \quad , \quad -\Delta w = f \quad \text{in} \quad \Omega''. \]

Since \( w < +\infty \) q.e. in \( \Omega'' \), there exists a constant \( t > 0 \) such that the measure \( f \) is not identically zero on \( \{ w \leq t \} \). Let \( \chi \) be the characteristic function of \( \{ w \leq t \} \) and let \( \theta \) be the solution to

\[ \theta \in H^1_0(\Omega'') \quad , \quad -\Delta \theta = k \chi f \quad \text{in} \quad \Omega'', \]

where \( k = m_{\Omega''}/t \). Since \( k \chi f \neq 0 \) in \( \Omega'' \), by the strong maximum principle we have

\[ \inf_{\Omega'} \theta = C^{'}_{\Omega'} > 0 \]

Since \( \chi f \leq f \), an easy comparison argument shows that \( \theta \leq kw \) q.e. in \( \Omega'' \), hence \( \theta \leq m_{\Omega''} \) q.e. in \( \{ w \leq t \} \). We claim that \( \theta \leq m_{\Omega''} \) q.e. in \( \Omega'' \). This is a consequence of the Domination Principle of potential theory (see [D.], 1. V. 10). An alternative variational proof is the following: if we multiply by \((\theta - m_{\Omega''})^+\) both sides of the equation which defines \( \theta \), we get

\[ \int_{\Omega''} |\nabla (\theta - m_{\Omega''})^+|^2 \, dx = \langle -\Delta \theta, (\theta - m_{\Omega''})^+ \rangle \]

\[ = \langle k \chi f, (\theta - m_{\Omega''})^+ \rangle = k \int_{\{w \leq t\}} (\theta - m_{\Omega''})^+ \, df = 0, \]

hence \((\theta - m_{\Omega''})^+ = 0\) q.e. in \( \Omega'' \), which implies \( \theta \leq m_{\Omega''} \) q.e. in \( \Omega'' \). Now, if we denote by \( \tilde{\theta} \) the extension of \( \theta \) by 0 outside \( \Omega'' \), we easily get (see the Appendix, Lemma A)

\[ -\Delta \tilde{\theta} \leq k \chi f \leq f \quad \text{in} \quad \Omega. \]

Since \( \tilde{\theta} \leq m_{\Omega''} \leq \Psi \) a.e. in \( \Omega'' \) and \( \tilde{\theta} = 0 \leq \Psi \) a.e. in \( \Omega' \subset \subset \Omega'' \), we have \( \tilde{\theta} \in K^\Psi \). Since \( z \) is the largest subsolution of the equation \( -\Delta v = f \) which belongs to \( K^\Psi \) (see [K.S.]), we obtain \( z \geq \tilde{\theta} \) a.e. in \( \Omega \), hence

\[ u \geq z \geq \tilde{\theta} \geq C^{'}_{\Omega'} > 0 \quad \text{a.e. in} \quad \Omega'. \]

This concludes the proof of (2.3). The assertions regarding the measure \( \mu \) follow from Proposition 2 and Remark 2.
Let us consider the solution $u_0$ to the problem
\[ u_0 \in H^1_0(\Omega) \ , \ \ -\Delta u_0 = f \ \text{in} \ \Omega. \] (2.6)

**Remark 5**: The function $u_0$ is the solution of (0.2) corresponding to $\mu = 0$. Note also that $u_0$ is the largest element of $K$. This is a consequence of the maximum principle, since for any $u \in K$ one has $-\Delta u \leq f = -\Delta u_0$. In particular for any $\mu \in M_0(\Omega)$ one has $u_\mu \leq u_0$ q.e. in $\Omega$.

If $E$ is a subset of $\Omega$, the essential closure of $E$ is the closed set
\[ \text{ess. cl } E = \{ x \in \Omega : \text{meas}(E \cap B(x, \rho)) > 0 \ \forall \ \rho > 0 \}. \]

Then, if $\text{Supp } \nu$ denotes the support of a measure $\nu$ one has:

**Theorem 3**: Assume (1.6). Let $u$ be a minimizer of (1.18) or (1.20). Then the Radon measure
\[ \nu = \Delta u + f \]
satisfies
\[ \text{Supp } \nu \subseteq \text{ess. cl } \{ u_0 \geq \Psi \}. \]
(We do not assume here necessarily that (1.18), (1.20) are equivalent, but (1.20) is assumed to have a minimizer).

**Proof**: Set $U = \Omega \setminus \text{ess. cl } \{ u_0 \geq \Psi \}$. Then, $U$ is open and $u_0 < \Psi$ a.e. in $U$. Let $B$ be a ball included in $U$. Assume by contradiction that $\nu \neq 0$ on $B$ and introduce $\xi$ the solution to
\[ \xi \in H^1_0(B) \ , \ \ -\Delta \xi = \nu = \Delta u + f \ \text{in} \ B. \] (2.7)
(This makes sense since $\nu \in H^{-1}(B)$). Then, it is clear that $\xi \geq 0$ and if $\nu \neq 0$ on $B$ we have $\xi > 0$ on $B$ by the strong maximum principle. Denote by $\tilde{\xi}$ the extension of $\xi$ by 0 outside $B$. Since $\xi \geq 0$ and
\[ -\Delta \xi \leq \Delta u + f \ \text{on} \ B \]
one has (see the Appendix)
\[ -\Delta \tilde{\xi} \leq \Delta u + f \ \text{on} \ \Omega, \]
hence,
\[ u + \tilde{\xi} \in K^+ \text{ or } K \]
if $u \in K^+$ or $K$ respectively. Next, since $u + \tilde{\xi} \in K$, one has $u + \tilde{\xi} \leq u_0$, hence $u + \xi \leq u_0 < \Psi$ a.e. on $B$.
Then, since $\xi > 0$ on $B$,
\[ \int_\Omega g(x, u + \xi) \ dx = \int_B g(x, u + \xi) \ dx + \int_{\Omega \setminus B} g(x, u) \ dx < \int_\Omega g(x, u) \ dx \]
and a contradiction with the definition of \( u \). Thus \( \nu \equiv 0 \) on \( B \), and since this holds for any \( B \) the result follows.

In the same spirit, one has:

**Theorem 4**: Assume that \( u \) is a minimizer of (1.18) or (1.20). Then, for any \( \epsilon > 0 \), the Radon measure

\[
\nu = \Delta u + f
\]

satisfies

\[
\text{Supp } \nu \subset \text{ess.cl. } \{ u \geq \Psi - \epsilon \}.
\]

**Proof**: Set \( U_\epsilon = \Omega \setminus \text{ess.cl. } \{ u \geq \Psi - \epsilon \} \). Then \( U_\epsilon \) is open and \( u < \Psi - \epsilon \) a.e. in \( U_\epsilon \). Let \( B \) be a ball included in \( U_\epsilon \). We want to prove that \( \nu \equiv 0 \) on \( B \). If not, introduce the solution \( w \) to

\[
w \in H^1_0(B), \quad -\Delta w = \nu \quad \text{in} \quad B.
\]

Since \( w < +\infty \) q.e. in \( B \) and \( \nu \neq 0 \) in \( B \), there exists a constant \( t > 0 \) such that \( \nu \neq 0 \) in \( \{w \leq t\} \). Let \( \chi \) be the characteristic function of \( \{w \leq t\} \) and let \( \xi \) be the solution to

\[
\xi \in H^1_0(B), \quad -\Delta \xi = \chi \nu \quad \text{in} \quad B. \tag{2.8}
\]

Since \( \chi \nu \geq 0 \) and \( \chi \nu \neq 0 \) in \( B \), by the strong maximum principle we have \( \xi > 0 \) in \( B \). Since \( \chi \nu \leq \nu \), we have also \( \xi \leq w \) q.e. in \( B \), hence, in particular, \( \xi \leq t \) q.e. in \( \{w \leq t\} \). Arguing as in the proof of Theorem 2 we obtain \( \xi \leq t \) q.e. in \( B \).

Since \( -\Delta \xi \geq 0 \), for any \( 0 < \delta < 1 \) one has

\[
-\delta \Delta \xi \leq -\Delta \xi \leq \nu = \Delta u + f \quad \text{in} \quad B. \tag{2.9}
\]

If \( \tilde{\xi} \) denotes the extension of \( \xi \) by 0 outside \( B \), then, clearly, for \( \delta \) small enough, (i.e. \( 0 < \delta < \min(1, \epsilon/t) \)), one has

\[
u + \delta \tilde{\xi} \in K^+ \text{ or } K, \quad u + \delta \tilde{\xi} < \Psi \quad \text{a.e. on} \quad B \tag{2.10}
\]

(see the Appendix, Lemma A). Thus, if \( \nu \) does not vanish identically on \( B \), one has \( \xi > 0 \) on \( B \) and

\[
\int_{\Omega} g(x, u + \delta \tilde{\xi}) \, dx = \int_{B} g(x, u + \delta \tilde{\xi}) \, dx + \int_{\Omega \setminus B} g(x, u) \, dx < \int_{\Omega} g(x, u) \, dx
\]

and a contradiction with the definition of \( u \). Thus, \( \nu \equiv 0 \) on \( B \) and this ends the proof of the theorem.

As a consequence we have
Corollary 2: Assume that \( u \) is a minimizer of (1.18) or (1.20). Then, if \( u - \Psi \) is continuous, the Radon measure
\[
\nu = \Delta u + f
\]
satisfies
\[
\text{Supp} \, \nu \subset \{ u \geq \Psi \}.
\]

Proof: This follows from the fact that
\[
\text{ess.cl.} \, \{ u \geq \Psi - \epsilon \} \subset \{ u \geq \Psi - \epsilon \}.
\]
Thus
\[
\text{Supp} \, \nu \subset \cap_{\epsilon > 0} \{ u \geq \Psi - \epsilon \} = \{ u \geq \Psi \}.
\]

Let us now get some information on the set \( \{ u < \Psi \} \). For that, let us first prove:

Proposition 4: Assume \( u, \hat{\Psi} \in C^0(\Omega) \cap H^1(\Omega) \),
\[
\Delta \hat{\Psi} + f \geq 0 \, \text{ in } \Omega \ , \quad \Delta u + f \geq 0 \, \text{ in } \Omega.
\] (2.11)

Let \( C \) be a connected component of \( \{ u < \hat{\Psi} \} \) and let \( z = u + \chi_C(\hat{\Psi} - u) \), where \( \chi_C \) is the characteristic function of \( C \). Then
\[
z \in H^1(\Omega) \ , \quad \Delta z + f \geq 0 \, \text{ in } \Omega.
\] (2.12)

\( C^0(\Omega) \) is the space of continuous functions in \( \Omega \).

Proof: Let us prove that \( z \in H^1(\Omega) \). For every \( \epsilon > 0 \) and for every \( \Omega' \subset \subset \Omega \) let
\[
C_\epsilon = \{ x \in C \cap \Omega' : \hat{\Psi}(x) > u(x) + \epsilon \}.
\]

Since \( C_\epsilon \subset \subset C \), there exists \( \varphi_\epsilon \in C^1_0(\Omega) \) such that \( 0 \leq \varphi_\epsilon \leq 1 \) in \( \Omega \), \( \varphi_\epsilon = 1 \) in \( C_\epsilon \), and \( \varphi_\epsilon = 0 \) outside \( C \).

Then the function
\[
z_\epsilon = [(1 - \varphi_\epsilon)u + \varphi_\epsilon(\hat{\Psi} - \epsilon)] \lor u
\]
belongs to \( H^1(\Omega) \) (see [K.S.]). Using the fact that \( z_\epsilon = u \) in \( \Omega' \setminus C_\epsilon \) and \( z_\epsilon = \hat{\Psi} - \epsilon \) in \( C_\epsilon \), we obtain that \( z_\epsilon \) is bounded in \( H^1(\Omega') \). Since
\[
z_\epsilon \to z \quad \text{in} \quad L^2(\Omega') \quad \text{as} \quad \epsilon \to 0,
\]
we obtain \( z \in H^1(\Omega') \), hence \( z \in H^1_{loc}(\Omega) \). Since \( \nabla z = \nabla u \) a.e. in \( \Omega \setminus C \) and \( \nabla z = \nabla \hat{\Psi} \) a.e. in \( C \), we conclude that \( \nabla z \in L^2(\Omega) \), hence \( z \in H^1(\Omega) \).
Next, consider \( w \) the solution to

\[
w \in K^z = \{ v \in H^1(\Omega) \mid v \leq z \text{ a.e. in } \Omega, \ v - z \in H^1_0(\Omega) \}\]

\[
< -\Delta w - f, v - w > \geq 0 \quad \forall v \in K^z.
\] (2.13)

First, we have \( u \leq w \).

Indeed, plugging \( v = w + (u - w)^+ \), which belongs to \( K^z \), in (2.13), we get

\[
< -\Delta w - f, (u - w)^+ > \geq 0.
\]

Subtracting this to

\[
< -\Delta u - f, (u - w)^+ > \leq 0
\]

(note that \( (u - w)^+ \in H^1_0(\Omega) \)) we obtain

\[
< -\Delta (u - w), (u - w)^+ > \leq 0
\]

and thus \( u \leq w \).

Next, we prove that \( z = w \).

Indeed, taking \( v = z \) in (2.13) and using the inequality \( w \leq z \) together with (2.11) we get

\[
< -\Delta w - f, z - w > \geq 0
\]

\[
< -\Delta \hat{\Psi} - f, z - w > \leq 0.
\]

Subtracting the first equation from the second one we obtain

\[
< -\Delta (\hat{\Psi} - w), z - w > \leq 0,
\]

which reads also, since \( u \leq w \leq z \),

\[
< -\Delta (z - w), z - w > \leq 0.
\]

Hence \( z = w \). Since \( \Delta w + f \geq 0 \), the result follows.

A consequence of this proposition is:

**Theorem 5**: Assume that \( \hat{\Psi} \) is a function satisfying \( \hat{\Psi} \in C^0(\Omega) \cap H^1(\Omega) \), \( \Delta \hat{\Psi} + f \geq 0 \) in \( \Omega \), and

\[
g(x, s) > g(x, \hat{\Psi}(x)) \quad \forall s < \hat{\Psi}(x), \text{ for a.e. } x \in \Omega.
\] (2.14)

Let \( u \in C^0(\Omega) \) be a minimizer of (1.18) or (1.20). Then the set \( \{ u < \hat{\Psi} \} \) cannot have a connected component \( C \) such that \( C \subset \subset \Omega \).
Proof: Set \( z = u + \chi_C(\hat{\Psi} - u) \). Then, it results from the previous proposition that \( z \in K^+ \) or \( K \). Moreover,

\[
\int_{\Omega} g(x, z) \, dx = \int_{C} g(x, \hat{\Psi}) \, dx + \int_{\Omega \setminus C} g(x, u) \, dx < \int_{\Omega} g(x, u) \, dx.
\]

This contradicts the fact that \( u \) is a minimizer and the result follows.

Remark 6: The function \( \Psi \) defined by (2.1) could be a particular \( \hat{\Psi} \). However, as we just saw, properties like (2.14) are relevant here and are weaker than to impose, for instance, that \( g(x, .) \) is decreasing. We denote by \( \hat{\Psi} \) functions for which we assume a property of this type leaving the notation \( \Psi \) for the function defined by (2.1).

An other consequence is:

Theorem 6: Assume that \( \hat{\Psi} \) is a function satisfying \( \hat{\Psi} \in C^0(\Omega) \cap H^1(\Omega), \hat{\Psi} \geq 0, \Delta \hat{\Psi} + f \geq 0 \) in \( \Omega \), and

\[
g(x, s) > g(x, \hat{\Psi}(x)) \quad \forall s > \hat{\Psi}(x), \text{ for a.e. } x \in \Omega.
\]

(2.15)

Let \( u \in C^0(\Omega) \) be a solution to (1.18) or (1.20). Then the set \( \{u > \hat{\Psi}\} \) cannot have a connected component \( C \) such that

\[
C \subset \subset \{u \geq \hat{\Psi}\}^0.
\]

(2.16)

\( E^0 \) denotes the interior of the set \( E \).

Proof: Assume that \( C \) is a connected component of \( \{u > \hat{\Psi}\} \) such that (2.16) holds. Set

\[
\tilde{u} = \begin{cases} 
\hat{\Psi}(x) & \text{if } x \in C \\
u(x) & \text{if } x \in \Omega \setminus C.
\end{cases}
\]

On \( \{u \geq \hat{\Psi}\}^0 \) one has

\[
\tilde{u} = \vee_i \{\hat{\Psi} + \chi_{C_i}(u - \hat{\Psi})\}
\]

where \( C_i \) denote the connected components of \( \{u > \hat{\Psi}\} \) distinct from \( C \). Thus, (see (2.12))

\[
\Delta \tilde{u} + f \geq 0 \quad \text{on} \quad \{u \geq \hat{\Psi}\}^0.
\]

Moreover, if we denote by \( \overline{C} \) the closure of \( C \) in \( \Omega \), on \( \Omega \setminus \overline{C} \) one has

\[
\Delta \tilde{u} + f = \Delta u + f \geq 0.
\]

Since the two open sets \( \Omega \setminus \overline{C} \}, \{u \geq \hat{\Psi}\}^0 \) cover \( \Omega \), we have

\[
\Delta \tilde{u} + f \geq 0 \quad \text{in} \quad \Omega.
\]
Then, it is clear that $\tilde{u} \in K^+$ or $K$. So, we get
\[
\int_{\Omega} g(x, \tilde{u}) \, dx = \int_{C} g(x, \hat{\Psi}) \, dx + \int_{\Omega \setminus C} g(x, u) \, dx < \int_{\Omega} g(x, u) \, dx.
\]
This contradicts the fact that $u$ is a minimizer and the result follows.

3. THE ADJOINT PROBLEM

In this section we assume that $g(x, \cdot)$ is smooth for a.e. $x \in \Omega$. More precisely, we assume that this function is differentiable for a.e. $x \in \Omega$ and that the derivative
\[
g_s(x, s) \text{ is a Carathéodory function.}
\]  
Moreover, we assume that there exist $a_1 \in L^2(\Omega)$, and a constant $b_1$ for which
\[
|g_s(x, s)| \leq a_1(x) + b_1|s| \quad \text{for a.e. } x \in \Omega \quad \text{for all } s \in \mathbb{R}.
\]  
In this case, (see [B. DM.]), when $u$ denotes a minimizer of (1.20), it is useful to introduce the solution $v$ of the so called adjoint problem:
\[
v \in H_0^1(\Omega) \quad , \quad -\Delta v = g_s(x, u) \quad \text{in} \quad \Omega.
\]  
One has:

**Theorem 7** : Assume (1.4), (1.5), (1.6), (3.1), (3.2). If $u$ is a minimizer to (1.20) and $v$ is the solution to (3.3), then $(u, v)$ satisfies the complementarity problem
\[
\Delta u + f \geq 0 \quad , \quad v \leq 0 \quad \text{q.e. in} \quad \Omega \quad , \quad <\Delta u + f, v> = 0.
\]  
Conversely, if $(u, v) \in (H_0^1(\Omega))^2$ is a solution to (3.4) and
\[
g(x, \cdot) \text{ is convex for a.e. } x \in \Omega,
\]  
then $u$ is a minimizer to (1.20).

**Proof** : If $u$ is a minimizer of (1.20) on $K$, then for any $t \in (0, 1)$ one has
\[
\int_{\Omega} [g(x, u + t(w - u)) - g(x, u)] \, dx \geq 0 \quad \forall w \in K.
\]
Dividing by $t$ and letting $t \to 0$ we get
\[ \int_{\Omega} g_s(x, u)(w - u) \, dx \geq 0 \quad \forall w \in K. \]

Hence, by (3.3),
\[ \langle -\Delta v, w - u \rangle \geq 0 \quad \forall w \in K, \]
and thus
\[ \langle -\Delta w + \Delta u, v \rangle \geq 0 \quad \forall w \in K. \tag{3.6} \]
Taking $w = u_0$ in (3.6) we obtain
\[ \langle \Delta u + f, v \rangle \geq 0. \tag{3.7} \]

Next, if $\varphi \geq 0$, $\varphi$ smooth, denote by $w$ the solution to
\[ w \in H^1_0(\Omega), \quad -\Delta w = -\Delta u - \varphi. \]

One has clearly
\[ \Delta w + f = \Delta u + f + \varphi \geq 0 \]
so that $w \in K$. Plugging this $w$ in (3.6) we obtain
\[ \langle \varphi, v \rangle \leq 0 \quad \forall \varphi \geq 0, \quad \varphi \text{ smooth.} \]

Thus, $v \leq 0$ q.e. in $\Omega$. Since $\Delta u + f \geq 0$ in $\Omega$ we get
\[ \langle \Delta u + f, v \rangle \leq 0; \]
combining this with (3.7) we obtain
\[ \langle \Delta u + f, v \rangle = 0 \]
and (3.4).

If now (3.4) holds, then we deduce
\[ \langle -\Delta v, w - u \rangle = \langle -\Delta w + \Delta u, v \rangle = \langle -\Delta w - f, v \rangle \geq 0 \quad \forall w \in K. \]

Thus, if $g(x, \cdot)$ is convex:
\[ \int_{\Omega} [g(x, w) - g(x, u)] \, dx \geq \int_{\Omega} g_s(x, u)(w - u) \, dx = \langle -\Delta v, w - u \rangle \geq 0 \quad \forall w \in K. \]

This concludes.

As we are about to see, this function $v$ is related to the measure $\mu \in M_0(\Omega)$ such that $u = u_\mu$. 

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More precisely, recall that $\Psi$ is defined by (2.1) and thus one has in particular

$$g_s(x,s) \leq 0 \quad \forall s \leq \Psi(x) \quad \text{for a.e. } x \in \Omega.$$  \hspace{1cm} (3.8)

Recall also that when $\Psi(x) \geq 0$ for a.e. $x \in \Omega$ then the problems (1.17), (1.18), (1.20) are equivalent (see remark 4).

Then, we can prove :

**Theorem 8** : Assume (1.4), (1.5), (2.2), (3.1), (3.2) and that

$$\inf_{\Omega'} \Psi = m_{\Omega'} > 0 \quad \forall \Omega' \subset\subset \Omega.$$  \hspace{1cm} (3.9)

Let $\mu \in \mathcal{M}_0(\Omega)$ be a minimizer of (1.17). Denote by $u_\mu$ the solution to (0.2) and by $v_\mu$ the solution of (3.3) corresponding to $u = u_\mu$. Then, one has

$$v_\mu \leq 0 \quad \text{q.e. in } \Omega, \quad v_\mu = 0 \quad \mu - \text{a.e. in } \Omega.$$  \hspace{1cm} (3.10)

Conversely, when (3.5) holds, if $\mu$ is a measure in $\mathcal{M}_0(\Omega)$ which satisfies (3.10), then $\mu$ is a minimizer of (1.17).

**Proof** : Recall that in the case we are considering the problems (1.17), (1.18), (1.20) are equivalent (see Corollary 1 and Proposition 3). Moreover, it results from Theorem 2 that if $u$ is a minimizer to (1.18), then

$$u > 0 \quad \text{q.e. in } \Omega$$

so that the measure $\mu \in \mathcal{M}_0(\Omega)$ such that $u = u_\mu$ is uniquely determined (see Proposition 2).

Now, if $u_\mu$ is a minimizer of (1.20), then the inequality

$$v_\mu \leq 0 \quad \text{q.e. in } \Omega$$

results from Theorem 7. Moreover, if $\nu = \Delta u_\mu + f$, (3.4) gives also

$$< \nu, v_\mu >= 0.$$  \hspace{1cm} (3.14)

Thus, $v_\mu = 0 \quad \nu - \text{a.e. in } \Omega$ and thus also $\mu - \text{a.e. in } \Omega$ (see Proposition 2).

Conversely, assume that $\mu \in \mathcal{M}_0(\Omega)$ is such that (3.10) holds and let $u_\mu$ be the solution to (0.2). Since $v_\mu \in H^1_0(\Omega) \cap L^2(\Omega, \mu)$ and $v_\mu = 0 \mu$-a.e. in $\Omega$, (0.3) gives

$$< -\Delta u_\mu, v_\mu > = \int_{\Omega} \nabla u_\mu \nabla v_\mu \, dx + \int_{\Omega} u_\mu \, v_\mu \, d\mu = < f, v_\mu >.$$
Thus
\[ \langle \Delta u_\mu + f, v_\mu \rangle = 0 \]
and \((u_\mu, v_\mu)\) satisfies the complementarity conditions (3.4). The result is then a consequence of Theorem 7.

Corresponding to \(u_0\) defined by (2.6) one can associate the solution \(v_0\) to
\[ v_0 \in H^1_0(\Omega), \quad -\Delta v_0 = g_s(x, u_0) \quad \text{in} \quad \Omega. \quad (3.11) \]

Then, we can prove:

**Theorem 9**: Assume that \(\Omega\) is connected, \(\Psi\) satisfies (3.9) and
\[ g_s(x, s) < 0 \quad \forall s < \Psi(x), \quad \text{for a.e.} \quad x \in \Omega, \quad (3.12) \]
\[ g_s(x, s) > 0 \quad \forall s > \Psi(x), \quad \text{for a.e.} \quad x \in \Omega. \quad (3.13) \]
Let \(\mu \in M_0(\Omega)\) be a minimizer of (1.17). Then \(\mu\) is a Radon measure and

(i) if \(\mu(\Omega) = 0\), then \(u_\mu = u_0\) and \(v_0 \leq 0\) q.e. in \(\Omega\). Moreover, either \(u_0 = \Psi\) a.e. in \(\Omega\) or the set \(\{u_0 < \Psi\}\) has a positive Lebesgue measure.

(ii) if \(\mu(\Omega) > 0\), then either \(u_\mu = \Psi\) or the sets
\[ \{u_\mu < \Psi\}, \quad \{u_\mu > \Psi\} \]
have both a positive Lebesgue measure.

Moreover, if the function \(v_\mu\) associated to \(u_\mu\) by (3.3) is continuous, then
\[ u_\mu = \Psi \quad \text{a.e. in} \quad \text{Supp} \ \mu, \quad (3.14) \]
if \(U\) denotes the interior of the support of \(\mu\), we have
\[ \Psi \in H^1(U), \quad \Delta \Psi + f \geq 0 \quad \text{in} \quad U, \quad \mu\vert_U = (\Delta \Psi + f)/\Psi \quad \text{in} \quad U \]
(\(\mu\vert_U\) denotes the restriction of the measure \(\mu\) to \(U\) and the pointwise values of \(\Psi\) are determined according to (1.2)). The same result holds with \(U = \Omega\) provided \(u_\mu = \Psi\) a.e. in \(\Omega\).

**Proof**: The fact that \(\mu\) is a Radon measure was proved in Theorem 2.

(i) If \(\mu(\Omega) = 0\), then \(\mu = 0\) and \(u_\mu = u_0\). By (3.10) this implies \(v_0 \leq 0\) q.e. in \(\Omega\). If the assertion on \(u_0\) and \(\Psi\) is false, then \(u_0 \geq \Psi\) a.e. in \(\Omega\) and \(u_0 > \Psi\) on a set of positive Lebesgue measure. By (3.13) this implies \(g_s(x, u_0) \geq 0\) a.e. in \(\Omega\) and \(g_s(x, u_0) \neq 0\). Since
\[ -\Delta v_0 = g_s(x, u_0), \]

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by the strong maximum principle, \( v_0 > 0 \) in \( \Omega \). This contradicts \( v_0(x) \leq 0 \) q.e. in \( \Omega \).

(ii) Assume \( \mu(\Omega) > 0 \).

If our first assertion is wrong, then either

\[
\begin{align*}
\text{a) } & \quad u_\mu(x) \geq \Psi(x) \text{ a.e. in } \Omega \text{ and } \text{meas}\{u_\mu > \Psi\} > 0 \\
\text{b) } & \quad u_\mu(x) \leq \Psi(x) \text{ a.e. in } \Omega \text{ and } \text{meas}\{u_\mu < \Psi\} > 0
\end{align*}
\]

In the first case, taking (3.13) into account we get

\[
-\Delta v_\mu = g_s(x, u_\mu) \geq 0
\]

and \( g_s(x, u_\mu) \neq 0 \). By the strong maximum principle this implies \( v_\mu > 0 \) and a contradiction to (3.10).

In case b), by (3.12), we have

\[
-\Delta v_\mu = g_s(x, u_\mu) \leq 0
\]

and thus \( v_\mu < 0 \) q.e. in \( \Omega \), since \( g_s(x, u_\mu) \neq 0 \). But then (see (3.10)) we would have \( \mu(\Omega) = 0 \) and a contradiction.

To prove (3.14), note that by (3.10), since \( v_\mu \) is continuous, one has

\[
v_\mu = 0 \quad \text{a.e. on } \text{Supp } \mu,
\]

and since \( v_\mu \in H^2_{\text{loc}}(\Omega) \) (see [G.T.]) we get

\[
g_s(x, u_\mu) = -\Delta v_\mu = 0 \quad \text{a.e. on } \text{Supp } \mu.
\]

Thus, by (3.12) and (3.13),

\[
u_\mu(x) = \Psi(x) \quad \text{a.e. on } \text{Supp } \mu.
\]

The end of the proof is an easy consequence of Proposition 2 and Theorem 2.

**Remark 7**: The function \( v_\mu \) associated to \( u_\mu \) by (3.3) is continuous, in particular, when

\[
f \in W^{-1,s}(\Omega), \quad s > n/3, \quad a_1 \in L^p(\Omega), \quad p > n/2
\]

(see (3.2) for the definition of \( a_1 \) and recall that \( 0 \leq u_\mu \leq u_0 \)).
4. THE RADIAL CASE

In this section we assume that

\[ \Omega = B(0, R) = \{ x \in \mathbb{R}^n : |x| < R \} \]

and that our data \( f \) and \( g \) are radially symmetric.

More precisely, denote by \( O(n) \) the group of orthogonal transformations \( \rho \) of \( \mathbb{R}^n \), i.e. the group of linear transformations such that

\[ \rho^T \rho = \rho \rho^T = I. \]

(\( I \) denotes the identity of \( \mathbb{R}^n \), \( \rho^T \) the transpose of \( \rho \)). Moreover, for any distribution \( S \), denote by \( \rho S \) the distribution defined by

\[ \langle \rho S, \varphi \rangle = \langle S, \varphi \circ \rho \rangle \quad \forall \varphi \in \mathcal{D}(\Omega). \]  \hspace{1cm} (4.1)

(\( \mathcal{D}(\Omega) \) is the space of \( C^\infty \) function with compact support).

Then we assume that

\[ \rho f = f \quad \forall \rho \in O(n). \]  \hspace{1cm} (4.2)

Moreover, for every \( s \), we suppose that there exists a function \( \tilde{g}(., s) \) on \( (0, R) \) such that

\[ g(x, s) = \tilde{g}(|x|, s) \quad \forall s \in \mathbb{R}, \text{ for a.e. } x \in \Omega. \]  \hspace{1cm} (4.3)

As a consequence, the function \( \Psi \) defined in (2.1) satisfies

\[ \Psi(x) = \tilde{\Psi}(|x|) \quad \text{for a.e. } x \in \Omega \]  \hspace{1cm} (4.4)

for some function \( \tilde{\Psi} \) defined on \( (0, R) \). In what follows, for any radially symmetric function \( h \), we will denote by \( \tilde{h} \) the function defined on \( (0, R) \) such that

\[ h(x) = \tilde{h}(|x|). \]

It should be remarked that, if \( h \in H^1(\Omega) \), then \( \tilde{h} \in H^1(\epsilon, R) \) for any \( \epsilon > 0 \) and \( \tilde{h} \) is a continuous function on \( (0, R) \). Of course, it results that \( h \) itself is a continuous function on \( \Omega \\setminus \{0\} \).

Under these assumptions we do not know if any minimizer of (1.18) is radially symmetric. However, we have :

**Proposition 5**: Assume (1.4), (1.5), (1.6), (4.2), (4.3). If

\[ \tilde{g}(r, \cdot) \text{ is convex for a.e. } r \in (0, R), \]  \hspace{1cm} (4.5)
then there exists at least one radially symmetric minimizer to (1.18). If

\[ \tilde{g}(r,.) \text{ is strictly convex for a.e. } r \in (0, R), \]

then the unique minimizer of (1.18) is radially symmetric.

**Proof:** First, remark that, if \( u \in K^+ \), then

\[ \Delta(\rho u) + f = \rho(\Delta u + f) \geq 0 \quad \forall \rho \in O(n), \]

so that \( \rho u \in K^+ \). Moreover, if \( u \) is a minimizer of (1.18), so does \( \rho u \), since

\[ \int_{\Omega} g(x, \rho u(x)) \, dx = \int_{\Omega} \tilde{g}(|x|, u(\rho^{-1}(x))) \, dx = \int_{\Omega} g(x, u(x)) \, dx \quad (4.7) \]

(the last equality results from the change of variable \( y = \rho^{-1}(x) \) in the middle integral).

Now, if \( g(x, s) \) is strictly convex, then, by uniqueness of the minimizer of (1.18), (see remark 3), one has

\[ \rho u = u \quad \forall \rho \in O(n) \]

and thus \( u \) is radially symmetric.

If \( g(x, .) \) is only supposed to be convex, and if \( d\rho \) denotes the normalized Haar measure on \( O(n) \), i.e. the one such that \( \int_{O(n)} d\rho = 1 \), one deduces from (4.7) by the Fubini Theorem and Jensen’s Inequality:

\[ \int_{\Omega} g(x, u(x)) \, dx = \int_{O(n)} \int_{\Omega} g(x, u(x)) \, dx \, d\rho = \int_{O(n)} \int_{\Omega} g(x, \rho u(x)) \, dx \, d\rho = \]

\[ \int_{\Omega} \int_{O(n)} g(x, \rho u(x)) \, d\rho \, dx \geq \int_{\Omega} g(x, \int_{O(n)} \rho u \, d\rho) \, dx \geq \int_{\Omega} g(x, u(x)) \, dx. \]

Thus \( \int_{O(n)} \rho u \, d\rho \) is a radially symmetric minimizer to (1.18). This concludes the proof of the theorem. Note that the same type of result holds for \( K \) instead of \( K^+ \) provided (1.20) admits a minimizer.

In the rest of this section we would like to study more precisely the radially symmetric minimizers to (1.18). For that, consider a function \( \hat{\Psi} \) satisfying:

\[ \hat{\Psi} \text{ is radially symmetric } , \quad \hat{\Psi} \in H^1(\Omega) \quad , \quad \Delta \hat{\Psi} + f \geq 0 \text{ in } \Omega. \quad (4.8) \]

Then we have:

**Proposition 6:** In addition to the hypotheses of Proposition 5, assume that \( \hat{\Psi} \) satisfies (4.8) and that

\[ g(x, s) > g(x, \hat{\Psi}(x)) \quad \forall s < \hat{\Psi}(x) \quad , \quad \text{for a.e. } x \in \Omega. \quad (4.9) \]
Then, if \( u \) denotes a radially symmetric solution to (1.18) or (1.20), there exists \( r_2 \in [0, R] \) such that
\[
\tilde{u}(r) \geq \tilde{\Psi}(r) \quad \text{for} \quad 0 < r \leq r_2, \quad \tilde{u}(r) < \tilde{\Psi}(r) \quad \text{for} \quad r_2 < r < R.
\] (4.10)

**Proof:** Denote by \( r_2 \) the largest nonnegative number such that
\[
\tilde{u}(r) \geq \tilde{\Psi}(r) \quad \forall \ r < r_2.
\]
(Possibly \( r_2 = 0 \) or \( r_2 = R \)). Then one has \( \tilde{u}(r) < \tilde{\Psi}(r) \) for \( r > r_2 \). Indeed, otherwise the set \( \{ u < \tilde{\Psi} \} \) would have a connected component \( C \) such that \( C \subset \subset \Omega \), which contradicts Theorem 5.

**Remark 8:** If \( u \) and \( \tilde{\Psi} \) are radially symmetric then the assumption \( u, \tilde{\Psi} \in C^0(\Omega) \) can be dropped in Proposition 4 and in Theorems 5 and 6. In fact, this hypothesis was used only to deduce that the sets \( \{ u < \tilde{\Psi} \} \) and \( \{ u > \tilde{\Psi} \} \) are open and to prove that \( z \in H^1(\Omega) \). Since we have assumed \( n \geq 2 \) the same results can be obtained under the weaker hypothesis of continuity in \( \Omega \setminus \{0\} \), which is always satisfied in the radially symmetric case.

Moreover, we have:

**Proposition 7:** Under the assumptions of Proposition 6, suppose that \( \tilde{\Psi} \geq 0 \) and that
\[
g(x, s) > g(x, \tilde{\Psi}(x)) \quad \forall s < \tilde{\Psi}(x) \quad \text{for a.e.} \quad x \in \Omega,
\] (4.11)
\[
g(x, s) > g(x, \tilde{\Psi}(x)) \quad \forall s > \tilde{\Psi}(x) \quad \text{for a.e.} \quad x \in \Omega.
\] (4.12)

Then, if \( u \) denotes a radially symmetric solution to (1.18) or (1.20), there exist \( r_1, r_2 \), with \( 0 \leq r_1 \leq r_2 \leq R \), such that
\[
\tilde{u}(r) = \tilde{\Psi}(r) \quad \text{for} \quad 0 < r \leq r_1, \quad \tilde{u}(r) > \tilde{\Psi}(r) \quad \text{for} \quad r_1 < r < r_2,
\]
\[
\tilde{u}(r) < \tilde{\Psi}(r) \quad \text{for} \quad r_2 < r < R.
\] (4.13)

**Proof:** From Proposition 6 we know that there exists \( r_2 \) such that
\[
\tilde{u}(r) \geq \tilde{\Psi}(r) \quad \text{for} \quad 0 < r \leq r_2, \quad \tilde{u}(r) < \tilde{\Psi}(r) \quad \text{for} \quad r_2 < r < R.
\]

If, now, \( r_1 \) denotes the largest nonnegative number such that \( \tilde{u}(r) = \tilde{\Psi}(r) \) on \( (0, r_1) \), then one has
\[
\tilde{u}(r) > \tilde{\Psi}(r) \quad \text{on} \quad (r_1, r_2).
\]

Indeed, otherwise the set \( \{ u > \tilde{\Psi} \} \) would have a connected component \( C \) such that \( C \subset \subset \{ u \geq \tilde{\Psi} \}^0 \), which contradicts Theorem 6.

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Remark 9: When \( \hat{\Psi} = \Psi \), if \( r_1 = r_2 \), and if \( u \) is a minimizer of (1.20), then \( u \leq \hat{\Psi} \) and \( u \) is the largest function satisfying

\[
u \leq \hat{\Psi} , \quad \Delta u + f \geq 0.
\]

Indeed, otherwise, there would be a function \( w \) greater than \( u \) satisfying the same inequalities (see (1.10)) and such that \( w \geq u \) on a set of positive measure. This would imply that

\[
\int_\Omega g(x, w) \, dx < \int_\Omega g(x, u) \, dx
\]

(see (2.1)). Thus, in this case, the problem (1.20) admits a radially symmetric minimizer such that \( r_1 = r_2 \) if and only if this minimizer is the solution of the variational inequality

\[
z \in K^{\hat{\Psi}} = \{ v \in H^1_0(\Omega) : v(x) \leq \hat{\Psi}(x) \quad \text{a.e.} \quad x \in \Omega \} < -\Delta z + f, v - z > \geq 0 \quad \forall \, v \in K^{\hat{\Psi}}.
\]

In fact, under some additional assumptions, we can show that this cannot happen unless \( u = u_0 \) (see Theorem 10 below).

Next, we want to provide more information in the case where \( g \) is smooth.

Let \( \Psi \) be the function defined by (2.1). Assume that

\[
\Psi \in H^1(\Omega) , \quad \Delta \Psi + f \geq 0 \quad \text{in} \quad \Omega.
\]

First, remark that since \( \hat{\Psi} \in H^1(\epsilon, R) \) the limit

\[
\lim_{r \to R^-} \hat{\Psi}(r) = \hat{\Psi}(R)
\]

exists. If now this limit is 0, then, under the assumptions (3.12), (3.13) (which imply (4.11), (4.12) with \( \Psi \) instead of \( \hat{\Psi} \)), \( u = \Psi \) is the only minimizer of (1.20). So, to avoid this trivial case still remaining in the hypotheses of Theorem 9, let us assume here

\[
\Psi \geq \eta > 0 \quad \text{a.e. in} \quad \Omega,
\]

where \( \eta \) is a positive constant. Since, by (4.14), \( \Delta \Psi \) is a (signed) Radon measure on \( \Omega \), the second derivative \( \hat{\Psi}'' \) is a (signed) Radon measure on \((0, R)\). Therefore, the right and left derivatives \( \hat{\Psi}'(r^+) \) and \( \hat{\Psi}'(r^-) \) exist at any \( r \in (0, R) \). The same result holds for an arbitrary radially symmetric function \( u \) of \( K \).

Then we have:
Theorem 10: Assume (2.2), (4.2), (4.3), (4.14), (4.15). Moreover suppose that $g$ satisfies (1.4), (1.5), (3.1), (3.2), (3.12), (3.13). Then, if $u$ is a radially symmetric minimizer of (1.18) one has:

There exist $r_1$, $r_2$, with $0 \leq r_1 \leq r_2 < R$, such that

$$
\tilde{u}(r) = \tilde{\Psi}(r) \text{ for } 0 < r \leq r_1 \quad , \quad \tilde{u}(r) > \tilde{\Psi}(r) \quad \text{for } r_1 < r < r_2 \quad ,
$$

$$
\tilde{u}(r) < \tilde{\Psi}(r) \quad \text{for } r_2 < r < R.
$$

(4.16)

If $u \neq u_0$, then

$$
0 < r_1 < r_2 < R.
$$

(4.17)

In this case the measure $\mu \in \mathcal{M}_0(\Omega)$ such that $u = u_\mu$ is uniquely determined by

$$
\mu = \frac{\lambda|_{B(0, r_1)}}{\Psi} + \frac{f|_{\partial B(0, r_1)}}{\tilde{\Psi}(r_1)} H^{n-1}_{\partial B(0, r_1)} + \frac{\tilde{u}'(r_1^+) - \tilde{\Psi}'(r_1^-)}{\tilde{\Psi}(r_1)} H^{n-1}_{\partial B(0, r_1)},
$$

(4.18)

where $\lambda = \Delta \Psi + f$ (which is a Radon measure by (4.14)), $H^{n-1}_{\partial B(0, r_1)}$ denotes the $(n - 1)$-dimensional Hausdorff measure on $\partial B(0, r_1)$, and $\alpha|_E$ denotes the restriction of the measure $\alpha$ to the measurable set $E$, defined as $(\alpha|_E)(B) = \alpha(B \cap E)$ for any $B \in \mathcal{B}(\Omega)$.

Remark 10: Since the Radon measure $f$ is radially symmetric, we have

$$
f|_{\partial B(0, r_1)} = \frac{f(\partial B(0, r_1))}{H^{n-1}(\partial B(0, r_1))} H^{n-1}_{\partial B(0, r_1)}.
$$

If $f \in L^2(\Omega)$ the measure $\mu$ given by (4.18) reduces to

$$
\mu = \frac{\lambda|_{B(0, r_1)}}{\Psi} + \frac{\tilde{u}'(r_1^+) - \tilde{\Psi}'(r_1^-)}{\tilde{\Psi}(r_1)} H^{n-1}_{\partial B(0, r_1)}.
$$

By (4.14) and (4.18) the function $u - \Psi$ is superharmonic on $A = B(0, r_2) \setminus \overline{B}(0, r_1)$ and vanishes at $\partial A$ (assuming $u \neq u_0$). Therefore, the Hopf maximum principle implies $\tilde{u}'(r_1^+) > \tilde{\Psi}'(r_1^+)$. By (4.14), if $f \in L^2(\Omega)$ we have $\tilde{\Psi}'(r_1^+) \geq \tilde{\Psi}'(r_1^-)$, therefore $\tilde{u}'(r_1^+) - \tilde{\Psi}'(r_1^-) > 0$ and $\mu|_{\partial B(0, r_1)} \neq 0$. This last result holds also in the general case $f \in H^{-1}(\Omega)$, $f \geq 0$, because (4.14) implies

$$
[\tilde{\Psi}'(r_1^+) - \tilde{\Psi}'(r_1^-)] H^{n-1}_{\partial B(0, r_1)} + f|_{\partial B(0, r_1)} \geq 0,
$$

hence

$$
\mu|_{\partial B(0, r_1)} \geq \frac{\tilde{u}'(r_1^+) - \tilde{\Psi}'(r_1^+)}{\tilde{\Psi}(r_1)} H^{n-1}_{\partial B(0, r_1)}.
$$
Proof : (4.16) results from Proposition 7.

Assume now that \( u \neq u_0 \).

Let \( \mu \) be the measure corresponding to \( u \). Since \( u \neq u_0 \), we have \( \mu(\Omega) > 0 \). Since \( \Psi \notin H^1_0(\Omega) \) by (4.15), we have \( u \neq \Psi \), thus, by Theorem 9 (ii), we get \( 0 \leq r_1 < r_2 < R \). Next, introduce the function \( v \) associated to \( u \) by (3.3), i.e. such that

\[
v \in H^1_0(\Omega), \quad -v'' - \frac{n-1}{r} v' = \tilde{g}_s(r,u) \quad \text{on} \ (0,R).
\]

Clearly, see [G.T.], \( v \in H^2(\Omega) \) and \( \tilde{v} \in C^1(0,R) \), the space of continuously differentiable functions on \( (0,R) \). We know that \( \tilde{v} \leq 0 \) on \( (0,R) \) by (3.10).

We claim that \( \tilde{v} < 0 \) on \( (r_1,R) \).

First, \( \tilde{v} \) cannot vanish at \( r \in (r_2,R) \). Indeed, since \( u < \Psi \) in \( \Omega \setminus \overline{B}(0,r_2) \), by (3.12) we have

\[
-\Delta v = g_s(x,u) < 0 \quad \text{in} \quad \Omega \setminus \overline{B}(0,r_2),
\]

thus the strong maximum principle implies \( \tilde{v} < 0 \) on \( (r_2,R) \). If now \( \tilde{v} \) vanishes for some \( r \in (r_1,r_2) \) then

\[
v \in H^1_0(B(0,r)), \quad -\Delta v = g_s(x,u) \geq 0 \quad \text{in} \quad B(0,r).
\]

Since \( g_s(x,u) \neq 0 \) in \( B(0,r) \) by (3.13), the strong maximum principle implies \( v > 0 \) in \( B(0,r) \), which contradicts \( v \leq 0 \) q.e in \( \Omega \) (see (3.10)). So, \( \tilde{v} < 0 \) on \( (r_1,R) \). If now \( r_1 = 0 \), then \( v < 0 \) on \( \Omega \setminus \{0\} \) and by (3.10)

\[
\nu = \mu = 0 \quad \text{in} \quad \Omega \setminus \{0\}
\]

and thus also \( \mu = 0 \) in \( \Omega \). This contradicts \( u \neq u_0 \) and concludes the proof of (4.17).

Let us prove (4.18). We know, by Theorem 2, that \( \inf \Omega' \ u > 0 \) for every \( \Omega' \subset \subset \Omega \). This implies that \( 1/u \in H^1_{loc}(\Omega) \) and, by Proposition 2, that

\[
\mu = \frac{\Delta u + f}{u}.
\]

Thus for every \( \varphi \in \mathcal{D}(\Omega) \):

\[
< \mu, \varphi > = < \frac{\Delta u + f}{u}, \varphi > = < f, \frac{\varphi}{u} > + < \Delta u, \frac{\varphi}{u} > = \int_{\Omega} \frac{\varphi}{u} \ df - \int_{\Omega} \nabla u. \nabla (\frac{\varphi}{u}) \ dx
\]

\[
= \int_{B(0,r_1)} \frac{\varphi}{u} \ df - \int_{\Omega \setminus \overline{B}(0,r_1)} \nabla u. \nabla (\frac{\varphi}{u}) \ dx + \int_{\Omega \setminus \overline{B}(0,r_1)} \frac{\varphi}{u} \ df - \int_{\Omega \setminus \overline{B}(0,r_1)} \nabla u. \nabla (\frac{\varphi}{u}) \ dx
\]

\[
= \int_{B(0,r_1)} \frac{\varphi}{\Psi} \ df - \int_{\Omega \setminus \overline{B}(0,r_1)} \nabla \Psi. \nabla (\frac{\varphi}{\Psi}) \ dx + \int_{\Omega \setminus \overline{B}(0,r_1)} \frac{\varphi}{u} \ df - \int_{\Omega \setminus \overline{B}(0,r_1)} \nabla u. \nabla (\frac{\varphi}{u}) \ dx
\]

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\[
\begin{align*}
&= \int_{B(0,r_1)} \frac{\phi}{\tilde{\Psi}} \, df + \int_{\partial B(0,r_1)} \frac{\phi}{\tilde{\Psi}} \, d(\Delta \tilde{\Psi}) - \int_{\partial B(0,r_1)} \tilde{\Psi}'(r_1^-) \frac{\phi}{\tilde{\Psi}} \, dH^{n-1} \\
&\quad + \int_{\Omega \setminus B(0,r_1)} \frac{\phi}{u} \, d(\Delta u + f) + \int_{\partial B(0,r_1)} u'(r_1^+) \frac{\phi}{\tilde{\Psi}} \, dH^{n-1},
\end{align*}
\]
this last equality being due to the Green formula. Since \( v < 0 \) on \((r_1, R)\), by (3.10) we have \( \mu = 0 \) in \( \Omega \setminus \overline{B}(0,r_1) \), hence

\[\Delta u + f = 0 \quad \text{in} \quad \Omega \setminus \overline{B}(0,r_1).\]

This implies

\[< \mu, \phi > = \int_{B(0,r_1)} \frac{\phi}{\Psi} \, d\lambda + \frac{1}{\Psi(r_1)} \int_{\partial B(0,r_1)} \phi \, df + \frac{\tilde{\Psi}'(r_1^+)}{\Psi(r_1)} - \frac{\tilde{\Psi}'(r_1^-)}{\Psi(r_1)} \int_{\partial B(0,r_1)} \phi \, dH^{n-1}\]

and (4.18) follows.

**Remark 11:** Under the hypotheses of Theorem 10, if \( u = u_0 \), then three cases are possible:

1. \( 0 = r_1 = r_2 < R \), so that \( \bar{u}_0(r) < \tilde{\Psi}(r) \) \( \forall r \in (0,R) \);  
2. \( 0 = r_1 < r_2 < R \), so that \( \bar{u}_0(r) > \tilde{\Psi}(r) \) on \((0,r_2)\), \( \bar{u}_0(r) < \tilde{\Psi}(r) \) on \((r_2,R)\);  
3. \( 0 < r_1 = r_2 < R \), so that \( \bar{u}_0(r) = \tilde{\Psi}(r) \) on \((0,r_2]\), \( \bar{u}_0(r) < \tilde{\Psi}(r) \) on \((r_2,R)\).

The case \( 0 < r_1 < r_2 < R \) is excluded by the strong maximum principle, since \( u_0 - \Psi \) is superharmonic in \( B(0,r_2) \) and vanishes at the boundary \( \partial B(0,r_2) \). The case \( r_2 = R \) is excluded by the fact that \( u_0 \in H^1_0(\Omega) \) and \( \Psi \geq \eta > 0 \) in \( \Omega \).

**Examples:**

Set

\[g(x,s) = (s - \Psi(x))^2 \quad \text{(4.19)}\]

where \( \Psi \) is a radially symmetric function satisfying (4.14).

Then, clearly, \( g(x,.) \) is a strictly convex function for a.e. \( x \in \Omega \). Thus, the problem (1.18) corresponding to this \( g(x,.) \) admits a unique minimizer which is radially symmetric. Moreover, since \( \Psi(x) \) is a strict minimum for \( g(x,.) \), this minimizer satisfies for some \( r_2 \in [0,R] \) (see Proposition 6)

\[\bar{u}(r) \geq \bar{\Psi}(r) \quad \text{for} \quad 0 < r \leq r_2, \quad \bar{u}(r) < \tilde{\Psi}(r) \quad \text{for} \quad r_2 < r < R.\]

If, in addition, \( \Psi \geq 0 \) in \( \Omega \), this minimizer is such that

\[\bar{u}(r) = \tilde{\Psi}(r) \quad \text{for} \quad 0 < r \leq r_1, \quad \bar{u}(r) > \tilde{\Psi}(r) \quad \text{for} \quad r_1 < r < r_2\]
for some \( r_1 \) with \( 0 \leq r_1 \leq r_2 \) (see Proposition 7).

If one takes the fact that \( f \geq 0 \) into account, a suitable \( \Psi \) is then

\[
\Psi = a = \text{constant}.
\]

In this case, if \( a \leq 0 \), then the solution to (1.18) is given by \( u = 0 \). If \( a \geq \sup_{\Omega} u_0 \), where \( u_0 \) is defined by (2.6), then the solution to (1.18) is \( u = u_0 \). In the other cases the above results apply. Assume, for instance, that \( f \) satisfies (2.2) and (4.2). For any \( a \geq 0 \) let \( v_a \) be the solution of (3.3) corresponding to \( g(x,s) = (s-a)^2 \) and \( u = u_0 \), i.e.

\[
v_a \in H_0^1(\Omega), \quad -\Delta v_a = 2(u_0 - a) \quad \text{in} \quad \Omega.
\]

Note that, by the strong maximum principle, we have

\[
a < b \implies v_a > v_b \quad \text{in} \quad \Omega,
\]

\[
a \geq \sup_{\Omega} u_0 \implies v_a < 0 \quad \text{in} \quad \Omega.
\]

Let \( a_0 \) be the least constant \( a \geq 0 \) such that \( v_a \leq 0 \) in \( \Omega \) (\( a_0 = +\infty \) if no constant \( a \) has this property). Since \( f \neq 0 \), by the strong maximum principle, we have \( u_0 > 0 \) in \( \Omega \), hence, if \( v_0 \) is the solution \( v_a \) corresponding to \( a = 0 \), \( v_0 > 0 \) in \( \Omega \). This implies

\[
0 < a_0 \leq \sup_{\Omega} u_0.
\]

Let us prove that \( a_0 \leq \sup_{\Omega} u_0 \) if \( \sup_{\Omega} u_0 < +\infty \). First, we observe that \( \tilde{u}_0 \) is decreasing on \( (0, R) \) since \( u_0 \) is superharmonic in \( \Omega \). Let us fix \( r_0 \in (0, R) \) such that

\[
0 < \tilde{u}_0(r_0) < \sup_{\Omega} u_0
\]

and let \( c \) be a constant such that

\[
0 < c < \sup_{\Omega} u_0 - \tilde{u}_0(r_0).
\]

If \( \epsilon > 0 \) is small enough (so that \( c + \epsilon < \sup_{\Omega} u_0 - \tilde{u}_0(r_0) \)) and \( a \geq \sup_{\Omega} u_0 - \epsilon \), then

\[
u_0 - a \leq \epsilon \chi_{B(0,r_0)} - c \chi_{\Omega \setminus B(0,r_0)} \quad \text{in} \quad \Omega,
\]

where \( \chi_A \) denotes the characteristic function of \( A \). By a comparison argument we get

\[
v_a \leq 2w_{\epsilon} \quad \text{in} \quad \Omega,
\]

where \( w_{\epsilon} \) is the solution to

\[
w_{\epsilon} \in H_0^1(\Omega), \quad -\Delta w_{\epsilon} = \epsilon \chi_{B(0,r_0)} - c \chi_{\Omega \setminus B(0,r_0)} \quad \text{in} \quad \Omega,
\]

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and by a direct computation we obtain that \( w_\varepsilon \leq 0 \) in \( \Omega \) for \( \varepsilon > 0 \) small enough. Therefore \( v_\varepsilon \leq 0 \) in \( \Omega \) for \( a \geq \sup_{\Omega} u_0 - \varepsilon \). This gives
\[
0 < a_0 < \sup_{\Omega} u_0,
\]
provided \( \sup_{\Omega} u_0 < +\infty \).

If \( a \geq a_0 \), then \((u_0, v_\varepsilon)\) satisfies the complementarity problem (3.4), thus \( u = u_0 \) is the unique minimizer of (1.20) and (1.18). If \( 0 < a < a_0 \), then \((u_0, v_0)\) does not satisfy (3.4), hence \( u_0 \) is not a minimizer of (1.18). In this case Theorem 10 can be applied and one has
\[
0 < r_1 < r_2 < R.
\]

Moreover, the unique minimizer \( u = u_\mu \) of (1.18) satisfies
\[
u = a \quad \text{on} \quad B(0, r_1), \quad \mu = \frac{1}{a} f|_{\partial B(0, r_1)} + \frac{\tilde{u}'(r_1^+)}{a} \cdot H^{n-1}_{\partial B(0, r_1)}.
\]

This allows us to construct an example where the equality
\[
\mu = c \cdot H^{n-1}_{\partial B(0, r_1)}
\]
holds. Indeed, \( u \) being as above, introduce the solution \( \tilde{\Psi} \) to
\[
\tilde{\Psi} - a \in H^1_0(B(0, r_1)), \quad -\Delta(\tilde{\Psi} - a) = f \quad \text{in} \quad B(0, r_1).
\]
Since \( f \) is nonnegative one has clearly \( \tilde{\Psi} - a \geq 0 \) in \( B(0, r_1) \). If we extend this function by 0 outside \( B(0, r_1) \) - or if we still denote by \( \tilde{\Psi} \) the extension of \( \tilde{\Psi} \) by a outside \( B(0, r_1) \) - it results from Lemma A of the Appendix that
\[
\Delta \tilde{\Psi} + f \geq 0 \quad \text{in} \quad \Omega.
\]

Now, we claim that the solution of (1.18) corresponding to
\[
g(x, s) = (s - \tilde{\Psi}(x))^2 \tag{4.20}
\]
is given by
\[
\tilde{u} = \begin{cases} 
\tilde{\Psi} & \text{on} \quad B(0, r_1) \\
u & \text{on} \quad \Omega \setminus B(0, r_1). 
\end{cases}
\]
Indeed, first we claim that
\[
\Delta \tilde{u} + f \geq 0 \quad \text{in} \quad \Omega. \tag{4.21}
\]
This follows from the fact that
\[
\tilde{u} = \tilde{\Psi} \lor u \quad \text{in} \quad B(0, r_2),
\]
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hence (by (1.10))
\[ \Delta \hat{u} + f \geq 0 \quad \text{in} \quad B(0, r_2), \]
and
\[ \Delta \hat{u} + f = \Delta u + f \geq 0 \quad \text{in} \quad \Omega \setminus \overline{B}(0, r_1). \]

Next, introduce the solution \( \tilde{v} \) to
\[ \tilde{v} \in H^0_0(\Omega), \quad -\Delta \tilde{v} = g_s(x, \tilde{u}) = 2(\hat{u} - \tilde{\Psi}) \quad \text{in} \quad \Omega. \]

Since
\[ \hat{u} - \tilde{\Psi} = u - a \quad \text{in} \quad \Omega, \]
\( \tilde{v} \) coincides with the function \( v \) associated to \( u \) by (3.3) in the case \( g(x, s) = (s - a)^2 \). So we have
\[ \tilde{v} \leq 0 \quad \text{q.e. in} \quad \Omega, \quad \tilde{v} = 0 \quad \text{in} \quad B(0, r_1). \tag{4.22} \]

Moreover, since \( v \in H^2(\Omega) \),
\[ < \Delta \hat{u} + f, \tilde{v} > = < \Delta \hat{v}, \tilde{u} > + < f, \hat{v} > = < \Delta v, u > + < f, v > = < \Delta u + f, v > \geq 0. \]

Thus, (3.4) holds and by, Theorem 7, \( \hat{u} \) is the minimizer of (1.18) with \( g \) given by (4.20).

Taking into account the fact that
\[ \Delta \tilde{\Psi} + f = 0 \quad \text{in} \quad \Omega \setminus \partial B(0, r_1) \]
the measure \( \tilde{\mu} \) associated to \( \tilde{u} \) satisfies
\[ \tilde{\mu} = c \cdot H^{n-1}_{\partial B(0, r_1)} \]
with \( c > 0 \), since obviously \( \hat{u} \neq u_0 \).

In the special case \( n = 2 \), we can describe very easily a minimizing sequence for the optimal design problem
\[ \inf_A \int_\Omega (u_A - \tilde{\Psi})^2 \, dx \tag{4.23} \]
corresponding to the measure \( \tilde{\mu} = c \cdot H^1_{\partial B(0, r_1)} \) (see the introduction for the definition of \( u_A \)). We identify \( \mathbb{R}^2 \) with the complex plane \( \mathbb{C} \). For every integer \( h, k \) let
\[ E^k_h = \{ z \in \mathbb{C} : |z - r_1 e^{2\pi k i / h}| < e^{-h/cr_1} \}, \]
\[ E_h = \bigcup_{k=1}^{h} E^k_h, \quad A_h = \Omega \setminus E_h, \]
where \( i = \sqrt{-1} \). Then using the technique of Example 2.9 of [C.M.], we can prove that
\[ u_{A_h} \rightharpoonup \hat{u} \quad \text{weakly in} \quad H^1_0(\Omega). \]
By (0.5) this implies that \((A_h)\) is a minimizing sequence of (4.22).

Of course, the same analysis could be performed for
\[
g(x, s) = \varphi(s - \Psi(x))
\]
where \(\varphi\) is a strictly convex function which admits a minimum at 0 and provided \(\varphi\) satisfies some quadratic growth condition (see (1.5)).

**APPENDIX**

We prove here, for the reader’s convenience, a result that we have used without proof in the course of the paper.

**Lemma A**: Let \(\Omega'\) be an open subset of \(\Omega\), \(g \in H^{-1}(\Omega)\), \(g \geq 0\). Let \(\xi\) be a function of \(H^1_0(\Omega')\) satisfying:
\[
\xi \geq 0 \quad , \quad \Delta \xi + g \geq 0 \quad \text{in} \quad \Omega'.
\]
If we denote by \(\tilde{\xi}\) the extension of \(\xi\) by 0 outside of \(\Omega'\), then one has:
\[
\Delta \tilde{\xi} + g \geq 0 \quad \text{in} \quad \Omega.
\]

**Proof**: Set
\[
K_{\tilde{\xi}} = \{ v \in H^1_0(\Omega) : v \leq \tilde{\xi} \quad \text{a.e. in} \quad \Omega \}.
\]
Consider the solution \(w\) to
\[
w \in K_{\tilde{\xi}} \quad , \quad <-\Delta w - g, v - w \geq 0 \quad \forall v \in K_{\tilde{\xi}}.
\]
First, we claim that \(w \geq 0\) in \(\Omega\). Indeed, \(w\) is the largest subsolution to
\[
-\Delta u = g
\]
which belongs to \(K_{\tilde{\xi}}\) (see [K.S.]). Since 0 is such a subsolution, one has \(w \geq 0\) in \(\Omega\).

Next we claim that \(\tilde{\xi} = w\). For that, take \(v = \tilde{\xi}\) in (A.2). Thus:
\[
<-\Delta w - g, \tilde{\xi} - w \geq 0.
\]
Since \(0 \leq w \leq \tilde{\xi}\) in \(\Omega\), we have \(\tilde{\xi} - w = 0\) outside of \(\Omega'\), hence
\[
<-\Delta w - g, \xi - w \geq 0, \quad \Omega' \geq 0,
\]
where \(< , >_{\Omega'}\) denotes the duality bracket between \(H^{-1}(\Omega')\) and \(H^1(\Omega')\).
Since $\xi \geq w$ in $\Omega'$, from (A.1) we deduce

$$< -\Delta \xi - g, \xi - w >_{\Omega'} \leq 0.$$  

Subtracting (A.3) we obtain

$$< -\Delta (\xi - w), \xi - w >_{\Omega'} \leq 0$$

and thus, $\xi - w = 0$ in $\Omega'$. Since $0 \leq w \leq \tilde{\xi} = 0$ in $\Omega \setminus \Omega'$, we have proved that

$$w = \tilde{\xi} \quad \text{in} \quad \Omega,$$

and the result follows from the fact that

$$\Delta w + g \geq 0 \quad \text{in} \quad \Omega.$$  

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