On the rate of Poisson convergence

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1. Introduction

Let $X_1, \ldots, X_n$ be independent Bernoulli random variables, and let $p_i = P[X_i = 1]$, $\lambda = \sum_{i=1}^n p_i$ and $W = \sum_{i=1}^n X_i$. Successively improved estimates of the total variation distance between the distribution $\mathcal{L}(W)$ of $W$ and a Poisson distribution $P_\lambda$ with mean $\lambda$ have been obtained by Prohorov [5], Le Cam [4], Kerstan [3], Vervaat [8], Chen [2], Serfling [7] and Romanowska [6]. Prohorov, Vervaat and Romanowska discussed only the case of identically distributed $X_i$'s, whereas Chen and Serfling were primarily interested in more general, dependent sequences. Under the present hypotheses, the following inequalities, here expressed in terms of the total variation distance

$$d(\mu, \nu) = \sup_{A \subseteq \mathbb{Z}} |\mu(A) - \nu(A)|,$$

were established respectively by Le Cam, Kerstan and Chen:

\[
\begin{align*}
  d(\mathcal{L}(W), P_\lambda) &\leq \sum_{i=1}^n p_i^2; \\
  d(\mathcal{L}(W), P_\lambda) &\leq 1.05\lambda^{-1} \sum_{i=1}^n p_i^2, \quad \text{if } \max_i p_i \leq \frac{1}{6}; \\
  d(\mathcal{L}(W), P_\lambda) &\leq 5\lambda^{-1} \sum_{i=1}^n p_i^2. 
\end{align*}
\]

(Kerstan's published estimate of $2d \leq 1.2\lambda^{-1}\sum_{i=1}^n p_i^2$ ([3], p. 174, equation (1)) is a misprint for $2d \leq 2.1\lambda^{-1}\sum_{i=1}^n p_i^2$, the constant 2.1 appearing twice on p. 175 of his paper.)

Here, we use Chen's [2] elegant adaptation of Stein's method to improve the estimates given in (1.1), and we complement these estimates with a reverse inequality expressed in similar terms. Second order estimates, and the case of more general non-negative integer valued $X_i$'s, are also discussed.

In the latter case, it is natural to expect the distribution of $W$ to be almost Poisson only if the contribution to $W$ from $X_i$'s taking values other than 1 is in some sense small. As observed by Serfling, the $X_i$'s can be reduced to 0-1 random variables by replacing all values greater than 1 by 0, at a cost in total variation distance of no more than $\sum_{j=1}^n P[X_j \geq 2]$. Thus close approximation to the Poisson is possible when the chance of $\max_i X_i$ exceeding 1 is small. It is shown here that good Poisson approxi-
mation can also be achieved in another natural situation, in which the expected contribution to $W$ from those $X_i$'s greater than 1 is negligible compared to

$$\left\{ \sum_{j=1}^{n} P[X_j = 1] \right\}^{\frac{1}{2}},$$

the standard deviation of the approximating Poisson distribution; upper and lower distance estimates are derived to quantify this approximation.

2. Upper and lower bounds

Let $x$ be any real valued function on the non-negative integers. Then, as in Chen [2],

$$E(\lambda x(W + 1) - W x(W)) = \sum_{j=1}^{n} \{p_j E(x(W + 1) - E(X_j x(W)) \}
\begin{align*}
&= \sum_{j=1}^{n} p_j E[x(W + 1) - x(W_j + 1)] \\
&= \sum_{j=1}^{n} p_j^2 E[x(W_j + 2) - x(W_j + 1)],
\end{align*}

(2.1)

where $W_j = W - X_j$. The following theorem is derived from (2.1) by choosing a suitable function $x$.

**Theorem 1.** Under the hypotheses set out in the Introduction,

$$d(L(W), P_\lambda) \leq \lambda^{-1}(1 - e^{-\lambda}) \sum_{j=1}^{n} p_j^2. \quad (2.2)$$

**Proof.** For any $A \subseteq \mathbb{Z}$, define $x = x_{\lambda, A}$ by

$$x(0) = 0; \quad x(m + 1) = \lambda^{-m} e^m \left( [P_\lambda(A \cap U_m) - P_\lambda(A \cup U_m)] \right), \quad m \geq 0, \quad (2.3)$$

where $U_m = \{0, 1, \ldots, m\}$. For this $x$,

$$\lambda x(m + 1) - m x(m) = I[m \in A] - P_\lambda(A),$$

and so, from (2.1),

$$|P[W \in A] - P_\lambda(A)| \leq \sum_{j=1}^{n} p_j^2 E|x(W_j + 2) - x(W_j + 1)|. \quad (2.4)$$

It is shown in the appendix to Barbour and Eagleson [1], that, uniformly in $A$,

$$\|x\| \equiv \sup_{m \geq 0} |x(m)| \leq 1 \wedge (1 - 4\lambda^{-1}), \quad (2.5)$$

and

$$\Delta x \equiv \sup_{m \geq 0} |x(m + 1) - x(m)| \leq \lambda^{-1}(1 - e^{-\lambda}). \quad (2.6)$$

The theorem follows immediately from (2.4) and (2.6).

**Remark.** Theorem 1 improves upon each of the estimates given in (1.1). For $0 \leq \lambda \leq 1$, set $p_1 = \lambda; p_j = 0$, $2 \leq j \leq n$. Then

$$d(W, P_\lambda) = \lambda(1 - e^{-\lambda}) = \lambda^{-1}(1 - e^{-\lambda}) \sum_{j=1}^{n} p_j^2,$$

so that the inequality (2.2) is sharp in this case. For integral $\lambda \geq 1$, set $p_j = 1, 1 \leq j \leq \lambda; p_j = 0, \lambda < j \leq n$. Then

$$d(L(W), P_\lambda) = 1 - \lambda^2 e^{-\lambda}/\lambda! \approx 1 - 1/\sqrt{2\pi\lambda},$$
as compared to the right hand side of (2-2) which takes the value $1 - e^{-\lambda}$, both of which tend to one as $\lambda \to \infty$. Thus (2-2) also comes close to being sharp for large $\lambda$. There is, however, no reason to suppose that (2-2) could not be improved under added restrictions on the $p_i$’s; for instance, Romanowska’s[6] inequality for all $p_i$’s equal to $p$ is sharper than (2-2) when $\frac{1}{k}(1-p)^{-k} < 1 - e^{-\lambda}$.

As a complement to Theorem 1, we prove the following result: note that

$$(1 \land \lambda^{-1}) \leq \lambda^{-1}(1 - e^{-\lambda}).$$

**Theorem 2.** Under the hypotheses set out in the Introduction,

$$d(\mathcal{L}(W), P_\theta) \geq \frac{1}{2} (1 \land \lambda^{-1}) \sum_{j=1}^{n} p_j^2.$$  \hspace{1cm} (2-7)

**Proof.** Take $x$ defined by

$$x(m) = (m-\lambda)e^{-(m-\lambda)p^i\theta^j}, \quad m \geq 0,$$

in (2-1), where the constant $\theta$ will be chosen later. Since, for $P$ a Poisson variate with mean $\lambda$, $E\{x(P + 1) - Px(P)\} = 0$, Equation (2-1) yields the equation

$$E\{[\lambda x(W + 1) - W x(W)] - [\lambda x(P + 1) - Px(P)]\} = \sum_{j} p_j^2 E\{x(W_j + 2) - x(W_j + 1)\},$$

from which it follows that

$$2d(\mathcal{L}(W), P_\theta) \sup_j |\lambda x(j + 1) - j x(j)| \geq \sum_{j} p_j^2 E\{x(W_j + 2) - x(W_j + 1)\}.$$  \hspace{1cm} (2-9)

Our first task is to bound the supremum on the left in (2-9). Since

$$(d/dw) (we^{-w^{i\theta^j}}) = (1 - 2w^2/\theta \lambda) e^{-w^{i\theta^j}},$$

which takes only values in the interval $[-2e^{-\frac{3}{\theta}}, 1]$, then

$$-2e^{-\frac{3}{\theta}} \leq x(w + 1) - x(w) \leq 1,$$ \hspace{1cm} (2-10)

and so

$$|\lambda x(j + 1) - j x(j)| = |\lambda [x(j + 1) - x(j)] - (j - \lambda)^2 \exp \{-(j - \lambda)^2/\theta \lambda]\|$$

$$\leq \lambda \max (1, 2e^{-\frac{3}{\theta}} + \theta e^{-1}).$$ \hspace{1cm} (2-11)

Next we treat the series on the right hand side of (2-9). Now,

$$1 - e^{-\omega^i\theta^j}(1 - 2w^2/\theta \lambda) \leq 3w^2/\theta \lambda,$$

whence, writing $U_j = W_j - \lambda$,

$$1 - \{x(W_j + 2) - x(W_j + 1)\} \leq \int_{U_j+1}^{U_j+2} (3w^2/\theta \lambda) dw = (\theta \lambda)^{-1}(3U_j^2 + 3U_j + 7).$$ \hspace{1cm} (2-12)

Therefore

$$1 - E\{x(W_j + 2) - x(W_j + 1)\} \leq (\theta \lambda)^{-1}(3\lambda + 7),$$

since $E(U^2_j) = \sum_{i+t} p_j(1 - p_j) + p^j$ and $E(U_i) = -p_i$. Consequently,

$$\sum_{j} p_j^2 E\{x(W_j + 2) - x(W_j + 1)\} \geq \sum_{j} p_j^2 \{1 - (\theta \lambda)^{-1}(3\lambda + 7)\}.$$ \hspace{1cm} (2-13)

Combining (2-9), (2-11) and (2-13), we see that if $\theta \geq e$, $d(\mathcal{L}(W), P_\lambda) \geq k \sum_{j} p_j^2,$

where

$$k = \{1 - (\theta \lambda)^{-1}(3\lambda + 7)\}/2\lambda(2e^{-\frac{3}{\theta}} + \theta e^{-1}).$$ \hspace{1cm} (2-14)

If $\lambda \geq 1$ we take $\theta = 21$, which gives

$$\lambda k \geq (1 - 10/\theta)/2(2e^{-\frac{3}{\theta}} + \theta e^{-1}) \geq 1/32,$$
while if \( \lambda < 1 \) we take \( \theta = 21/\lambda \), obtaining
\[
k \geq \frac{(1 - 10/\theta \lambda)/2(2e^{-\theta} + \theta \lambda e^{-1})}{32}.
\]
Theorem 2 follows.

3. Second-order estimates

For many choices of \((p_j)_{j=1}^n\), the bounds given in (2-2) and (2-7) can usefully be replaced by an estimate of \(d(\mathcal{L}(W), P_\lambda)\) together with a bound on the error of the estimate. This approach was considered by Prohorov [5], Kerstan [3] and Chen [2]: our argument is similar to Chen’s, though the error estimate is improved. The method is to take, for any \( A \subseteq \mathbb{Z}^+ \), the function \( x \) defined in (2-3), and then to approximate the right hand side of (2-1) by \( E[\sum_{j=1}^n p_j^3]E[x(P + 2) - x(P + 1)] \), where \( P \), here and subsequently, denotes a Poisson variate with mean \( \lambda \). Let \( \Delta(A) \) denote the error in this approximation:
\[
\Delta(A) = \frac{1}{n} \sum_{j=1}^n p_j^3 E[x(P + 2) - x(P + 1)]
\]
\[
= P[W \in A] - P_\lambda(A) + \frac{1}{n} \sum_{j=1}^n p_j^3 E[I[P \in A] (P^2 - (2(\lambda + 1) P + \lambda^2))].
\]

\textbf{THEOREM 3.} For any \( A \subseteq \mathbb{Z}^+ \), under the hypotheses set out in the Introduction,
\[
|\Delta(A)| \leq 2\lambda^{-1}(1 - e^{-\lambda})(1 \wedge 1.4\lambda^{-1}) \sum_{j=1}^n p_j^3 + 2\lambda^{-1}(1 - e^{-\lambda}) \sum_{j=1}^n p_j^3.
\]

\textbf{Proof.} Taking in (2-1) the function \( x \) defined in (2-3), it follows immediately that
\[
|\Delta(A)| \leq 2\lambda^{-1}(1 - e^{-\lambda}) \sum_{j=1}^n p_j^3 d(\mathcal{L}(W_j), P_\lambda).
\]
However, analogously to (2-1), for any \( B \subseteq \mathbb{Z}^+ \),
\[
P[W_j \in B] - P_\lambda(B) = E(\lambda x_{\lambda, B}(W_j + 1) - W_j x_{\lambda, B}(W_j))
\]
\[
= p_j E x_{\lambda, B}(W_j + 1) + \sum_{k+j} p_k^3 E(x_{\lambda, B}(W_{jk}) + 2) - x_{\lambda, B}(W_{jk} + 1)),
\]
where \( W_{jk} = W_j - X_k \). Thus it follows from (2-5) and (2-6) that
\[
d(\mathcal{L}(W_j), P_\lambda) \leq p_j (1 \wedge 1.4\lambda^{-1}) + \lambda^{-1}(1 - e^{-\lambda}) \sum_{k+j} p_k^3,
\]
establishing (3-2).

\textbf{Remark.} Since, by Schwarz’s inequality, \(\lambda^{-1}(\sum_{j=1}^n p_j^3)^2 \leq \sum_{j=1}^n p_j^3 \), equation (3-2) implies that \( |\Delta(A)| \leq 4(1 - e^{-\lambda})(1 \wedge 1.4\lambda^{-1}) \lambda^{-1} \sum_{j=1}^n p_j^3 \), which improves on Chen’s estimate of \( (12 + 48\sqrt{2}) \lambda^{-1} \sum_{j=1}^n p_j^3 \).

\textbf{COROLLARY.} Let \( \delta(\lambda) \equiv \frac{1}{n} \sum_{j=1}^n p_j^3 E((P^2 - (2\lambda + 1) P + \lambda^2)^{-1}) \).

Then
\[
|d(\mathcal{L}(W), P_\lambda) - \lambda^{-1}\left(\sum_{j=1}^n p_j^3\right) \delta(\lambda)| \leq 2\lambda^{-1}(1 - e^{-\lambda})(1 \wedge 1.4\lambda^{-1}) \sum_{j=1}^n p_j^3
\]
\[
+ 2\left(\lambda^{-1}(1 - e^{-\lambda}) \sum_{j=1}^n p_j^3\right)^2.
\]

\textbf{Remark.} Kerstan gives instead the upper bound
\[
0.65\lambda^{-1} \sum_{j=1}^n p_j^3 + 1.95 \left(\lambda^{-1} \sum_{j=1}^n p_j^3\right)^2, \quad \text{if } \max_{1 \leq j \leq n} p_j \leq \frac{1}{4},
\]
which is sometimes better than that of the Corollary, and sometimes worse.
The quantity $\delta(\lambda)$ is not in general very neatly expressible, except for moderately small values of $\lambda$: for example,

$$\delta(\lambda) = \begin{cases} \lambda(1 - \frac{1}{2}\lambda)e^{-\lambda}, & 0 \leq \lambda \leq 2 - \sqrt{2} \\ \lambda \left(1 + \frac{1}{2} - \frac{\lambda^2}{4}\right)e^{-\lambda}, & 2 - \sqrt{2} \leq \lambda \leq 3 - \sqrt{3}. \end{cases}$$

However, $y^2 - (2\lambda + 1)y + \lambda^2 \geq - (\lambda + \frac{1}{2})$ for all $y$, and so

$$\delta(\lambda) \leq \frac{1}{2} + \frac{1}{8\lambda},$$

for all $\lambda$. Furthermore, as $\lambda \to \infty$, $\delta(\lambda) \sim 0.242$.

The Corollary enables some further evaluation of the relative precision of the bounds in Theorems 1 and 2 to be made, in the following sense. Suppose that $\{X_{jm}, 1 \leq j \leq n < \infty\}$ is a double array of Bernoulli random variables, independent within rows, and set $p_{jn} = P[X_{jm} = 1]$, $\lambda_n = \sum_{j=1}^{n} p_{jn}$ and $W_n = \sum_{j=1}^{n} X_{jn}$. Suppose also that as $n \to \infty$, $d(\mathcal{L}(W_n), P_{\lambda_n}) \to 0$, or, equivalently, that $\lambda_n^{-1}(1 - e^{-\lambda_n}) \sum_{j=1}^{n} p_{jn}^2 \to 0$.

Then the error estimate given in the Corollary is of asymptotically smaller order as $n \to \infty$ than $\lambda_n^{-1}\delta(\lambda_n) \sum_{j=1}^{n} p_{jn}^2$, so that

$$d(\mathcal{L}(W_n), P_{\lambda_n}) \sim \lambda_n^{-1}\delta(\lambda_n) \sum_{j=1}^{n} p_{jn}^2,$$

as $n \to \infty$. Thus, for example, if also $\lambda_n \to \infty$,

$$u_n \equiv \lambda_n d(\mathcal{L}(W_n), P_{\lambda_n})/\left(1 - e^{-\lambda_n}\right) \sum_{j=1}^{n} p_{jn}^2 \to 0.242,$$

whereas Theorems 1 and 2 guarantee that $\frac{1}{2} \leq u_n \leq 1$. Actually, the choice of $\theta = 6.55$ in (2.14) gives $u_n \geq \frac{1}{2}$, provided that $\lambda$ is sufficiently large, which is not too great a deviation from 0.242.

Similar comparisons when $\lambda_n \leq 2 - \sqrt{2}$ yield

$$u_n \sim \lambda_n(1 - e^{-\lambda_n})^{-1}(1 - \frac{1}{2}\lambda_n)e^{-\lambda_n},$$

and so $u_n \to 1$ if $\lambda_n \to 0$. Again, for small enough $\lambda$, the choice of $\theta(= 14.6)$ in (2.14) improves the bound given by Theorem 2, but this time to $u_n \geq \frac{1}{2}$.

4. Non-negative integer varieties

Let $\{Y_j\}_{j=1}^{n}$ be independent non-negative integer valued random variables, and let $p_j = P[Y_j = 1]$, $q_j = P[Y_j \geq 2]$, $\lambda = \sum_{j=1}^{n} p_j$ and $V = \sum_{j=1}^{n} Y_j$. Define the zero-one random variables $(X_j)_{j=1}^{n}$ by

$$X_j = \begin{cases} Y_j & \text{if } Y_j = 0 \text{ or } 1; \\ 0 & \text{otherwise}, \end{cases}$$

and set $W = \sum_{j=1}^{n} X_j$. Note that the $X_j$'s satisfy the conditions outlined in the Introduction. Then, as observed by Serfling[7],

$$d(\mathcal{L}(V), \mathcal{L}(W)) \leq \sum_{j=1}^{n} q_j,$$

and so (2.2) and (2.7) yield the inequalities

$$d(\mathcal{L}(V), P_\lambda) \leq \lambda^{-1}(1 - e^{-\lambda}) \sum_{j=1}^{n} p_{j}^2 + \sum_{j=1}^{n} q_j, \quad (4.1)$$

and

$$d(\mathcal{L}(V), P_\lambda) + \sum_{j=1}^{n} q_j \geq \frac{1}{2}(1 - \lambda^{-1}) \sum_{j=1}^{n} p_{j}^2. \quad (4.2)$$
If the $Y_j$'s have finite second moments, the Stein–Chen method can be applied to get alternatives to (4·1) and (4·2). Let $\nu_j = EY_j$, $\kappa_j = E(Y_j(Y_j - 1))$, $\mu_j = E(Y_j I[Y_j \geq 2])$ and $\nu = \sum_{j=1}^{n} \nu_j$.

**Theorem 4.** If $(Y_j)_{j=1}^{n}$ satisfy the above hypotheses, and if $\nu < \infty$,

$$d(\mathcal{L}(V), P_{\nu}) \leq \nu^{-1}(1 - e^{-\nu}) \sum_{i=1}^{n} \nu_{i} p_{i}$$

$$+ \sum_{i=1}^{n} \nu_{i} \left[ \{2(1 \wedge 1 \cdot 4 \nu^{-1}) q_{i} \} \wedge \{\nu^{-1}(1 - e^{-\nu}) \mu_{i} \} \right]$$

$$+ \left\{ \{2(1 \wedge 1 \cdot 4 \nu^{-1}) \sum_{i=1}^{n} \mu_{i} \} \wedge \{\nu^{-1}(1 - e^{-\nu}) \sum_{i=1}^{n} \kappa_{i} \} \right\}$$

$$\leq \nu^{-1}(1 - e^{-\nu}) \left( \sum_{i=1}^{n} \nu_{i} p_{i} + \sum_{i=1}^{n} \kappa_{i} \right)$$

(4·3)

and

$$d(\mathcal{L}(V), P_{\lambda}) \leq \lambda^{-1}(1 - e^{-\lambda}) \sum_{i=1}^{n} p_{i}^{2}$$

$$+ \sum_{i=1}^{n} p_{i} \left[ \{2(1 \wedge 1 \cdot 4 \lambda^{-1}) q_{i} \} \wedge \{\lambda^{-1}(1 - e^{-\lambda}) \mu_{i} \} \right] + (1 \wedge 1 \cdot 4 \lambda^{-1}) \sum_{i=1}^{n} \mu_{i}$$

$$+ \left\{ \{2(1 \wedge 1 \cdot 4 \lambda^{-1}) \sum_{i=1}^{n} \mu_{i} \} \wedge \{\lambda^{-1}(1 - e^{-\lambda}) \sum_{i=1}^{n} \kappa_{i} \} \right\}$$

(4·5)

$$\leq \lambda^{-1}(1 - e^{-\lambda}) \left( \sum_{i=1}^{n} \nu_{i} p_{i} + \sum_{i=1}^{n} \kappa_{i} \right) + (1 \wedge 1 \cdot 4 \lambda^{-1}) \sum_{i=1}^{n} \mu_{i}.$$  

(4·6)

**Remark.** Estimate (4·1) is clearly better than (4·5) if $\nu = \infty$. On the other hand, for large $\lambda$, (4·5) can improve upon (4·1), in circumstances where $\sum_{j=1}^{n} q_{j}$ is not small but $\lambda^{-\frac{1}{2}} \sum_{j=1}^{n} \mu_{j}$ is: the latter condition is natural, in that it is simply requiring that the expected contribution to $V$ from $Y$'s not taking the values 0 or 1 should be small when compared to the spread $\lambda^{\frac{1}{2}}$ of $P_{\lambda}$. If the $Y_j$'s take only the values 0, 1 and 2, estimate (4·4) reduces to $\nu^{-1}(1 - e^{-\nu}) \left( \sum_{i=1}^{n} p_{i}^{2} + 2 \sum_{i=1}^{n} q_{i}(1 + p_{i}) \right)$, and estimate (4·6) to

$$\lambda^{-1}(1 - e^{-\lambda}) \left( \sum_{i=1}^{n} p_{i}^{2} + 2 \sum_{i=1}^{n} q_{i}(1 + p_{i}) \right) + 2(1 \wedge 1 \cdot 4 \lambda^{-1}) \sum_{i=1}^{n} q_{i},$$

enabling comparison with (4·1) to be easily made.

Estimate (4·4) is typically smaller than (4·6), because $\nu^{-1}(1 - e^{-\nu}) \leq \lambda^{-1}(1 - e^{-\lambda})$ and, usually, $\nu^{-1} \sum_{i=1}^{n} \mu_{i} \nu_{i} \leq (1 \wedge 1 \cdot 4 \lambda^{-1}) \sum_{i=1}^{n} \mu_{i}$. However, in view of (4·1), this does not necessarily imply that it is better to use $P_{\nu}$ than $P_{\lambda}$ to approximate the distribution of $V$.

**Proof.** Pick any $A \in \mathbb{Z}^{+}$, set $V = V - Y_{j}$, define $x$ as in (2·3) but with $\nu$ for $\lambda$, and observe that

$$P[V \in A] - P_{\nu}(A) = \sum_{j=1}^{n} \nu_{j} E[x(V) - x(V_{j} + 1)]$$

$$+ \sum_{j=1}^{n} E[Y_{j}(x(V_{j} + 1) - x(V))]$$

$$= \sum_{j=1}^{n} \nu_{j} p_{j} E[x(V_{j} + 2) - x(V_{j} + 1)]$$

$$+ \sum_{j=1}^{n} \nu_{j} E[(x(V + 1) - x(V_{j} + 1)) I[Y_{j} \geq 2]]$$

$$+ \sum_{j=1}^{n} E[Y_{j}(x(V_{j} + 1) - x(V))].$$
The three terms are now estimated using (2-5) and (2-6), again with \( v \) for \( \lambda \), giving (4-3).

The proof of (4-5) is similar, starting from the equation

\[
P[\mathcal{V} \in \mathcal{A}] - P_{\lambda}(\mathcal{A}) = \sum_{j=1}^{n} p_j \mathbb{E}[x(V_j + 1) - x(V_j) + x(V_j + 1)]
\]

It is also possible to adapt the proof of Theorem 2 so as to get lower bounds for \( d(\mathcal{L}(\mathcal{V}), P_{\lambda}) \) and \( d(\mathcal{L}(\mathcal{V}), P_{\lambda}) \), to contrast with (4-2). The following Theorem establishes such a result, without, however, retaining the elegance of (2-7) or (4-2).

**Theorem 5.** If the hypotheses of Theorem 4 are satisfied, and if, in addition, \( \kappa_j < \infty \), \( 1 \leq j \leq n \), then

\[
d(\mathcal{L}(\mathcal{V}), P_{\lambda}) + \frac{1}{2} \nu^{-1}[2e^{-\frac{\lambda}{2}} + 21e^{-1} (1 + \nu)^{-1}]^{-1} \times \left( 2e^{-\frac{\lambda}{2}} \sum_{j=1}^{n} \mu_j \nu_j + \sum_{j=1}^{n} \kappa_j + \frac{1}{2} (1 + \nu)^{-1} \right)
\times \left[ \left( \sum_{j=1}^{n} \nu_j p_j \right) \left( \sum_{j=1}^{n} \kappa_j - \nu_j^2 \right) + 2 \sum_{j=1}^{n} p_j \nu_j^2 (\nu_j - 2) \right] \geq \frac{1}{32} (1 + \nu)^{-1} \sum_{j=1}^{n} \nu_j p_j, \tag{4-7}
\]

and

\[
d(\mathcal{L}(\mathcal{V}), P_{\lambda}) + \frac{1}{2} \nu^{-1}[2e^{-\frac{\lambda}{2}} + 21e^{-1} (1 + \lambda)^{-1}]^{-1} \times \left( 2e^{-\frac{\lambda}{2}} \sum_{j=1}^{n} \mu_j \nu_j + \sum_{j=1}^{n} \kappa_j \right)
\times \left( \sum_{j=1}^{n} \nu_j p_j \right) \left( \sum_{j=1}^{n} \kappa_j - \nu_j^2 \right)
+ 2 \sum_{j=1}^{n} p_j \nu_j^2 (\nu_j - 2) \right] \geq \frac{1}{48} (1 + \lambda)^{-1} \sum_{j=1}^{n} \nu_j^2. \tag{4-8}
\]

**Remark.** If the \( \gamma_j \)'s are in fact \( 0-1 \) random variables, the complicated additional term on the left hand side of (4-7) is negative: it only becomes important when the \( \gamma_j \)'s are too far from being \( 0-1 \) variates. Some such term has to be present, since, if each \( \gamma_j \) is a Poisson variate, \( d(\mathcal{L}(\mathcal{V}), P_{\lambda}) = 0 \), whereas the right hand side of (4-7) is positive.

**Proof.** Take \( x \) as defined in (2-8), but with \( v \) for \( \lambda \), and deduce, from the proofs of Theorems 2 and 4, that, for \( \theta \geq e \),

\[
2d(\mathcal{L}(\mathcal{V}), P_{\lambda}) \nu(2e^{-\frac{\lambda}{2}} + \theta e^{-1}) \geq \sum_{j=1}^{n} \nu_j p_j \mathbb{E}[x(\gamma_j + 2) - x(\gamma_j + 1)]
\]

Thus, from the argument leading to (2-10), it follows that

\[
2d(\mathcal{L}(\mathcal{V}), P_{\lambda}) \nu(2e^{-\frac{\lambda}{2}} + \theta e^{-1}) + 2 e^{-\frac{\lambda}{2}} \sum_{j=1}^{n} \nu_j \mu_j + \sum_{j=1}^{n} \kappa_j
\geq \sum_{j=1}^{n} \nu_j p_j \mathbb{E}[x(\gamma_j + 2) - x(\gamma_j + 1)]
\geq \sum_{j=1}^{n} \nu_j p_j \{ 1 - (\theta v)^{-1} (3E(\gamma_j - v)^2 + 9E(\gamma_j - v) + 7) \}. \tag{4-9}
\]
where the last line is a consequence of (2-12). Evaluating the moments of $V_j - \nu$ enables the right hand side of (4-9) to be estimated as no smaller than

$$\sum_{j=1}^{n} \nu_j p_j \left\{ 1 - (\theta \nu)^{-1} (3\nu + 7) \right\} - 3(\theta \nu)^{-1} \sum_{j=1}^{n} \nu_j p_j \left\{ \sum_{j=1}^{n} (\kappa_j - \nu_j^2) + 2\nu_j (\nu_j - 2) \right\},$$

and the proof of (4-7) is concluded in the same way as the proof of Theorem 2, taking $\theta = 2(1 \wedge \nu)^{-1}$. The upper bound (4-8) is proved in a similar way.

Some simplification of (4-7) and (4-8) can often be achieved. The next Corollary, which follows directly from (4-7), illustrates the possibilities. The inequality obtained reduces to (2-7) when the $Y_j$'s are $0$–$1$ random variables.

**Corollary.** If, in addition to the conditions of Theorem 5, $\nu_j \leq 2$ for all $j$,

$$d(\mathcal{L}(V), P_\nu) + \frac{3}{100} \left\{ \sum_{j=1}^{n} \mu_j \nu_j + \frac{9}{4} \left( 1 + \frac{1}{7} \sum_{j=1}^{n} p_j \nu_j \right) \sum_{j=1}^{n} \kappa_j \right\} \geq \frac{1}{32} \sum_{j=1}^{n} \nu_j p_j \quad \text{if} \quad \nu < 1$$

and

$$d(\mathcal{L}(V), P_\nu) + \frac{11\nu^{-1}}{400} \left\{ \sum_{j=1}^{n} \mu_j \nu_j + \frac{9}{4} \left( 1 + \frac{\nu^{-1}}{7} \sum_{j=1}^{n} p_j \nu_j \right) \sum_{j=1}^{n} \kappa_j \right\} \geq \frac{\nu^{-1}}{32} \sum_{j=1}^{n} \nu_j p_j \quad \text{if} \quad \nu \geq 1.$$

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**REFERENCES**


