A lifting result for local cohomology of graded modules

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1. Introduction

In this paper we prove a lifting result for local cohomology. As a special case we get the following result for the Serre-cohomology over a projective variety:

**Proposition (1-1).** Let $\mathcal{F}$ be a coherent sheaf over $X$, where $X$ is a projective variety over an algebraically closed field $k$. Let $i \geq 0$ and assume that there is a pencil $\mathcal{P}$ of hyperplane sections which is in general position with respect to $\mathcal{F}$ (which means that $x \notin \mathcal{H}$, $\forall x \in \text{Ass}(\mathcal{F}), \forall H \in \mathcal{P}$), and such that for each $H \in \mathcal{P}$, $H^i(X, \mathcal{F}|_H(n)) = 0$, $\forall n \leq 0$. Then $H^{i+1}(X, \mathcal{F}(n)) = 0$, $\forall n \leq 0$.

As an application of our result we give a completely elementary proof of Serre’s finiteness theorem (7) for cohomology of coherent sheaves over projective varieties.

This also gives a special case of Grothendieck’s finiteness theorem (5), namely the graded case. In fact, we shall prove this result for local cohomology supported in the homogeneous maximal ideal of a graded algebra of finite type over a discrete valuation ring.

The proof of (1-1) will furnish a number $\rho \in \mathbb{Z}$ such that $H^i(X, \mathcal{F}|_H(n)) = 0$, $\forall n \leq \rho$. It seems interesting to know, how small $n$ must be to guarantee that $H^{i+1}(X, \mathcal{F}(n)) = 0$. In (4-8) a corresponding bound for $n$ is given; this bound is obtained by following the arguments of the proof. But except in very special cases, it is far from being optimal.

For general references (in particular with respect to notations) see: (1), (4) concerning ideal transforms, (5) for local cohomology, (6), (7) for Serre cohomology, (6) for general algebraic geometry, (8) for general commutative algebra.

2. A lifting principle for graded modules

**Proposition (2-1).** Let $A = A_0 \oplus A_1 \oplus A_2 \oplus \ldots$ be a graded ring such that $A_0 = k$ is an algebraically closed field. Let $H = \bigoplus _{n \in \mathbb{Z}} H_n$ be a graded $A$-module. Let $V \subseteq A_1$ be a 2-dimensional $k$-vector space satisfying:

(a) For all $h \in H$ there is a $v \in N$ with $V^v h = 0$.

(b) For all $x \in V - (0)$, $(0: x)_H$ is an $A$-module of finite length.

Then $H$ is an $A$-module of finite length.

**Proof:** Let $x, y$ be a $k$-basis of $V$. As $(0: x)_H$ is of finite length, there is a $c \in \mathbb{N}$ such that $(0: x)_{H_n} = 0$ as soon as $|n| > c$. Let $n > c$. We want to show that $H_n = 0$. So let $h \in H_n$. By hypotheses (a) we find a least integer $\mu \geq 0$ such that $x^\mu h = 0$. If $\mu = 0$, clearly $h = 0$. The case $\mu > 0$ is excluded as $x^{\mu-1} h \in (0: x)_{H_{n+\mu-1}} = 0$. So we get $H_n = 0$, $\forall n > 0$.

This implies in particular that $H_c \subseteq (0: x)_H$. By hypotheses (b) we conclude that $\dim_k (H_c) < \infty$. Moreover we have $\dim_k (0: x)_{H_n} < \infty$, $\forall n \leq c$. So, the sequences
0 \to (0: x)_{H_n} \to H_n \to H_{n+1} \to 0 \text{ show (by descending induction) that } \dim_k (H_n) < \infty \text{ for all } n. \text{ Moreover } \dim_k (H_n) \text{ takes a constant value } d \text{ for all } n \leq 0, \text{ as }

\dim_k (H_{n-1}) \leq \dim_k (H_n)

for \( n \leq c \), which follows as \( x: H_{n-1} \to H_n \) then is injective. It remains to show that \( d = 0 \).

To show this, we set \( U_n = \{(c_1, c_2) \in k^2 | xc_1 + yc_2: H_{n-1} \to H_n \text{ is injective} \} \). Let \( e_1, \ldots, e_s \) be a basis of \( H_{n-1} \); \( f_1, \ldots, f_t \) a basis of \( H_n \). For appropriate elements \( \alpha_{i,j}, \beta_{i,j} \in k \) (\( i = 1, \ldots, s; j = 1, \ldots, t \)) we have \( xe_i = \sum_j \alpha_{i,j} f_j, \ y e_i = \sum_j \beta_{i,j} f_j, \) thus

\[(xc_1 + yc_2)e_i = \sum_j (\alpha_{i,j}c_1 + \beta_{i,j}c_2)f_j \quad (i = 1, \ldots, s).\]

Therefore \((c_1, c_2) \in U_n \) iff the rank of the matrix \((\alpha_{i,j}c_1 + \beta_{i,j}c_2 | i = 1, \ldots, s; j = 1, \ldots, t)\) equals \( s \). But this means that at least one of the \( s \times s \) minors of the matrix in question does not vanish. This shows that \( U_n \) is a \((\text{Zariski})\) open subset of \( k^2 \).

Now, let \((c_1, c_2) \in U_n\), with \( n < c \). Then, putting \( z = xc_1 + yc_2, z: H_{n-1} \to H_n \) is injective. Moreover \( x: H_{n-2} \to H_{n-1} \) is injective. So, by the diagram

\[
\begin{array}{ccc}
H_{n-2} & \xrightarrow{x} & H_{n-1} \\
\downarrow & & \downarrow \\
H_{n-1} & \xrightarrow{x} & H_n \\
\end{array}
\]

\( z: H_{n-2} \to H_{n-1} \) is injective too. It follows \((c_1, c_2) \in U_{n-1}\). We thus get the following chain of inclusions:

\[ U_c \subseteq U_{c-1} \subseteq U_{c-2} \subseteq \ldots \]

Finally let \((c_1, c_2) \in k^2 - \{(0, 0)\} \). Then \( z = xc_1 + yc_2 \) belongs to \( V - (0) \). Thus, for all \( n \leq 0, z: H_{n-1} \to H_n \) is injective. This shows that \((c_1, c_2) \in U_n \) for some \( n \leq c \). Consequently we have \( k^2 - \{(0, 0)\} = \bigcup_{n \leq c} U_n \). As \( k^2 - \{(0, 0)\} \) is quasicompact we find an \( n_0 < c \) such that \( k^2 - \{(0, 0)\} = U_{n_0} \). By decreasing \( n_0 \) if necessary, we may assume that \( \dim_k (H_{n_0-1}) = \dim_k (H_{n_0}) = d \). By our choice of \( n_0, r: H_{n_0-1} \to H_{n_0} \) then becomes an isomorphism for all \( z \in V - (0) \).

Assume now that \( d > 0 \). Let \( \phi: H_{n_0} \to H_{n_0-1} \) be the inverse of \( x: H_{n_0-1} \to H_{n_0} \). Then \( \phi \circ y \) is an automorphism of \( H_{n_0-1} \), which has an Eigenvalue \( \lambda \in k \). So, for a \( h \in H_{n_0-1} - (0) \) we have \( \phi \circ y(h) = \lambda h \), thus \( y(h) = x \circ \phi \circ y(h) = \lambda x(h) \), thus \( (y - \lambda x) h = 0 \). As \( y - \lambda x \in V - (0, 0) \) we get a contradiction.

**Corollary (2-2).** Let \((A_0, m_0)\) be noetherian and local. Let \( A = A_0 \oplus A_1 \oplus A_2 \oplus \ldots \) be a graded \( A_0 \)-algebra and assume that \( A_0/m_0 \) is algebraically closed. Let \( H = \bigoplus_{n \in \mathbb{Z}} H_n \) be a graded \( A \)-module and assume that there is a set \( U \subseteq A_1 \) satisfying the following properties:

\[(a) \quad U = \{u = u \mod m_0 A_1 \mid u \in U\} \text{ is of the form } V - (0), \text{ where } V \subseteq A_1/m_0 A_1 \text{ is a } k \text{-vectorspace of dimension } 2.\]

\[(b) \quad \text{For all } h \in H \text{ there is a } v \in \mathbb{N} \text{ with } U^*h = 0.\]

\[(c) \quad \text{For all } x \in U, (0: x)_{H} \text{ is an } A \text{-module of finite length. Then } H \text{ is an } A \text{-module of finite length.}\]
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Proof. By the arguments of the proof of (2-1) we first see, that there exists a \( c \in \mathbb{N} \) such that \( H_n = 0 \) for all \( n > c \), that \( \ell_{d_0}(H_n) \leq \infty \) for all \( n \) and that \( \ell_{d_0}(H_n) \) takes a constant value \( d \) for \( n \leq 0 \). Thereby \( \ell_{d_0} \) stands for the \( A_0 \)-length. Of course we have to show again that \( d = 0 \). As \( H_n \) is of finite length over \( A_0 \), we have \( m_0 H_n = 0 \) for an appropriate \( \nu \in \mathbb{N} \). This shows that \( H_n = 0 \) if and only if \((0 : m_0)_{H_n} = 0 \). As

\[
(0 : m_0)_{H_n} = ((0 : m_0 A)_H)_n,
\]

we have to show that \( H' \colon (0 : m_0 A)_H \) is of finite length. Clearly \( H' \) also satisfies our hypotheses. So we may replace \( H \) by \( H' \), this assuming that \((m_0 A)_H = 0 \). Now, \( H \) becomes a graded module over \( A' \colon A/m_0 A = \bigoplus_{n \geq 0} A_n/m_0 A_n \). Replacing \( A \) by \( A' \) and \( U \) by its canonical image \( \overline{U} \) does not affect our hypotheses. So we may replace \( A \) by \( A' \), thus assuming that \( A_0 = k \) is an algebraically closed field. But now we are in the situation of (2-1), and our conclusion is clear.

Example (2-3). We show that (2-1) needs not to hold, if \( k \) is not algebraically closed. Let \( k = \mathbb{R} \), \( W = \mathbb{R}^2 \). Let \( \phi, \psi \colon W \to W \) the linear maps given with respect to the canonical basis of \( \mathbb{R}^2 \) by the matrices \( A_\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( A_\psi = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} \). Then for all \( \lambda, \mu \in \mathbb{R}^2 \) we have \( \det(\lambda \phi + \mu \psi) = \det \begin{pmatrix} \lambda + \mu & 2\mu \\ -2\mu & \lambda - \mu \end{pmatrix} = \lambda^2 - \mu^2 + 4\mu^2 = \lambda^2 + 3\mu^2 \). So, \( \lambda \phi + \mu \psi \) is an isomorphism whenever \( (\lambda, \mu) \neq (0, 0) \). Now, put

\[
A = \mathbb{R}[X, Y], \quad H = \oplus W \oplus W \oplus \ldots \oplus W,
\]
such that \( H_n = W \), \( \forall n \leq 0 \), \( H_n = 0 \), \( \forall n > 0 \). Now, defining \( X \colon H_{n-1} \to H_n \) resp. \( Y \colon H_{n-1} \to H_n \) by \( \phi \) resp. \( \psi \) if \( n \leq 0 \), and by \( 0 \) if \( n > 0 \), \( H \) becomes a graded \( A \)-module.

Now, let \( V = A_1 = \mathbb{R}X + \mathbb{R}Y \). By the previous remark we have \((0 : z)_H = H_0 = W \) for all \( z \in V - (0) \). So (a) of (2-1) holds. (b) clearly also is satisfied. But obviously \( H \) is not of finite length.

Remark (2-4). There is a version of (2-1), which holds for arbitrary ground fields \( k \). We namely only need to assume that (a) and (b) hold ‘universally’, e.g. that these properties remain valid under finite algebraic extensions of the ground field. A similar statement holds for (2-2) (see (2)).

3. Application to Cohomology

In this section let \((A_0, m_0)\) be a noetherian local ring. Let \( A = A_0 \oplus A_1 \oplus A_2 \oplus \ldots \) be a graded \( A_0 \)-algebra of finite type such that \( A = A_0[A_1] \). By \( M \) we denote the maximal homogeneous ideal \( m_0 \oplus A_1 \oplus A_2 \oplus \ldots \).

Let \( M = \oplus_{n \in Z} M_n \) be a graded \( A \)-module. Then the local cohomology modules \( H^\delta_n(M) \) are canonically graded. This follows from the fact that \( \text{Ext}_A^n(A/m^n) \) calculated in the category of graded \( A \)-modules, coincides with the ordinary corresponding functor (see (2), (3)).

More precisely: If \( M \) and \( N \) are graded \( A \)-modules, \( M = \oplus_n M_n \), \( N = \oplus_n N_n \), we say that an \( A \)-homomorphism \( \alpha \colon M \to N \) is homogeneous of degree \( d \) if \( \alpha(M_n) \subseteq N_{n+d} \), for all \( n \in Z \). Expressing this fact by writing \( M \overset{\alpha}{\longrightarrow} N \) we have (see (2)): 

\[
(0 : m_0)_{H_n} = ((0 : m_0 A)_H)_n,
\]
Lemma (3-1). Let \( 0 \rightarrow M \xrightarrow{(d)} N \xrightarrow{(s)} P \rightarrow 0 \) be a short exact sequence of graded \( A \)-modules. Then, the cohomology sequence takes the form

\[
\begin{array}{ccccccc}
0 & \rightarrow & H^0_m(M) & \rightarrow & H^0_m(N) & \rightarrow & H^0_m(P) & \rightarrow & H^1_m(M) \\
& & \rightarrow & H^1_m(N) & \rightarrow & H^1_m(P) & \rightarrow & H^2_m(M) & \rightarrow & \ldots
\end{array}
\]

In the sequel, assume that \( M \) is finitely generated and graded over \( A \). An element \( x \in A \) is said to be filter-regular (see (9)) \( \text{(f-regular)} \) with respect to \( M \), if it is regular with respect to \( M/\Gamma_m(M) \), where \( \Gamma_m(M) \) stands for the \( m \)-torsion

\[
\bigcup_f (0: m^f)_M = \text{lim Hom}_d (A/m^f, M) = H^0_m(M).
\]

It is equivalent to say that \( x \notin \bigcup_f \text{Ass}(M) - \{m\} \).

In particular, if \( A_0 = k \) is a field and if \( x \) is a form of degree \( d(x \in A_0) \), \( x \) is \( f \)-regular iff it is quasi-regular with respect to \( M/M_A(M) \). This latter means that \( (0: x)_M \) vanishes in all large degrees or, equivalently, that the \( A_0 \)-homomorphisms \( M_n \rightarrow M_{n+d} \) are injective for all large \( n \). Geometrically this is expressed by the fact that the hypersurface \( V(x) = \text{Proj}(A/xA) \subseteq X = \text{Proj}(A) \) does not meet \( \text{Ass}(M) \), \( M \) being the (coherent) \( \mathcal{O}_x \)-sheaf induced by \( M \).

In general if \( A_0 \) is not necessarily a field, \( f \)-regularity implies quasi-regularity, but not conversely.

A set \( U \subseteq A_1 \) is called a pencil if it is a system of representatives of a set \( V - \{0\} \), where \( V \subseteq A_1/m_0 A_1 \) is a \( A_0/m_0 \)-vector space of dimension 2.

Lemma (3-2). Let \( |A_0/m_0| = \infty \). Let \( M \) be a graded \( A \)-module which is finitely generated. Assume that \( \dim (A_0/m_0 \otimes_{A_0} A/\mathfrak{f}) > 1 \) for all \( \mathfrak{f} \in \text{Ass}(M) - \{m\} \). Then there is a pencil \( U \subseteq A_1 \) which consists of elements \( x \) which are \( f \)-regular with respect to \( M \).

Proof. Let \( \{f_1, \ldots, f_s\} = \text{Ass}(M) - \{m\} \). By hypothesis we have

\[
\dim (B_t = A_0/\mathfrak{m}_0 \otimes_{A_0} A/\mathfrak{f}_t) > 1
\]

for all \( t \in \{1, \ldots, s\} \). As \( B_t = A_0/\mathfrak{m}_0 [A_1/(\mathfrak{m}_0 A_1 + f_t \cap A_1)] \), \( A_1/(\mathfrak{m}_0 A_1 + f_t \cap A_1) \) must be a \( k = A_0/m_0 \) vector space of dimension > 1 for all \( t \) in question. As \( k \) is infinite, we therefore find a subspace \( V \) of \( A_1/m_0 A_1 \) such that

\[
V \cap [\bigcup_t (f_t \cap A_1 + m_0 A_1)/m_0 A_1] = \{0\}.
\]

Choosing a system \( U \) of representatives of \( V - \{0\} = A_1 \) we must have \( x \notin \bigcup_{t=1}^s f_t \) for all \( x \in U \). But this is the requested property.

A pencil \( U \) as in (3-2) will be called a pencil of \( f \)-regular elements with respect to \( M \).

Remark (3-3). The proof of (3-2) shows that we may even choose

\[
U \subseteq A_1 - q_1 \cup \ldots \cup q_r,
\]

where \( \{q_1, \ldots, q_r\} \) is a set of homogeneous primes of \( A \) satisfying

\[
\dim (A_0/m_0 \otimes_{A_0} A/q_i) > 1 \quad \text{for } i = 1, \ldots, r.
\]

Remark (3-4). Let \( A_0 = k \) be algebraically closed. Let \( X = \text{Proj}(A) \) and let \( \mathcal{F} \) be a coherent \( \mathcal{O}_x \)-sheaf. Then \( U \subseteq A_1 \) is a pencil of \( f \)-regular elements with respect to
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Let \( \mathcal{F} = \hat{M} \), if \( \mathcal{U}_U = \{ \text{Proj}(A/xA) : H_x[x \in U] \} \) is in general position with respect to \( \mathcal{F} \) in the sense of (1·1) (3·2) states, that \( \mathcal{F} \) admits pencils \( \mathcal{P} \) of hyperplane sections if \( \text{Ass}(\mathcal{F}) \) contains no closed points of \( X \). In fact, as shown by the proof of (3·2), in this case the generic pencil of hyperplane sections is in general position with respect to \( \mathcal{F} \).

**Proposition (3·5).** Assume that \( A_0/m_0 \) is algebraically closed. Let \( M \) be a finitely generated and graded \( A \)-module. Let \( U \subseteq A_1 \) be a pencil of \( f \)-regular elements with respect to \( M \). Let \( i \in \mathbb{N} \) and assume that \( H^{i-1}(M/xM) \) is of finite length (over \( A \)) for all \( x \in A \). Then \( H^i_m(M) \) is of finite length.

**Proof.** Let \( \bar{M} = M/\Gamma_m(M) \). Then it holds \( H^i_m(M) \cong H^i_m(\bar{M}) \). If \( i > 1 \), it also holds \( H^{i-1}_m(M/xM) \cong H^{i-1}_m(\bar{M}/x\bar{M}) \). If \( i = 1 \), \( H^{i-1}_m(\bar{M}/x\bar{M}) = \Gamma_m(\bar{M}/x\bar{M}) \) is of finite length. All \( x \in U \) are \( \hat{M} \)-regular, thus in particular \( f \)-regular with respect to \( M \). So we may replace \( M \) by \( \bar{M} \) to prove our claim. Therefore we may assume that the elements \( x \in U \) are all regular with respect to \( M \). This leads to exact sequences

\[
0 \rightarrow M \xrightarrow{x} M \xrightarrow{x} M/xM \rightarrow 0.
\]

So, in cohomology we obtain exact sequences \( H^{i-1}_m(M/xM) \rightarrow H^i_m(M) \rightarrow H^i_m(M) \) for all \( x \in U \). Now we conclude by (2·2).

**Remark (3·6).** To prove (1·1), write \( \mathcal{F} = \hat{M} \) and use the well known relations

\[
\bigoplus_n H^i(X, \mathcal{F}(n)) = H^{i+1}_m(M) \quad (i > 0)
\]

\[
\bigoplus_n H^0(X, \mathcal{F}(n)) = D_m(M)
\]

(7) and observe (3·4).

(For the definition of \( D_m \) and its basic properties see (1) or (4).)

4. **The finiteness theorem**

We begin with a preliminary result:

**Lemma (4·1).** Let \( (B, \mathfrak{n}) \) be a discrete valuation ring. Then there is an integral flat extension \( B' \) of \( B \) such that \( (B', \mathfrak{n}B') \) is again a discrete valuation ring and such that \( B'/\mathfrak{n}B' \) is the algebraic closure of \( B/\mathfrak{n} \).

**Proof.** Let \( K \) be the quotient field of \( B \), \( L \) its algebraic closure. Let \( \mathfrak{B} \) be the family of all integral extension rings \( C \subseteq L \) of \( B \) for which \( (C, \mathfrak{n}C) \) is a discrete valuation ring. Then \( B \in \mathfrak{B} \). Moreover \( B \rightarrow C \) is flat by (8), 21.D) for all \( C \in \mathfrak{B} \). Let \( C = \cup C_i \), where \( C_i (i \in I) \) is an ascending chain of \( \mathfrak{B} \). Then \( C \) is integral over \( B \) and \( \mathfrak{n}C \) is the unique maximal ideal of \( C \), where \( \mathfrak{n} = \mathfrak{n}B \). If \( C_i \subseteq C_i' (i, \iota \in I) \), \( C_i' \) is faithfully flat over \( C_i \). This shows that \( \mathfrak{n}C \cap C_i = \mathfrak{n}C_i \) for all \( n \). We thus must have \( \cap \mathfrak{n}C = 0 \). So \( (C, \mathfrak{n}C) \) is again a discrete valuation ring. This shows that \( \mathfrak{B} \) has a maximal member, say \( B' \).

We have to show that \( B'/\mathfrak{n}B' \) is algebraically closed. Assuming the opposite, there is a unitary polynomial \( f \in B'[X] \) whose image modulo \( \mathfrak{n}B' \) is irreducible and of degree \( > 1 \). Put \( B'' = B'[X]/(f) \). \( B'' \) is finite over \( B' \). \( (\pi, f) \) is the unique maximal ideal of \( B'[X] \) which contains \( f \) and \( \pi \). So \( \pi B'' \) is the unique maximal ideal of \( B'' \).

There is a \( c \in L - B' \) with \( f(c) = 0 \). So \( \bar{B} = \bar{B} \neq B' \) is a homomorphic image of \( B'' \), by means of an epimorphism \( \lambda: B'' \rightarrow \bar{B} \), which sends \( X \mod f(X) \) to \( c \) and which acts
identically on $B'$. So $\pi \bar{B} = \lambda(\pi B')$ is the unique maximal ideal of $\bar{B}$. As $\lambda(\pi) \neq 0$, we have $\pi \bar{B} \neq 0$. So $\lambda$ is an isomorphism. Thus $\bar{B} \in \mathfrak{B}$, which contradicts the supposed maximality of $B'$.

**Lemma (4.2).** Let $A$ be a homomorphic image of a regular ring. Let $J \subseteq A$ be an ideal and let $M$ be a finitely generated $A$-module. Then $H^1_A(M)$ is finitely generated iff $ht((J + \mathfrak{f})/(\mathfrak{f})) + 1$ for all $\mathfrak{f} \in \text{Ass}(M)$.

**Proof.** See (1), (3-1)) and use the sequence

$$M \rightarrow D_\mathfrak{f}(M) \rightarrow H^1_\mathfrak{f}(M) \rightarrow 0.$$

Now, let $A = A_0 \oplus A_1 \oplus \ldots$ be as in the previous section. For a finitely generated $A$-module $M$ which is graded, we introduce the following three invariants:

(4-3) $e_m(M) = \min \{i | H^i_m(M) \text{ not finitely generated}\}$.

(4-4) $c_m(M) = \min \{\text{depth } (M_i) + ht(m/\mathfrak{f}) | \mathfrak{f} \in \text{Spec } (A), \mathfrak{f} \subseteq m\}$.

(4-5) $\check{c}_m(M) = \min \{\text{depth } (M_i) + ht(m/\mathfrak{f}) | \mathfrak{f} \in \text{Spec } (A), m \not\subseteq \mathfrak{f}, \mathfrak{f} \text{ homogeneous}\}$.

We now give the announced elementary proof of the homogeneous finiteness theorem which (using the conventions $\min \{\varnothing\} = \infty$, depth $(0) = \infty$) may be stated as

**Proposition (4-6).** Let $A_0$ be a field or a discrete valuation ring. Let

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \ldots$$

be a noetherian graded ring with $A = A_0[A_1]$. Let $M$ be a finitely generated and graded $A$-module. Then

$$e_m(M) = c_m(M) = \check{c}_m(M).$$

**Proof.** Obviously we have $c_m(M) \leq \check{c}_m(M)$. So it remains to prove $e_m(M) \leq c_m(M)$ and $e_m(M) \geq \check{c}_m(M)$. Putting $\bar{M} = M/\mathfrak{m}_M(M)$ we clearly have $e_m(\bar{M}) = e_m(M)$, $c_m(\bar{M}) = c_m(M)$ and $\check{c}_m(\bar{M}) = \check{c}_m(M)$. Generally it also is clear that $e_m(M) \geq 1$, $c_m(M) \geq 1$. To prove $e_m(M) \leq c_m(M)$ we assume that $H^i_m(M)$ is finitely generated for $i = 0, \ldots, h$ $(h \geq 0)$ and prove that $h < c_m(M)$. Thereby $M$ needs not to be graded. For $h = 0$, there is nothing to prove. So let $h > 0$. As remarked above we may replace $M$ by $\bar{M}$, thus assuming that $\Gamma_m(M) = 0$. Now let $\mathfrak{f} \in \text{Spec } (A)$ such that $\mathfrak{f} \subseteq m$. We have to show that depth $(M_i) + ht(m/\mathfrak{f}) > h$. If $\mathfrak{f} \notin \text{Ass } (M)$, we choose an $M$-regular element $x \in m$ (such an $x$ exists as $m \notin \text{Ass } (M)$). We then find a minimal prime divisor $q$ of $x A + \mathfrak{f}$ such that $q \subseteq m$. As $h > 0$ $H^h_m(M)$ is finitely generated. So, by (4.2), $ht(m/\mathfrak{f}) > 1$. By Krull's lemma we have $ht(q/\mathfrak{f}) = 1$. So clearly $q \not\subseteq m$. By (t3), p. 99) we have $q \in \text{Ass } (M/x M)$. As $x$ is $M$-regular, we have a regular sequence

$$0 \rightarrow M \rightarrow M \rightarrow M/x M \rightarrow 0,$$

which implies in cohomology $H^i_m(1) \rightarrow H^i_m(1) \rightarrow H^i_m(1)$, thus showing that $H^h_m(1/x M)$ is finitely generated for all $i \leq h$.

Therefore we have, by induction, $ht(m/q) = ht(m/q) + \text{depth } ((M/x M)q) > h - 1$. It follows $ht(m/\mathfrak{f}) > ht(m/q) > h$. If $\mathfrak{f} \notin \text{Ass } (M)$ we either find an $\mathfrak{f}_0 \not\subseteq m$ such that
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\[ \text{depth } (M_i) + \text{ht}(m/\mathfrak{f}) > \text{ht}(m/\mathfrak{f}_0) = \text{ht}(m/\mathfrak{f}_0) + \text{depth } (M_i) > h. \]

In the second case the regularity of \( z \) induces a sequence

\[ 0 \to M_m \to M_m \to (M/xM)_m \to 0, \]

which induces exact sequences

\[ H^{i-1}_m(M_m) \to H^{i-1}_m((M/xM)_m) \to H^i_m(M_m). \]

As each element of \( H^i_m(M_m) \) is annihilated by some power of \( m \) it follows

\[ H^i_m(M'_m) = A_m \otimes_A H^i_m(M_m). \]

So the above sequences show that \( H^{i-1}_m(M/xM) \) is finitely generated for all \( i \leq h \).

Applying induction we get

\[ \text{depth } (M_i) + \text{ht}(m/\mathfrak{f}) = \text{depth } ((M/xM)_i) + \text{ht}(m/\mathfrak{f}) + 1 > h - 1 + 1 = h. \]

Thus we have shown the inequality \( e_m(M) \leq c_m(M) \).

To prove \( e_m(M) = c_m(M) \) we apply (4-1) and let \( A'_0 \) be the algebraic closure of \( A_0 \), if \( A_0 \) is a field, resp. \( (A_0, m_0, A'_0) \) be an integral flat extension of \( (A_0, m_0) \) which is a discrete valuation ring with algebraically closed residue field, in case \( (A_0, m_0) \) is a discrete valuation ring. Now let

\[ A' = A'_0 \otimes_A A, \quad M' = A'_0 \otimes_A M = A' \otimes_A M, \quad m' = m_0 A'_0 \oplus (A'_0 \otimes_A A_1) \oplus \ldots. \]

Then \( A' \) is again of the requested type, and \( M' \) is finitely generated and graded over \( A' \). As \( A' \) is integral over \( A \) we have \( \sqrt{m} A' = m' \). So, using the flat base change property of local cohomology, we get

\[ H^i_m(M'_m) = \text{depth } (A'_0 \otimes_A M) = A' \otimes_A H^i_m(M_m). \]

As \( A \to A' \) is faithfully flat we obtain \( e_m(M') = e_m(M) \). Next, let \( \mathfrak{f} \not\subseteq m \), \( \mathfrak{f}' \not\subseteq m' \) be primes of \( A \) resp. \( A' \) such that \( \mathfrak{f}' \cap A = \mathfrak{f} \). Then, as \( A \to A' \) is flat and integral, we have

\[ \text{ht}(m'/\mathfrak{f}') = \text{ht}(m/\mathfrak{f}) \quad \text{and} \quad \text{depth } (M'_i) = \text{depth } (A'_0 \otimes_A M_i) = \text{depth } (m_i). \]

Therefore we have

\[ \text{depth } (M'_i) + \text{ht}(m'/\mathfrak{f}') = \text{depth } (M_i) + \text{ht}(m/\mathfrak{f}). \]

If \( \mathfrak{f} \not\subseteq m \) is a prime, there is a minimal prime \( \mathfrak{f}' \) of \( \sqrt{m} A' \) with \( \mathfrak{f}' \subseteq m' \). As \( A \to A' \) is flat it follows

\[ \mathfrak{f}' \cap A = \mathfrak{f} \quad \text{and} \quad \mathfrak{f}' \subseteq m'. \]

If \( \mathfrak{f}' \subseteq m' \), clearly \( \mathfrak{f} = \mathfrak{f}' \cap A \) and \( \mathfrak{f}' \subseteq m' \). Moreover, if \( \mathfrak{f}' \cap A = \mathfrak{f}, \mathfrak{f}' \) is homogeneous iff \( \mathfrak{f} \subseteq m \). Combining these statements, we see that \( e_m(M') = e_m(M) \). So we may replace \( A \) and \( M \) by \( A' \) resp. \( M' \), thus assuming that \( A_0/m_0 \) is algebraically closed. As \( A_0 \) is a field or a discrete valuation ring we find a \( \pi \in A_0 \) with \( m_0 = \pi A_0 \).

To show our statement we assume that

\[ \text{depth } (M_i) + \text{ht}(m/\mathfrak{f}) \geq h \]

for all homogeneous primes \( \mathfrak{f} \not\subseteq m \) and shown that \( H^i_m(M) \) is finitely generated for all \( i < h \). For \( h < 1 \) this is clear. So let \( h > 1 \). Again we may replace \( M \) by \( M' \), thus assuming \( \Gamma_m(M) = \emptyset \), hence \( m \not\in \text{Ass}(M) \). As \( M \) is graded, it follows \( \mathfrak{f} \not\subseteq m \) for all \( \mathfrak{f} \in \text{Ass}(M) \).

So, we have

\[ \text{ht}(m/\mathfrak{f}) = \text{depth } (M_i) + \text{ht}(m/\mathfrak{f}) \geq h \]

for all such \( \mathfrak{f} \). Using (4-2) we see that \( H^i_m(M) \) is finitely generated. This settles in particular the case \( h = 2 \). So let \( h > 2 \).

Then, by the previous remark, it suffices to show that \( H^i_m(M) \) is finitely generated over \( A \) for all \( i \leq h - 1 \). As \( h > 2 \), we have \( \text{ht}(m/\mathfrak{f}) > 2 \) for all \( \mathfrak{f} \in \text{Ass}(M) - \{m\} \). It follows

\[ \dim (A_0/m_0 \otimes_A \mathfrak{f}) = \dim (A_0/m_0 \mathfrak{f}) = \text{ht}(m/\mathfrak{f}) - 1 > 1. \]
for all such \( f \). Thus by (3-2) there is a pencil \( U \subseteq A_1 \) of \( f \)-regular elements with respect to \( M \). As \( \Gamma_m(M) = 0 \), all \( x \in U \) are even regular with respect to \( M \). Therefore clearly 
\[
\text{depth} \left( (M/xM) \right) + \text{ht}(m/f) - 1 \geq \text{ht} - 1 \text{ for all homogeneous primes } f \nsubseteq m.
\]
So, by induction \( H_m^{-1}(M/xM) \) is finitely generated for all \( i < h \) and all \( x \in U \). As each element of \( H_m^i(.) \) is annihilated by some power of \( m \), we have: 
\[
\ell(H_m^i(.)) < \infty \Leftrightarrow H_m^i(.) \text{ is finitely generated}.
\]
Therefore we may finish our proof by (3-5).

Remark (4.7). Let \( A_0 = k \) be algebraically closed and let \( X \) be the projective \( k \)-variety \( \text{Proj}(A) \). Let \( \mathcal{F} \) be a coherent \( \mathcal{O}_X \)-sheaf such that \( \mathcal{F} = \mathcal{M} \), where \( M \) is finitely generated and graded. Then \( e_m(M) = \min \{ \text{depth} (\mathcal{F}_x) + \dim \{ x \} | x \in X \} + 1 \). To see this, choose \( x_1, \ldots, x_r \in A_1 \) such that \( A_1 = \Sigma x_i k \). Then the open sets
\[
X_{i_1} = X - \text{Proj} \left( A/x_i A \right)
\]
form an open affine covering of \( X \) and it holds \( \Gamma(X_{i_1}, \mathcal{F}) = (M_{x_i})_0 \), where \( M_{x_i} \) is furnished with its canonical grading. So \( \Gamma(X_{i_1}, \mathcal{F}) [T, T^{-1}] \approx M_{x_i} \). Now, using the relations mentioned in (3-6) we get the following result of Serre(7):
\[
H^i(X, \mathcal{F}(n)) = 0, \quad \forall n < 0 \Leftrightarrow i \leq \min \{ \text{depth} (\mathcal{F}_x) + \dim \{ x \} | x \in X \}.
\]

Remark (4-8). In the above notations, let \( \rho \in \mathbb{Z} \) be such, that \( H^i(X, \mathcal{F}|_H(n)) = 0 \) for all \( n \leq \rho \) and all \( H \in \mathcal{B} \) (such a \( \rho \) exists under the hypotheses of (1-1), as is seen from the proof of (2-1)). Then, analyzing the proofs of (2-1) and of (3-5) we get the following estimates
\[
\begin{align*}
(i) & \quad \dim_k(H^{i+1}(X, \mathcal{F}(n))) \leq \max \left[ 0, \sum_{m > n} \dim_k(H^i(X, \mathcal{F}|_H(m)) + \min(0, n - \rho) \right], H \in \mathcal{B}, \\
(ii) & \quad H^{i+1}(X, \mathcal{F}(n)) = 0, \quad \text{if } n < \rho - \sum_{m > \rho} \dim_k(H^i(X, \mathcal{F}|_H(m)).
\end{align*}
\]
In view of these estimates, it is appropriate to choose
\[
\sigma_n(H) = \sum_{m > n} \dim_k(H^i(X, \mathcal{F}|_H(m))
\]
as small as possible, in choosing \( H \) appropriately. Choosing \( H \in \mathcal{B} \) we have exact sequences
\[
\begin{align*}
\bigoplus_{n} H^i(X, \mathcal{F}(n)) & \rightarrow \bigoplus_{n} H^i(X, \mathcal{F}|_H(m)) \rightarrow \bigoplus_{n} H^{i+1}(X, \mathcal{F}(n-1)) \\
& \rightarrow \bigoplus_{n} H^{i+1}(X, \mathcal{F}(n))
\end{align*}
\]
(\( x_H \) is the one-form defining \( H \)).

These sequences show, that \( \dim_k(H^i(X, \mathcal{F}|_H(n))) \) takes its minimal value for a general \( H \in \mathcal{B} \). So \( \sigma_n(H) \) takes its minimal value for all but eventually finitely many \( H \in \mathcal{B} \). So, in (i) and (ii) we may choose \( H \) general to get the best estimates.

Assume that \( H^i(X, \mathcal{F}(n)) = 0, \forall n \). Then the above sequences show that in (i) we have equality as long as \( n \geq \rho \), provided \( \rho \) is chosen maximally. In this case we conclude inductively that \( \dim_k(H^i(X, \mathcal{F}|_H(n))) \) is the same for all \( H \in \mathcal{B} \). So the same holds for all numbers \( \sigma_n(H) \). In particular, we then need to consider only one \( H \) to determine \( \rho \) and the sequence \( \{ \sigma_n(H) | n \in \mathbb{Z} \} \). If moreover \( \sigma_\rho(H) = 1 \), (i) holds for all \( n \).

This is the special case in which
\[
H^i(X, \mathcal{F}(n)) = 0, \quad \forall n; \quad H^i(X, \mathcal{F}|_H(n)) \simeq \begin{cases} 0, n \neq \rho + 1, & H^{i+1}(X, \mathcal{F}(n)) \simeq \{ 0, n = \rho \}. \end{cases}
\]
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