Solutions to the Korteweg-de Vries Equation with Initial Profile in $L_1^1 (\mathbb{R}) \cap L_N^1 (\mathbb{R}^+)$. 

Cohen, A
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Abstract

The Cauchy problem for the Korteweg-de Vries equation is considered with initial profile integrable against $(1 + |x|)dx$ on $\mathbb{R}$ and against $(1 + |x|)^N dx$ on $\mathbb{R}^+$. Classical solutions are constructed for $N \geq \frac{11}{4}$. Under mild additional hypotheses the solution evolves in $L^2 (\mathbb{R})$. 
SOLUTIONS TO THE KORTEWEG–DE VRIES EQUATION WITH INITIAL PROFILE IN $L^1(\mathbb{R}) \cap L^N(\mathbb{R}^+)^*$

AMY COHEN† AND THOMAS KAPPELER‡

Abstract. The Cauchy problem for the Korteweg–deVries equation is considered with initial profile integrable against $(1 + |x|)^N \, dx$ on $\mathbb{R}$ and against $(1 + |x|)^N \, dx$ on $\mathbb{R}^+$. Classical solutions are constructed for $N \geq 11/4$. Under mild additional hypotheses the solution evolves in $L^2(\mathbb{R})$.

Key words. Korteweg–de Vries equation, inverse scattering method

AMS(MOS) subject classification. 35Q20

1. Introduction and summary of results. This paper considers the initial value problem for the Korteweg–deVries equation (KdV),

\begin{align}
& u_t - 6uu_x + u_{xxx} = 0, \\
& u(x, 0) = U(x),
\end{align}

under the hypothesis that

\begin{align}
& \int_{-\infty}^{\infty} |U(x)| \, (1 + |x|) \, dx < \infty, \\
& \int_{0}^{\infty} |U(x)| \, (1 + |x|)^N \, dx < \infty.
\end{align}

No differentiability is assumed at all. The goal is to find the range of $N$ such that the problem (1.1), (1.2) has a solution. Our existence theorem is based on a construction suggested by the inverse scattering method. We show that if $N \geq 11/4$, then a classical solution exists in $t > 0$ which approaches its initial profile in an appropriate distribution sense as $t \to 0^+$.

These results improve considerably on earlier work of the first author [3], which required that $U$ be at least piecewise $C^1$ as well as that $U$ be integrable against $(1 + |x|)^N \, dx$ on $\mathbb{R}$ for large enough $N$. By using Kappeler’s new $L^2$ inverse scattering result [8], we are also able to get control over our solution as $x \to -\infty$, at least for $U$ satisfying a rather mild additional hypothesis. These results also improve on work of Sachs [14], who requires that $U(x)$ be integrable against $(1 + |x|)^N \, dx$ on all of $\mathbb{R}$ with $N > 11/4$ rather than only on $\mathbb{R}^+$ with $N \geq 11/4$. Sachs claims convergence to initial profile in a weighted $L^1$ norm on each halfline $[a, +\infty)$; it appears that his proof of this point is flawed.

There is no direct comparison between our results and the very interesting paper of Kruzhkov and Faminskii [11], in which they prove the existence of a weak solution to KdV with arbitrary $L^2$ initial data, and show that the solution is classical if the datum is not only $L^2$ on $\mathbb{R}$ but also $L^2$ with respect to $(1 + |x|)^3 \, dx$ on $\mathbb{R}^+$. While Sachs’ paper uses a different inverse scattering construction from ours (Deift and Trubowitz [5] rather than Faddeev [6]), Kruzhkov and Faminskii use a different approach altogether: they cut off and mollify their initial profile, apply results of Yakupov [19] and Shabat [16] solving KdV with data in $C^0_0(\mathbb{R})$, and then take limits.

* Received by the editors October 8, 1984; accepted for publication (in revised form) April 7, 1986.
† Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903.
‡ Forschungsinstitut für Mathematik, ETH-Zentrum, CH-8092 Zürich, Switzerland. The author was supported in part by the Swiss National Foundation.
In their pioneering paper [7], Gardner, Greene, Kruskal and Miura showed that if \( u(x, t) \) solved KdV and evolved in the Schwartz class \( \mathcal{S} \), then the scattering data of the Schrödinger equation

\[
-\psi'' + u(x, t)\psi = k^2\psi
\]

evolved according to simple first order linear o.d.e.'s in the variable \( t \). By appealing to Faddeev's inverse scattering theory [6], they showed that \( u(x, t) \) for \( t > 0 \) could be recovered from \( u(x, 0) \). This idea has been the basis for a succession of existence theorems [17], [3], [13], [14] employing progressively weaker hypotheses on the initial profile \( U \).

Rather than give a detailed exposition of the forward scattering theory of (1.4) we refer the reader to Cohen's paper [4].

In § 2 we analyze the scattering data associated to (1.4) under the hypothesis \( U \in L^1_n(\mathbb{R}) \), i.e., \( U \) is integrable with respect to \( (1 + |x|^N) \) \( dx \), with \( N \geq 1 \). The main result is Proposition 2.5 which says that generically the reflection coefficient \( R_+ \) is in \( C^{N-1}(\mathbb{R}) \cap C^{N}(\mathbb{R} \sim \{0\}) \) and \( \lim_{k \to 0} kR_+^{(N)}(k) \) exists—but that if \( U \) is exceptional, then \( R_+ \) is only in \( C^{N-2}(\mathbb{R}) \cap C^{N-1}(\mathbb{R} \sim \{0\}) \) and \( \lim_{k \to 0} kR_+^{(N-1)}(k) \) exists.

In § 3 we analyze the kernels \( \Omega_+(x, t) \) and \( \Omega_-(x, t) \) used in the Marchenko equations

\[
(M+) \quad B_+(x, y, t) + \Omega_+(x + y, t) + \int_0^\infty B_+(x, z, t)\Omega_+(x + y + z, t) \, dz = 0,
\]
\[
(M-) \quad B_-(x, y, t) + \Omega_-(x + y, t) + \int_{-\infty}^0 B_-(x, z, t)\Omega_-(x + y + z, t) \, dz = 0.
\]

What Gardner, Green, Kruskal and Miura showed was that if \( u(x, t) \) solves KdV, and \( \Omega_+ \) are as defined below, then

\[
u(x, t) = -\partial_x B_+(x, 0, t) = -\partial_x B_-(x, 0, t).
\]

The kernels are defined as follows:

\[
\Omega_+(x, t) = F_+(x, t) + 2 \sum_i c_{i+j} \exp(-2i\kappa_jx + 8i\kappa_j^2t)
\]

where

\[
F_+(x, t) = \pi^{-1} \int_{-\infty}^\infty R_+(k) \exp(2ikx + 8ik^3t) \, dk
\]

and

\[
\Omega_-(x, t) = F_-(x, t) + 2 \sum_j c_{-i-j} \exp(2i\kappa_jx - 8i\kappa_j^3t)
\]

where

\[
F_-(x, t) = \pi^{-1} \int_{-\infty}^\infty R_-(k) \exp(-2ikx - 8ik^3t) \, dk.
\]

Clearly the existence, regularity and decay of the \( B_\pm(x, y, t) \) depend on the regularity and decay of the \( \Omega_\pm \). In § 3, we show that for each fixed \( t > 0 \), \( \partial_x^2\Omega_+(x, t) \) is continuous for \( 0 \leq \nu \leq 2N + 3/2 \) and establish algebraic decay rates as \( x \to +\infty \) for these derivatives. We also analyze the decay and regularity of \( \Omega_+(x, t) \) using the properties of \( R_+ \) proved in Proposition 2.5. To study \( F_- \), we note just that \( R_- \) is quite similar to \( R_+ \) in its regularity and decay. Then we see that the decay of \( \Omega_- \) is controlled by that of \( F_- \) and that the integral for \( F_- \) has stationary points when \( x < 0 \). Nonetheless we find that
if $U \in L_N^1$, $N \geq 5$, and $R^{(n)}(k) = O(k^{-\lambda})$ for $\lambda \geq 5/2$, then $\partial_x \Omega_-(x, t) = O(|x|^{-\lambda/2 + 1/4})$ as $x \to -\infty$.

In § 5 we prove sharper versions of the following results.

**Result 1.** Suppose $U$ satisfies (1.3a) and (1.3b) with $N \geq 11/4$. Then there is a classical solution $u(x, t)$ of KdV in $t > 0$ such that

$$u(x, t) \to U(x) \quad \text{in } H^{-1}(\mathbb{R}).$$

**Result 2.** Suppose that $U \in L_2^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and that $R^{(n)}(k) = O(|k|^{-\lambda})$ as $k \to \pm \infty$ for some $\lambda > 5/2$, and $n = 0, 1, 2$. Then the solution given by Result 1 evolves in $L^2(\mathbb{R})$ for $t > 0$.

**Result 3.** Suppose that $N \geq 3$ and that $U \in L_N^1(\mathbb{R})$ if $U$ is generic but that $U \in L_{N+1}^1(\mathbb{R})$ if $U$ is nongeneric. Suppose further that $(1 + |x|)^{N-1} U(x)$ is in $L^2(\mathbb{R})$. If $u(x, t)$ is the solution to KdV given by Result 1, then $x^n u(x, t) \to x^n U(x)$ in $L^2(+\infty)$ as $t \to 0$ for $\alpha = 0$ and $\alpha = N - 1$.

We should also remark that the question of uniqueness is still largely open. Uniqueness is known for the initial value problem for KdV if the initial profile is in $H^s$ with $s \geq 3/2$ [2], [10], [15]. Uniqueness is also known within the class of solutions $u(x, t)$ such that $u(x, t)$ and $u_t(x, t)$ go to 0 as $x \to \pm \infty$ and $u_{xx}(x, t)$ is bounded as $x \to \pm \infty$ [12]. Kruzhkov and Faminskii [11] show that the problem (1.1), (1.2) is well posed in the class of functions $U$ which are $L^2$ on $\mathbb{R}$ and $L^2$ with respect to a weight on $\mathbb{R}^+$. Unless we add to our minimal hypotheses we cannot show that our solution $u(x, t)$ evolves in a class where either of these uniqueness theorems applies.

**Notational conventions.** The operator $\partial_x$ denotes the partial derivative with respect to the subscript variable.

$f^*(x, k)$ the complex conjugate of $f(x, k)$.

In dealing with functions of $x$ and $k$, prime (') always denotes the $x$-derivative and dot (·) always denotes the $k$-derivative; thus

$$f'(x, k) = \partial_x f(x, k), \quad f^*(x, k) = \partial_k f(x, k).$$

The space $L^1_N(\mathbb{R})$ consists of functions $g(x)$ such that

$$\int_x^\infty |g(x)| (1 + |x|)^N dx < \infty \quad \text{for all finite } X.$$

The space $L^2(\mathbb{R})$ consists of functions $g(x)$ such that

$$\int_x^\infty |g(x)|^2 dx < \infty \quad \text{for all finite } X.$$

We use $a \wedge b$ to denote max {$a, b$}.

**2. Analysis of the initial scattering data.**

2.1. **The Jost functions.** Suppose that $U(x)$ belongs to $L_N^1(\mathbb{R})$ with $N \geq 1$. Then the Jost functions for

$$-y'' + U(x)y = k^2 y$$

are the solutions $f_+(x, k)$ and $f_-(x, k)$ with the asymptotic behavior

$$f_+(x, k) \sim e^{ikx} \quad \text{as } x \to +\infty, \quad f_-(x, k) \sim e^{-ikx} \quad \text{as } x \to -\infty.$$

These exist for Im $k \geq 0$ and can be represented as

$$f_+(x, k) = e^{ikx} h_+(x, k), \quad f_-(x, k) = e^{-ikx} h_-(x, k)$$

where $h_+(x, k)$ and $h_-(x, k)$ are bounded for $k \to \pm \infty$.
where

\[ h_+(x, k) = 1 + \int_0^\infty B_+(x, y) e^{2iky} dy, \quad h_-(x, k) = 1 + \int_{-\infty}^0 B_-(x, y) e^{-2iky} dy \]

where, in turn,

\[ B_\pm(x, \cdot) \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+) \subset L^2(\mathbb{R}^+), \]

\[ B_\pm(x, y) \text{ is continuous on } \mathbb{R} \times \mathbb{R}^+. \]

Here \( \mathbb{R}^+ = [0, \infty) \) and \( \mathbb{R}^- = (-\infty, 0] \). Moreover, the maps \( x \mapsto B_\pm(x, \cdot) \) are absolutely continuous and Fréchet differentiable from \( \mathbb{R} \) to \( L^1(\mathbb{R}^+) \). The following estimates are valid since \( U \in L^1(\mathbb{R}) \):

\[ \int_{x+y}^\infty U(t) dt, \int_x^\infty U(t) dt \]

\[ \int_{x+y}^\infty U(t) dt, \int_x^\infty U(t) dt \]

\[ \int_{x+y}^\infty U(t) dt, \int_x^\infty U(t) dt \]

Analogous bounds (2.5\text{--}), (2.6\text{--}) and (2.7\text{--}) hold for \( B_-(x, y) \), \( \partial_x B_-(x, y) - U(x+y) \) and \( \partial_y B_-(x, y) - U(x+y) \) in terms of integrals over left-half-lines. See [1], [3]-[6] for details. Applying these bounds to the forms (2.3), (2.4), one obtains the following.

**Proposition 2.1.** For any fixed \( x \), the functions \( y^n B_+(x, y), y^n \partial_x B_+(x, y), \) and \( y^n \partial_y B_+(x, y) \) are integrable over \( 0 < y < \infty \) for \( 0 \leq n \leq N-1 \). Similar results hold for \( B_- \) with integrability over \( -\infty < y < 0 \).

It follows that \( h_+(x, k) \) and \( \partial_x h_+(x, k) \) are \( (N-1) \) times continuously differentiable with respect to \( k \). Indeed, if \( 1 \leq n \leq N-1 \) and \( \text{Im } k \geq 0 \), then

\[ \partial^n_x h_+(x, k) = \int_0^\infty (2iy)^n B_+(x, y) e^{2iky} dy. \]

If in addition \( k \neq 0 \), then an integration by parts yields

\[ \partial^n_x h_+(x, k) = \left[ \frac{1}{2ik} \right]_{-1}^1 [n(2iy)^n-1 B_+(x, y) + (2iy)^n \partial_y B_+(x, y)] e^{2iky} dy. \]

Further

\[ \partial^n_x \partial^m_x h_+(x, k) = \int_0^\infty (2iy)^m \partial_x B_+(x, y) e^{2iky} dy. \]

Thus for \( 1 \leq n \leq N-1 \) and for each finite \( X, k \partial^n_x h_+(x, k) \) and \( \partial^n_x \partial^m_x h_+(x, k) \) are uniformly bounded on \( \{(x, k) : x \geq X \text{ and } \text{Im } k \geq 0\} \).

It is possible to get better information about the regularity of \( h_+(x, k) \) by using the approach of Deift and Trubowitz [5, p. 130]. Let

\[ D_k(y) = \int_0^y e^{2ikt} dt = (e^{2ikt} - 1)/2ik. \]
Deift and Trubowitz show that \( h_+(x, k) = 1 + \int_{x}^{\infty} D_k(t-y) U(t) h_+(t, k) \, dt \). The next several propositions are similar to results in \([4]-[6]\).

**Proposition 2.2.** Assume that \( U \in L_1^N(\mathbb{R}) \) with \( N \geq 1 \). As functions of \( k \) with \( x \) fixed in \( \mathbb{R} \), \( h_+(x, k) \) and \( \partial_r h_+(x, k) \) are \( C^{N-1} \) on \( \{ k : \text{Im} \, k \rightarrow 0 \} \) and \( C^N \) on \( \{ k : \text{Im} \, k \geq 0, k \neq 0 \} \). Further \( k \partial^N h_+(x, k) \) and \( k \partial^N \partial_r h_+(x, k) \) extend continuously to \( k = 0 \). Moreover there are nonincreasing functions \( K(x) \) such that for \( 0 \leq n \leq N, \text{Im} \, k \geq 0, \text{and} \, x \leq t < \infty \)

\[
\begin{align*}
(i) & \quad |k \partial^n h_+(t, k)| \leq K(x), \\
(ii) & \quad \left| \frac{|k|}{|k| + 1} \partial^N \partial^1 h_+(t, k) \right| \leq K(x), \text{ and} \\
(iii) & \quad k \partial^N h_+(x, k) \rightarrow 0 \text{ and } \partial^N \partial_r h_+(x, k) \rightarrow 0 \text{ as } |k| \rightarrow \infty, \text{ uniformly in } \text{Im} \, k \geq 0.
\end{align*}
\]

**Proof.** We have already noted the claimed regularity on \( \{ k : \text{Im} \, k \geq 0 \} \). To get the \( N \)th derivative away from \( k = 0 \), we differentiate (2.10) \( N \) times formally and multiply by \( k \). Thus if \( \partial^N h_+(x, k) \) exists then \( w = k \partial^N h_+(x, k) \) satisfies the integral equation

\[
(2.11) \quad (I-T)w = r
\]

where

\[
T[g](x) = \int_{x}^{\infty} D_k(t-x) U(t) g(t) \, dt
\]

and

\[
r(x) = \sum_{\nu=1}^{N} C_\nu \int_{x}^{\infty} k \partial^\nu [D_k(t-x)] U(t) \partial^{N-\nu} h_+(t, k) \, dt
\]

for easily computable \( C_\nu \). For any finite \( X, T \) is a bounded operator on \( L^\infty(X, \infty) \); indeed for \( m \in \mathbb{N} \)

\[
\| T^m g \| \leq \| g \| \left( \int_{X}^{\infty} |U(t)| \, dt / |k| \right)^m.
\]

Since \( |k \partial^\nu [D_k(y)]| \leq |2y|^\nu \) for all \( \nu \geq 0 \), it follows that

\[
\| r \| \leq \sum_{\nu=1}^{N} C_\nu \int_{x}^{\infty} |2(t-x)|^\nu |U(t)| A(x) \, dt
\]

where

\[
A(x) = \sup \{ |\partial^\mu h_+(t, k)| : \text{Im} \, k \geq 0, 0 \leq \mu \leq N-1, x \leq t \leq \infty \}.
\]

Note that \( A(x) \) is finite and nonincreasing. One can also verify that

\[
\int_{x}^{\infty} |2(t-x)|^\nu |U(t)| \, dt \leq K_\nu(x) \quad \text{for } 1 \leq \nu \leq N
\]

where \( K_\nu \) is the nonincreasing function

\[
K_\nu(x) = \begin{cases} 
\int_{\infty}^{x} 2|t|^\nu |U(t)| \, dt + 2x |U(t)| \int_{x}^{\infty} |U(t)| \, dt & \text{if } x < 0, \\
\int_{-\infty}^{x} 2|t|^\nu |U(t)| \, dt & \text{if } x \geq 0.
\end{cases}
\]

So \( \| r \| \in L^\infty(X, \infty) \) is bounded by a nonincreasing function \( B(X) = K(X) \sum_{\nu=1}^{N} C_\nu K_\nu \).

It follows that the solution of (2.11) is given by

\[
w = \sum_{m=0}^{\infty} T^m r
\]
and that \(w\) is continuous in \(x\) and \(k\) in \(\mathbb{R} \times \{\text{Im } k \geq 0\}\). Further analysis reveals that \(w(x, k)/k\) is indeed \(\mathcal{O}_h(x, k)\) and that

\[
|k\delta^N_k h_+(x, k)| \leq \exp \left( \int_x^\infty |U(t)| \frac{dt}{|k|} \right) B(x).
\]

Continuing in this vein one finds that \(k\delta^N_k h_+(x, k)\) has a derivative with respect to \(x\) in distribution sense, and then that \(\delta^N_k[k\delta^N_k h_+(x, k)]\) is a classical derivative as well, and satisfies (ii) and (iii). 

**Remark.** The factor \((1 + |k|)^{-1}\) in (ii) is necessary because the term with \(n = N\) in the sum for \(r(x)\) involves \(kh_+(x, k)\), which grows like \(|k|\) as \(|k| \to \infty\).

### 2.2. Regularity and decay of \(W[f_-, f_+]\) and \(W[f^*_+, f^-_+]\)

Let \(W(k)\) and \(V(k)\) be defined on \(\text{Im } k \geq 0\) by \(W(k) = W[f_-, f_+]\) and \(V(k) = W[f^*_+, f^-_+]\). Since \(f_-, f_+\), and \(f^*_+\) solve (2.1), these Wronskians are independent of \(x\). Evaluating at \(x = 0\), we get

\[
W(k) = h_-(0, k)h'_+(0, k) - h'_-(0, k)h_+(0, k) + 2ikh_-(0, k)h_+(0, k)
\]

and

\[
V(k) = h^*_+(0, k)h'_+(0, k) - h_+(0, k)h^*_+(0, k).
\]

Where ambiguity is possible we reserve prime (') for \(\partial/\partial x\) and dot (\(\cdot\)) for \(\partial/\partial k\).

The following propositions follow immediately from the results of § 2.1.

**PROPOSITION 2.3.** Assume \(U \in L^1_N(\mathbb{R})\) with \(N \geq 1\). Then \(W \in C^{N-1}(\mathbb{R}) \cap C^N(\mathbb{R} \sim \{0\})\). Moreover \(k\delta^N_k [W(k)]\) extends continuously to \(k = 0\). For all \(n\) with \(0 \leq n \leq N\), \(\lim_{|k| \to \infty} \delta^N_k[W(k)] = 0\).

**PROPOSITION 2.4.** Assume \(U \in L^1_N(\mathbb{R})\) with \(N \geq 1\). Then \(V \in C^{N-1}(\mathbb{R}) \cap C^N(\mathbb{R} \sim \{0\})\); \(k\delta^N_k[V(k)]\) extends continuously to \(k = 0\); and \(\lim_{|k| \to \infty} \delta^N_k[V(k)] = 0\) for \(0 \leq n \leq N\).

### 2.3. Regularity and decay of \(R_+(k), R_-(k)\)

Recall that the reflection coefficients \(R_+\) and \(R_-\) are defined for \(k \neq 0\) by

\[
R_+(k) = \frac{V(k)}{W(k)}, \quad R_-(k) = \frac{V^*(k)}{W(k)}.
\]

We concentrate on \(R_+(k); R_-(k)\) can be analyzed by the same methods. Note that

\[
W(k)R_+(k) = V(k)
\]

so that formally

\[
W(k)R_+(k)^{(n)}(k) = V^{(n)}(k) - \sum_{\nu=0}^{n-1} \binom{n-1}{\nu} R_+^{(\nu)}(k)W^{(n-\nu)}(k).
\]

**PROPOSITION 2.5.** Assume that \(U \in L^1_N(\mathbb{R})\) with \(N \geq 1\). Then \(R_+ \in C^N(\mathbb{R} \sim \{0\})\) and \(\lim_{|k| \to \infty} kR_+^{(n)}(k) = 0\) for \(0 \leq n \leq N\).

**Furthermore**

(A) If \(U\) is of generic type, then \(R_+ \in C^{N-1}(\mathbb{R})\) and \(kR_+^{(N)}(k)\) extends continuously to \(k = 0\).

(B) If \(U\) is of exceptional type and \(N \geq 2\), then \(R_+ \in C^{N-2}(\mathbb{R})\) and both \(kR_+^{(N-1)}(k)\) and \(k^2R_+^{(N)}(k)\) extend continuously to \(k = 0\).

**Proof.** The regularity away from \(k = 0\) and the decay as \(k \to \pm \infty\) follow from Propositions 2.3 and 2.4.

If \(U\) is generic then \(W(k)\) is nonzero on \(\mathbb{R}\) and (A) follows by an induction using (2.14). Suppose next that \(U\) is exceptional and that \(N \geq 2\). Then instead of treating \(R_+\) as the ratio \(V/W\) we treat \(R_+\) as the quotient of \(V/k\) and \(W/k\). In this case it is
known that $W/k$ is continuous on $\mathbb{R}$ and never zero. Since $V/k$ and $W/k$ are $C^{N-2}$ on $\mathbb{R}$, so is $R$. Using (2.14) it is easy to complete the proof of (B).

We now turn to results involving $L^2$ hypotheses as well as $L^1$ assumptions on $U$.

**Lemma 2.6.** Suppose that $y^n U(y) \in L^2(\mathbb{R})$ and $U(y) \in L^1_{\nu+1}(\mathbb{R})$ for $0 \leq \nu \leq n$. Then $V(\nu) \in L^2$ for $0 \leq \nu \leq n$.

**Proof.** Deift and Trubowitz [5, p. 159] have proved that

$$V(k) = \int_{-\infty}^{\infty} \Pi_1(y) e^{-2iky} dy$$

where there is a constant $K$ such that

$$|\Pi_1(y)| \leq |U(y)| + KL(y)$$

for

$$L(y) = \int_{-\infty}^{\infty} \left| U(t) \right| dt \quad \text{if } y \geq 0, \quad L(y) = \int_{-\infty}^{y} \left| U(t) \right| dt \quad \text{for } y < 0.$$ 

To show that $V(\nu) \in L^2$, it suffices to show that $y^n \Pi_1(y) \in L^2$. Since

$$|\Pi_1(y)|^2 \leq (1 + K^2)(|U(y)|^2 + L(y)^2)$$

it follows that

$$\int_{0}^{\infty} |y^n \Pi_1(y)|^2 \, dy \leq (1 + K^2) \int_{0}^{\infty} |y^n U(y)|^2 \, dy + (1 + K^2) \int_{0}^{\infty} y^{2n} L(y)^2 \, dy.$$

The first term is finite since $y^n U(y) \in L^2$. Further

$$\int_{0}^{\infty} y^{2n} L(y)^2 \, dy = \int_{0}^{\infty} \left( y^n \int_{s=0}^{\infty} |U(s)| \, ds \right) \left( y^n \int_{t=y}^{\infty} \left| U(t) \right| \, dt \right) \, dy$$

$$\leq \int_{0}^{\infty} \left( \int_{s=0}^{\infty} s^n |U(s)| \, ds \right) \left( y^n \int_{t=y}^{\infty} \left| U(t) \right| \, dt \right) \, dy$$

$$= \int_{0}^{\infty} s^n |U(s)| \, ds \int_{t=0}^{\infty} t^{\nu+1} \left| U(t) \right| \, dt < \infty.$$

Thus $y^n \Pi_1(y)$ is in $L^2$ on $\mathbb{R}^+$; the proof that it is in $L^2$ on $\mathbb{R}^-$ is similar. □

**Proposition 2.7.** Suppose that $U \in L^1_{\nu}(\mathbb{R})$ and that $y^n U(y) \in L^2(\mathbb{R})$ for $0 \leq n \leq M$.

(A) If $U$ is of generic type, then $R_{\nu}^{(n)} \in L^1(\mathbb{R})$ and $kR_{\nu}^{(n)}(k) \in L^2(\mathbb{R})$ for $0 \leq n \leq \min\{M, N-1\}$.

(B) If $U$ is of exceptional type, then $R_{\nu}^{(n)} \in L^1(\mathbb{R})$ and $kR_{\nu}^{(n)}(k) \in L^2(\mathbb{R})$ for $0 \leq n \leq \min\{M, N-2\}$.

**Proof.** The proof is an induction based on the formula

$$R_{\nu}^{(n)}(k) = \left[ V^{(n)}(k) - \sum_{\nu=1}^{n} \binom{n}{\nu} R_{\nu}^{(n-\nu)}(k) W^{(\nu)}(k) \right] / W(k).$$

We discuss (A) first. Since $W(k)$ is continuous, never zero, and grows like $|k|$ at $\pm\infty$ it follows that $1/W$ is in $L^2$ and that $k/W \in L^1$. Since $V \in L^2$, it follows that $R_{\nu} = V/W$ is both $L^1$ and $L^2$, and that $kR_{\nu} \in L^2$.

Keep $0 \leq n \leq \min\{M, N-1\}$. We then know that $V^{(n)} \in L^2$ and that $W^{(\nu)} \in L^\infty$ for $1 \leq \nu \leq n$, then $R_{\nu}^{(n)} \in L^1 \cap L^2$ and $kR_{\nu}^{(n)} \in L^2$. Result (A) now follows by induction.
The induction for result (B) is similar, except that in the exceptional case we have only $R_+ \in C^{N-2}$.  

3. Regularity and decay of $\Omega_+(x, t)$ and $\Omega_-(x, t)$ for $t > 0$.

3.1. Properties of $F_+(x, t)$. Recall from the introduction that the kernel of the Marchenko equation $(M+)$ is

$$\Omega_+(x, t) = F_+(x, t) + G_+(x, t)$$

where

$$F_+(x, t) = \pi^{-1} \int_{-\infty}^{\infty} R_+(k) \exp (2ikx + 8ik^3t) \, dk$$

and

$$G_+(x, t) = 2 \sum_{j \in \mathbb{J}} c_{+j} \exp (-2\kappa_j x + 8\kappa_j^3 t).$$

Since $G_+(x, t)$ is $C^\infty$ and decays exponentially as $x \to +\infty$ for fixed $t > 0$, we need to concentrate on the properties of $F_+$. In the first part of this subsection we use a representation of $F_+(x, t)$ in terms of $F_+(x)$ and the Airy function to find out as much as possible about $F_+(x, t)$ without using differentiability of $R_+(k)$. Later we report on what can be said of $F_+(x, t)$ using derivatives of $R_+(k)$ by a careful extension of the methods of Cohen in [3]. For convenience, we set

$$F_+(x) := F_+(x, 0) = \pi^{-1} \int_{-\infty}^{\infty} R_+(k) e^{2ikx} \, dk.$$ 

**Lemma 3.1.** If $U(x) \in L^1(\mathbb{R})$, then $R_+(k) \in L^2(\mathbb{R})$ and $F_+(x) \in L^2()$.

**Proof.** These results are well known; see [5].

**Lemma 3.2.** Suppose $U(x) \in L^1(\mathbb{R}) \cap L^1_N(\mathbb{R}^+)$ for some $N$ with $N \geq 11/4$. Then

(a) $\int_0^\infty |F_+(x)|(1 + x)^{-N} \, dx < \infty$,

(b) $\int_0^\infty |\partial_+ F_+(x)|(1 + x)^N \, dx < \infty$.

**Proof.** Because of the exponential decay of $G_+(x, 0)$ as $x \to +\infty$, it is enough to prove the analogues of (a) and (b) for $\Omega_+$. By Faddeev [6, p. 155], we know that

$$|\Omega_+(x)| \leq C(x) \int_x^\infty |U(z)| \, dz$$

and

$$|\partial_+ \Omega_+(x) - U(x)| \leq C(x) \left[ \int_x^\infty |U(z)| \, dz \right]^2$$

where each $C(x)$ is a nonincreasing function of $x$. Now by (3.1)

$$\int_0^\infty |\Omega_+(x)| x^{N-1} \, dx \leq \int_{x=0}^\infty C(0) \int_x^\infty |U(z)| \, dz x^{N-1} \, dx$$

$$= C(0) \int_{z=0}^\infty |U(z)| \int_{x=0}^z x^{N-1} \, dx \, dz$$

$$= C(0) N^{-1} \int_{z=0}^\infty |U(z)| z^N \, dz < \infty.$$
Thus (a) follows. For (b), use (3.2):

\[ \int_0^\infty |\partial_x \Omega_+(x)| x^N \, dx \leq \int_0^\infty |U(z)| x^N \, dx + \int_0^\infty C(x) \left( \int_x^\infty |U(z)| \, dz \right)^2 x^N \, dx. \]

The first term is finite by hypothesis. For the second,

\[ \int_0^\infty \left[ \int_x^\infty |U(z)| \, dz \right]^2 x^N \, dx \leq \int_0^\infty \left[ \int_x^\infty \int_z^\infty |U(z)| \, dz \right] \left[ \int_x^\infty \int_z^\infty |U(z)| \, dz \right] \, dx \]

\[ \leq \int_0^\infty |U(z)| \, dz \int_0^\infty \left[ \int_x^\infty \int_z^\infty |U(z)| \, dz \right] \, dx \]

\[ \leq \int_0^\infty |U(z)| \, dz \left[ \int_0^\infty |U(z)| \, dz \right] \left[ \int_0^\infty x^{N-1} \, dx \right] \]

\[ \leq \left[ \int_0^\infty |U(z)| \, dz \right] \left[ \int_0^\infty |U(z)| z \, dz \right] \left[ \int_0^\infty |U(z)| z^N \, dz \right] \]

\[ < \infty \]

since each factor is finite. □

By Lemmas 3.1 and 3.2 we know that \( F_+(x) \) is a real valued function such that

\[ F_+(x) \in L^2(\mathbb{R}) \cap L_{N-1}(\mathbb{R}^+), \quad \partial_x F_+(x) \in L_{100}(\mathbb{R}) \cap L_1^\infty(\mathbb{R}^+). \]

To analyze \( F_+(x, t) \) with \( t > 0 \), we use the observation [13] that \( F_+(x, t) \) is essentially a convolution of \( F_+(x) \) with an Airy function:

\[ F_+(x, t) = (3t)^{-1/3} \int_{-\infty}^\infty F_+(y) \text{Ai} \left( \frac{x-y}{(3t)^{1/3}} \right) \, dy. \]

We use the following properties of the Airy function [13]:

\[ |\text{Ai}(z)| < 1 \quad \forall z \in \mathbb{R}, \]

\[ \text{Ai}(z) \in C^0(\mathbb{R}) \quad \text{and} \quad \text{Ai}''(z) = z \text{Ai}(z), \]

\[ |\text{Ai}^{(n)}(z)| \leq C_n z^{n/2-1/4} \quad \text{as} \quad z \to -\infty, \]

\[ |\text{Ai}^{(n)}(z)| \leq C_n z^{n/2-1/4} \exp(-2|z|^{3/2}/3) \quad \text{as} \quad z \to +\infty. \]

Because of the different behavior at \( +\infty \) and \( -\infty \), it is convenient to divide the integral in (3.3) into pieces. To this end let \( \zeta_1(x) \) denote a nonincreasing \( C^\infty \) function such that

\[ \zeta_1(x) = 1 \quad \text{on} \quad -\infty < x \leq \frac{1}{2}, \quad \zeta_1(x) = 0 \quad \text{on} \quad \frac{1}{2} \leq x < \infty. \]

Let \( \xi_2(x) := 1 - \zeta_1(x) \). Next set \( F_i(x) = F_+(x)\xi_i(x) \) for \( i = 1, 2 \). Next set

\[ F_i(x, t) = \begin{cases} (3t)^{-1/3} \int_{-\infty}^\infty F_i(y) \text{Ai} \left( \frac{x-y}{(3t)^{1/3}} \right) \, dy & \text{if} \ t > 0, \\ F_i(x) & \text{if} \ t = 0. \end{cases} \]

Note that \( F_+(x, t) = F_1(x, t) + F_2(x, t) \).

**Lemma 3.3.**

(a) \( F_i(x, t) \) is \( C^\infty \) in \( \mathbb{R} \times \mathbb{R}^+ \);

(b) \( \lim_{x \to \pm \infty} x^n \partial_x^n F_i(x, t) = 0 \) for nonnegative integers \( n, j \);

(c) \( \int_0^\infty |\partial_x^j F_i(x, t)| x^m \, dx < \infty \) for all \( j, m \).
Proof. Part (c) follows immediately from (a) and (b). The regularity (a) follows from (3.7), the rapid decay of all $\text{Ai}^{(j)}(z)$ as $z \to +\infty$, and the fact that $\text{supp} \ F_1 \subseteq (-\infty, 1]$. Now

$$\partial_x^j F_1(x, t) = (3t)^{-1/3} \int_{-\infty}^{\infty} F_1(y) \text{Ai}^{(j)} \left( \frac{x-y}{(3t)^{1/3}} \right) (3t)^{-j/3} \, dy$$

and

$$|\partial_x^j F_1(x, t)| \leq (3t)^{-(j+1)/3} \left[ \int_{-\infty}^{1} |F_1(y)|^2 \, dy \right]^{1/2} \left[ \int_{-\infty}^{1} |\text{Ai}^{(j)} \left( \frac{x-y}{(3t)^{1/3}} \right)|^2 \, dy \right]^{1/2}.$$ 

Setting $\xi = (x-y)/(3t)^{1/3}$ we see that the second integral is

$$I(x, t) = \int_{(x-1)(3t)^{1/3}}^{\infty} |\text{Ai}^{(j)}(\xi)|^2 (3t)^{1/3} \, d\xi.$$

Since $\text{Ai}^{(j)}(\xi)$ decays faster than exponentially as $\xi \to +\infty$, it follows that $I(x, t)$ decays at least exponentially fast as $x \to +\infty$, and (b) follows.

We next analyze $F_2(x, t)$. Note that $\text{supp} \ F_2 \subseteq [0, +\infty]$ and that $\text{Ai} \left( \frac{(x-y)}{(3t)^{1/3}} \right)$ is less well behaved as $y \to +\infty$. A technical remark precedes the analysis.

Lemma 3.4. There is a constant $C$ such that

$$|\text{Ai}(-\xi)| \leq C(1+\xi)^{-1/4} \quad \text{for all real } \xi.$$

Proof. We know that $|\text{Ai}(z)| < 1$ for all $z$, and that there is a $K$ such that

$$|\text{Ai}(z)| \leq K(1+|z|)^{-1/4} \quad \text{for all } z \leq 0.$$

Choose $C = \max \{1, K\}$. If $\xi \leq 1$, then $(1+\xi)^{-1/4} = 1$ and

$$|\text{Ai}(-\xi)| \leq (1+\xi)^{-1/4} \leq C(1+\xi)^{-1/4}.$$

If $\xi > 1$, then $(1+\xi)^{-1/4} = \xi$ and

$$|\text{Ai}(-\xi)| \leq K(1+\xi)^{-1/4} \leq C(1+\xi)^{-1/4}$$

since $(1+\xi)/(1+\xi) \equiv 1$ for $\xi > 1$. □

Lemma 3.5. (a) $F_2(x, t)$ is continuous in $\mathbb{R} \times (0, \infty)$.

(b) If $0 \leq n \leq N-1$, then $F_2(x, t) = o(x^{-n})$ as $x \to +\infty$.

(c) If $0 \leq n \leq N-7/4$, then $\int_{0}^{\infty} x^n \left| F_2(x, t) \right| \, dx < \infty$.

Proof. Because of the support of $F_2(x)$ we have

$$F_2(x, t) = (3t)^{-1/3} \int_{0}^{\infty} F_2(y) \text{Ai} \left( \frac{x-y}{(3t)^{1/3}} \right) \, dy.$$ 

The integrand is continuous in $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ for each $y$. By its definition, $F_2(y) \in L^1(\mathbb{R})$. By (3.4) the integrand is bounded by $|F_2(y)|$. Thus (a) follows by Lebesgue's dominated convergence theorem.

For (b) we assume $0 \leq n \leq N-1$, keep $x \geq 2$, and fix $t > 0$. Let $A(x, t)$ denote the part of the integral in (3.8) over $[0, x/2]$, and $B(x, t)$, the part over $[x/2, \infty)$. We need to show that $x^n A(x, t)$ and $x^n B(x, t)$ go to 0 as $x \to +\infty$. Now

$$x^n A(x, t) = x^n \int_{0}^{x/2} F_2(y) \text{Ai} \left( \frac{x-y}{(3t)^{1/3}} \right) \, dy = x^n \int_{x/2(3t)^{1/3}}^{x/(3t)^{1/3}} \text{Ai} \left( \xi \right) F_2(x-(3t)^{1/3}\xi)(3t)^{1/3} \, d\xi.$$
Note $\xi > x/2(3t)^{1/3}$ implies $x \leq 2(3t)^{1/3}\xi$. So

$$|x^nA(x, t)| \leq 2^n(3t)^{(n+1)/3} \int_{x/(2(3t)^{1/3})}^{x/(3t)^{1/3}} \xi^n|\text{Ai}(\xi)|^{2}d\xi \leq 2^n(3t)^{(n+1)/3}\left[\int_{-\infty}^{x/(3t)^{1/3}} |F(s)|^2 ds\right]^{1/2}\left[\int_{x/(2(3t)^{1/3})}^{x/(3t)^{1/3}} \xi^n|\text{Ai}(\xi)|^2 d\xi\right]^{1/2}.$$  

The decay rate (3.6) of $\text{Ai}$ at $+\infty$ is such that $\xi^2n|\text{Ai}(\xi)|^2$ is integrable on $\mathbb{R}^+$. Thus $x^nA(x, t) \to 0$ as $x \to +\infty$.

Next, since $|\text{Ai}(s)| \leq 1$ for all $s$,

$$|x^nB(x, t)| = \left|\int_{x/2}^{\infty} F_2(y)\text{Ai}\left(\frac{x-y}{(3t)^{1/3}}\right) dy\right| \leq 2^n \int_{x/2}^{\infty} y^n|F_2(y)| dy.$$

Since $F \in L^1_{N-1}(\mathbb{R}^+)$ and $n \leq N-1$, $x^nB(x, t) \to 0$ as $x \to +\infty$. Thus (b) is proved. For (c) note that

$$\int_0^\infty x^n|F_2(x, t)| dx = \int_0^\infty x^n(3t)^{-1/3} \int_{y=0}^{x} F_2(y) \text{Ai}\left(\frac{y-x}{(3t)^{1/3}}\right) dy \ dx$$

$$\leq (3t)^{-1/3} \int_0^\infty x^n \int_{y=0}^{x} |F_2(y)| \text{Ai}\left(\frac{y-x}{(3t)^{1/3}}\right) dy \ dx$$

$$+ (3t)^{-1/3} \int_0^\infty x^n \int_{y=0}^{x} |F_2(y)| \text{Ai}\left(\frac{y-x}{(3t)^{1/3}}\right) dy \ dx.$$

Call these terms $T_1$ and $T_2$. It suffices to show $T_1$ and $T_2$ are finite. Now by Lemma 3.4

$$T_1 \leq (3t)^{-1/3} \int_0^\infty x^n \int_{y=0}^{x} |F_2(y)|C\left(1 + \left\{\frac{y-x}{(3t)^{1/3}}\right\}\right)^{-1/4} dy \ dx$$

$$= C(3t)^{-1/3} \int_0^\infty |F_2(y)| \int_{x=0}^{y} x^n\left(1 + \left\{\frac{y-x}{(3t)^{1/3}}\right\}\right)^{-1/4} dx \ dy$$

$$\leq C(3t)^{-1/3} \int_0^\infty |F_2(y)| \int_{0}^{y} x^n\left(\frac{y-x}{(3t)^{1/3}}\right)^{-1/4} dx \ dy.$$

It is easy to prove by induction on $n$ that

$$\int_0^y x^n\left(\frac{y-x}{(3t)^{1/3}}\right)^{-1/4} dx \leq (3t)^{-1/12} K_n y^{3/4}.$$

Thus there is a function $C = C(t)$ such that

$$T_1 \leq C(t) \int_{y=0}^{\infty} |F_2(y)| y^{n+3/4} dy.$$

Since $n + \frac{3}{4} \leq N-1$ and $F_2 \in L^1_{N-1}(\mathbb{R}^+)$, $T_1 < \infty$. Next

$$T_2 = (3t)^{-1/3} \int_0^\infty |F_2(y)| \int_{x=y}^{x} x^n|\text{Ai}\left(\frac{y-x}{(3t)^{1/3}}\right)| dx \ dy.$$

Consider the inside integral $I(y)$ and let $\xi = (x-y)/(3t)^{1/3}$. Thus

$$I(y) = \int_{0}^{\infty} (y + (3t)^{1/3}\xi)^n|\text{Ai}(\xi)|(3t)^{1/3} d\xi \leq 2^n(3t)^{1/3} \int_{0}^{\infty} (y^n + (3t)^{n/3}\xi^n)|\text{Ai}(\xi)| d\xi \leq C_1(t)y^n + C_2(t)$$

Thus there is a function $C = C(t)$ such that

$$T_2 \leq C(t) \int_{y=0}^{\infty} |F_2(y)| y^{n+3/4} dy.$$
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for positive functions $C_1(t), C_2(t)$, since $\xi^j |\text{Ai}(\xi)| \in L^1(\mathbb{R}^+)$ for all $j \geq 0$. Now

$$T_2 \leq (3t)^{-1/3} \int_{y=0}^{\infty} |F_2(y)|(C_2(t) + y^n C_1(t)) \, dy < \infty$$

because $n \leq N - 7/4 < N - 1$ and $F_2 \in L^1_{N-1}(\mathbb{R}^+)$. □

**Lemma 3.6.** Suppose $0 \leq j \leq 2N + 1/2$. Then

(a) $O(\text{J}_{x} \text{J}_{x} F_2(x, t))$ is continuous in $\mathbb{R} \times (0, \infty)$,

(b) $\partial_x F_2(x, t) = o(x^{-n})$ as $x \to +\infty$ for $0 \leq n \leq N$. If $j \equiv 1$, then $\partial_x \partial_x F_2(x, t) = o(x^{-n})$

(c) $\int_0^\infty \int_0^\infty |\text{J}_{x} \text{J}_{x} F_2(x, t)| \, dx \, dy < \infty$ for $0 \leq n \leq N - 3/4 - j/2.$

**Proof.** From (3.3) we see

$$(3.9) \quad \partial_x F_2(x, t) = (3t)^{-2/3} \int_0^\infty F_2'(y) \text{Ai}' \left( \frac{y-x}{(3t)^{1/3}} \right) \, dy.$$  

The continuity of $\partial_x F_2(x, t)$ follows from the Lebesgue dominated convergence theorem

and the facts $F_2(y) \in L^1_{N-1}$ with $N \geq 11/4$ and $\text{Ai}'(\xi) = O(|\xi|^{1/4})$ as $\xi \to -\infty$. To show

continuity of $\partial_x \partial_x F_2$ with $j \equiv 1$, we integrate by parts in (3.9) getting

$$(3.10) \quad \partial_x F_2(x, t) = - (3t)^{-1/3} \int_0^\infty F_2'(y) \text{Ai} \left( \frac{x-y}{(3t)^{1/3}} \right) \, dy.$$  

Now fix $j \equiv 1$

$$(3.11) \quad \partial_x F_2(x, t) = (3t)^{-(j+1)/3} \int_0^\infty F_2'(y) \text{Ai}^{(j)} \left( \frac{x-y}{(3t)^{1/3}} \right) \, dy.$$  

We need to show the continuity of $\partial_x \partial_x F_2$ at $x_0 \in \mathbb{R}, t_0 > 0$. We keep $x \equiv x_0 - 1, t \equiv t_0/3.$

The integrand is continuous in $(x, t)$ for almost all $y$. Further it is uniformly bounded

for $x \equiv x_0 - 1$:

$$\left| F_2'(y) \text{Ai}^{(j)} \left( \frac{x-y}{(3t)^{1/3}} \right) \right| \leq C |F_2'(y)| \left( 1 + \frac{y-x}{(3t)^{1/3}} \right)^{j/2-1/4} \leq C |F_2'(y)| \left( 1 + \frac{y-x_0}{t_0^{1/3}} \right)^{j/2-1/4}$$

since $j/2 - 1/4 > 0$ when $j \equiv 1$. Further this bound is integrable on $\mathbb{R}^+$ since $F_2 \in L^1_N$ and

the hypothesis on $j$ implies $j/2 - 1/4 \leq N$. This completes (a).

For the remainder of the proof fix $j$ so $0 \leq j \leq 2N + 1/2$. For part (b) we pick $n$ so

$0 \leq n \leq N + 1/4 - j/2$ and keep $x \equiv 2$. From (3.11) we get

$$|x^n \partial_x F_2(x, t)| \leq (3t)^{-(j+1)/3} x^n \int_0^\infty \left| F_2'(y) \text{Ai}^{(j)} \left( \frac{x-y}{(3t)^{1/3}} \right) \right| \, dy.$$  

Let $J_1$ and $J_2$ be the two terms obtained by splitting the integral at $y = x/2$. Note that

$y > x/2$ implies $x^n \leq 2^n y^n$.

$$J_1 = (3t)^{-(j+1)/3} x^n \int_0^{x/2} \left| F_2'(y) \text{Ai}^{(j)} \left( \frac{x-y}{(3t)^{1/3}} \right) \right| \, dy$$

$$= (3t)^{-j/3} x^n \int_{-x/2}^{x/2} \left| F_2'(x-(3t)^{1/3} \xi) \text{Ai}^{(j)} (\xi) \right| \, d\xi$$

after setting $\xi = (x-y)/(3t)^{1/3}$. Note $0 \leq y \leq x/2$ implies $x^n \leq 2^n (3t)^{n/3} x^n$. Thus

$$J_1 \leq (3t)^{-n/3} 2^n \int_{-x/2}^{x/2} \xi^n \left| F_2'(x-(3t)^{1/3} \xi) \text{Ai}^{(j)} (\xi) \right| \, d\xi.$$
By (3.6) we can find a constant $K$ such that $|\xi^n \text{Ai}^{(j)}(\xi)| \leq K e^{-\xi}$ for $\xi \geq 0$ and in particular $|\xi^n \text{Ai}^{(j)}(\xi)| \leq K \exp(-x/2(3t)^{1/3})$ for $\xi \geq x/2(3t)^{-1/3}$. Thus

$$J_1 \leq (3t)^{(j-n)/2} K e^{-x/2(3t)^{1/3}} \int_0^{x/2} |F'_2(y)| \, dy.$$ 

Since $F'_2 \in L^1(\mathbb{R}^+)$, $J_1 \to 0$ as $x \to +\infty$.

It remains to deal with

$$J_2 = J_{2,j}(x) = (3t)^{-(j+1)/3} x^n \int_{x/2}^{\infty} \left| F'_2(y) \text{Ai}^{(j)} \left( \frac{x-y}{(3t)^{1/3}} \right) \right| \, dy.$$ 

If $j = 0$ and $0 \leq n \leq N$, then

$$J_{2,0}(x) \equiv (3t)^{-1/3} n \int_{x/2}^{\infty} y^n |F'_2(y)| \, dy,$$

which goes to zero as $x \to +\infty$ because $F'_2 \in L^1$.

Note that (3.6) implies that there is a constant $A_j$ such that

$$|\text{Ai}^{(j)}(-\xi)| \leq A_j(1 + |\xi|^{j/2-1/4})$$

for real $\xi$. Thus when $j \geq 1$ we get

$$J_{2,j}(x) \leq (3t)^{-(j+1)/2} n \int_{x/2}^{\infty} y^n |F'_2(y)| A_j \left( 1 + \frac{y-x}{(3t)^{1/3}} \right)^{j/2-1/4} \, dy$$

$$\leq C_1(t) \int_{x/2}^{\infty} y^n |F'_2(y)| \, dy + C_2(t) \int_{x/2}^{\infty} y^n |F'_2(y)|(y-x)^{j/2-1/4} \, dy$$

where $C_1(t)$ and $C_2(t)$ are positive functions of $t$. The first integral goes to zero as $x \to +\infty$ because $n \leq N - \frac{1}{3} < N$. In the second integral note $(y-x)^{j/2-1/4}$ is a decreasing function of $x$ since $j \geq 1$. Keeping $x \geq 1$, the second integral is bounded by

$$\int_{x/2}^{\infty} y^n |F'_2(y)|(y-1)^{j/2-1/4} \, dy,$$

which goes to zero as $x \to +\infty$ since we assume $n+j/2 - \frac{1}{4} \leq N$ and know $F'_2 \in L^1$. This finishes (b).

We finally turn to part (c). Keep $0 \leq n \leq N - \frac{1}{3} - j/2$ and note $n < N$ for all $j \geq 0$.

Now

$$\int_0^{\infty} x^n |\partial_x^2 \partial_y F_2(x, t)| \, dx \leq \int_0^{\infty} x^n \left( \int_0^x + \int_x^{\infty} \right) (3t)^{-(j+1)/3} |\text{Ai}^{(j)} \left( \frac{x-y}{(3t)^{1/3}} \right) F'_2(y) | \, dy \, dx.$$ 

Let $K_1$ denote the integral $\int_0^x \int_0^{x_{(3t)^{1/3}}} \cdots \, dy \, dx$, and let $K_2$ denote the other. Set $\xi = (x-y)/(3t)^{1/3}$ in $K_1$. We get

$$K_1 = (3t)^{-j/3} \int_{x_{(3t)^{1/3}}}^{\infty} x^n \left( \int_0^{x/(3t)^{1/3}} |\text{Ai}^{(j)}(\xi) F'_2(x-(3t)^{1/3} \xi) \, d\xi \right) \, dx$$

$$= (3t)^{-j/3} \int_{\xi=0}^{\infty} |\text{Ai}^{(j)}(\xi)| \int_{x=(3t)^{1/3} \xi}^{\infty} x^n |F'_2(x-(3t)^{1/3} \xi) | \, dx \, d\xi$$

$$= (3t)^{-j/3} \int_{\xi=0}^{\infty} |\text{Ai}^{(j)}(\xi)| \int_{y=0}^{\infty} |F'_2(y)| (y+(3t)^{1/3} \xi)^n \, dy \, d\xi$$

$$\leq (3t)^{-j/3} n \int_{\xi=0}^{\infty} |\text{Ai}^{(j)}(\xi)| \int_{y=0}^{\infty} y^n |F'_2(y)| \, dy \, d\xi$$

$$+ (3t)^{n/3} \int_{\xi=0}^{\infty} |\text{Ai}^{(j)}(\xi)| |\xi^n \int_{y=0}^{\infty} |F'_2(y)| \, dy| < \infty.$$
since $F'_j \in L^1_N$ with $N \geq 11/4$ and $\text{Ai}^{(j)}(\xi)$ has faster than exponential decay as $\xi \to +\infty$, and $|y + (3t)^{1/3} \xi|^n \leq 2^n |y^n + (3t)^{n/3} \xi^n|$.  

$$K_2 = (3t)^{-(j+1)/3} \int_{x=0}^\infty x^n \int_{y=x}^\infty \left| F'_j(y) \right| |\text{Ai}^{(j)}(x-y/(3t)^{1/3})| \, dy \, dx.$$  

By (3.6)  

$$K_2 \leq (3t)^{-(j+1)/3} \int_{x=0}^\infty x^n \int_{y=x}^\infty \left| F'_j(y) \right| C_j^+(1 + y-x)^{-j/2-1/4} \, dy \, dx$$  

$$\leq C_j^+(3t)^{-(j+1)/3} \int_{y=0}^\infty \left| F'_j(y) \right| (1+y)^{n+1+j/2-1/4} \, dy < \infty$$  

since $n+\frac{1}{2}-j/2 \leq N$.  

In case $j \geq 1$ we have $j/2 - \frac{j}{2} \geq 0$. So for $x \geq 1$  

$$K_2 \leq C_j^+(3t)^{-(j+1)/3} \int_{y=0}^\infty \left| F'_j(y) \right| \left(1 + y-x \right)^{j/2-1/4} \, dx \, dy$$  

$$\leq C(3t)^{-(j+1)/3} \int_{y=0}^\infty \left| F'_j(y) \right| (1+y)^{n+1+j/2-1/4} \, dy < \infty$$  

since $n+\frac{1}{2}-j/2 \leq N$.  

If $j = 0$, then $j/2 - \frac{j}{2} < 0$ and this argument fails. However, if $j = 0$ we get  

$$K_2 \leq C_j^+(3t)^{-1/3} \int_{y=0}^\infty \left| F'_j(y) \right| \left(1 + y-x \right)^{-1/4} \, dx \, dy.$$  

By induction one shows that the inner integral is $O(y^{n+(3/4)})$. Thus  

$$K_2 \leq C(3t)^{-1/3} \int_{y=0}^\infty \left| F'_j(y) \right| (1+y)^{n+3/4} \, dy < \infty$$  

since we are assuming $n \leq N - \frac{1}{2}$ when $j = 0$.  

The results of Lemmas 3.5 and 3.6 rely on the fact that $\partial_y F_+(x,0)$ is in $L^1(+\infty)$.  

By contrast our next result does not use estimates on $\partial_y F_+(x,0)$. The next result is used by Kappeler in [9] where he considers KdV with certain measures as initial data.  

Up to this point we have used the Airy function strenuously; the rest of our results in this section rely on the type of analysis found in Cohen's paper [3]. Also by way of contrast, note that the distinction between generic and nongeneric data does not arise in the Airy function approach, whereas it does arise using the method of [3].  

**Proposition 3.7.** Suppose that $U \in L^1_M(\mathbb{R})$ with $M \geq 3$. If $U$ is generic, set $N = M$; otherwise, set $N = M - 1$. Let $R_+$ be the reflection coefficient of $U(x)$. Then a function  

$$F_+(x,t) = \pi^{-1} \int_{-\infty}^\infty R_+(k) \exp(2ikx + 8ik^3t) \, dk$$  

may be well defined on $\mathbb{R} \times (0, \infty)$ as an improper Riemann integral. Further for each fixed $t > 0$, $F_+(x,t)$ is $(2N-1)$-times continuously differentiable with respect to $x$. Moreover for arbitrarily small $\varepsilon(0 < \varepsilon \ll 1/2)$ there are functions $K_0,N(t)$ and $K_{1,N}(t)$ such that  

$$|F_+(x,t)| \leq K_0,N(t) x^{-N+1+\varepsilon},$$  

$$|\partial_x^j F_+(x,t)| \leq K_{j,N}(t) x^{-N+j/2+\varepsilon} \text{ for } 1 \leq j \leq 2N - 1$$  

whenever $x > 12t > 0$. $K_{j,N}(t)$ can be chosen nonincreasing, bounded as $t \uparrow +\infty$, and $O(t^{-j/2-\varepsilon})$ as $t \downarrow 0$.  


Proof. Use Proposition 2.7 and a careful adaptation of the methods of [3].

3.2. Properties of $F_-(x, t)$. Recall that the crucial term in the kernel of the left-side Marchenko equation is

$$F_-(x, t) = \pi^{-1} \int_{-\infty}^{\infty} R_-(k) \exp (-2ikx - 8ik^3 t) \, dk.$$ 

Since $R_-(k) = R_+(k)$, this may be rewritten as

$$F_-(x, t) = \pi^{-1} \int_{-\infty}^{\infty} R_+(k) \exp (2ikx + 8ik^3 t) \, dk.$$ 

Because $R_+(k) = V(k)/W(k)$, the analysis of $R_+$ is easily adapted to $R_-$ and the regularity of $F_-$ is the same as that of $F_+$. The decay of $F_-$ and its derivatives as $x \to -\infty$ requires different treatment because there will be stationary points when $x < 0$, namely $k = \pm (|x|/12t)^{1/2}$.

The purpose of this subsection is to identify conditions on $R_+(k)$ sufficient to verify the hypotheses on $F_-$ in Kappeler's $L^2$ inverse scattering theorem [8]. The crucial point is to see when $\partial_t^j F_+(x, t)$ and $|x|^{1/2} \partial_x^j F_+(x, t)$ are in $L^2(-\infty, X)$ for finite $X$ and $j = 0, 1, \cdots, 4$. We formulate the results in two ways to allow some flexibility as to whether we ask $R_-$ to have many derivatives of slower decay or fewer derivatives of faster decay.

This subsection will not be used until late in § 5.

Lemma 3.8. Suppose the function $g$ has property $A(\lambda, N)$, namely

$$g \in C^N(\mathbb{R}) \quad \text{for } N \geq 2,$$

$$g^{(n)}(k) = O(|k|^{-\lambda}) \quad \text{as } |k| \to \infty \quad \text{for } n = 0, 1, 2,$$

$$g^{(n)}(k) = O(|k|^{-\lambda(n)}) \quad \text{as } |k| \to \infty \quad \text{for } 3 \leq n \leq N$$

where

$$\lambda \geq 1, \quad \lambda(n) \geq \max \{1, \lambda - n\} \quad \text{and} \quad \lambda \geq \lambda(3) \geq \cdots \geq \lambda(N).$$

Let $G(x, t)$ be defined by

$$G(x, t) = \int_{-\infty}^{\infty} g(k) e^{8ik^3 t + 2ikx} \, dk.$$ 

If $N \geq \lambda + 3/2$, then for $t > 0$

(i) $\partial_x G(x, t) = O(|x|^{-(\lambda - j)/2 - 1/4})$ as $x \to -\infty$ for $j = 0, 1, 2$.

If $N > (\lambda + 1)/2$, then for $t > 0$ there is a $\delta$ such that $0 < \delta < 1$ and

(ii) $\partial_x G(x, t) = O(|x|^{1/2(\lambda - j)/2 - \delta})$ as $x \to -\infty$, $j = 0, 1, 2$.

Proof. This proof requires the careful extension and correction of the Appendix B of [13], i.e., a careful analysis by the method of stationary phase. □

Remark. Result (ii) gets a weaker result but requires less regularity in $g$ for fixed $\lambda$. The following applications will be used in discussion of the $L^2$ inverse scattering problem in § 5.

Application 1(i). Suppose $g$ satisfies $A(\lambda, N)$ with $5/2 < \lambda \leq 7/2$, $N = 5$, and $\lambda(n) = 1$ for $3 \leq n \leq 5$. Then $N \geq \lambda + 3/2$. Part (i) of Lemma 3.8 tells us that for $t > 0$

$$\partial_x G(x, t) = O(|x|^{-(\lambda - 1)/2 - 1/4}) = O(|x|^{-(\lambda - 2)/2 + 1/4}) \quad \text{as } x \to -\infty.$$ 

Since $-\lambda/2 + 1/4 < -1$, it follows that both $\partial_x G(x, t)$ and $|x|^{1/2} \partial_x G(x, t)$ are in $L^2(-\infty)$. 

Proof. Use Proposition 2.7 and a careful adaptation of the methods of [3]. □

3.2. Properties of $F_-(x, t)$. Recall that the crucial term in the kernel of the left-side Marchenko equation is

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The purpose of this subsection is to identify conditions on $R_+(k)$ sufficient to verify the hypotheses on $F_-$ in Kappeler's $L^2$ inverse scattering theorem [8]. The crucial point is to see when $\partial_t^j F_+(x, t)$ and $|x|^{1/2} \partial_x^j F_+(x, t)$ are in $L^2(-\infty, X)$ for finite $X$ and $j = 0, 1, \cdots, 4$. We formulate the results in two ways to allow some flexibility as to whether we ask $R_-$ to have many derivatives of slower decay or fewer derivatives of faster decay.

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Lemma 3.8. Suppose the function $g$ has property $A(\lambda, N)$, namely

$$g \in C^N(\mathbb{R}) \quad \text{for } N \geq 2,$$

$$g^{(n)}(k) = O(|k|^{-\lambda}) \quad \text{as } |k| \to \infty \quad \text{for } n = 0, 1, 2,$$

$$g^{(n)}(k) = O(|k|^{-\lambda(n)}) \quad \text{as } |k| \to \infty \quad \text{for } 3 \leq n \leq N$$

where

$$\lambda \geq 1, \quad \lambda(n) \geq \max \{1, \lambda - n\} \quad \text{and} \quad \lambda \geq \lambda(3) \geq \cdots \geq \lambda(N).$$

Let $G(x, t)$ be defined by

$$G(x, t) = \int_{-\infty}^{\infty} g(k) e^{8ik^3 t + 2ikx} \, dk.$$ 

If $N \geq \lambda + 3/2$, then for $t > 0$

(i) $\partial_x G(x, t) = O(|x|^{-(\lambda - j)/2 - 1/4})$ as $x \to -\infty$ for $j = 0, 1, 2$.

If $N > (\lambda + 1)/2$, then for $t > 0$ there is a $\delta$ such that $0 < \delta < 1$ and

(ii) $\partial_x G(x, t) = O(|x|^{1/2(\lambda - j)/2 - \delta})$ as $x \to -\infty$, $j = 0, 1, 2$.

Proof. This proof requires the careful extension and correction of the Appendix B of [13], i.e., a careful analysis by the method of stationary phase. □

Remark. Result (ii) gets a weaker result but requires less regularity in $g$ for fixed $\lambda$. The following applications will be used in discussion of the $L^2$ inverse scattering problem in § 5.

Application 1(i). Suppose $g$ satisfies $A(\lambda, N)$ with $5/2 < \lambda \leq 7/2$, $N = 5$, and $\lambda(n) = 1$ for $3 \leq n \leq 5$. Then $N \geq \lambda + 3/2$. Part (i) of Lemma 3.8 tells us that for $t > 0$

$$\partial_x G(x, t) = O(|x|^{-(\lambda - 1)/2 - 1/4}) = O(|x|^{-(\lambda - 2)/2 + 1/4}) \quad \text{as } x \to -\infty.$$ 

Since $-\lambda/2 + 1/4 < -1$, it follows that both $\partial_x G(x, t)$ and $|x|^{1/2} \partial_x G(x, t)$ are in $L^2(-\infty)$. 

Proof. Use Proposition 2.7 and a careful adaptation of the methods of [3]. □
Application 1(ii). Suppose \( g \) satisfies \( A(\lambda, N) \) with \( \lambda = 3, N = 3, \lambda(3) = 1 \). This requires more decay but fewer derivatives than the previous application. Note that \( N > (\lambda + 1)/2 \). By part (ii) of Lemma 3.8, if \( t > 0 \), then

\[
\partial_x G(x, t) = O(\|x\|^{-(\lambda - 1)/2 - 8}) = O(\|x\|^{-1 - 8})
\]

for some very small positive \( \delta \). Thus, in this case also, both \( \partial_x G(x, t) \) and \( \|x\|^{1/2} \partial_x G(x, t) \) are in \( L^2(-\infty) \).

Application 2(i). Suppose that \( g \in C^8(\mathbb{R}) \) and that

\[
g^{(n)}(k) = O(|k|^{-\lambda_0}) \text{ as } |k| \to \infty \text{ for } n = 0, 1, 2,
\]

\[
g^{(n)}(k) = O(|k|^{-\lambda_0(n)}) \text{ as } |k| \to \infty \text{ for } 3 \leq n \leq 8
\]

where

\[
\lambda_0 = \frac{1}{2} + \varepsilon, \quad 0 < \varepsilon < \frac{1}{2},
\]

\[
\lambda_0(n) \equiv \max \{1, \lambda_0 - n\} \text{ for } 3 \leq n \leq 8,
\]

\[
4 = \lambda_0(3) \equiv \lambda_0(4) \equiv \cdots \equiv \lambda_0(8).
\]

Let \( g_0 = g \) and \( g_1 = g' \). Then it is easy to see that \( g_0 \) satisfies \( A(\lambda, N) \) with \( \lambda = \lambda_0 \) and \( N = 8 \). One can also verify that \( g_1 \) satisfies \( A(\lambda, N) \) with \( \lambda = \lambda_1 = 4 \) and \( N = 7 \). Since \( 8 > \lambda_0 + 3/2 \) we can apply Lemma 3.8(i) to get

\[
\partial_x^j G_0(x, t) = O(|x|^{-(\lambda_0 - j)/2 - 1/4}) \quad \text{as } x \to -\infty, \quad j = 0, 1, 2.
\]

Since \( 7 > \lambda_1 + 3/2 \), we can also obtain

\[
\partial_x^j G_1(x, t) = O(|x|^{-(\lambda_1 - j)/2 - 1/4}) \quad \text{as } x \to -\infty, \quad j = 0, 1, 2.
\]

Recall that

\[
\partial_x^2 G_0(x, t) = \frac{1}{6it} G_1(x, t) + \frac{x}{3t} G_0(x, t),
\]

\[
\partial_x^2 G_0(x, t) = \frac{1}{6it} \partial_x^2 G_1(x, t) + \frac{1}{3t} G_0(x, t) + \frac{x}{3t} \partial_x G_0(x, t),
\]

\[
\partial_x^4 G_0(x, t) = \frac{1}{6it} \partial_x^4 G_1(x, t) + \frac{2}{3t} G_0(x, t) + \frac{x}{3t} \partial_x^2 G_0(x, t).
\]

It follows that

\[
\partial_x^3 G_0(x, t) = O(|x|^{1 - (3/4)}) \quad \text{as } x \to -\infty
\]

and

\[
\partial_x^4 G_0(x, t) = O(|x|^{1 - 1/2}) \quad \text{as } x \to -\infty.
\]

We can conclude that for \( j = 0, 1, \cdots, 4 \) both \( \partial_x^j G(x, t) \) and \( \|x\|^{1/2} \partial_x^j G(x, t) \) belong to \( L^2(-\infty) \).

Application 2(ii). Suppose that \( g \) satisfies \( A(\lambda, N) \) with \( \lambda = 6, N = 4, \lambda(3) = 4, \) and \( \lambda(4) = 2 \). Then since \( 4 > (6 + 1)/2 \), Lemma 3.8(ii) gives us

\[
\partial_x G_1(x, t) = O(|x|^{-(6-j)/2 - 5}) \quad \text{as } x \to -\infty, \quad j = 0, 1, 2
\]

for some small positive \( \delta \). Also \( g' \) satisfies \( A(\lambda, N) \) with \( \lambda = 4 \) and \( N = 3 \). Since \( 3 > (4 + 1)/2 \), Lemma 3.8(i) gives us

\[
\partial_x G_1(x, t) = O(|x|^{1 - (4-j)/2 - 1/4}) \quad \text{as } x \to -\infty, \quad j = 0, 1, 2.
\]
Thus
\[
\begin{align*}
\delta_x^j G(x, t) &= O(|x|^{-3+j/2-\delta}) \quad \text{as } x \to -\infty, \quad j = 0, 1, 2, \\
\delta_x^3 G(x, t) &= O(|x|^{-3/2-\delta}) \quad \text{as } x \to -\infty, \\
\delta_x^4 G(x, t) &= O(|x|^{-1-\delta}) \quad \text{as } x \to -\infty.
\end{align*}
\]

It follows that for \( j = 0, 1, \cdots, 4 \) both \( \delta_x^j G(x, t) \) and \( |x|^{1/2} \delta_x^j G(x, t) \) are in \( L^2(-\infty) \).

4. The regularity of the solutions of the Marchenko equation. The right-hand side Marchenko equation is
\[
B_+(x, y) + \Omega_+(x + y) + B_+(x, z)\Omega_+(x + y + z) \, dz = 0.
\]

It is well known [1], [6] that if \( \Omega_+ \in L^1(+\infty) \cap L^\infty(+\infty) \) and if \( \Omega_+ \in L^1(+\infty) \), then \( \Omega_+ \) generates a compact operator from \( L^1(\mathbb{R}^+) \) to \( L^1(\mathbb{R}^+) \) and from \( L^2(\mathbb{R}^+) \) to \( L^2(\mathbb{R}^+) \) for each \( x \) by
\[
\Omega_+[f](y) = \int_0^\infty f(z)\Omega_+(x + y + z) \, dz.
\]

Theorems 4.1 and 4.2 are similar to results in [4]-[6]. They are stated here in the form used later.

**THEOREM 4.1.** Suppose \( n \geq 1 \). Suppose that \( \Omega_+ \in C^{n+1}(\mathbb{R}) \) and that for all finite \( X \)
\[
\begin{align*}
(i) & \quad \|\Omega_+(s)(1 + |s|)\| ds < \infty; \\
(ii) & \quad \text{If } 0 \leq k \leq n, \text{ then } \int_X |\Omega_+^{(k)}(s)| ds < \infty; \\
(iii) & \quad \text{If } 0 \leq k \leq n + 1, \text{ then } \sup \{|\Omega_+^{(k)}(X + s)| : s \geq 0\} < \infty; \\
(iv) & \quad \int_X |\Omega_+^{(n+1)}(s)|^2 ds < \infty.
\end{align*}
\]

Then (4.1) has a unique solution \( B_+(x, \cdot) \) in \( L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \). Further \( x \mapsto B_+(x, \cdot) \) is \( (n + 1) \)-times Fréchet-differentiable as a function from \( \mathbb{R} \) to \( L^2(\mathbb{R}^+) \). Moreover \( \delta_x^k B_+(x, y) \) is continuous in \( \mathbb{R} \times [0, \infty) \) for \( k \leq n + 1 \). Finally if \( k \leq n \), then \( \delta_x^k B_+(x, \cdot) \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+) \).

**Proof.** Consider first the case \( n = 1 \):

The existence of \( B_+(x, \cdot) \) in \( L^1(\mathbb{R}^+) \) is well known [1], [6]. We need to show (a) that \( x \mapsto B_+(x, \cdot) \) is twice differentiable as a function from \( \mathbb{R} \) to \( L^2(\mathbb{R}^+) \), (b) that \( \delta_x^2 B_+(x, y) \) is continuous on \( \mathbb{R} \times \mathbb{R}^+ \) for \( k = 1, 2, 3 \), and (c) that \( \delta_x^k B_+(x, \cdot) \in L^1(\mathbb{R}^+) \) for \( k = 0, 1, 2, 3 \).

Since \( \Omega_+ \in L^1(+\infty) \) it is easy to check that \( x \mapsto \Omega_{+x} \) is continuous in the uniform operator norm on both \( L^1(\mathbb{R}^+) \) and \( L^\infty(\mathbb{R}^+) \). Thus \( (I + \Omega_{+x})^{-1} \) also depends continuously on \( x \) in the uniform norm in \( \mathcal{L}(L^1(\mathbb{R}^+)) \). It is also easy to see that \( \Omega_{+x} \) maps \( L^1(\mathbb{R}^+) \) into \( L^\infty(\mathbb{R}^+) \):
\[
\sup_{y \geq 0} |\Omega_{+x}[g](y)| \leq \sup_{y \geq 0} \int_0^\infty |g(z)| |\Omega_+(x + y + z)| \, dz \leq \sup_{s > x} \sup_{y \geq 0} |\Omega_+(s)| \cdot \int_0^\infty |g(z)| \, dz.
\]

It now follows from (4.1) that \( B_+(x, \cdot) \) is in \( L^\infty(\mathbb{R}^+) \) as well as in \( L^1(\mathbb{R}^+) \). To see the continuity of \( B_+(x, y) \) we note
\[
B_+(x_1, y_1) - B_+(x_2, y_2) = -\int_0^\infty B_+(x_1, z)\{\Omega_+(x_1 + y_1 + z) - \Omega_+(x_2 + y_2 + z)\} \, dz
\]
\[
- \int_0^\infty \{B_+(x_1, z) - B_+(x_2, z)\}\{\Omega_+(x_2 + y_2 + z)\} \, dz.
\]
Thus
\[
|B_+(x_1, y_1) - B_+(x_2, y_2)| \lesssim |\Omega_+(x_1 + y_1) - \Omega_+(x_2 + y_2)| + \sup_{s \leq s_0} |\Omega'_+(s)| |x_1 + y_1 - x_2 - y_2| \|B_+(x_1, \cdot)\|_{L^1} + \sup_{s \leq s_0} |\Omega_+(s)| \cdot \|B_+(x_1, \cdot) - B_+(x_2, \cdot)\|_{L^1}
\]
where \(s_0 \equiv \min \{x_1 + y_1, x_2 + y_2\}\). The continuity of \(B_+(x, y)\) as a map \(\mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}\) now follows easily.

For the rest of this section we will omit the subscripts "\(\cdot\)" from \(B_+, \Omega_+, \) and \(\Omega_+\). Where we intend \(B_-\) and \(\Omega_-\), the subscript "\(\cdot\)" will appear.

Next we ask whether \(x \mapsto B(x, \cdot)\) is differentiable as a map \(\mathbb{R} \mapsto L^1(\mathbb{R}^+)\).

For \(h \neq 0\), set
\[
\Phi_h(x, y) = \frac{\Omega(x + y + h) - \Omega(x + y)}{h} + \int_{z=0}^{\infty} B(x, z) \frac{\Omega(x + y + z + h) - \Omega(x + y + z)}{h} dz
\]
and for \(h = 0\), set
\[
\Phi_0(x, y) = \Omega'(x + y) + \int_0^\infty B(x, z) \Omega'(x + y + z) dz.
\]

Note that
\[
\frac{B(x + h_1, y) - B(x, y)}{h} = -(1 + \Omega_+(x + h_1))^{-1}[\Phi_h(x, \cdot)](y).
\]
Clearly \(\Phi_h(x, \cdot) \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)\) for all \(h\). Further \(\Phi_h(x, \cdot) \mapsto \Phi_0(x, \cdot)\) in both \(L^1(\mathbb{R}^+)\) and \(L^\infty(\mathbb{R}^+)\) as \(h \to 0\). Thus
\[
\lim_{h \to 0^+} \frac{B(x + h, y) - B(x, y)}{h} = -(1 + \Omega_+(x))^{-1}[\Phi_0(x, \cdot)] \text{ in } L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+).
\]

Thus
\[
\partial_+ B(x, y) + \int_{z=0}^{\infty} \partial_+ B(x, z) \Omega(x + y + z) dz = -\Phi_0(x, y)
\]
\[
= -\Omega'(x + y) - \int_0^\infty B(x, z) \Omega'(x + y + z) dz.
\]

It now follows that \(\partial_+ B(x, y)\) depends continuously on \(x\) and \(y\).

Finally consider the map \(x \mapsto \partial_+ B(x, \cdot)\) as going from \(\mathbb{R}\) to \(L^2(\mathbb{R}^+)\). We must show that it is differentiable. Write \(B^{(1,0)}(x, y)\) for \(\partial_+ B(x, y)\). For \(h \neq 0\), set
\[
\psi_h(x, \cdot) = -(1 + \Omega_+(x + h)) \left[ B^{(1,0)}(x + h, \cdot) - B^{(1,0)}(x, \cdot) \right].
\]
Since all \(B^{(1,0)}(x, \cdot)\) are in \(L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)\) it follows that \(\psi_h(x, \cdot)\) is in \(L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)\). Computation shows that
\[
\psi_h(x, y) = \frac{\Omega'(x + y + h) - \Omega'(x + y)}{h} + \int_0^\infty \frac{B(x + h, z) - B(x, z)}{h} \Omega'(x + y + h + z) dz
\]
\[
+ \int_0^\infty B^{(1,0)}(x, z) \frac{\Omega'(x + y + h + z) - \Omega'(x + y + z)}{h} dz
\]
\[
+ \int_0^\infty B(x, z) \frac{\Omega'(x + y + h + z) - \Omega'(x + y + z)}{h} dz.
\]
For \( h = 0 \) set
\[
\psi_0(x, y) = \Omega''(x + y) + 2 \int_0^\infty B^{(1,0)}(x, z)\Omega'(x + y + z) \, dz + \int_0^\infty B(x, z)\Omega''(x + y + z) \, dz.
\]

Our hypotheses and earlier results tell us that
\[
\psi_0(x, \cdot) \in L^2(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+).
\]

Now we verify that \( \psi_h(x, \cdot) \rightarrow \psi_0(x, \cdot) \in L^2(\mathbb{R}^+) \). As a function of \( y \) in \( \mathbb{R}^+ \), \( \{\Omega'(x + y + h) - \Omega'(x + y)\}/h \) converges to \( \Omega''(x + y) \) in \( L^2(\mathbb{R}^+) \) as \( h \rightarrow 0 \). The remaining three terms in \( \psi_h(x, y) \) are essentially convolutions. It is straightforward to verify the convergence of the factors in these convolutions in \( L^2(\mathbb{R}^+) \). Thus, still in \( L^2(\mathbb{R}^+) \), \( \psi_h(x, \cdot) \rightarrow \psi_0(x, \cdot) \) as \( h \rightarrow 0 \). Thus
\[
\lim_{h \rightarrow 0} \frac{B^{(1,0)}(x + h, \cdot) - B^{(1,0)}(x, \cdot)}{h} = -\lim_{h \rightarrow 0} (I + \Omega(x+h))^{-1}\psi_h(x, \cdot) = -(I + \Omega_x)^{-1}\psi_0(x, \cdot).
\]

So \( \partial_x^2 B(x, \cdot) \) exists in \( L^2(\mathbb{R}^+) \) and satisfies
\[
\partial_x^2 B(x, y) + \int_0^\infty \partial_x^2 B(x, z)\Omega(x + y + z) \, dz = -\Omega''(x + y) - 2 \int_0^\infty B^{(1,0)}(x, z)\Omega'(x + y + z) \, dz - \int_0^\infty B(x, z)\Omega''(x + y + z) \, dz.
\]

The continuity of \( \partial_x^2 B(x, y) \) follows from analysis of this equation.

This proves the theorem for \( n = 1 \). The method extends in the obvious way to cases where \( n > 1 \). □

**Theorem 4.2.** Suppose that \( \Omega(x, t) \) has the following properties:

(i) For fixed \( t > 0 \), \( \Omega(x, t) \) is a \( C^1 \) function of \( x \), and
\[
\int_X |\partial_x \Omega(x, t)|(1 + |x|) \, dx < \infty \quad \text{for all finite } X.
\]

(ii) The mapping \( t \mapsto \Omega(\cdot, t) \) is differentiable both as a map from \((0, \infty)\) to \( L^1(\mathbb{R}^+) \) and from \((0, \infty)\) to \( L^\infty(\mathbb{R}^+) \); \( \int_X |\partial_x \Omega(x, t)| \, dx < \infty \), for finite \( X \).

(iii) The mapping \( t \mapsto \Omega(\cdot, t) \) is differentiable as a map from \((0, \infty)\) to \( L^2(\mathbb{R}^+) \),
\[
\int_X |\partial_t \partial_x \Omega(x, t)|^2 \, dx < \infty \quad \text{for finite } X.
\]

(iv) For fixed \( t > 0 \), the functions \( \Omega(x, t), \partial_x \Omega(x, t), \partial_t \Omega(x, t), \) and \( \partial_t \partial_x \Omega(x, t) \) are in \( L^\infty(\mathbb{R}^+) \).

(v) The functions mentioned in (iv) are continuous on \( \mathbb{R} \times (0, \infty) \). For each \( t > 0 \), let \( B(x, \cdot, t) \) denote the solution of
\[
B(x, y, t) + \Omega(x + y, t) + \int_0^\infty B(x, z, t)\Omega(x + y + z, t) \, dz = 0,
\]
which is the Marchenko equation with \( \Omega = \Omega(x, t) \). Then

(a) The map \( t \mapsto B(x, \cdot, t) \) is differentiable both as a map \((0, \infty) \rightarrow L^1(\mathbb{R}^+) \) and as a map \((0, \infty) \rightarrow L^2(\mathbb{R}^+) \). Further both \( B(x, y, t) \) and \( \partial_t B(x, y, t) \) are continuous in \( \mathbb{R} \times (0, \infty) \times (0, \infty) \).

(b) The map \( t \mapsto \partial_x B(x, \cdot, t) \) is differentiable as a map \((0, \infty) \rightarrow L^2(\mathbb{R}^+) \), and \( \partial_t \partial_x B(x, y, t) \) is continuous in \( \mathbb{R} \times (0, \infty) \times (0, \infty) \).
Proof. By hypotheses we have $\Omega(x, t) \in L^1(\mathbb{R}^+) \text{ and } \partial_t \Omega(x, t) \in L^1(\mathbb{R}^+)$ for each fixed $t > 0$. Therefore the solutions $B(x, \cdot, t)$ exist in $L^1(\mathbb{R}^+)$. The continuity of $B(x, y, t)$ follows immediately, as does the existence and continuity of $\partial_x B(x, y, t)$.

Let $\Omega_x$ denote the operator $\Omega_x[g](y) = \int_0^\infty \Omega_x(x + y + z, t)g(z) \, dz$.

We use again the methods of the previous theorem. For $h \neq 0$, one gets

$$\frac{B(x, y, t + h) - B(x, y, t)}{h} + \int_0^\infty \frac{B(x, z, t + h) - B(x, z, t)}{h} \Omega(x + y + z, t + h) \, dz$$

$$= -\Phi_h(x, y, t)$$

where

$$\Phi_h(x, y, t) = \frac{\Omega(x + y, t + h) - \Omega(x + y, t)}{h} + \int_0^\infty B(x, z, t) \left\{ \frac{\Omega(x + y + z, t + h) - \Omega(x + y + z, t)}{h} \right\} \, dz.$$ 

We have assumed that the map $t \mapsto \Omega(\cdot, t)$ is differentiable in $L^1(\mathbb{R}^+)$. Therefore as $h \to 0$, $\Phi_h(x, \cdot, t)$ converges in $L^1(\mathbb{R}^+)$ to

$$\Phi_0(x, y, t) = \partial_t \Omega(x + y, t) + \int_0^\infty B(x, z, t) \partial_t \Omega(x + y + z, t) \, dz.$$ 

It is easy to see that $\Phi_h(x, \cdot, t) \to \Phi_0(x, \cdot, t)$ also in $L^\infty(\mathbb{R}^+)$, whence in $L^2(\mathbb{R}^+)$ as well. Now we have

$$\frac{B(x, \cdot, t + h) - B(x, \cdot, t)}{h} = (\mathbb{I} + \Omega_x^{t+h})^{-1}[-\Phi_h(x, \cdot, t)].$$

The operator $(\mathbb{I} + \Omega_x^{t+h})^{-1}$ depends continuously on $h$ in the operator norms on both $L^1(\mathbb{R}^+)$ and $L^2(\mathbb{R}^+)$. So

$$\lim_{h \to 0} \frac{B(x, \cdot, t + h) - B(x, \cdot, t)}{h} = (\mathbb{I} + \Omega_x^{t})^{-1}[-\Phi_0(x, \cdot, t)]$$ 

in both spaces; equivalently $\partial_x B(x, \cdot, t)$ exists in $L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ and satisfies

$$\partial_x B(x, y, t) = -\partial_t \Omega(x + y, t) - \int_0^\infty B_x(x, z, t) \Omega(x + y + z, t) \, dz$$

$$- \int_0^\infty B(x, z, t) \partial_z \Omega(x + y + z, t) \, dz.$$ 

From this it follows that $\partial_x B(x, \cdot, t)$ is in $L^\infty(\mathbb{R}^+)$ and that $\partial_x B(x, y, t)$ is continuous in $\mathbb{R} \times [0, \infty) \times (0, \infty)$.

Next we study the map $t \mapsto \partial_x B(x, \cdot, t)$. Set

$$-\Psi_h(x, \cdot, t) = (\mathbb{I} + \Omega_x^{t+h})^{-1} \left[ \frac{B(x, \cdot, t + h) - B(x, \cdot, t)}{h} \right].$$

Computation shows that

$$\Psi_h(x, y, t) = \frac{\Omega^{(1,0)}(x + y, t + h) - \Omega^{(1,0)}(x + y, t)}{h} + \int_0^\infty B_x(x, z, t) \left\{ \Omega(x + y + z, t + h) - \Omega(x + y + z, t) \right\} h^{-1} \, dz$$

$$+ \int_0^\infty B(x, z, t) \left\{ \Omega^{(1,0)}(x + y + z, t + h) - \Omega^{(1,0)}(x + y + z, t) \right\} h^{-1} \, dz.$$
Clearly as \( h \to 0 \), we get the convergence \( \psi_h(x, \cdot, t) \to \psi_0(x, \cdot, t) \) in \( L^2(\mathbb{R}^+) \), where

\[
\psi_0(x, y, t) = \partial_t \partial_x \Omega(x + y, t) + \int_0^\infty B_x(x, z, t) \Omega(x + y + z, t) \, dz
\]

Thus \( t \mapsto B_x(x, \cdot, t) \) is differentiable in \( L^2(\mathbb{R}^+) \) and

\[
\partial_t \partial_x B(x, y, t) + \int_0^\infty \partial_t \partial_x B(x, z, t) \Omega(x + y + z, t) \, dz = -\psi_0(x, y, t)
\]

whence \( \partial_t \partial_x B(x, y, t) \) is continuous and belongs to \( L^\infty(\mathbb{R}^+) \) as a function of \( y \).

We next investigate the consequences of a stronger decay assumption on \( \Omega'(x) \), namely that there is an \( \alpha \geq 1 \) such that

\[
\int_x^\infty |\Omega'(x)|(1 + |x|^\alpha) \, dx < \infty \quad \text{for all } x \in \mathbb{R}.
\]

We make use of an inequality of Faddeev's [6, p. 160]

\[
|\Omega(x) + \partial_x B(x, 0)| \leq C(x) \left[ \int_x^\infty |\Omega'(t)| \, dt \right]^2
\]

where \( C(x) \) is a nonincreasing function of \( x \).

**Lemma 4.3.** Suppose that \( \Omega \in L^1(+\infty) \) and \( \Omega' \in L^\alpha_1(+\infty) \) with \( \alpha \geq 1 \). Let \( B(x, y) \) be the solution of (4.1) and set \( u(x) = -\partial_x B(x, 0) \). Then \( u \in L^\alpha_1(+\infty) \).

**Proof.** Because of (4.2) it suffices to prove that

\[
Q = \int_{y=x}^\infty y^\alpha \left( \int_{t=y}^\infty |\Omega'(t)| \, dt \right)^2 \, dy < \infty
\]

for \( x \geq 0 \).

Now

\[
Q = \int_{y=x}^\infty y^\alpha \left( \int_{t=y}^\infty |\Omega'(t)| \, dt \right) \left( \int_{s=y}^\infty |\Omega'(s)| \, ds \right) \, dy
\]

\[
= \int_{t=x}^\infty |\Omega'(t)| \int_{s=x}^t |\Omega'(s)| \int_{y=x}^s y^\alpha \, dy \, ds \, dt + \int_{t=x}^\infty |\Omega'(t)| \int_{s=t}^\infty |\Omega'(s)| \int_{y=x}^t y^\alpha \, dy \, ds \, dt
\]

\[
\leq \frac{1}{\alpha + 1} \int_{t=x}^\infty |\Omega'(t)| t^{(\alpha + 1)/2} \int_{s=x}^t \Omega'(s) s^{(\alpha + 1)/2} \, ds \, dt
\]

\[
< \infty
\]

since \( (\alpha + 1)/2 \leq \alpha \). \( \square \)

**5. The existence theorem for KdV; properties of the solution.** In this section we describe the properties of the function \( u(x, t) \) constructed by the inverse scattering method and establish the sense in which it solves the problem (1.1), (1.2).
THEOREM 5.1. Suppose that $U \in L^1_1(\mathbb{R})$ and that $U \in L^1_N(\mathbb{R}^+)$ for some $N \geq 11/4$. Then there is a classical solution $u(x, t)$ of KdV in $t > 0$ such that

(i) $\partial_x^j \partial_t^k u(x, t)$ is continuous in $x$ for each positive $t$ when $0 \leq j + 3k \leq 2N - 3$;

(ii) $u(\cdot, t) \to U$ in $H^{-1}(+\infty)$ as $t \to 0$.

(iii) $x^n \partial_x^j u(x, t) \to 0$ as $x \to +\infty$ for $0 \leq n \leq N + 1 - (j + 1)/2$.

Proof. For each fixed positive $t$ we consider the Marchenko equation

$$B_+(x, y, t) + \Omega_+(x + y, t) + \int_{z=0}^{\infty} \Omega_+(x + y + z, t) B_+(x, z, t) \, dz = 0$$

where

$$\Omega_+(x, t) = F_+(x, t) + 2 \sum_{j \in J} c_{+j} e^{-2jx + 8\kappa j^2 t}$$

and

$$F_+(x, t) = \pi^{-1} \int_{-\infty}^{\infty} R_+(k) e^{8k\kappa t + 2ikx} \, dk.$$ 

This $F_+(x, t) = F_1(x, t) + F_2(x, t)$ as in § 3. By appealing to Lemmas 3.3, 3.4, 3.5 and to the hypothesis $N \geq 11/4$ we conclude that

(a) $\partial_x \Omega_+(x, t) \in L^1_1(+\infty)$,

(b) $\partial_x^2 \Omega_+(x, t) \in L^1(+\infty)$ for $0 \leq \nu \leq 2N - 1/2$,

(c) $\partial_x^3 \Omega_+(x, t) \in L^\infty(+\infty)$ for $0 \leq \nu \leq 2N + 1/2$ and

(d) $\partial_x^{n+1} \Omega_+(x, t) \in L^2(+\infty)$ for $0 \leq \nu \leq 2N - 3/2$.

Note that $N \geq 11/4$ implies that $2N - 3/2 \geq 4$. Thus our kernel $\Omega_+(x, t)$ satisfies the hypotheses of Theorem 4.1 with $1 \leq n \leq 2N - 3/4$. So we obtain a solution $B_+(x, y, t)$ to (5.1) such that

$$B_+(x, \cdot, t) \in L^1_1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+).$$

$$\partial_x \Omega_+(x, \cdot, t) \text{ is continuous for } (x, y) \in \mathbb{R} \times (0, \infty) \text{ if } \nu \leq n + 1.$$ 

Let $u(x, t) = -\partial_x B_+(x, 0, t)$. We must now show that $u(x, t)$ is the desired solution of KdV.

In addition to properties (a)–(d) of $\Omega_+$ we know that in the distribution sense

$$\partial_t \Omega_+(x, t) + \partial_x^2 \Omega_+(x, t) = 0 \quad \text{for } t > 0, \quad x \in \mathbb{R}.$$ 

Since $N \geq 11/4$, and thus $2N - 1 \geq 4$ we conclude that $\Omega_+(x, t)$ and $\partial_x \Omega_+(x, t)$ are continuously differentiable with respect to $t$, and that

$$\partial_t \Omega_+(\cdot, t) = -\partial_x^2 \Omega_+(\cdot, t) \in L^1(+\infty) \cap L^\infty(+\infty),$$

$$\partial_t^2 \partial_x \Omega_+(\cdot, t) = -\partial_x^3 \Omega_+(\cdot, t) \in L^2(+\infty) \cap L^\infty(+\infty).$$

In order to apply Theorem 4.2 we need finally to check that $t \mapsto \Omega_+(\cdot, t)$ is differentiable in $L^\infty(+\infty)$ for $t > 0$; but this follows from the continuity and decay rate of $\Omega_t = -\Omega_{xxx}$.

By applying Theorem 4.2 we now learn that

$$\partial_t B_+(x, \cdot, t) \in L^1_1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+),$$

$$\partial_t \partial_x B_+(x, \cdot, t) \in L^2_1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+),$$

and further that all $\partial_x^j \partial_t^k B_+(x, y, t)$ are continuous in $\mathbb{R} \times [0, \infty) \times (0, \infty)$ for $j + 3k \leq 2N - 2$. 
For fixed positive $t$, it is clear that $u$ has the regularity (i) and the decay rate (iii). The proof that this function $u(x, t)$ satisfies the KdV equation (1.1) in $t > 0$ follows Tanaka’s argument in [17]. The condition $N \geq 11/4$ gives enough regularity to justify the formal argument.

To prove (ii) we show that $u(x, t) \to U(x)$ in $H^{-1}([X, \infty))$ as $t \to 0$ for each finite $X$. Since $B_+(x, 0, t) = \int_0^\infty u(s, t) \, ds$ for $t \geq 0$ we must show that $B_+(x, 0, t) \to B_+(x, 0, 0)$ in $L^2([X, \infty))$ as $t \to 0$. From the Marchenko equation (4.1) we obtain

$$B_+(x, 0, t) - B_+(x, 0, 0) = -Q_1(x, t) - Q_2(x, t) - Q_3(x, t)$$

where

$$Q_1(x, t) = \Omega_+(x, t) - \Omega_+(x, 0),$$

$$Q_2(x, t) = \int_0^\infty \{B_+(x, z, t) - B_+(x, z, 0)\} \Omega_+(x + z, t) \, dz,$$

$$Q_3(x, t) = \int_0^\infty B_+(x, z, 0) \{\Omega_+(x + z, t) - \Omega_+(x + z, 0)\} \, dz.$$

One easily sees that $Q_1(x, t) \to 0$ in $L^2([X, \infty))$ as $t \to 0$.

Next we show $Q_3(x, t) \to 0$ in $L^2([X, \infty))$ as $t \to 0$ by showing that

$$Q_3(x, t) \to 0$$

where

$$Q_3(x, t) \to 0$$

For any $h \in L^2([X, \infty))$

$$\left| \int_X h(x) Q_3(x, t) \, dx \right| = \left| \int_{x=X}^\infty h(x) \int_{x=x}^\infty B_+(x, s - x, 0) \{\Omega_+(s, t) - \Omega_+(s, 0)\} \, ds \, dx \right|$$

$$\leq \left| \int_{s=X}^\infty \Omega_+(s, t) - \Omega_+(s, 0) \left\{ \int_{x=x}^s h(x) B_+(x, s - x, 0) \, dx \right\} \, ds \right|$$

$$\leq \left\| \Omega_+(\cdot, t) - \Omega_+(\cdot, 0) \right\|_{L^2([X, \infty))} \cdot \left\{ \int_{s=X}^\infty \left( \int_{x=x}^s \left| h(x) B_+(x, s - x, 0) \right| \, dx \right)^2 \, ds \right\}^{1/2}.$$
whence \( \Phi \leq C \| h \|_{L^2([-\infty, \infty])} \) where

\[
C = \frac{1}{2} K \left\{ \int_X |U(w)| \, dw \int_{z=X}^{\infty} (z-X)^2 |U(z)| \, dz \right\}^{1/2} < \infty.
\]

Now (5.2) follows since \( h \) was arbitrary.

Finally we look at \( Q_2(x,t) \): One finds

\[
\| Q_2(x,t) \|_{L^2([-\infty, \infty])} \leq \left\{ \int_X^{\infty} \int_{z=X}^{\infty} \left| B_+(x,z,t) - B_+(x,z,0) \right|^2 \, dz \, dx \right\} ^{1/2} \| Q_1(\cdot,t) \|_{L^2([-\infty, \infty])}.
\]

The second factor is bounded as \( t \to 0 \), so we look at the first factor, \( \Psi(t) \).

\[
B_+(x,z,t) - B_+(x,z,0) = -Q_4(x,z,t) - Q_5(x,z,t)
\]

where

\[
Q_4(x,\cdot,t) = (I + \Omega_x^+)^{-1} \left[ \Omega_+(x + [\cdot], t) - \Omega_+(x + [\cdot], 0) \right]
\]

and

\[
Q_5(x,\cdot,t) = \left[ (I + \Omega_x^+)^{-1} - (I + \Omega_x^0) \right] \Omega_+(x + [\cdot], 0).
\]

Thus

\[
\psi(t) = \int_X^{\infty} \| Q_4(x,\cdot,t) + Q_5(x,\cdot,0) \|_{L^2(\mathbb{R}^+)} \, dx \leq 2 \int_X^{\infty} \| Q_4(x,\cdot,t) \|^2 + \| Q_5(x,\cdot,t) \|^2 \, dx.
\]

There is a bound \( M \) such that

\[
\| (I + \Omega_x^+)^{-1} \|_{op,L^2(\mathbb{R}^+)} \leq M \quad \text{for } X \leq x \leq \infty, \quad 0 \leq t \leq 1
\]

because the operator depends continuously on \( (x,t) \) and the kernel decays fast enough as \( x \to +\infty \). Thus

\[
\| Q_4(x,\cdot,t) \|_{L^2(\mathbb{R}^+)} \leq M \int_X^{\infty} |\Omega_+(s,t) - \Omega_+(s,0)|^2 \, ds
\]

and

\[
\int_X^{\infty} \| Q_4(x,\cdot,t) \|^2 \, dx \leq M \int_X^{\infty} (s-X) |\Omega_+(s,t) - \Omega_+(s,0)|^2 \, ds.
\]

By the form of \( \Omega_+, \Omega_+ = F_+ + G_+ \) in §3.1, we need only show

\[
\int_X^{\infty} (s-X) |F_+(s,t) - F_+(s,0)|^2 \, ds \to 0 \quad \text{as } t \to 0.
\]

We already know \( F_+(s,t) \to F_+(s,0) \) in \( L^2(\mathbb{R}) \) so it suffices to show \( \int_X^{\infty} s^2 |F_+(s,t) - F_+(s,0)|^2 \, ds \to 0 \). By Proposition 2.5 we know \( R_+ \in C^1(\mathbb{R}) \) and \( R_+(k) = O(k^{-1}) \) as \( k \to \pm\infty \). We may therefore follow Kappeler's proof of Theorem 3.1(iv) in [9] to conclude that \( sF_+(s,t) \to sF_+(s,0) \) in \( L^2(\mathbb{R}^+) \) as \( t \to 0 \). It remains to make \( \int_X^{\infty} \| Q_5(x,\cdot,t) \|^2 \, dx \to 0 \) as \( t \to 0 \). But, using \( L^2 \) norms on \( \mathbb{R}^+ \),

\[
\| Q_5(x,\cdot,t) \| \leq \| (I + \Omega_x^+)^{-1} (\Omega_x^0 - \Omega_x^0)(I + \Omega_x^0)^{-1} \|_{op} \| \Omega_+(x + [\cdot], t) \|
\]

\[
\leq M^2 \| \Omega_x^0 - \Omega_x^0 \|_{op} \| \Omega_+(x + [\cdot], t) \|.
\]
The last factor is bounded as $t \to 0$ so
\[ \int_{x}^{\infty} \|Q_{s}(x, \cdot, t)\|^2 \, dx \leq C \int_{x}^{\infty} \|\Omega_{s}^{0} - \Omega_{x}^{1}\|_{op}^2 \, dx \]
\[ \leq C \int_{x}^{\infty} \int_{x}^{\infty} |\Omega_{+}(s, t) - \Omega_{+}(s, 0)|^2 \, ds \, dx \]
\[ \leq C \int_{x}^{\infty} (s - X)|\Omega_{+}(s, t) - \Omega_{+}(s, 0)|^2 \, ds, \]
which we have already seen goes to 0 with $t$.

This completes the proof of Theorem 5.1. □

Under certain additional hypotheses we can also study $u(x, t)$ as $x \to -\infty$.

**Theorem 5.2.** (i) Assume that $U \in L_{1}^{1}(\mathbb{R})$, that $N \geq 5$, and that $R_{\lambda}^{(n)}(k) = O(\lambda^{-\lambda})$ as $k \to \pm \infty$ for $n = 0, 1, 2$ and some $\lambda > 5/2$. Let $u(x, t)$ be the solution to KdV with initial profile $U$ in the sense of Theorem 5.1. Then $u(\cdot, t)$ evolves in $L^{2}(-\infty, \infty)$ for $t > 0$. If $N \geq \lambda + 2$, then $\int_{-\infty}^{\infty} |u(x, t)|^{2}\lambda x^{2\alpha} \, dx < \infty$ for any $\alpha$ with $\lambda - 3/2 > 2\alpha \geq 1$.

(ii) Suppose that $U$ satisfies the hypothesis of Theorem 5.1, that $R_{\lambda}^{(n)}(k) = O(k^{-3})$ for $n = 0, 1, 2$, and that $R_{\lambda}^{(\lambda)}(k) = O(k^{-1})$. Let $u(x, t)$ be the solution of KdV given by Theorem 5.1. Then $u(x, t)$ evolves in $L^{2}(\mathbb{R})$ for $t > 0$.

**Remark 1.** The purpose of the extra hypothesis is to allow use of the left-side Marchenko equation as well as the right-side one, and thereby to study $u(x, t)$ as $x \to -\infty$. Sachs did not treat this point. Kruzhkov and Faminskii also show evolution in $L^{2}(\mathbb{R})$, but they consider weighted $L^{2}$ norms only on $\mathbb{R}^{+}$, and their construction is not conducive to analysis of the long time asymptotics of their solution.

**Remark 2.** For $U$ in $L_{1}^{1}(\mathbb{R})$ it is known that additional regularity of $U$ is a sufficient condition for additional decay of $R_{\lambda}(k)$. For example, from [4] one learns that if $U(x)$ is absolutely continuous, $U'(x)$ is piecewise absolutely continuous, $U \in L_{1}^{1}(\mathbb{R})$, $U' \in L_{1}^{1}(\mathbb{R})$, and $U'' \in L_{1}^{1}(\mathbb{R})$, then the hypothesis of (ii) is satisfied. Another example [20] shows that if $x^{m}U'(x)$ is in $L^{2}(\mathbb{R})$ for $0 \leq m \leq 4$ and $0 \leq j \leq 4$, then the hypothesis of (ii) is also satisfied.

**Proof.** (i) We consider the solution $u(x, t)$ provided by Theorem 5.1. We know from §3 that for $t > 0$
\[ \frac{\partial}{\partial x} F_{+}(x, t) = O(|x|^{-N+1/2+\varepsilon}) \quad \text{as} \quad x \to +\infty, \]
\[ \frac{\partial}{\partial x} F_{-}(x, t) = O(|x|^{-1/2+1/4}) \quad \text{as} \quad x \to -\infty. \]
The same decay rates hold for $\partial_{x} \Omega_{+}, \partial_{x} \Omega_{-}$. By the forward scattering theory [5], [6] all hypotheses of Theorem 3.3 in [8] are satisfied at each $t > 0$. Thus we conclude by Theorem 3.9 of [8] that
\[ u(\cdot, t) = -\partial_{x} B_{+}(\cdot, 0, t) = \partial_{x} B_{-}(\cdot, 0, t) \]
in $L_{1}^{1}(\mathbb{R})$ for each fixed positive $t$, where $B_{+}$ and $B_{-}$ are the solutions of the Marchenko equations
\[ B_{+}(x, y, t) + \Omega_{+}(x + y, t) + \int_{z=0}^{\infty} B_{+}(x, z, t) \Omega_{+}(x + y + z, t) = 0, \]
\[ B_{-}(x, y, t) + \Omega_{-}(x + y, t) + \int_{z=-\infty}^{0} B_{-}(x, z, t) \Omega_{-}(x + y + z, t) = 0 \]
in $L^{1}(\mathbb{R}^{+})$ and $L^{1}(\mathbb{R}^{-})$, respectively.
In the generic case \( u(x, t) = O(\sqrt{x^{-N+1+\varepsilon}}) \) as \( x \to \infty \) for \( 0 < \varepsilon \ll 1/2 \). In the exceptional case \( u(x, t) = O(x^{-N+2+\varepsilon}) \) as \( x \to \infty \) for \( 0 < \varepsilon \ll 1/2 \). Since \( N \geq 5 \), \( u(\cdot, t) \in L^2(\mathbb{R}^+) \).

If we know that both \( \lambda > 5/2 \) and \( N \geq \lambda + 2 \), then we can pick \( \alpha \) so \( \lambda - 3/2 > 2\alpha \geq 1 \) and conclude that

\[
\int_0^\infty |s|^{2\alpha} |u(s, t)|^2 \, ds < \infty
\]

since \( 2\alpha + 2(-N+2+\varepsilon) < -1 \).

If we take \( \alpha \) so \( \lambda - 3/2 > 2\alpha \geq 1 \), then by Kappeler's Theorem 3.9 [8] we get

\[
\int_0^t |s|^{2\alpha} |u(s, t)|^2 \, ds < \infty \quad \text{for } t > 0.
\]

Since \( 2\alpha \geq 1 \) we get \( \int_{-\infty}^{-1} |u(s, t)|^2 \, ds < \infty \) also. But since \( u(x, t) \) is in \( L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+) \) we can conclude that \( \int_0^\infty |u(s, t)|^2 \, ds < \infty \), and further that \( u(\cdot, t) \in L^2(\mathbb{R}^-) \).

The proof of (i) is completed by combining results on \( \mathbb{R}^+ \), \( \mathbb{R}^- \); the proof of (ii) is similar.

\textbf{Theorem 5.3.} For \( N \geq 3 \), assume that \( U \in L^1_\alpha(\mathbb{R}) \) if \( U \) is generic, but that \( U \in L^1_{N+1}(\mathbb{R}) \) if \( U \) is nongeneric. Let \( u(x, t) \) be the solution of KdV provided by Theorem 5.1. Recall \( B_+ (x, 0, t) = \int_{-\infty}^x u(z, t) \, dz \).

(i) For \( 0 \leq n \leq N-1 \), \( x^n B_+(x, 0, t) \to x^n B_+(x, 0, 0) \) in \( L^2(\mathbb{R}^+) \) as \( t \to 0 \);

(ii) For each \( n \) with \( 1 \leq n \leq N-1 \), if \( (1+x^n)U(x) \) is in \( L^2(\mathbb{R}) \), then \( x^n u(x, t) \to x^n U(x) \) in \( L^2(\mathbb{R}^+) \) as \( t \to 0 \) for all \( \alpha \) with \( 0 \leq \alpha \leq n \);

(iii) For each \( n \) with \( 1 \leq n \leq N-1 \), if both \( (1+x^n)U(x) \) and \( (1+x^n)U'(x) \) are in \( L^2(\mathbb{R}^+) \), then also \( x^n \partial_x u(x, t) \to x^n U'(x) \) in \( L^2(\mathbb{R}^+) \) as \( t \to 0 \) for all \( \alpha \) with \( 0 \leq \alpha \leq n \).

\textbf{Proof.} Because \( R_+(k) \) is at least \( C^1 \) and \( R_+^{(1)}(k) \) is \( O(k^{-1}) \) as \( k \to \pm \infty \), the proof of (i) may be taken over from the proof of Kappeler's [9, Thm. 3.1]. We prove (ii) below; the proof of (iii) is similar.

Start with the representation

\[
u(x, t) - U(x) = \partial_x \Omega_+(x, t) - \partial_x \Omega_+(x, 0)
\]

\[
+ \int_0^\infty \{ B_+(x, z, t) - B_+(x, z, 0) \} \partial_x \Omega_+(x+z, t) \, dz
\]

\[
+ \int_0^\infty B_+(x, z, 0) \{ \partial_x \Omega_+(x+z, t) - \partial_x \Omega_+(x+z, 0) \} \, dz
\]

\[
+ \int_0^\infty \{ \partial_x B_+(x, z, t) - \partial_x B_+(x, z, 0) \} \Omega_+(x+z, t) \, dz
\]

\[
+ \int_0^\infty \partial_x B_+(x, z, 0) \{ \Omega_+(x+z, t) - \Omega_+(x+z, 0) \} \, dz
\]

which is based on the Marchenko equation. Call the five terms on the right \( T_\nu(x, t) \) for \( \nu = 1, \ldots, 5 \). We must show that \( x^\alpha T_\nu(x, t) \to 0 \) in \( L^2([X, \infty)) \) as \( t \to 0 \) for arbitrary \( X \) and \( \nu = 1, \ldots, 5 \). We do this by assuming three technical points which will be stated when first used but not proved until the end.

Since \( x^\alpha U(x) \in L^2(\mathbb{R}) \) for \( 0 \leq \alpha \leq n \), Proposition 2.7 tells us that \( kR_+^{(\alpha)}(k) \) is in \( L^2(\mathbb{R}) \) for \( 0 \leq \alpha \leq n \), and that \( R_+^{(\alpha)}(k) \) is in \( L^2(\mathbb{R}) \) for \( 0 \leq \alpha \leq N \). Thus

\[
x^\alpha F_+(x) \in L^2(\mathbb{R}) \quad \text{for } 0 \leq \alpha \leq N,
\]
and
\[ x^n \partial_x F_+ (x) \in L^2(\mathbb{R}) \quad \text{for } 0 \leq \alpha \leq n. \]

By Kappeler's method of proof [9, Thm. 3.1] one may show that if \( 0 \leq \beta \leq n \), then
\[(5.3a) \quad x^\beta \Omega_+(x, t) \to x^\beta \Omega_+(x, 0) \quad \text{in } L^2(\mathbb{R}^+) \quad \text{as } t \to 0\]
and that
\[(5.3b) \quad \partial_x \Omega_+(x, t) \to \partial_x \Omega_+(x, 0) \quad \text{in } L^2(\mathbb{R}^+) \quad \text{as } t \to 0.\]

The convergence of \( x^n T_1(x, t) \) to 0 in \( L^2([X, \infty)) \) as \( t \to 0 \) is one part of our first technical result:

**Point 1.** If \( 0 \leq \alpha \leq n \) and \( X \) is fixed, then
\[(5.4) \quad x^n \partial_x \Omega_+(x, t) \to x^n \partial_x \Omega_+(x, 0) \quad \text{in } L^2([X, \infty)) \quad \text{as } t \to 0,\]
\[(5.5) \quad \int_{X}^\infty |x|^{2\alpha} \left( \int_{X}^\infty |\partial_x \Omega_+(s, t) - \partial_x \Omega_+(s, 0)|^2 \, ds \right)^{1/2} \, dx \to 0 \quad \text{as } t \to 0.\]

Looking at \( T_2(x, t) \) we see
\[
\int_{X}^\infty |x|^{2\alpha} \left( \int_{X}^\infty |\partial_x \Omega_+(x+z, t) - \partial_x \Omega_+(x+z, 0)|^2 \, dz \right)^{1/2} \, dx \to 0 \quad \text{as } t \to 0.
\]

The second factor is bounded as \( t \to 0 \) by (5.4). Part of our next technical point tells us that the first factor vanishes as \( t \to 0 \):

**Point 2.** Fix \( X \). Then, as \( t \to 0 \),
\[(5.6) \quad B_+(x, \cdot, t) \to B_+(x, \cdot, 0) \quad \text{in } L^2(\mathbb{R}^+) \quad \text{uniformly for } x \geq X,\]
\[(5.7) \quad \int_{X}^\infty |x|^{2\alpha} \left( \int_{0}^\infty |B_+(x, z, t) - B_+(x, z, 0)|^2 \, dz \right)^{1/2} \, dx \to 0 \quad \text{for } 0 \leq \alpha \leq n.
\]

Looking at the third term we see
\[
\int_{X}^\infty |x|^{2\alpha} \left( \int_{0}^\infty |\partial_x \Omega_+(x, t) - \partial_x \Omega_+(x, 0)|^2 \, dz \right)^{1/2} \, dx \to 0 \quad \text{as } t \to 0.
\]
The first factor is finite by (5.6). The second factor vanishes because of the form of \( \Omega_+ \) and result (5.5).

For the fourth term, we see
\[
\int_x |x^\alpha T_4(x, t)|^2 \, dx \leq \int_x |x|^\alpha \left\{ \int_0^\infty |\partial_x B_+(x, z, t) - \partial_x B_+(x, z, 0)|^2 \, dz \right\} \cdot \left\{ \int_0^\infty |\Omega_+(x + z, t)|^2 \, dz \right\} \cdot \left\{ \left\{ \int_0^\infty |x|^2 \left( \int_0^\infty |\Omega_+(x + z, t)|^2 \, dz \right)^2 \, dx \right\}^{1/2} \right. \cdot \left\{ \left( \int_0^\infty |x|^2 \left( \int_0^\infty |\Omega_+(x + z, t)|^2 \, dz \right)^2 \, dx \right\}^{1/2}.
\]

The second factor is bounded as \( t \to 0 \) by Point 1. The first factor vanishes by the final technical result:

**Point 3.** As \( t \to 0 \)

(5.8) \( \partial_x B_+(x, z, t) \to \partial_x B_+(x, z, 0) \) in \( L^2(\mathbb{R}^+) \) uniformly for \( x \geq X \),

(5.9) \( \int_x |x|^{2\alpha} \left( \int_0^\infty |\partial_x B_+(x, z, t) - \partial_x B_+(x, z, 0)|^2 \, dz \right)^2 \, dx \to 0 \) for \( 0 \leq \alpha \leq n \).

Finally,
\[
\int_x |x^\alpha T_3(x, t)|^2 \, dx \leq \int_x |x|^\alpha \left\{ \int_0^\infty |\partial_x B_+(x, z, 0)|^2 \, dz \right\} \cdot \left\{ \int_0^\infty |\Omega_+(x + z, t) - \Omega_+(x + z, 0)|^2 \, dz \right\} \, dx.
\]

\[
\leq \left\{ \int_x |x|^2 \left( \int_0^\infty |\partial_x B_+(x, z, 0)|^2 \, dz \right)^2 \, dx \right\}^{1/2} \cdot \left\{ \int_x |x|^2 \left( \int_0^\infty |\Omega_+(x + z, t) - \Omega_+(x + z, 0)|^2 \, dz \right)^2 \, dx \right\}^{1/2}.
\]

Point 3 says the first factor is finite; the second factor vanishes by (5.3a).

To complete the proof of Theorem 5.3 we must now prove three technical points. Recall that

(5.4') \( \partial_x F_+(x, t) = F_+(x, t) + 2 \sum_j c_{+j} \exp (8\kappa_j t - 2\kappa_j x) \).

Thus for Point 1 it suffices to prove

**Point 1'.** If \( 0 \leq \alpha \leq n \), \( 1 \leq n \leq N - 1 \) and \( X \in \mathbb{R} \), then as \( t \to 0 \)

(5.4) \( x^\alpha \partial_x F_+(x, t) \to x^\alpha \partial_x F_+(x, 0) \) in \( L^2([X, \infty)) \),

(5.5') \( \int_x |x|^{2\alpha} \left( \int_x |\partial_x F_+(s, t) - \partial_x F_+(s, 0)|^2 \, ds \right)^2 \, dx \to 0 \).

**Proof of (5.4').** It suffices to treat \( \alpha = 0 \) and \( \alpha = n \). Recall that

(5.10) \( \partial_x F_+(x, t) = \pi^{-1} \int_{-\infty}^\infty 2ikR_+(k) e^{8ik^2 t + 2ikx} \, dk \).
By Proposition 2.7 we know $2ikR_{\alpha}(k)$ is in $L^2(\mathbb{R})$. It follows that $\partial_x F_+(x, t) \to \partial_x F_+(x, 0)$ in $L^2(\mathbb{R})$ as $t \to 0$. It is the exponential terms in $\partial_x \Omega_+(x, t)$ that restrict the convergence (5.4) to halflines. This takes care of the case $\alpha = 0$. Next we take $1 \leq \alpha = n \leq N - 1$. By the case $\alpha = 0$ we know

$$x^\alpha \partial_x F_+(x, t) \to x^\alpha \partial_x F_+(x, 0) \quad \text{in } L^2_{\text{loc}}.$$

Thus it suffices to prove

$$x^\alpha \partial_x F_+(x, t) \to x^\alpha \partial_x F_+(x, 0) \quad \text{in } L^2(2, \infty)).$$

Set $R(k) = 2ikR_+(k)$ in (5.10). Note that

$$\partial_x F_+(x, t) = (-1)^{\alpha-1} \int_{-\infty}^{\infty} \mathcal{F}[R(k)] e^{8ik^3t + 2ikx} dk.$$

Repeating this procedure one finds

$$\partial_x F_+(x, t) = (-1)^{\alpha-1} \int_{-\infty}^{\infty} \mathcal{F}[R(k)] e^{8ik^3t + 2ikx} dk,$$

where $\mathcal{F}[g] = \partial_x [g/12i(12k^2t + x)]$. Now $\mathcal{F}[R]$ is a linear combination of terms of the form

$$\frac{R^{(\lambda)}(k)k^\mu t^\nu}{(12k^2t + x)^{\alpha + \nu}} \quad \text{where } 0 \leq \lambda \leq \alpha, 0 \leq \mu \leq \nu, 0 \leq \lambda + \nu \leq \alpha.$$

Since $x^\alpha \partial_x F_+(x, t)$ is a linear combination of the terms

$$I_{\lambda, \mu, \nu}(x, t) = t^\nu \pi^{-1} \int_{-\infty}^{\infty} \mathcal{R}^{(\lambda)}(k) \frac{k^\mu x^\alpha}{(12k^2t + x)^{\alpha + \nu}} e^{8ik^3t + 2ikx} dk,$$

it will suffice to show that as $t \to 0$

$$I_{\lambda, \mu, \nu}(x, t) \to I_{\lambda, \mu, \nu}(x, 0) \quad \text{in } L^2(2, \infty)).$$

Case $\nu > 0$. Here we need $I_{\lambda, \mu, \nu}(x, t) \to 0$ in $L^2(2, \infty)$. Since $\alpha = n \leq N - 1$ we know $R^{(\lambda)}(k) \in L^2(\mathbb{R})$. For each $t$ let $W_\lambda(x, t)$ denote the inverse Fourier transform of $R^{(\lambda)}(k) \exp(8ik^3t)$. Now we can see $I_{\lambda, \mu, \nu}$ as the result of a pseudo-differential operator acting on $W_\lambda$. The symbol

$$p_{\mu, \nu}(x, k) = \frac{k^\mu x^\alpha}{(12k^2t + x)^{\alpha + \nu}}$$

has the property that there is a $C$ such that

$$|\partial_x p_{\mu, \nu}(x, k)| \leq C \quad \text{for } x \geq 1, \quad 0 \leq t \leq 1.$$

Choose a nonnegative $C^\infty$ cutoff function $\zeta(x)$ such that $\zeta(x) = 0$ for $x \leq 0$, $\zeta(x) = 1$ for $x \geq 2$, and $|\partial_x^m \zeta(x)| \leq M_0$ for all $m \in \mathbb{N}$. Now for $x \geq 2$

$$I_{\lambda, \mu, \nu}(x, t) = t^\nu \mathcal{F}^{-1}[\zeta(x)p_{\mu, \nu}(x, t)\mathcal{F}[W_\lambda(x, t)]].$$
By a result of Calderon and Vaillancourt (as present in [18, Thm. 3.1, p. 347]) we see there is a constant $M_1$ independent of $t$ for $0 \leq t \leq 1$ such that
\[
\| t^\nu F^{-1} \left[ \phi(x) p_{\mu_n}(x, t) F \left[ W_n(x, t) \right] \right] \|_{L^2(-\infty < x < \infty)} \leq t^\nu M_1 \| F \left[ W_n(x, t) \right] \|_{L^2(-\infty < x < \infty)} = t^\nu M_1 \| F \|_{L^2(-\infty < k < \infty)}.
\]
Thus, as required, $I_{\lambda, \mu_n}(x, t) \to 0$ in $L^2([2, \infty))$ as $t \to 0$.

Case $\nu = 0$. Since $0 \leq \mu \leq \nu$, we must show
\[
I_{\lambda, \mu, 0}(x, t) \to I_{\lambda, \mu, 0}(x, 0) \text{ in } L^2([2, \infty)) \text{ as } t \to 0.
\]

Now
\[
I_{\lambda, \mu, 0}(x, t) - I_{\lambda, \mu, 0}(x, 0) = \pi^{-1} \int_{-\infty}^{\infty} \mathcal{R}(k) \frac{x}{(12k^2 t + x)^\alpha} \left[ e^{8ik^3 t} - 1 \right] e^{2ikx} \, dk
\]
\[+ \pi^{-1} \int_{-\infty}^{\infty} \mathcal{R}(k) \left( \frac{x}{(12k^2 t + x)^\alpha} - 1 \right) e^{2ikx} \, dk
\]
and $\mathcal{R}(k)[e^{8ik^3 t} - 1]$ is in $L^2(\mathbb{R})$. Apply the same result of Calderon and Vaillancourt to obtain
\[
\pi^{-1} \int_{-\infty}^{\infty} \mathcal{R}(k)[e^{8ik^3 t} - 1] \phi(x) \left( \frac{x}{12k^2 t + x} \right)^\alpha e^{2ikx} \, dk
\]
\[\leq M_1 \| \mathcal{R}(k)[e^{8ik^3 t} - 1] \|_{L^2(-\infty < k < \infty)},
\]
the right side of which clearly goes to 0 as $t \to 0$. The second term in the difference $I_{\lambda, \mu, 0}(x, t) - I_{\lambda, \mu, 0}(x, 0)$ is treated similarly to complete the proof of (5.4').

In proving (5.5') we may assume $X \equiv 1$. Let $E(x, t)$ denote $\int_x^\infty |\partial_x F_+(s, t) - \partial_x F_+(s, 0)|^2 \, ds$. We must show
\[
\int_x^\infty |x|^{2\alpha} E(x, t)^2 \, dx \to 0 \text{ as } t \to 0.
\]

Divide the integral at $x = 1$. Clearly
\[
\int_1^\infty \cdots \, dx \equiv \max \{1, |x|^{2\alpha}\} \int_x^\infty E(x, t)^2 \, dx
\]
\[\leq \max \{1, |x|^{2\alpha}\} E(X) \int_x^\infty E(x, t) \, dx
\]
\[\leq KE(X) \int_x^\infty \int_x^\infty |\partial_x F_+(s, t) - \partial_x F_+(s, 0)|^2 \, ds \, dx
\]
\[\leq KE(X) \int_x^\infty (s-x)|\partial_x F_+(s, t) - \partial_x F_+(s, 0)|^2 \, ds,
\]
which goes to zero by (5.4'). Next
\[
\int_1^\infty \cdots \, dx = \int_1^\infty \{x^{2\alpha} E(x, t)\} E(x, t) \, dx
\]
\[\leq \int_1^\infty \left\{ \int_x^\infty s^{2\alpha} |\partial_x F_+(s, t) - \partial_x F_+(s, 0)|^2 \, ds \right\} E(x, t) \, dx
\]
\[= \left\{ \int_1^\infty s^{2\alpha} |\partial_x F_+(s, t) - \partial_x F_+(s, 0)|^2 \, ds \right\} \int_1^\infty E(x, t) \, dx.
\]
The first factor goes to zero by (5.4'). The second also does since
\[
\int_1^\infty E(x, t) \, dx = \int_1^\infty (s-1) |\partial_s F_+(s, t) - \partial_s F_+(s, 0)|^2 \, ds.
\]
This completes the proof of Point 1.

**Point 2.**
(i) \(B_+(x, \cdot, t) \to B_+(x, \cdot, 0)\) in \(L^2(\mathbb{R}^+)\) as \(t \to 0\) uniformly in \(x \geq X\);
(ii) \(\int_1^\infty |x|^{2\alpha} \|B_+(x, \cdot, t) - B_+(x, \cdot, 0)\|^4_{L^2(\mathbb{R}^+)} \, dx \to 0\) as \(t \to 0\) for \(0 \leq \alpha \leq n\).

**Proof.** We have
\[
\|B_+(x, \cdot, t) - B_+(x, \cdot, 0)\|_{L^2(\mathbb{R}^+)}
\leq C_0(x) \|\Omega_+(x + [\cdot], t) - \Omega_+(x + [\cdot], 0)\|_{L^2(\mathbb{R}^+)}
+ \| (I + \Omega_+^0)^{-1} - (I + \Omega_+^0)^{-1} \|_{op} \|\Omega_+(x + [\cdot], 0)\|_{L^2(\mathbb{R}^+)}
\]
where \(C_0(x) = \sup \{ \| (I + \Omega_+^0)^{-1} \|_{op} : x \leq w, 0 \leq t \leq 1 \}\). \(C_0(x)\) is finite and nonincreasing.

We show that each term on the right of (5.11) vanishes as \(t \to 0\). First
\[
\sup \{ \|\Omega_+(x + [\cdot], t) - \Omega_+(x + [\cdot], 0)\|^2_{L^2(\mathbb{R}^+)} : x \geq X \} = \int_X^\infty \Omega_+(s, t) - \Omega_+(s, 0)^2 \, ds,
\]
which, as we have already seen, vanishes as \(t \to 0\). Second,
\[
\| (I + \Omega_+^0)^{-1} - (I + \Omega_+^0)^{-1} \|_{op} \leq C_0(x)^2 \| (I + \Omega_+^0)^{-1} \|_{op}
\leq C_0(x)^2 \left( \int_0^\infty (y - x) |\Omega_+(y, t) - \Omega_+(y, 0)|^2 \, dy \right)^{1/2}.
\]
Thus
\[
\sup \{ \| (I + \Omega_+^0)^{-1} - (I + \Omega_+^0)^{-1} \|_{op} : x \geq X \}
\leq C_0(X) \left( \int_{y=x}^\infty (y - X) |\Omega_+(y, t) - \Omega_+(y, 0)|^2 \, dy \right)^{1/2}
\]
which vanishes as \(t \to 0\) by Point 1. This completes (i). Because of (i) it suffices to prove (ii) for \(X = 0\). Now by (5.11)
\[
\|B_+(x, \cdot, t) - B_+(x, \cdot, 0)\|^4
\leq C_1(x) \left( \int_X^\infty |\Omega_+(s, t) - \Omega_+(s, 0)|^2 \, ds \right)^2
+ C_0(x) \left( \int_X^\infty (y - x) |\Omega_+(y, t) - \Omega_+(y, 0)|^2 \, dy \right)^2 \left( \int_X^\infty |\Omega_+(s, 0)|^2 \, ds \right)^2
\]
where \(C_1(x)\) and \(C_2(x)\) are nonincreasing. Now first
\[
\int_0^\infty |x|^{2\alpha} \left( \int_X^\infty |\Omega_+(s, t) - \Omega_+(s, 0)|^2 \, ds \right)^2 \, dx \equiv \int_0^\infty \left( \int_X^\infty s^\alpha |\Omega_+(s, t) - \Omega_+(s, 0)| \, ds \right)^2 \, dx
= \int_0^\infty \int_X^\infty s^\alpha |\Omega_+(s, t) - \Omega_+(s, 0)|^2 \, ds \, dx \cdot \int_X^\infty s^\alpha |\Omega_+(s, t) - \Omega_+(s, 0)|^2 \, ds
\]
which goes to 0 as \( t \to 0 \) by (5.3a) since \( \alpha \leq N - 1 \). Next, we see

\[
\int_0^\infty |x|^{2\alpha} \left( \int_x^\infty (y-x)|\Omega_+(y, t) - \Omega_+(y, 0)|^2 \, dy \right)^2 \left( \int_x^\infty |\Omega_+(s, 0)|^2 \, ds \right)^2 \, dx
\]

\[
\leq \int_0^\infty \left( \int_x^\infty (y-x)|\Omega_+(y, t) - \Omega_+(y, 0)|^2 \, dy \right)^2 \left( \int_x^\infty s^\alpha|\Omega_+(s, 0)|^2 \, ds \right)^2 \, dx
\]

\[
\leq K^2 \int_0^\infty y|\Omega_+(y, t) - \Omega_+(y, 0)|^2 \, dy \int_0^\infty \int_x^\infty y|\Omega_+(y, t) - \Omega_+(y, 0)|^2 \, dy \, dx
\]

\[
= K^2 \int_0^\infty y|\Omega_+(y, t) - \Omega_+(y, 0)|^2 \, dy \int_0^\infty y^2|\Omega_+(y, t) - \Omega_+(y, 0)|^2 \, dy.
\]

Both of the last two factors go to 0 with \( t \to 0 \) by (5.3a). This completes part (ii).

**Point 3.**

(i) \( \partial_x B_+(x, \cdot, t) \to \partial_x B_+(x, \cdot, 0) \) in \( L^2(\mathbb{R}^+) \) as \( t \to 0 \) uniformly in \( x \geq X \);

(ii) \( \int_X^\infty |x|^{2\alpha} \|\partial_x B_+(x, \cdot, t) - \partial_x B_+(x, \cdot, 0)\|_{L^2(\mathbb{R}^+)}^2 \, dx \to 0 \) as \( t \to 0 \) for \( 0 < \alpha \leq n \).

**Proof.** One may verify that

\[
(I + \Omega_x^\alpha)[\partial_x B_+(x, \cdot, t) - \partial_x B_+(x, \cdot, 0)] = - \sum_{\nu=1}^4 Q_\nu(x, \cdot, t)
\]

where

\[
Q_1(x, \cdot, t) = [(I + \Omega_x^\alpha) - (I + \Omega_x^0)]\partial_x B_+(x, \cdot, 0),
\]

\[
Q_2(x, y, t) = \partial_x \Omega_+(x+y, t) - \partial_x \Omega_+(x+y, 0),
\]

\[
Q_3(x, y, t) = \int_0^\infty \{B_+(x, z, t) - B_+(x, z, 0)\} \partial_x \Omega_+(x+y+z, t) \, dz,
\]

\[
Q_4(x, y, t) = \int_0^\infty B_+(x, z, 0)\{\partial_x \Omega_+(x+y+z, t) - \partial_x \Omega_+(x+y+z, 0)\} \, dz.
\]

Since \( \|(I + \Omega_x^\alpha)^{-1}\|_{op} \leq C_0(X) \) for all \( x \geq X, 0 \leq t \leq 1 \), it suffices to prove for each \( \nu \) that as \( t \to 0 \)

(i) \( Q_\nu(x, \cdot, t) \to 0 \) in \( L^2(\mathbb{R}^+) \) uniformly for \( x \geq X \), and

(ii) \( \int_X^\infty |x|^{2\alpha} \|Q_\nu(x, \cdot, t)\|_{L^2(\mathbb{R}^+)}^2 \, dx \to 0 \).

**Case \( \nu = 1 \).**

\[
\|Q_1(x, \cdot, t)\|_{L^2(\mathbb{R}^+)}^2 = \|\Omega_x^\alpha - \Omega_x^0\|_{op}^2 \|\partial_x B_+(x, \cdot, 0)\|_{L^2(\mathbb{R}^+)}^2.
\]

So

\[
\sup \{\|Q_1(x, \cdot, t)\|_{L^2(\mathbb{R}^+)}^2; x \geq X\} \leq K \int_{s-x}^\infty (s-X)|\Omega_+(s, t) - \Omega_+(s, 0)|^2 \, ds,
\]

which goes to 0 with \( t \) by (5.3). Thus (i) holds.
\[
\int_0^\infty |x|^{2\alpha} \|Q_1(x, \cdot, t)\|^4 \, dx \equiv K \int_0^\infty x^{2\alpha} \left( \int_{s=x}^\infty s|\Omega_+(s, t) - \Omega_+(s, 0)|^2 \, ds \right)^2 \, dx
\]
\[
\equiv K \int_0^\infty \left\{ x^2 \left( \int_{s=x}^\infty s|\Omega_+(s, t) - \Omega_+(s, 0)|^2 \, ds \right) \right\} \cdot \left\{ x^{2\alpha-2} \left( \int_{s=x}^\infty s|\Omega_+(s, t) - \Omega_+(s, 0)|^2 \, ds \right) \right\} \, dx
\]
\[
\equiv K \int_{s=0}^\infty s^3|\Omega_+(s, t) - \Omega_+(s, 0)|^2 \, ds
\]
\[
\cdot \int_{s=0}^\infty s^{2\alpha-1}|\Omega_+(s, t) - \Omega_+(s, 0)|^2 \, ds \, dx
\]
\[
\equiv K \int_{s=0}^\infty s^3|\Omega_+(s, t) - \Omega_+(s, 0)|^2 \, ds \int_{s=0}^\infty s^{2\alpha}|\Omega_+(s, t) - (s, 0)|^2 \, ds,
\]
which goes to zero with \( t \) by (5.3) since \( N \geq 3 \) says \( 3/2 \leq n \leq N - 1 \). This finishes (ii).

Case \( \nu = 2 \). This follows directly from Point 1.

Case \( \nu = 3 \).

\[
\|Q_3(x, \cdot, t)\|^2_{L^2(\mathbb{R}^+)} = \left( \int_{y=0}^\infty \left( \int_{z=0}^\infty \{ B_+(x, z, t) - B_+(x, z, 0)\} \partial_x \Omega_+(x + y + z, t) \, dz \right)^2 \, dy \right)^{1/2}
\]
\[
\equiv \int_{z=0}^\infty \| B_+(x, \cdot, t) - B_+(x, \cdot, 0)\|^2_{L^2(\mathbb{R}^+)} \int_{s=x+y}^\infty |\partial_x \Omega_+(s, t)|^2 \, ds \, dy
\]
\[
\equiv \| B_+(x, \cdot, t) - B_+(x, \cdot, 0)\|^2 \int_{s=x}^\infty (s-x)|\partial_x \Omega_+(s, t)|^2 \, ds.
\]

Convergence (i) follows by Point 1 and Point 2. For (ii) note
\[
\int_0^\infty |x|^{2\alpha} \|Q_3(x, \cdot, t)\|^4 \, dx
\]
\[
\equiv \int_0^\infty |x|^{2\alpha} \|B_+(x, \cdot, t) - B_+(x, \cdot, 0)\|^4 \int_{s=x}^\infty (s-x)|\partial_x \Omega_+(s, t)|^2 \, ds \right)^2 \, dx.
\]

Since \( B_+(x, \cdot, t) \to B_+(x, \cdot, 0) \) in \( L^2(\mathbb{R}^+) \) uniformly in \( x \geq 0 \) we check the boundedness of the rest:
\[
\int_0^\infty |x|^{2\alpha} \left( \int_{s=x}^\infty (s-x)|\partial_x \Omega_+(s, t)|^2 \, ds \right)^2 \, dx
\]
\[
\equiv \int_0^\infty \left\{ x^2 \int_{s=x}^\infty s|\partial_x \Omega_+(s, t)|^2 \, ds \right\}^2 \left\{ x^{2\alpha-2} \left( \int_{s=x}^\infty s|\partial_x \Omega_+(s, t)|^2 \, ds \right) \right\} \, dx
\]
\[
\equiv \int_{s=0}^\infty s^3|\partial_x \Omega_+(s, t)|^2 \, ds \int_{s=0}^\infty s^{2\alpha}|\partial_x \Omega_+(s, t)|^2 \, ds.
\]

Both factors are bounded for \( 0 \leq t \leq 1 \) by Point 1.
Case $\nu = 4$.

$$\| Q_4(x, \cdot, t) \|^2 = \int_{y = 0}^\infty \left( \int_0^\infty B_+(x, z, 0) \{ \partial_x \Omega_+(x + y + z, t) - \partial_x \Omega_+(x + y + z, 0) \} \, dz \right)^2 \, dy$$

$$\leq \int_0^\infty \left( \int_0^\infty |B_+(x, z, 0)|^2 \, dz \right) \cdot \left( \int_0^\infty \left| \partial_x \Omega_+(x + y + z, t) - \partial_x \Omega_+(x + y + z, 0) \right|^2 \, dz \right) \, dy$$

$$\leq \int_0^\infty |B_+(x, z, 0)|^2 \, dz \int_{y = 0}^\infty \int_{s = x+y}^\infty \left| \partial_x \Omega_+(s, t) - \partial_x \Omega_+(s, 0) \right|^2 \, ds \, dy$$

$$\leq \int_0^\infty |B_+(x, z, 0)|^2 \, dz \int_{s = x}^\infty \left( s - x \right) \left| \partial_x \Omega_+(s, t) - \partial_x \Omega_+(s, 0) \right|^2 \, ds$$

Now

$$\sup \{ \| Q_4(x, \cdot, t) \|^2 : x \equiv X \} \leq \sup \{ \| B_+(x, \cdot, 0) \|^2 : x \equiv X \}$$

$$\cdot \int_{x = X}^\infty \left( s - X \right) \left| \partial_x \Omega_+(s, t) - \partial_x \Omega_+(s, 0) \right|^2 \, ds$$

which goes to 0 by Point 1. Further

$$\int_0^\infty \left| x \right|^{2\alpha} \| Q_4(x, \cdot, t) \|^4 \, dx$$

$$\leq \int_0^\infty \left| x \right|^{2\alpha} \left( \int_0^\infty |B_+(x, z, 0)|^2 \, dz \right) \left( \int_{s = x}^\infty \left( s - x \right) \left| \partial_x \Omega_+(s, t) - \partial_x \Omega_+(s, 0) \right|^2 \, ds \right) \, dx$$

$$\leq \int_{s = x}^\infty s \left| \partial_x \Omega_+(s, t) - \partial_x \Omega_+(s, 0) \right|^2 \, ds \int_0^\infty x^{2\alpha} \int_0^\infty |B_+(x, z, 0)|^2 \, dz \, dx$$

The second factor is finite and the first goes to 0 by Point 1.

This finishes the proof of the last of the three technical points needed to complete the proof of Theorem 5.3. $\square$

Acknowledgments. The authors are grateful to the University of California at Berkeley for its hospitality during the time this work was begun. We also gratefully acknowledge several very helpful discussions with S. Agmon.

REFERENCES


