On the speed of convergence in Strassen’s law of the iterated logarithm

Bolthausen, E
ON THE SPEED OF CONVERGENCE IN STRASSEN’S LAW OF THE ITERATED LOGARITHM

BY E. BOLTHAUSEN

University of Konstanz

Here there is derived a condition on sequences $\varepsilon_n \downarrow 0$ which implies that
$P[\{W(t) : t \geq 0, \omega \in \Omega\} \not\subset K_{\varepsilon} \text{ i.o.}] = 0$, where $W$ is the Wiener process
and $K$ is the compact set in Strassen’s law of the iterated logarithm. A similar result for random walks is also given.

1. Introduction. Let $\{W(t, \omega) : t \geq 0, \omega \in \Omega\}$ be a Brownian motion process
defined on some probability space $(\Omega, \mathcal{F}, P)$. We assume that $W(t, \omega)$ is con-
tinuous in $t$ for each $\omega \in \Omega$. Let $C[0, 1]$ be the space of continuous real valued
function on the interval $[0, 1]$. $\parallel \parallel$ the sup-norm on $C$ and $\mathcal{B}$ the class of Borel
sets in $C[0, 1]$. For $n \in \mathbb{N}$ let $W_n(\omega) \in C[0, 1]$ be defined by $W_n(\omega)(t) = W(nt, \omega)/n^{\frac{1}{2}}$.
It is well known that the $W_n$ all have the same distribution on $(C[0, 1], \mathcal{B})$. For
$r > 0$ let $K_r = \{f \in C[0, 1] : f' \text{ exists and } \int (f'(t))^2 dt \leq r\}$. $K_r$ is a compact
set in $C[0, 1]$ (e.g., [2] page 282). Let $H = \bigcup_{r > 0} K_r$. If $A \subset C[0, 1]$ and $\varepsilon > 0$
let $A' = \{f \in C[0, 1] : \exists g \in A \text{ with } \|f - g\| < \varepsilon\}$. One half of Strassen’s law
of iterated logarithm [3] states that for each $\varepsilon > 0$

\begin{equation}
P(\lim \sup_{n \to \infty} \{W_n/(2 \log n)^{\frac{1}{2}} \notin K_{\varepsilon}'\}) = 0 .
\end{equation}

Here and elsewhere in this paper we set $L_x = \max (\log x, 1)$. (1.1) is sharpened in this paper in the following way:

**Theorem 1.** If $\alpha < \frac{1}{2}$ and $\varepsilon_n = (L_x n)^{-\alpha}$ then

\begin{equation}
P(\lim \sup_{n \to \infty} \{W_n/(2 \log n)^{\frac{1}{2}} \notin K_{\varepsilon_n}'\}) = 0 .
\end{equation}

The result should be seen as an attempt to prove “stronger” forms of infinite
dimensional log log laws, that is, to give conditions on increasing sequences of
sets $A_n \subset C[0, 1]$ such that $P(W_n \notin A_n \text{ infinitely often}) = 0$. But of course the
above theorem is far away from providing a complete solution.

All proofs of infinite dimensional log log laws consist in approximating the
random variables in question by elements of the compact set which appears in
the theorem to be proven. The approximation used in the proof of Theorem 1
is the following: We take a lattice on the state space with suitable span and
approximate $W_n/(2 \log n)^{\frac{1}{2}}$ by the linear interpolation between the time points
where the process passes through a lattice point. One should compare this
method with the approximation used by Strassen. He takes a grid on the time
axis and interpolates the process between the points of the grid. One would
think it possible to obtain a theorem like the above by simultaneously refining

Received May 23, 1977.

AMS 1970 subject classifications. Primary 60F15; Secondary 60J15.

Key words and phrases. Brownian motion, Strassen’s law of iterated logarithm.

668
the grid with the growth of \( n \). This is indeed the case but the result which can be obtained is less good. In fact with this method one can prove that (1.2) hold for \( \varepsilon_\alpha = (L \ln n)^{-\alpha} \) for \( \alpha < 1/4 \). Strassen's approximation is more adapted to the "Gaussian character" of the Wiener process where the former approximation uses the fact that Brownian motion is a Markov process.

With the use of Skorohod imbedding one can easily derive an invariance theorem. Let \( \{X_n: n \in \mathbb{N}\} \) be independent identically distributed real random variables with \( EX_n = 0, 0 < \sigma^2 = \text{Var}(X_n) < \infty \) and define \( \{Y_n(t): 0 \leq t \leq 1\} \) as the linear interpolation of the chain \( Y_n(k/n) = n^{-1} \sum_{i=1}^{k} X_i \).

**Theorem 2.** If \( E|X_n|^{\alpha + \delta} < \infty \) for some \( \delta > 0 \) and \( \alpha \) and \( \varepsilon_\alpha \) are as in Theorem 1 then

\[
P(\limsup_{n \to \infty} \{Y_n/(2 L \ln n)^{1/2} \notin K_{\varepsilon^n}\}) = 0 .
\]

**2. Proofs.** For \( \delta > 0 \) let \( T_{\varepsilon}^t = \inf\{t: |W(t)| = \delta\} \) and inductively

\[
T_{\varepsilon}^t = \inf\{t: |W(t + \sum_{i=1}^{t} T_{i}^t) - W(\sum_{i=1}^{t} T_{i}^t)| = \delta\}.
\]

By the strong Markov property the \( T_{\varepsilon}^t \) are independent and identically distributed. Using a standard transformation argument one sees that \( T_{\varepsilon}^t \) has the distribution of \( \delta^2 T_{\varepsilon}^{-1} \). We will write \( T_{i} \) for \( T_{\varepsilon}^t \). \( T_{i} \) and \( 1/T_{i} \) have absolutely continuous distributions on the positive real numbers. Let \( g \) be the density of \( T_{i} \) and \( h \) the density of \( 1/T_{i} \).

From well-known expressions for the distribution of the \( T_{i} \) (see [1], page 330) and some elementary calculations one obtains

\[
g(x) = \left( \frac{2}{\pi} \right)^{1/4} x^{-1/4} \sum_{k=0}^{\infty} (-1)^k (2k + 1) \exp(-(2k + 1)^2/2x) \]

(2.1)

\[
h(x) = \left( \frac{2}{\pi} \right)^{1/4} x^{-1/4} \sum_{k=0}^{\infty} (-1)^k (2k + 1) \exp(-(2k + 1)^2x/2) \]

for \( x \geq 0 \).

**Lemma 1.** If \( t \leq 1 \) and \( k \in \mathbb{N} \) then

\[
P(\sum_{i=1}^{t} T_{i} \leq t) \leq \left( \frac{t}{k} \right)^{-3k/2} \exp(-k^3/2t).
\]

**Proof.** Let \( \Delta_k = \{(x_1, \ldots, x_k) \in \mathbb{R}^k: 0 \leq x_i \leq 1, \sum_{i=1}^{k} x_i \leq t\} \) and \( \lambda^k \) be Lebesgue measure on \( \mathbb{R}^k \). Then \( P(\sum_{i=1}^{k} T_{i} \leq t) = \int_{\Delta_k} g(x_1) \cdots g(x_k) \lambda^k(d(x_1, \ldots, x_k)) \). If \( x_i \leq 1 \) then \( g(x_i) \leq (2/\pi)^{1/4} x_i^{-1/4} \exp(-1/2x_i) \). So

\[
P(\sum_{i=1}^{k} T_{i} \leq t) \leq \left( \frac{2}{\pi} \right)^{k/2} \lambda^k(\Delta_k) \sup_{(x_1, \ldots, x_k) \in \Delta_k} g(x_1) \cdots g(x_k)
\]

\[
\leq \left( \frac{t}{k} \right)^{-3k/2} \exp(-k^3/2t).
\]

**Lemma 2.** There exists a number \( c > 0 \) such that for all \( k \in \mathbb{N} \) and \( d > 1 \)

\[
P\left( \frac{1}{k} \sum_{i=1}^{k} 1/T_{i} \geq d \right) \leq (2d + c)^k \exp\left(\frac{-k}{2 (d - 1)}\right).
\]
PROOF. Let $\tau = (d - 1)/2d$ and $\mu$ be the distribution of $1/T_t - d$. For $x \geq 1$
$h(x) \leq (2/\pi)^{1/2} x^{-1} \exp(-x/2)$.

$$
\rho = E(\exp(\tau_1 (1/T_t - d))) = \int_0^\infty \exp(\tau_1 (x - d)) h(x) \, dx \\
\leq (e^1 + \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \exp(x - i t) \exp(-x/2) \, dx) \exp(-d) \\
= (e^1 + 2/(1 - 2i)) \exp(-d) \\
= (e^1 + 2d^2) \exp(-d) < \infty.
$$

Then

$$
P\left(\frac{1}{k} \sum_{i=1}^{k} (1/T_t) \geq d\right) = P(\sum_{i=1}^{k} (1/T_t) - d) = 0
$$

$$
= P(\exp(\tau(\sum_{i=1}^{k} (1/T_t) - d)) \geq 1) \\
\leq E(\exp(\tau(\sum_{i=1}^{k} (1/T_t) - d))) \\
= \rho^k.
$$

The proof of the lemma is finished.

Let $\tau_{k^4} = \sum_{i=1}^{k^4} T_{i^4}$; $\tau_{0^4} = 0$. For $\delta > 0$ we define \(W_{\delta}(t): 0 \leq t \leq 1\) as follows: Let

\[ m(\delta) = \max \{k: \tau_{k^4} \leq 1\}, \]
\[ W_{\delta}(\tau_{k^4}) = W(\tau_{k^4}) \quad \text{if} \quad k \leq m(\delta), \]
\[ W_{\delta}(1) = W(\tau_{m(\delta)}) \]

and $W_{\delta}(t)$ linearly interpolated elsewhere. Clearly

\begin{equation}
\sup_{0 \leq t \leq 1} |W_{\delta}(t) - W(t)| \leq 2\delta
\end{equation}

\begin{equation}
W_{\delta} \in H \quad \text{and} \quad \frac{1}{2} \int_0^1 (W_{\delta}'(t))^2 \, dt = \sum_{i=1}^{m(\delta)} \frac{\delta^2}{T_{i^4}}.
\end{equation}

PROOF OF THEOREM 1. Let $\rho > 0$, $\delta > 1$, $K \in \mathbb{N}$ such that $\rho^2/K > 1$. If $W$ is
as above but restricted to $[0, 1]$, then

\begin{equation}
P(W \notin K^2) \leq P(W \notin K_0) = P(\sum_{i=1}^{m(\delta)} \delta^2/T_{i^4} \geq \rho^2) \\
= P\left(\bigcup_{k=1}^{\infty} \left(\sum_{i=1}^{k} \delta^2/T_{i^4} \geq \rho^2 \cap \{\sum_{i=1}^{k} T_{i^4} \leq 1\}\right)\right) \\
\leq \sum_{k=1}^{\infty} P\left(\sum_{i=1}^{k} 1/T_{i^4} \geq \rho^2, \sum_{i=1}^{k} T_{i^4} \leq 1/\rho^2\right) \\
\leq K P(\sum_{i=1}^{K^4} 1/T_{i^4} \geq \rho^2) + \sum_{k=K+1}^{\infty} P(\sum_{i=1}^{k} T_{i^4} \leq 1/\rho^2) \\
\leq K(2\rho/K^4 + C) \exp \left(-\frac{K}{2} (\rho^4/K - 1)\right) \\
+ \sum_{k=K+1}^{\infty} \left(\frac{1}{\delta^2 k}\right)^{\frac{1}{2}} \exp \left(-\frac{\delta^2 k^2}{2}\right)
\end{equation}

by Lemmas 1 and 2

\[ = I_{k(\rho)} + H_{k(\delta)}, \quad \text{say}. \]

If now $s > 0$ and $\varepsilon(s) = s^{-\alpha'}$ for $2\alpha < \alpha' < 1$ ($\alpha$ as in the statement of the
ON STRASSEN’S LOG LOG LAW

\[ P(W \notin sK_1^{(s)}) \leq P(W \notin K_1^{(s)}(1 + \varepsilon/2)) \]
\[ \leq I_{[s]}^{(1+\varepsilon/2)} + II_{[s]}^{(1-\varepsilon/4)} \]

for \( s \) sufficiently large, where \([s]\) as usual denotes the integer part of \( s \).

Here we used the relation

\[ K_i \supseteq K_i^{(s)} \]

Indeed if \( f \in K_i^{(s)} \) there exists \( g \in K_{i+\varepsilon/2} \) with \( \|f - g\| < \varepsilon/2 \). Then \( g/(1 + \varepsilon/2) \in K_i \) and \( \|f-g/(1+\varepsilon/2)\| \leq \varepsilon/2 + |1-(1+\varepsilon/2)^{-1}||g|| < \varepsilon \) because \( \sup \{||g|| : g \in K_{i+\varepsilon/2}\} = 1 + \varepsilon/2 \). So (2.9) and then (2.8) follow. Now by some elementary calculations one obtains

\[ I_{[s]}^{(1+\varepsilon/2)} \leq \exp(-\delta^2 s(1 + s^{-\varepsilon}(1 + o(1))) \]
\[ II_{[s]}^{(1-\varepsilon/4)} \leq \exp(-\delta^2 s) \]

for some \( \delta > 0 \) and \( s \) sufficiently large, where \( o(1) \) in (2.10) is understood to hold for \( s \to \infty \).

For \( m \in \mathbb{N} \) we take \( n_m = \lfloor \exp(m/L m) \rfloor \), \( s_m = (2 \text{ LL } n_m)^\delta \). Then using (2.8), (2.10) and (2.11) and the fact that \( \sum_{m=1}^{\infty} (m/L m)^{-(1+c(L m)^{-\gamma})} < \infty \) for \( c > 0, \gamma < 1 \) one obtains

\[ \sum_{m=1}^{\infty} P(W \notin s_m K_1^{(s_m)}) < \infty \]

Now \( W_n \) has the same distribution as \( W \), so we obtain from (2.12)

\[ P(\limsup_{m \to \infty} (W \notin s_m K_1^{(s_m)}) = 0 \).

For \( \omega \in \Omega \) not belonging to this exceptional set, there is a \( N(\omega) \in \mathbb{N} \) such that for \( m \geq N(\omega) \)

\[ W_{n_m} \in (2 \text{ LL } n_m)^{1/4} K_1^{(s_m)} \]

that is, there exists \( f_m \in K_1 \) such that with the abbreviation \( b(n) = (2n \text{ LL } n)^\delta \)

\[ \sup_{0 \leq t \leq 1} |W(n_m t)/b(n_m) - f_m(t)| \leq (\text{ LL } n_m)^{-\varepsilon/2}. \]

If \( n_m \leq n < n_{m+1} \) and \( m \geq N(\omega) \) let \( g_m(t) = f_{m+1}(m/n_{m+1}) \). Then

\[ ||W(n_{m_+})/b(n) - f_{m+1}|| \leq \|f_{m+1} - g_m\| + (b(n_{m+1})/b(n_m))||W(n_{m})/b(n_{m+1}) - g_m|| \]
\[ + ((b(n_{m+1})/b(n_m)) - 1)||g_m|| \]
\[ \leq (1 - n_m/n_{m+1})^{\delta} + (b(n_{m+1})/b(n_m))(\text{ LL } n_{m+1})^{-\varepsilon/2} \]
\[ + ((b(n_{m+1})/b(n_m)) - 1) \]
\[ = O((\text{ LL } n_{m+1})^{-\varepsilon/2}) = O((\text{ LL } n)^{-\varepsilon/2}) \leq (\text{ LL } n)^{-\varepsilon} \]

for \( n \) sufficiently large, where we used the fact that for \( f \in K_1 \)

\[ |f(t) - f(s)| \leq (t - s)^\delta. \]

The theorem is proved.

Proof of Theorem 2. Under the hypothesis of the theorem there exists a Brownian motion \( W(t) \) and a sequence \( \{X_t', i \in \mathbb{N}\} \) with the same distribution
as the sequence \( \{X_i : i \in \mathbb{N} \} \) and \( 0 \leq \rho < \frac{1}{2} \) such that

\[
T = \sup_{t \geq 0} \{ t : |W(t) - Y'(t)| \geq t^{1-\rho} \} < \infty \quad \text{w.p. 1}
\]

where \( Y'(k) = \sum_{i=1}^{k} X_i \) and linearly interpolated elsewhere (see Theorem 4.6 of [4]). If \( \omega \) is not in the exceptional set where \( T = \infty \) we have for \( n \geq T(\omega) \)

\[
\sup_{0 \leq t \leq n^t} \left| \frac{W(nt)}{n^t} - \frac{Y'(nt)}{n^t} \right| = \sup_{0 \leq t \leq n^t} \left| \frac{W(t)}{n^t} - \frac{Y'(t)}{n^t} \right|
\leq \sup_{0 \leq t \leq T(\omega)} |W(t) - Y'(t)|/n^t + n^{-\rho} = O(n^{-\rho}).
\]

Theorem 2 follows from Theorem 1, the above relation and the fact that the sequence \( \{Y_n(t) : n \in \mathbb{N} \} \) has the same distribution as \( \{Y'(nt)/n^t : n \in \mathbb{N} \} \).

3. **Concluding remarks.** It seems to be difficult to obtain lower class statements, e.g., to derive conditions on \( \varepsilon_n \downarrow 0 \) such that (with notation of Theorem 1) \( P(W_n/(2 \text{ LL} n)^{1/2}) \notin K_{1}^{\ast} \) infinitely often) = 1. It would be interesting to know if \( \varepsilon_n = (\text{ LL} n)^{-\alpha} \) for \( \alpha \geq \frac{1}{2} \) belongs to that class. It is a fairly trivial consequence of the well-known integral tests for lower-class functions of sums of independent random variables that \( \varepsilon_n = (\text{ LL} n)^{-1} \) is lower class. Indeed if \( W_n/(2 \text{ LL} n)^{1/2} > 1 + \varepsilon_n \) then \( W_n/(2 \text{ LL} n)^{1/2} \notin K_{1}^{\ast} \), but from the Kolmogorov–Petrovskii–Erdös test it follows that \( P(W_n(1) > (2 \text{ LL} n)^{1/2}(1 + 1/\text{ LL} n) \text{ i.o.}) = 1 \). On the other hand \( P(W_n(1) > (2 \text{ LL} n)^{1/2}(1 + (\text{ LL} n)^{-\alpha}) \text{ i.o.}) = 0 \) for \( \alpha < 1 \). But of course \( W_n \) may escape from \( (2 \text{ LL} n)^{1/2} K_{1}^{\ast} \) in another way than “through” \( W_n(1) \). So there remains a gap for \( \frac{1}{2} \leq \alpha < 1 \).

**REFERENCES**


