Year: 2010

Competitive equilibria in semi-algebraic economies

Kubler, F; Schmedders, K

Kubler, F; Schmedders, K (2010). Competitive equilibria in semi-algebraic economies. Journal of Economic Theory, 145(1):301-330.
Postprint available at:
http://www.zora.uzh.ch

Originally published at:
Journal of Economic Theory 2010, 145(1):301-330.

Competitive Equilibria in Semi-Algebraic Economies*

Felix Kubler
Swiss Banking Institute
University of Zurich
and
Swiss Finance Institute
kubler@isb.uzh.ch

Karl Schmedders
Institute for Operations Research
University of Zurich
and
Swiss Finance Institute
karl.schmedders@ior.uzh.ch

September 16, 2009

Abstract

This paper develops a method to compute the equilibrium correspondence for exchange economies with semi-algebraic preferences. Given a class of semi-algebraic exchange economies parameterized by individual endowments and possibly other exogenous variables such as preference parameters or asset payoffs, there exists a semi-algebraic correspondence that maps parameters to positive numbers such that for generic parameters each competitive equilibrium can be associated with an element of the correspondence and each endogenous variable (i.e. prices and consumptions) is a rational function of that value of the correspondence and the parameters.

This correspondence can be characterized as zeros of a univariate polynomial equation that satisfy additional polynomial inequalities. This polynomial as well as the rational functions that determine equilibrium can be computed using versions of Buchberger's algorithm which is part of most computer algebra systems. The computation is exact whenever the input data (i.e. preference parameters etc.) are rational. Therefore, the result provides theoretical foundations for a systematic analysis of multiplicity in applied general equilibrium.

JEL classification numbers: C02, D51, D52, D58.

Keywords: Semi-algebraic preferences, equilibrium correspondence, polynomial equations, Gröbner bases, equilibrium multiplicity.

*We thank seminar participants at various universities and conferences and in particular Alex Citanna, Egbert Dierker, George Mailath, Andreu Mas-Colell, Paolo Siconolfi and Harald Uhlig for helpful comments. We thank Gerhard Pfister for help with SINGULAR and are indebted to Gerhard Pfister and Bernd Sturmfels for patiently answering our questions on computational algebraic geometry. We are grateful to two anonymous referees for excellent reports on earlier drafts.

1 Introduction

This paper examines the equilibrium correspondence of exchange economies with semi-algebraic preferences. For a typical economy all equilibria are among the finitely many solutions of a square system of polynomial equations. We apply methods from computational algebraic geometry to obtain a univariate polynomial that describes a semi-algebraic correspondence from exogenous parameters to the positive real line. Equilibrium allocations and prices are rational functions of parameters and values of the correspondence. Since the semi-algebraic correspondence is described by a univariate polynomial we call this equivalent characterization of the equilibrium correspondence the *univariate polynomial representation*.

We repeat the summary of our main results more formally. Consider a class of semi-algebraic exchange economies parameterized by elements in a set $\Xi \subset \mathbb{R}^m$. Our univariate polynomial representation then consists of a semi-algebraic correspondence $\rho : \Xi \rightrightarrows \mathbb{R}$ (described by a univariate polynomial and polynomial inequalities) as well as rational functions mapping from $\Xi \times \mathbb{R}$ to the endogenous variables (allocations and prices). For generic parameters, $\xi \in \Xi$, prices and consumption allocations are a competitive equilibrium if and only if there is an element $y \in \rho(\xi)$ such that the rational functions evaluated at the vector $(\xi, y) \in \Xi \times \mathbb{R}$ yield exactly these prices and allocations. Moreover, for fixed ξ the rational functions are polynomial in y.

Our proof of these results proceeds in two main steps. We first show that the assumption of semi-algebraic preferences allows us to characterize all equilibria as those solutions to a square polynomial system of equations that also satisfy a finite number of polynomial inequalities. In the second main step we then apply the 'Shape Lemma' from computational algebraic geometry (see Cox et al. (1997) or Sturmfels (2002)) to our system. The Shape Lemma states conditions under which a square polynomial system of equations can be transformed to an equivalent system of polynomial equations that has the same set of solutions but is of much simpler form. In particular, the new simple system consists of one polynomial equation in one selected variable and then sets each remaining variable equal to a polynomial expression of this selected variable.

The new representation of the original polynomials is a so-called *Gröbner basis*. We verify that for our polynomial system of equilibrium equations the conditions for the Shape Lemma hold for almost all parameter values. We therefore obtain the univariate polynomial representation of the equilibrium system as part of a Gröbner basis of this system. We can compute Gröbner bases in finitely many steps by Buchberger's algorithm (see Cox et al. (1997)). This algorithm is a cornerstone of computational algebraic geometry. In this paper we use a variation of this algorithm as implemented in the computer algebra system SINGULAR (see Greuel et al. (2005)), which is available free of charge at www.singular.uni-kl.de. We compute a variety of examples to illustrate our results.

A nice aspect of the implementation of Buchberger's algorithm is that all computations are exact if all input parameters are rational numbers. We obtain the Gröbner basis for our system without any rounding errors. This feature enables us to use the computed Gröbner bases in mathematical proofs. We encounter rounding errors only when we compute the roots of the univariate polynomial in order to explicitly compute equilibrium values.

Although the main contribution of this paper is theoretical, we emphasize its practical relevance. Applied general equilibrium models are ubiquitous in many areas of modern economics, in particular in macroeconomics, public finance and international trade. The results of this paper can be employed to explore uniqueness of equilibria in these models. The usefulness of the predictions of general equilibrium models and the ability to perform sensitivity analysis are seriously challenged in the presence of multiple equilibria. It is now well understood in general equilibrium analysis that sufficient assumptions for the global uniqueness of competitive equilibria are too restrictive to be applicable to models used in practice. However, it remains an open problem whether multiplicity of equilibria is a problem that is likely to occur in realistically calibrated models. With the univariate polynomial representation at hand the computation of all competitive equilibria reduces to finding all roots of a univariate polynomial – numerically a very simple task. Moreover, examining this univariate polynomial, we can sometimes find relatively tight bounds on the maximal number of equilibria.

For simplicity we first present our results for a static finite Arrow-Debreu economy and limit parameters to be individual endowments. We then show how the results extend to arbitrary parameters as long as these contain endowments of at least one agent. We can also easily apply the developed tools to expanded versions of the model that are often of more interest to applied researchers. To illustrate this point we apply our tools to exchange economies under uncertainty with possibly incomplete asset markets (see e.g. Magill and Quinzii (1996)). In this model it is notoriously difficult to approximate even one equilibrium with standard numerical techniques (see e.g. Brown et al. (1996) and Kubler and Schmedders (2000)). Somewhat surprisingly, we can avoid these problems for semi-algebraic economies.

The paper is organized as follows. In Section 2 we describe the basic semi-algebraic economy and motivate the assumptions. Section 3 contains the main result and its proof. Section 4 illustrates the main result with some examples. In Section 5 we show how to extend the analysis to models with incomplete asset markets. Section 6 concludes.

2 Semi-algebraic Exchange Economies

Before we describe the economic model we first need to review some mathematical facts from algebraic geometry. The first subsection defines polynomials and summarizes those properties of semi-algebraic functions that are relevant for the model description. We refer the reader to the excellent book by Bochnak et al. (1998) for an exhaustive treatment of real algebraic geometry. The second subsection then defines semi-algebraic economies and discusses the economic implications of our preference assumption (see Blume and Zame (1992) for an early application of this assumption).

2.1 Mathematical Preliminaries I: Polynomial Rings and Semi-Algebraic Sets and Functions

For the description of a polynomial f in the n variables $x_1, x_2, \ldots, x_n$ we first define monomials. A monomial in $x_1, x_2, \ldots, x_n$ is a product $x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_n^{\alpha_n}$ where all exponents $\alpha_i,\ i = 1, 2, \ldots, n$, are non-negative integers. It will be convenient to write a monomial as $x^\alpha \equiv x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_n^{\alpha_n}$ with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, the set of non-negative integer vectors of dimension n. A polynomial is a linear combination of finitely many monomials with coefficients in a field $\mathbb{K}$. We can write a polynomial f as

$$f(x) = \sum_{\alpha \in S} a_\alpha x^\alpha, \quad a_\alpha \in \mathbb{K}, \quad S \subset \mathbb{Z}_+^n \text{ finite.}$$

We denote the collection of all polynomials in the variables $x_1, x_2, \ldots, x_n$ with coefficients in the field $\mathbb{K}$ by $\mathbb{K}[x_1, \ldots, x_n]$, or, when the dimension is clear from the context, by $\mathbb{K}[x]$. The set $\mathbb{K}[x]$ satisfies the properties of a commutative ring and is called a polynomial ring. Commonly used examples of fields are the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the field of complex numbers $\mathbb{C}$. A function h is called rational if there are polynomials $f, g \in \mathbb{K}[x_1, \ldots, x_n]$ such that $h = f/g$ where g is not the zero polynomial.

A polynomial $f \in \mathbb{K}[x]$ is irreducible over $\mathbb{K}$ if f is non-constant and is not the product of two non-constant polynomials in $\mathbb{K}[x]$. Every non-constant polynomial $f \in \mathbb{K}[x]$ can be written uniquely (up to constant factors and permutations) as a product of irreducible polynomials over $\mathbb{K}$, see Cox et al. (1997). Once we collect the irreducible polynomials which only differ by constant multiples of one another, we can write f in the form $f = f_1^{a_1} \cdot f_2^{a_2} \cdots f_s^{a_s}$, where the polynomials f_i, $i = 1, \ldots, s$, are distinct irreducible polynomials and the exponents satisfy $a_i \geq 1$, $i = 1, \ldots, s$. Being distinct means that for all $i \neq j$ the polynomials f_i and f_j are not constant multiples of each other. The polynomial f is called square-free if $a_1 = a_2 = \ldots = a_s = 1$.

A subset $A \subset \mathbb{R}^n$ is a semi-algebraic subset of $\mathbb{R}^n$ if it can be written as the finite union and intersection of sets of the form $\{x \in \mathbb{R}^n : g(x) > 0\}$ or $\{x \in \mathbb{R}^n : f(x) = 0\}$ where f and g are polynomials in x with coefficients in $\mathbb{R}$, that is, $f, g \in \mathbb{R}[x]$. More valuable for our purposes than this definition is the following lemma. It is a special case of Proposition 2.1.8 in Bochnak et al. (1998) and provides a useful characterization of semi-algebraic sets.

Lemma 1 *Every semi-algebraic subset of $\mathbb{R}^n$ can be written as the finite union of semi-algebraic sets of the form*

$$\{x \in \mathbb{R}^n : f_1(x) = \cdots = f_l(x) = 0,\ g_1(x) > 0, \ldots, g_m(x) > 0\}, \tag{1}$$

where $f_1, \ldots, f_l, g_1, \ldots, g_m \in \mathbb{R}[x]$.

Sets of the form (1) are called basic semi-algebraic sets.

A function (correspondence) $\phi : A \to \mathbb{R}^m$ is semi-algebraic if its graph $\{(x, y) \in A \times \mathbb{R}^m : y = \phi(x)\}$ is a semi-algebraic subset of $\mathbb{R}^{n+m}$. Semi-algebraic functions have many nice properties. For example, if $\phi : A \to \mathbb{R}^m$ is a semi-algebraic mapping then the image $\phi(S)$ of a semi-algebraic subset

$S \subset A$ is also semi-algebraic. Similarly, the preimage $\phi^{-1}(T)$ of a semi-algebraic subset $T \subset \mathbb{R}^m$ is also semi-algebraic. And A itself must be a semi-algebraic set.

A semi-algebraic set A can be decomposed into a finite union of disjoint semi-algebraic sets $(A_i)_{i=1}^p$ where each A_i is (semi-algebraically) homeomorphic to an open hypercube $(0,1)^{d_i}$ for some $d_i \geq 0$, with $(0,1)^0$ being a point, see e.g. Bochnak et al. (1998, Theorem 2.3.6). This decomposition property of semi-algebraic sets naturally motivates the definition of the dimension of such sets. The dimension of the semi-algebraic set A is $\dim(A) = \max\{d_1, \ldots, d_p\}$. For any two semi-algebraic sets A and B it holds that $\dim(A \times B) = \dim(A) + \dim(B)$.

We say a property holds generically in a semi-algebraic set A if it holds everywhere except in a closed lower-dimensional subset of A. This notion of genericity is slightly different than the notion based on transversality theory typically used in economic theory, see for example Mas-Colell (1985).

The definition of dimension allows us now to state another important property of semi-algebraic functions. Let $A \subset \mathbb{R}^n$ and $\phi : A \to \mathbb{R}$ be a semi-algebraic function. Then ϕ is real analytic, hence infinitely often differentiable (smooth) outside a semi-algebraic subset of A of dimension less than n.

2.2 Semi-algebraic Economies

We consider standard finite Arrow-Debreu exchange economies with H individuals, $h \in \mathcal{H} = \{1, 2, \ldots, H\}$, and L commodities, $l = 1, 2, \ldots, L$. Consumption sets are $\mathbb{R}^L_+$, prices are denoted by $p \in \mathbb{R}^L_+$. Each individual h is characterized by endowments, $e^h \in \mathbb{R}^L_{++}$, and a utility function, $u^h : \mathbb{R}^L_+ \to \mathbb{R}$.

A competitive equilibrium consists of prices p and an allocation $(c^1, \ldots, c^H)$ such that

$$c^h \in \arg\max_{c \in \mathbb{R}^L_+} u^h(c) \text{ s.t. } p \cdot (c - e^h) \leq 0, \qquad \text{for all } h \in \mathcal{H},$$

and

$$\sum_{h \in \mathcal{H}} (c^h - e^h) = 0.$$

We denote the profile of endowments across individuals by $e^{\mathcal{H}} = (e^1, \ldots, e^H) \in \mathbb{R}^{HL}_{++}$ and similarly denote allocations by $c^{\mathcal{H}} \in \mathbb{R}^{HL}_+$ and define $\lambda^{\mathcal{H}} = (\lambda^1, \ldots, \lambda^H)$. We assume that for each agent $h \in \mathcal{H}$, u^h is C^1 on $\mathbb{R}^L_{++}$, strictly increasing and strictly concave. We also assume that for each agent h the gradient $\partial_c u^h(c) \gg 0$ is a semi-algebraic function. We discuss this assumption in detail below. As Blume and Zame (1992) point out, one can show generic local uniqueness of equilibrium prices in semi-algebraic economies without the assumption of differentiability or strict concavity. However, it is easy to see in an economy with Leontief utility that consumption allocations are not generically locally unique without strict concavity. Since we work on the system of first order conditions and market clearing equations involving both consumptions and prices we cannot dispense with strict concavity.

We define an interior Walrasian equilibrium to be a strictly positive solution $\left(c^{\mathcal{H}}, \lambda^{\mathcal{H}}; p\right)$ of the following system of equations.

$$\begin{aligned}
\partial_c u^h(c^h) - \lambda^h p &= 0, \quad \forall h \in \mathcal{H} && (2)\\
p \cdot (c^h - e^h) &= 0, \quad \forall h \in \mathcal{H} && (3)\\
\sum_{h \in \mathcal{H}} (c_l^h - e_l^h) &= 0, \quad l = 1, \ldots, L-1 && (4)\\
\sum_{l=1}^{L} p_l - 1 &= 0 && (5)
\end{aligned}$$

Equations (2) and (3) are the first-order conditions for the agents' utility maximization problem, equations (4) are the market-clearing conditions for all but the last good, and equation (5) is a standard price normalization.

An economy is called regular if at all Walrasian equilibria, $\partial_c u^h(c^h)$ is differentiable for all h and if the Jacobian of this system of equations (2)–(5) has full rank.

From now on, we use the terms equilibria and interior equilibria exchangeably. We again emphasize that we only focus on interior equilibria of a standard finite Arrow-Debreu exchange economy for ease of exposition. The ideas and results of this paper apply to more general models, see the examples in Section 5.

2.2.1 Economic Implications of Semi-algebraic Marginal Utility

How general is the premise of semi-algebraic marginal utility? From the practical point of view of applied modeling, Cobb-Douglas and CES utility functions with elasticities of substitution being rational numbers, are semi-algebraic utility functions. Therefore, a large number of interesting applied economic models satisfy our assumption.

From a theoretical point of view, note that if a function is semi-algebraic, so are all its derivatives (the converse is not true, as the example $f(x) = \log(x)$ shows). It follows from Blume and Zame (1992) that semi-algebraic preferences (i.e. better sets are semi-algebraic sets) implies semi-algebraic utility.

Also note that by Afriat's theorem (Afriat (1967)) any finite number of observations on Marshallian individual demand that can be rationalized by arbitrary non-satiated preferences can be rationalized by a piecewise linear, hence semi-algebraic function. While Afriat's construction does not yield a semi-algebraic, C^1, and strictly concave function, we can modify the construction in Chiappori and Rochet (1987) for our framework to obtain the following lemma.

Lemma 2 *Given N observations $(c^n, p^n) \in \mathbb{R}^{2L}_{++}$ with $p^i \neq p^j$ for all $i \neq j = 1, \ldots, N$, the following are equivalent.*

(1) There exists a strictly increasing, strictly concave and continuous utility function u such that

$$c^n = \arg\max_{c \in \mathbb{R}^L_+} u(c) \text{ s.t. } p^n \cdot c \leq p^n \cdot c^n.$$

(2) There exists a strictly increasing, strictly concave, semi-algebraic and C^1 utility function v such that

$$c^n = \arg\max_{c \in \mathbb{R}^L_+} v(c) \textit{ s.t. } p^n \cdot c \leq p^n \cdot c^n.$$

To prove the lemma, observe that if statement (1) holds, the observations must satisfy the condition 'SSARP' from Chiappori and Rochet (1987). Given this fact one can follow their proof closely to show that there exists a C^1 semi-algebraic utility function that rationalizes the data. The only difference to their proof is that in the proof of their Lemma 2, one needs to use a polynomial 'cap'-function which is at least C^1. In particular, the argument in Chiappori and Rochet goes through if one replaces C^∞ everywhere with C^1 and uses the cap-function $\rho(c) = \max(0, 1 - \sum_l c_l^2)^2$. Since the integral of a polynomial function is polynomial, the resulting utility function is piecewise polynomial, i.e. semi-algebraic.

Mas-Colell (1977) shows, in light of the theorems of Sonnenschein, Mantel and Debreu, that for any compact (non-empty) set of positive prices $P \subset \Delta^{L-1}$ there exists an exchange economy (without uncertainty) with (at least) L households, $((u^h)_{h=1}^L, (e^h)_{h=1}^L)$, with u^h strictly increasing, strictly concave and continuous such that the equilibrium prices of this economy coincide precisely with P. Given Lemma 2 above, this result directly implies that for any finite set of prices $P \subset \Delta$, there exists an exchange economy $((u^h)_{h=1}^L, (e^h)_{h=1}^L)$, with u^h strictly increasing, strictly concave, semi-algebraic and C^1 such that the set of equilibrium prices of this economy contains P. Therefore, the abstract assumption of semi-algebraic preferences imposes *no* restrictions on multiplicity of equilibria. Mas-Colell (1977) also shows that if the number of equilibria is odd, one can construct a regular economy and that there exist open sets of individual endowments for which the number of equilibria can be an arbitrary odd number.

Finally note that the results we obtain below are robust with respect to perturbations of preferences outside of the semi-algebraic class: If a semi-algebraic utility is C^2, and a regular economy has n equilibria, it follows from Smale (1974) that there is a C^2 Whitney-open neighborhood around the profile of utilities for which the number of equilibria is n.

In sum, our key assumption of semi-algebraic utility offers little if any room for objection. Much applied work in economics assumes semi-algebraic utility. Utility functions derived from demand observations are semi-algebraic.

2.2.2 Tarski-Seidenberg Principle

Semi-algebraic economies are theoretically appealing because of the Tarski-Seidenberg Principle (see e.g. Brown and Kubler (2008) for applications of real algebraic methods in economics). The principle, see e.g. Bochnak et al. (1998, Chapter 5), implies that it is 'decidable' whether a given semi-algebraic economy has one or multiple equilibria. Algorithmic quantifier elimination (see Basu et al. 2003) provides an algorithm to do so. In this subsection we explain how *theoretically* algorithmic quantifier elimination can be used to compute the number of competitive equilibria for any semi-algebraic economy. However, it is practically infeasible to implement this theoretical algorithm even for very small problems. This fact motivates us to reformulate the problem of determining the number of equilibria to solving a system of polynomial equations and to consider algorithms from computational algebraic geometry that find all solutions of polynomial systems of equations.

The next lemma follows directly from the Tarski-Seidenberg Principle, see e.g. Bochnak et al. (1998, Chapter 5).

LEMMA 3 *Given any semi-algebraic set* X, *with* $(x_0, x_1) \in X \subset \mathbb{R}^{l_0} \times \mathbb{R}^{l_1}$, *define*

$$\Phi = \{x_0 \mid \exists\ x_1\ [(x_0, x_1) \in X]\}.$$

The set Φ *is itself a semi-algebraic set and can therefore be written as the finite union of basic semi-algebraic sets of the form (1) as in Lemma 1.*

The lemma implies immediately that in our framework demand functions are semi-algebraic; their graphs can be described by $\{(c, p) \mid \exists\ \lambda\ [\partial_c u(c) - \lambda p = 0 \text{ and } p \cdot (c - e^h) = 0]\}$. Of course, in this case it is trivial to eliminate the quantifier by simply eliminating λ.

More interestingly, the lemma also implies that for each $n = 2, 3, \ldots$ the set

$$E_i = \{e^{\mathcal{H}} \in \mathbb{R}^{HL}_{++}\ :\ \exists\ \left(c_i^{\mathcal{H}}, \lambda_i^{\mathcal{H}}, p_i\right), i = 1, \ldots, n, \text{ that solve (2) – (5)} \\ \text{and } \left(c_i^{\mathcal{H}}, \lambda_i^{\mathcal{H}}, p_i\right) \neq \left(c_{i'}^{\mathcal{H}}, \lambda_{i'}^{\mathcal{H}}, p_{i'}\right) \text{ for all } i, i'\}$$

is a semi-algebraic set. If we knew the sets E_i we could easily determine the number of Walrasian equilibria for the economy with endowments $e^{\mathcal{H}}$. While quantifier elimination provides an algorithm for computing these sets, this approach is hopelessly inefficient. Surprisingly it turns out that, using tools from computational algebraic geometry, we can be much more efficient. This insight provides the basis of our strategy to finding all Walrasian equilibria. First we need to characterize equilibria by a system of polynomial equations.

3 The Main Result

In this section we state and prove two versions of our main result. First, we consider the case where the set of exogenous parameters consists of the profile of individual endowments. In Section 3.4 we extend the theorem to arbitrary parameters and state a corollary that links these results to the motivation in the introduction.

To simplify the statement of the following theorem, let $M = H(L + 1) + L$ and associate with $x \in \mathbb{R}^M$ the vector $(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p)$.

THEOREM 1 *For generic profiles of individual endowments,* $e^{\mathcal{H}} \in \mathbb{R}^{HL}_{++}$, *every competitive equilibrium* x^* *of the economy along with an accompanying positive number* y^* *is among the finitely many common zeros of the polynomials in a set* $\mathcal{G}$ *of the shape*

$$\mathcal{G} = \left\{x_1 - v_1(e^{\mathcal{H}}; y),\ \ldots\ ,\ x_M - v_M(e^{\mathcal{H}}; y),\ r(e^{\mathcal{H}}; y)\right\}. \tag{6}$$

The non-zero polynomial $r \in \mathbb{R}[e^{\mathcal{H}}; y]$ is not constant in y. Moreover, each v_i, $i = 1, \ldots, M$, is a polynomial in y of degree less than the degree of r. The coefficients of this polynomial are rational functions of $e^{\mathcal{H}}$.

We refer to the set $\mathcal{G}$ as the univariate polynomial representation (UPR) of the equilibrium correspondence. The set $\mathcal{G}$ consists of one univariate polynomial r in the single variable y and M very simple polynomials in two unknowns. The polynomial $x_i - v_i(e^{\mathcal{H}}; y)$ is linear in the variable x_i and consists otherwise only of the univariate polynomial v_i in the variable y. An immediate consequence of this special structure of the UPR is that once a positive solution y to the equation $r(e^{\mathcal{H}}; y) = 0$ has been determined the accompanying values for all variables $x_1, x_2, \ldots, x_M$ can be read off the functions $v_i(e^{\mathcal{H}}; y)$, $i = 1, \ldots, M$.

Note that the conclusions of Theorem 1 do not hold for all endowment profiles $e^{\mathcal{H}} \in \mathbb{R}^{HL}_{++}$ but only for generic endowments, that is, according to our convention, for all endowments in the complement of a closed lower-dimensional subset $E_0 \subset \mathbb{R}^{HL}_{++}$.

In the remainder of this section we prove and extend Theorem 1. The proof proceeds in two main steps. We first transform equations (2) – (5) into a square polynomial system of equations and some additional polynomial inequalities. Subsequently we use results from computational algebraic geometry to transform the derived square polynomial system into an equivalent univariate polynomial representation $\mathcal{G}$.

3.1 From Competitive Equilibrium to Polynomial Equations

For the transformation of equations (2) – (5) into a square polynomial system of equations and some additional polynomial inequalities we need some results from algebraic geometry. We first state these results and then apply them to our equilibrium equations.

3.1.1 Mathematical Preliminaries II: Some Results from Real Algebraic Geometry

Lemma 4 *Let $A \subset \mathbb{R}^n$ and $\phi : A \to \mathbb{R}$ be a semi-algebraic function. Then there exists a nonzero polynomial $f(x, y)$ in the variables $x_1, \ldots, x_n, y$ with $f \in \mathbb{R}[x, y]$ such that for every $x \in A$ it holds that $f(x, \phi(x)) = 0$.*

Proof. Lemma 1 states that the graph of the semi-algebraic function $\phi : A \to \mathbb{R}$ is the finite union of basic semi-algebraic sets, each of which is of the form

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R} : f_1(x, y) = \cdots = f_l(x, y) = 0,\ g_1(x, y) > 0, \ldots, g_m(x, y) > 0\}.$$

Note that in each basic semi-algebraic set at least one of the polynomials f_i must be nonzero, since otherwise the set would be open, which in turn would imply that the graph of ϕ contains a nonempty open subset of $\mathbb{R}^{n+1}$. But that would contradict the fact that ϕ is a function. Now consider the product f of all nonzero polynomials f_i across all basic semi-algebraic sets. This product is itself a nonzero polynomial and it satisfies $f(x, \phi(x)) = 0$. □

The next lemma is a simple consequence of Hardt's Triviality Theorem, see Bochnak et al. (1998, Theorem 9.3.2) or Basu et al. (2003, Theorem 5.45). For applications of this theorem in economics, see Blume and Zame (1992).

Lemma 5 *Let $A \subset \mathbb{R}^n$ and $\phi : A \to \mathbb{R}^k$ be a continuous semi-algebraic function. Then there is a finite partition of $\mathbb{R}^k$ into semi-algebraic sets $C_1, \ldots, C_m$ such that for each C_i and every $b \in C_i$*

$$\textit{dim}\ \phi^{-1}(b) = \textit{dim}\ \phi^{-1}(C_i) - \textit{dim}\ (C_i) \leq \textit{dim}\ (A) - \textit{dim}\ (C_i),$$

where negative dimension means the set is empty. In fact, the partition can be chosen such that the union of all C_i with dim $(C_i) < k$ is a closed subset of $\mathbb{R}^k$.

We also need a special case of the semi-algebraic version of Sard's theorem (see e.g. Bochnak et al. (1998), Theorem 9.6.2 for a general statement of the result).

Lemma 6 *Let $N \subset \mathbb{R}^n$ be open and $\phi : N \to \mathbb{R}^n$ be a C^∞ semi-algebraic function. Then the set of $y \in \mathbb{R}^n$ for which there exists an $x \in N$ with $\phi(x) = y$ and $\det(\partial_x \phi(x)) = 0$ is a semi-algebraic subset of $\mathbb{R}^n$ of dimension strictly smaller than n.*

As an application of the Tarski-Seidenberg Principle in combination with Hardt Triviality we prove the following result which is used in our analysis below.

Lemma 7 *Let $E \subset \mathbb{R}^l$ be an open semi-algebraic set. Suppose that a semi-algebraic function $\phi : E \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with $n \geq 1$ has finitely many zeros for each $e \in E$. Then for each μ outside a closed lower-dimensional subset $D_0 \subset \Delta^{n-1}$ there exists a closed lower-dimensional subset $E_0 \subset E$ such that for all $e \in E \setminus E_0$ there cannot be $(x', y') \neq (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $\phi(e; x, y) = \phi(e; x', y') = 0$ and*

$$\sum_{i=1}^n \mu_i x_i = \sum_{i=1}^n \mu_i x_i', \quad \sum_{i=1}^n \mu_i y_i = \sum_{i=1}^n \mu_i y_i'.$$

Proof. Consider the set

$$A = \Big\{ e \in E, \mu \in \Delta^{n-1} \ : \ \exists (x, y) \neq (x', y') \quad \phi(e; x, y) = \phi(e; x', y') = 0 \quad \text{and} \quad \sum_{i=1}^n \mu_i x_i = \sum_{i=1}^n \mu_i x_i', \quad \sum_{i=1}^n \mu_i y_i = \sum_{i=1}^n \mu_i y_i' \Big\}.$$

Lemma 3 implies that the set A is semi-algebraic. Under the assumption that $\phi(e; \cdot, \cdot)$ has only finitely many zeros it follows that $\dim(A) \leq l + n - 2$. Consider the projection of A onto Δ^{n-1}, $g : A \to \Delta^{n-1}$ with $g(e, \mu) = \mu$. This is a continuous semi-algebraic function. Lemma 5 ensures that Δ^{n-1} can be partitioned into a finite collection of semi-algebraic sets C_j, $j = 1, 2, \ldots, m$, such that for each $\mu \in C_j$, $\dim\left(g^{-1}(\mu)\right) \leq \dim(A) - \dim(C_i)$. Let D_0 be the union of all C_j with $\dim(C_j) < n - 1$. Lemma 5 states that D_0 is closed and that for any $\mu \in \Delta^{n-1} \setminus D_0$ it holds that

$\dim\left(g^{-1}(\mu)\right) \leq \dim(A) - \dim(\Delta^{n-1} \setminus D_0) \leq l + n - 2 - (n-1) = l - 1$. Therefore, the dimension of the corresponding set of parameters e must be less than l. Define the set E_0 as its closure. Proposition 2.8.2 in Bochnak et al. (1998) ensures that the closure has the same dimension less than l. □

3.1.2 Polynomial Equilibrium Equations

The central objective of this paper is to characterize equilibria as solutions to a polynomial system of equations. Recall that interior Walrasian equilibria of our model are defined as solutions to the system of equations (2)–(5). Obviously equations (2) are often not polynomial – even under our fundamental assumption that marginal utilities are semi-algebraic functions. This assumption, however, allows us to transform these equations into polynomial expressions. Unfortunately this transformation comes at the price of some technical difficulties.

The marginal utility $\partial_{c_l} u^h : \mathbb{R}^L_{++} \to \mathbb{R}$ is semi-algebraic. Lemma 4 then ensures the existence of a nonzero polynomial $m^h_l(c, y)$ with $m^h_l \in \mathbb{R}[c, y]$ such that for every $c \in \mathbb{R}^L_{++}$,

$$m^h_l(c, \partial_{c_l} u^h(c)) = 0. \tag{7}$$

Without loss of generality we can assume the polynomial m^h_l to be square-free. In a slight abuse of notation we define $m^h(c, \partial_c u^h(c)) = (m^h_1(c, \partial_{c_1} u^h(c)), \ldots, m^h_L(c, \partial_{c_L} u^h(c)))$. We use the implicit representation (7) of marginal utility to transform each individual equation of system (2),

$$\partial_{c_l} u^h(c^h) - \lambda^h p_l = 0, \tag{8}$$

into the polynomial equation

$$m^h_l(c^h, \lambda^h p_l) = 0. \tag{9}$$

By construction any solution to (8) also satisfies (9). Define the polynomial $F \in \mathbb{R}[c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p]$ by

$$F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) = \begin{cases} m^h(c^h, \lambda^h p), & h \in \mathcal{H} \\ p \cdot (c^h - e^h), & h \in \mathcal{H} \\ \sum_{h \in \mathcal{H}} (c^h_l - e^h_l), & l = 1, \ldots, L-1 \\ \sum_l p_l - 1 & \end{cases}$$

Instead of focusing on the equilibrium system (2)–(5) our attention now turns to the system of equations $F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) = 0$. This system has the original equations (2) replaced by polynomial equations of the form (9) but otherwise continues to include the original equations (3)–(5). Therefore, this system consists only of polynomial equations.

Note that the polynomials $m^h_l(c, y)$ are derived from marginal utilities using the construction in the proof of Lemma 4. Thus, for a solution of $F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) = 0$ to be an equilibrium it must also satisfy any inequalities that are suppressed in the construction process of the polynomials $m^h_l(c, y)$. Clearly checking many inequalities for an elaborately defined marginal utility will result in additional combinatorial complexity of finding equilibria.

For the proof of the main result we need to ensure that the Jacobian matrix $\partial_{c^{\mathcal{H}},\lambda^{\mathcal{H}},p}F$ has full rank at all Walrasian equilibria. We establish this fact in a generic sense. This property does not follow directly from Debreu's theorem on generic local uniqueness because we have replaced the marginal utilities by the polynomials m^h. To prove our result, we establish in Proposition 1 that for all consumption values c outside a lower-dimensional "bad" set $\partial_y m_l^h(c, \partial_{c_l} u^h(c)) \neq 0$ and the implicit function theorem can be applied. Proposition 2 then establishes that for almost all endowments all Walrasian equilibrium allocations lie outside the bad set. This property finally allows us to prove Proposition 3 which states that for almost all endowment vectors all Walrasian equilibria are regular solutions of the polynomial system $F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) = 0$.

Proposition 1 *Consider square-free nonzero polynomials m_l^h satisfying equation (7) for $l = 1, \ldots, L$, $h \in \mathcal{H}$. Then the following statements hold.*

(1) The dimension of the set $V_l^h = \left\{(c, y) \in \mathbb{R}_{++}^L \times \mathbb{R} : m_l^h(c, y) = 0\right\}$ is L.

(2) The set

$$S_l^h = \{(c, y) \in \mathbb{R}_{++}^L \times \mathbb{R} : m_l^h(c, y) = \partial_{c_1} m_l^h(c, y) = \partial_{c_2} m_l^h(c, y) = \ldots \\ \ldots = \partial_{c_L} m_l^h(c, y) = \partial_y m_l^h(c, y) = 0\}$$

is a closed semi-algebraic subset of $\mathbb{R}_{++}^L \times \mathbb{R}$ with dimension of at most $L - 1$. The projection of S_l^h on $\mathbb{R}_{++}^L$ is also a closed semi-algebraic subset with dimension of at most $L - 1$.

(3) The set

$$\bigcup_{l=1}^{L} \left\{c \in \mathbb{R}_{++}^L : \partial_y m_l^h(c, \partial_{c_l} u^h(c)) = 0\right\}$$

is a closed semi-algebraic subset of $\mathbb{R}_{++}^L$ with a dimension of at most $L - 1$. Put differently, at every point of the complement of a closed lower-dimensional semi-algebraic subset of $\mathbb{R}_{+}^L$ it holds that $\partial_y m_l^h(c, \partial_{c_l} u^h(c)) \neq 0$ for all $l = 1, \ldots, L$, and therefore u^h is C^∞.

(4) The set

$$B^h = \left\{c \in \mathbb{R}_{++}^L : \det\left(\partial_c m^h(c, \partial_c u^h(c))\right) = 0\right\}$$

is a closed semi-algebraic subset of $\mathbb{R}_{++}^L$ with a dimension of at most $L - 1$.

Proof. Statement (1) follows by construction of m_l^h since the marginal utility function $\partial_{c_l} u^h$ is defined for all $c \in \mathbb{R}_{++}^L$. Thus, for all $c \in \mathbb{R}_{++}^L$ there is a $y \in \mathbb{R}$ satisfying $m_l^h(c, y)$. The dimension of V_l^h cannot be $L + 1$ since m_l^h is a nonzero polynomial. Statement (2) follows from m_l^h being square-free and the fact that the projection of a semi-algebraic set is itself semi-algebraic.

Marginal utility $\partial_c u^h$ is a semi-algebraic function and thus C^∞ at every point of the complement of a closed semi-algebraic subset of $\mathbb{R}_{++}^L$ of dimension less than L. The implicit function theorem implies that at a point $\bar{c}$ with $\partial_y m_l^h(\bar{c}, \partial_{c_l} u^h(\bar{c})) \neq 0$ the function $\partial_{c_l} u^h$ is C^∞. The implicit function theorem also implies that at a point $\bar{c}$ with $\partial_y m_l^h(\bar{c}, \partial_{c_l} u^h(\bar{c})) = 0$ the function $\partial_{c_l} u^h$ can be C^∞

only if $\partial_{c_k} m_l^h(c, \partial_{c_l} u^h(c)) = 0$ for all $k = 1, \ldots, L$. Statement (2) implies that this property can hold only in a semi-algebraic set with dimension of at most $L-1$. The finite union of semi-algebraic sets of dimension less than L is again just that, a semi-algebraic sets with dimension of at most $L-1$. Thus, Statement (3) holds.

Utility u^h is strictly concave and so $\partial_c u^h$ is strictly decreasing. Moreover, outside a closed lower-dimensional set u^h is differentiably strictly concave, that is, the Hessian $\partial_{cc} u^h$ is negative definite. Statement (3) and the implicit function theorem then imply rank $[\partial_{cc} u^h] =$ rank $[\partial_c m^h] = L$ and thus Statement (4). □

We illustrate some of the possible complications in the proof of Proposition 1 in the context of an example.

EXAMPLE 1 *Consider the continuous function*

$$u'(c) = \begin{cases} \frac{4}{\sqrt{c}} & 0 < c \leq 1, \\ 6 - 2c & 1 < c \leq 2, \\ \frac{4}{c} & 2 < c. \end{cases}$$

The polynomial $m(c, y) = (16 - cy^2)(6 - 2c - y)(4 - cy)$ *satisfies* $m(c, u'(c)) = 0$ *for all* $c > 0$.

Unfortunately, for all values of c the equation $m(c, y) = 0$ allows positive solutions other than $y = u'(c)$. For example, for $c = 4$ not only $y = u'(4) = 1$ but also $y = 2$ yields $m(4, y) = 0$. Intuitively, the solution $(4, 2)$ is on the "wrong" branch of the function. At $(4, 2)$ the term $(16 - cy^2)$ is zero but the domain for this term is only $(0, 1]$. For each value of $c \in \mathbb{R}_{++}$ there are altogether four (real) solutions to the equation $m(c, y) = 0$.

The system $m(c, y) = \partial_c m(c, y) = \partial_y m(c, y) = 0$ has three solutions, $(1, 4)$, $(2, 2)$, and $(4, -2)$. For each value of $c \in \mathbb{R}_{++}$ the partial derivative term $\partial_y m(c, y)$ is a cubic polynomial in y with at most three real solutions. So, the set B of ill-behaved points in the sense of Proposition 1, Statement (4), is finite and thus of dimension $L - 1 = 0$.

This last fact would not be true if the polynomial $m(c, y)$ were not square-free. The polynomial $\tilde{m}(c, y) = (16 - cy^2)(6 - 2c - y)(4 - cy)^2$ has the identical zero set as $m(c, y)$. But note that $\partial_c \tilde{m}(c, y) = \partial_y \tilde{m}(c, y) = 0$ whenever $(4 - cy) = 0$. So, the fact that the polynomials m_l^h are square-free is crucial for our results.

We collect the first two sets of polynomial expressions in $F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p)$ in the 'demand system' and define for each $h \in \mathcal{H}$,

$$D^h(c, \lambda, p) = \begin{pmatrix} m^h(c, \lambda p) \\ p \cdot (c - e^h) \end{pmatrix}.$$

PROPOSITION 2 *For generic profiles of individual endowments,* $e^{\mathcal{H}} \in \mathbb{R}^{HL}_{++}$, *all Walrasian equilibria* $(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p)$ *have the property that for each* $h \in \mathcal{H}$, *the rank of the matrix*

$$\left[\partial_{(c,\lambda)} D^h\left(c^h, \lambda^h, p\right)\right]$$

is $(L+1)$ *and thus is full.*

To simplify the proof of the proposition we make use of individual demand functions. For this purpose we introduce the following notation. The positive price simplex is $\Delta_{++}^{L-1} = \{p \in \mathbb{R}_{++}^L : \sum_l p_l = 1\}$. Individual demand of agent h at prices p and income τ is

$$d^h(p,\tau) = \arg\max_{c \in \mathbb{R}_+^L} u^h(c) \text{ s.t. } p \cdot c = \tau.$$

Individual demand functions are continuous. Lemma 3 ensures that the continuous function $d^h : \Delta_{++}^{L-1} \times \mathbb{R}_{++} \to \mathbb{R}^L$ is also semi-algebraic.

Proof. The individual demand $d^h(p,\tau)$ of agent h is determined by the agent's first-order conditions,

$$\begin{aligned} \partial_c u^h(c^h) - \lambda^h p &= 0, \\ p \cdot c^h - \tau &= 0. \end{aligned}$$

Since $p \in \Delta_{++}^{L-1}$ these equations are equivalent to

$$\begin{aligned} \frac{\partial_c u^h(c^h)}{\sum_l \partial_{c_l} u^h(c^h)} &= p, \\ \sum_l c_l^h \frac{\partial_{c_l} u^h(c^h)}{\sum_{l'} \partial_{c_{l'}} u^h(c^h)} &= \tau. \end{aligned}$$

The function $G : \mathbb{R}_+^L \to \Delta_{++}^{L-1} \times \mathbb{R}_{++}$ given by the expressions on the left-hand side

$$G(c^h) = \begin{cases} \frac{\partial_c u^h(c^h)}{\sum_l \partial_{c_l} u^h(c^h)} \\ \sum_l c_l^h \frac{\partial_{c_l} u^h(c^h)}{\sum_{l'} \partial_{c_{l'}} u^h(c^h)} \end{cases}$$

is a continuous semi-algebraic function. Consider the set B^h from Statement (4) of Proposition 1. This set has dimension of at most $L-1$ and so the same must be true for the semi-algebraic set

$$G\left(B^h\right) = \left\{(p,\tau) \in \Delta_{++}^{L-1} \times \mathbb{R}_{++} : G(c^h) = (p,\tau) \text{ for some } c^h \in B^h\right\}.$$

Next consider the following function from Blume and Zame (1992),

$$H(p,\tau,e^2,\ldots,e^H) = \begin{cases} d^1(p,\tau) + \sum_{h=2}^H (d^h(p, p \cdot e^h) - e^h) \\ e^2 \\ \vdots \\ e^H \end{cases} \tag{10}$$

for $H : G(B^1) \times \mathbb{R}_{++}^{(H-1)L} \to \mathbb{R}_{++}^{HL}$. Note that the domain of H is a semi-algebraic subset with dimension at most $HL-1$. Lemma 5 then ensures the existence of a finite partition of $\mathbb{R}_{++}^{HL}$ into semi-algebraic subsets $C_1, \ldots, C_m$ such that for all subsets C_i of dimension HL and $e \in C_i$ it holds that $H^{-1}(e)$ is empty.

Thus, only for a closed lower-dimensional (that is, generic) subset of endowments it will be true that $c^1 \in B^1$. This argument works for all agents $h \in \mathcal{H}$. The finite union of semi-algebraic subsets of dimension less than HL is again a semi-algebraic subset of dimension less than HL. Therefore, for generic endowment vectors $(e^1, \ldots, e^H)$ all Walrasian equilibria have consumption allocations such that $c^h \notin B^h$ for all $h \in \mathcal{H}$. For such consumption allocation the standard argument for showing that

$$\partial_{(c,\lambda)} D^h \left(c^h, \lambda^h, p \right)$$

has full rank now goes through. □

The following proposition is a consequence of Proposition 2 and Lemma 6, the semi-algebraic version of Sard's Theorem.

Proposition 3 *All Walrasian equilibria are solutions to the system of polynomial equations*

$$F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) = 0. \tag{11}$$

For generic profiles of individual endowments, $e^{\mathcal{H}} \in \mathbb{R}^{HL}_{++}$, all Walrasian equilibria have the property that the rank of the matrix $\left[\partial_{c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p} F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p)\right]$ is $H(L+1) + L$ and thus is full.

Proof. Simply by construction all solutions to (2)–(5) are solutions to system (11).

Proposition 2 and its proof imply that there exists a subset of $\Delta^{L-1}_{++} \times \mathbb{R}_{++} \times \mathbb{R}^{(H-1)L}_{++}$ such that the function H as defined by Equation (10) is C^∞ on this set and the complement of its image in $\mathbb{R}^{HL}_{++}$ is closed and has dimension less than HL. By Lemma 6, the semi-algebraic version of Sard's Theorem, there is a semi-algebraic set $\bar{E} \subset \mathbb{R}^{HL}_{++}$ whose complement is lower dimensional and closed such that for each $e^{\mathcal{H}} \in \bar{E}$, if p is a W.E. price then the matrix $\partial_p \begin{pmatrix} \sum_{h \in \mathcal{H}} d^h(p, p \cdot e^h) \\ 1 - \sum_{l=1}^{L} p_l \end{pmatrix}$ has full rank L. Since by the implicit function theorem and by Proposition 2, at these points for each h,

$$\partial_p d^h(p, p \cdot e^h) = - \left(\partial_{c,\lambda} D^h(c^h, \lambda^h, p) \right)^{-1} \partial_p D^h(c^h, \lambda^h, p),$$

the result follows from a standard argument showing that an equilibrium is regular in the extended system (2) – (5) if and only if it is regular for the demand system $\sum_{h \in \mathcal{H}} (d^h(p, p \cdot e^h) - e^h) = 0$ and $1 - \sum_l p_l = 0$. □

Note that Proposition 3 does not imply that for all $e^{\mathcal{H}} \in \bar{E}$ the Jacobian of F has full rank at all complex solutions. Equation (11) may have complex solutions at which the Jacobian is singular.

3.1.3 All-Solution Homotopy Methods for Polynomial Systems of Equations

We have seen that all Walrasian equilibria of our economic model are among the solutions of a (square) system of polynomial equations. In our analysis below we apply Gröbner basis methods to examine such systems. Before we do so, it is worth noting that other methods for solving systems of polynomial equations exist, most notably elimination methods using resultants, see Cox et al. (1997) and Sturmfels (2002), and homotopy continuation methods, see Sommese and Wampler (2005). Particularly homotopy continuation methods have been successfully applied to finding all solutions of systems of polynomial equations. The currently leading software packages implementing homotopy methods are PHCpack (Verschelde, 1999) and Bertini (Bates et al., 2008).

Gröbner basis methods and homotopy algorithms use distinctly different approaches to finding all solutions and both types of methods have distinct advantages and disadvantages. Homotopy algorithms are purely numerical solution methods which use floating point operations, while Gröbner basis methods use exact computation with rational numbers. As a result homotopy methods usually are much faster and can solve much larger systems, while Gröbner basis methods can be used to derive theoretical results. Homotopy methods cannot handle parameters, while Gröbner basis methods can be used to analyze parameterized systems. In this paper we are interested in theoretical results, particularly in characterizing the equilibrium manifold, and thus choose Gröbner basis methods for the analysis of our problems.

3.2 Mathematical Preliminaries III: Gröbner Bases and the Shape Lemma

The study of systems of polynomial equations using Gröbner basis methods requires us to considerably change the mathematical focus of our discussion. So far our analysis relied heavily on fundamental results from the mathematical discipline of 'Real Algebraic Geometry', notably the Tarski-Seidenberg Principle and the Hardt Triviality Theorem. We now move into the discipline of '(Computational) Algebraic Geometry' and use Gröbner Bases to complete the proof of Theorem 1.

3.2.1 Polynomial Ideals and Varieties

Recall that the set of all polynomials in n variables with coefficients in some field $\mathbb{K}$ forms a ring which we denote by $\mathbb{K}[x] = \mathbb{K}[x_1, \ldots, x_n]$. A subset I of the polynomial ring $\mathbb{K}[x]$ is called an ideal if it is closed under sums, $f + g \in I$ for all $f, g \in I$, and it satisfies the property that $h \cdot f \in I$ for all $f \in I$ and $h \in \mathbb{K}[x]$. For given polynomials $f_1, \ldots, f_k$, the set

$$I = \{\sum_{i=1}^{k} h_i f_i : h_i \in \mathbb{K}[x]\} = \langle f_1, \ldots, f_k \rangle,$$

is an ideal. It is called the ideal generated by $f_1, \ldots, f_k$. This ideal $\langle f_1, \ldots, f_k \rangle$ is the set of all linear combinations of the polynomials $f_1, \ldots, f_k$, where the "coefficients" in each linear combination are themselves polynomials in the polynomial ring $\mathbb{K}[x]$. The Hilbert Basis Theorem, see Cox et al. (1997), states that for any ideal $I \subset \mathbb{K}[x]$ there exist finitely many polynomials that generate I. A set of such polynomials generating the ideal I is called a basis of I.

The *radical* of an ideal I is defined as $\sqrt{I} = \{f \in \mathbb{K}[x] : \exists m \geq 1 \text{ such that } f^m \in I\}$. The radical $\sqrt{I}$ is itself an ideal and contains I, $I \subset \sqrt{I}$. An ideal I is called radical if $I = \sqrt{I}$.

The notion of ideals is fundamental to solving polynomial equations. While the coefficients in our polynomial equations are real numbers much of the study of polynomial equations is done on algebraically closed fields, that is, on fields where each non-constant univariate polynomial has a zero. The field $\mathbb{R}$ of real numbers is not algebraically closed but the field $\mathbb{C}$ of complex numbers is. The set of common complex zeros of the polynomials $f_1, f_2, \ldots, f_k \in \mathbb{K}[x]$,

$$V(f_1, f_2, \ldots, f_k) = \{x \in \mathbb{C}^n : f_1(x) = \ldots = f_k(x) = 0\}$$

is called the complex variety defined by $f_1, \ldots, f_k$. For the remainder of this paper we only consider complex varieties, but allow the coefficients of polynomials to be in an arbitrary field $\mathbb{K}$. The variety does not change if we replace the polynomials $f_1, \ldots, f_k$ by another basis $g_1, \ldots, g_l$ generating the same ideal. That is, the notion of affine variety can be defined for ideals and not just for a set of polynomials. For an ideal $I = \langle f_1, \ldots, f_k \rangle = \langle g_1, \ldots, g_l \rangle$ we can write

$$V(I) = V(f_1, f_2, \ldots, f_k) = V(g_1, g_2, \ldots, g_l).$$

Let us emphasize this point. The set of common zeros of a set of polynomials $f_1, f_2, \ldots, f_k$ is identical to the common set of zeros of all (infinitely many!) polynomials in the ideal $I = \langle f_1, f_2, \ldots, f_k \rangle$. In particular, any other basis of I has the same zero set. If the set $V(I)$ is finite and thus zero-dimensional, we call the ideal I itself zero-dimensional.

At this point of our discussion the reader may already have guessed a promising strategy for analyzing and solving a system of polynomial equations. Considering that the set of solutions to a system $f_1(x) = \ldots = f_k(x) = 0$ is the same for any basis of the ideal $I = \langle f_1, f_2, \ldots, f_k \rangle$, we ask whether we can find a basis that has "nice" properties and which makes describing the solution set $V(I)$ straightforward. Put differently, our question is: Can we transform the original system $f_1(x) = \ldots = f_k(x) = 0$ into a new system $g_1(x) = \ldots = g_l(x) = 0$ that can be easily solved, particularly if the solution set is zero-dimensional?

3.2.2 The Shape Lemma

'Gröbner Bases' are such bases that have desirable algorithmic properties for solving polynomial systems of equations. Specifically, the 'reduced Gröbner basis $\mathcal{G}$ in the lexicographic term order' is ideally suited for solving systems of polynomial equations. A proper definition of the relevant notions of Gröbner basis, reduced Gröbner basis, and lexicographic term order is rather tedious. But the main mathematical result that is useful for our purposes is easily understood without many additional mathematical definitions. Therefore we do not give all these definitions here and instead refer the interested reader to the books by Cox et al. (1997) and Sturmfels (2002). The following lemma, the so-called Shape Lemma, is central for our analysis and describes the properties of a Gröbner basis that are important for Theorem 1. For a proof of the Shape Lemma see Becker et al. (1994).

LEMMA 8 (SHAPE LEMMA)
Let $I = \langle f_1, \ldots, f_n \rangle$ be a zero-dimensional radical ideal in $\mathbb{K}[x_1, \ldots, x_n]$ with $\mathbb{Q} \subset \mathbb{K}$ such that all d

elements of $V(I)$ have distinct values for the last coordinate x_n. Then the reduced Gröbner basis of I (in the lexicographic term order) has the shape

$$\mathcal{G} = \{x_1 - v_1(x_n), \ldots, x_{n-1} - v_{n-1}(x_n), r(x_n)\}$$

where r is a polynomial of degree d and the v_i are polynomials of degree strictly less than d.

The Shape Lemma provides conditions under which the zero set $V(f_1, f_2, \ldots, f_n)$ of a system of polynomial equations $f_1(x) = f_2(x) = \ldots = f_n(x) = 0$ is also the solution set to another equivalent system of polynomial equations having a very simple form. The equivalent system consists of one univariate equation $r(x_n) = 0$ in the last variable x_n and $n - 1$ equations, each of which depends only on a (different) single variable x_i and the last variable x_n. These equations are linear in their respective x_i, $i = 1, 2, \ldots, n - 1$.

Some simple examples shed some light on the assumptions of the Shape Lemma. Consider the system of equations $x_1^2 - x_2 = 0,\ x_2 - 4 = 0$ and its solutions $(2, 4)$ and $(-2, 4)$. Both solutions have the same value for the last coordinate x_2. Clearly, no polynomial of the form $x_1 - v_1(x_2)$ can yield the two possible values -2 and 2 for x_1 when $x_2 = 4$. The linearity in x_1 prohibits this from being possible. Next consider the system $x_1^2 - x_2 + 1 = 0,\ x_2 - 1 = 0$ and its solution $(0, 1)$. Observe that for $x_2 = 1$ the first equation yields $x_1^2 = 0$ and so 0 is a multiple zero of this equation. There cannot be a Gröbner basis linear in x_1 that yields a multiple zero. For polynomial systems with zero-dimensional solution sets, multiple zeros are ruled out by the Shape Lemma's assumption that I is a radical ideal. A multiple zero requires the ideal to contain a polynomial of the form f_i^m with $m \geq 2$ but not to contain f_i. This cannot happen for a radical ideal. (Note that this simple intuition is only correct for zero-dimensional ideals and does not generalize to higher dimensions.)

There is a large literature on the computation of a Gröbner basis for arbitrary sets of polynomials. In particular, *Buchberger's algorithm* always produces a Gröbner basis in finitely many steps. We refer the interested reader to the book by Cox et al. (1997).

Before we can apply the Shape Lemma to our polynomial system of equations we need to address two key issues. First, the lemma rests on the assumption that the ideal $I = \langle f_1, \ldots, f_n \rangle$ is zero-dimensional and radical. For our economic equations we need a sufficient condition that ensures this property. Secondly, for our economic model we do not only want to analyze a single system of polynomial equations characterizing an economic equilibrium. Instead, we often think of our economy being parameterized by a set of parameters and so would like to make statements about the equilibrium manifold. Economic parameters lead to polynomial systems with coefficients being polynomials in the parameters. These two issues motivate us to state a specialized version of the Shape Lemma.

3.2.3 A Sufficient Condition for the Shape Lemma

In order to state a simple sufficient condition for the shape lemma, we need to restrict attention to polynomials $f \in \mathbb{K}[x]$ with $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}\}$ and $x \in \mathbb{C}^n$. Given a polynomial function $g : \mathbb{C}^n \to \mathbb{C}$ one can define partial derivatives with respect to complex numbers in the usual way. Write

$$g = c_0(x_{-j}) + c_1(x_{-j})x_j + \ldots + c_d(x_{-j})x_j^d,$$

where the c_i are polynomials in the variables $x_{-j} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$. Then,

$$\frac{\partial g}{\partial x_j} := c_1(x_{-j}) + \ldots + dc_d(x_{-j})x_j^{d-1}.$$

Given a system of polynomial equations with $f_i : \mathbb{C}^n \to \mathbb{C}^n$, $i = 1, \ldots, n$, the Jacobian $\partial_x f(x)$ is defined as usual as the matrix of partial derivatives.

The next lemma states a sufficient condition for the ideal $I = \langle f_1, \ldots, f_n \rangle$ to be zero-dimensional and radical and follows from Cox et al. (1998, Chapter 4.2).

LEMMA 9 *Let $f_1, \ldots, f_n \in \mathbb{K}[x]$ with $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}\}$ and $x \in \mathbb{C}^n$. The ideal $I = \langle f_1, \ldots, f_n \rangle$ is zero-dimensional and radical if for all $x \in V(I)$ it holds that $det(\partial_x(f_1(x), \ldots, f_n(x))) \neq 0$.*

3.2.4 Parameterized Shape Lemma

We can allow for parametric coefficients, ξ, for the polynomials $f_1, \ldots, f_n$ by choosing as the field $\mathbb{K}(\xi)$. Buchberger's algorithm then yields a set of polynomials $v_1, \ldots, v_{n-1}, r$ with coefficients that are themselves rational functions of the parameters. For a generic $\bar{\xi}$, this set of polynomials forms a Gröbner basis for the ideal $\langle f_1, \ldots, f_n \rangle$, where all f_i are evaluated at this $\bar{\xi}$.

This is only true for all values of the parameters outside the union of the solution sets to finitely many polynomial equations because intuitively, Buchberger's algorithm performs many divisions by polynomials in the parameters and so for some parameter values a division by zero would occur. In that case the Gröbner basis would be different since Buchberger's algorithm performed an ill-defined division.

We restate the Shape Lemma in parameterized form and let coefficients be from the field $\mathbb{K} = \mathbb{Q}$ or the field $\mathbb{K} = \mathbb{R}$.

LEMMA 10 (PARAMETERIZED SHAPE LEMMA)
Let $\Xi \subset \mathbb{R}^m$ be an open set of parameters and let $f_1, \ldots, f_n \in \mathbb{K}[\xi_1, \ldots, \xi_m; x_1, \ldots, x_n]$ with $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}\}$ and $(x_1, \ldots, x_n) \in \mathbb{C}^n$. Suppose that for each $\bar{\xi} = (\bar{\xi}_1, \ldots, \bar{\xi}_m) \in \Xi$ the Jacobian matrix $D_x f(\bar{\xi}; x)$ has full rank n whenever $f(\bar{\xi}; x) = 0$ and all d solutions have a distinct last coordinate x_n. Then there exist $r, v_1, \ldots, v_{n-1} \in \mathbb{K}[\xi; x_n]$ and $w_1, \ldots, w_{n-1} \in \mathbb{K}[\xi]$ such that for all $\bar{\xi}$ outside a closed lower-dimensional subset Ξ_0 of Ξ,

$$\begin{aligned} & \{x \in \mathbb{C}^n : f_1(\bar{\xi}; x) = \ldots = f_n(\bar{\xi}; x) = 0\} \\ = \; & \{x \in \mathbb{C}^n : w_1(\bar{\xi})x_1 = v_1(\bar{\xi}; x_n), \ldots, w_{n-1}(\bar{\xi})x_{n-1} = v_{n-1}(\bar{\xi}; x_n);\; r(\bar{\xi}; x_n) = 0\}. \end{aligned}$$

The degree of r in x_n is d, the degrees of $v_1, \ldots, v_{n-1}$ in x_n are at most $d - 1$.

Note that Lemma 9 immediately implies that for fixed $\bar{\xi}$, there exist some polynomial functions in x, $\tilde{v}_1, \ldots, \tilde{v}_{n-1}, \tilde{r} \in \mathbb{K}[x]$ such that the Shape Lemma representations holds,

$$\begin{aligned} & \{x \in \mathbb{C}^n : f_1(\bar{\xi}; x) = \ldots = f_n(\bar{\xi}; x) = 0\} \\ = & \{x \in \mathbb{C}^n : x_1 = \tilde{v}_1(x_n), \ldots, x_{n-1} = \tilde{v}_{n-1}(x_n);\ \tilde{r}(x_n) = 0\}. \end{aligned}$$

The fact that the coefficients of these polynomials are rational functions of ξ follows from the observations that the set of rational functions, $\mathbb{K}(\xi)$, forms a field and that each ideal has a unique (reduced) Gröbner basis (see Cox et al. (1997)). For given generic $\bar{\xi}$, the unique (reduced) Gröbner basis of the ideal $\langle f_1(\xi; x), \ldots, f_n(\xi; x)\rangle \subset \mathbb{K}(\xi)[x_1, \ldots, x_n]$, which has the form of Lemma 10, must specialize to the unique (reduced) Gröbner basis of the ideal $\langle f_1(\bar{\xi}; x), \ldots, f_n(\bar{\xi}; x)\rangle \subset \mathbb{K}[x_1, \ldots, x_n]$.

3.3 Proof of Theorem 1 and Discussion

We are now in the position to complete the proof of Theorem 1.

Proof. We view equations (11) as a system of equations in complex space. Recall that to simplify the notation we let $M = H(L+1) + L$ and associate with $x \in \mathbb{C}^M$ the vector $(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p)$. In our economic models we cannot prohibit multiple equilibria to have identical values for one or several variables and so in general we cannot assume that the assumption of the Shape Lemma on the distinct values of the last variable x_M is satisfied. To circumvent this problem we introduce a new last variable y and a linear equation $y = \sum_i \mu_i x_i$ with random coefficients μ_i relating all existing variables to the new variable. And so we are now concerned with the system of $M+1$ polynomial equations

$$F(e^{\mathcal{H}}; x) = 0, \tag{12}$$

$$y - \sum_{i=1}^{M} \mu_i x_i = 0, \tag{13}$$

with parameters $\mu = (\mu_1, \ldots, \mu_M) \in \Delta^{M-1}$ and the variables $x \in \mathbb{C}^M$ and $y \in \mathbb{C}$.

Proposition 3 implies that for generic $e^{\mathcal{H}}$ equations (12) together with the condition

$$1 - t\ det[\partial F(e^{\mathcal{H}}; x)] = 0 \tag{14}$$

generate a zero-dimensional radical ideal in $\mathbb{K}[e^{\mathcal{H}}; x, t]$. The system (12),(14) consists of $M+1$ equations in the $M+1$ complex variables $x_1, \ldots, x_n, t$. We can identify a complex number $z \in \mathbb{C}$ with the vector $(Re(z), Im(z)) \in \mathbb{R}^2$ consisting of its real part $Re(z)$ and its imaginary part $Im(z)$. Then we can view the left-hand sides of these equations as a system of semi-algebraic functions $g : \mathbb{R}^{2M+2} \to \mathbb{R}^{2M+2}$. For generic $e^{\mathcal{H}}$ this function has finitely many zeros. Lemma 7 implies that for a generic element $\mu \in \Delta^{M-1}$ the set of $e^{\mathcal{H}}$ for which there are two distinct solutions $x \neq x' \in \mathbb{R}^{2M}$ with $g(Re(x), Im(x)) = g(Re(x'), Im(x')) = 0$ and $\sum_{i=1}^{M} \mu_i (Re(x)_i - Re(x')_i) = \sum_{i=1}^{M} \mu_i (Im(x)_i - Im(x')_i) = 0$ is lower-dimensional and closed. Therefore, the polynomials in the equations (12), (13), and (14) form a zero-dimensional radical ideal in $\mathbb{K}[e^{\mathcal{H}}; x, t, y]$. Thus, for generic endowments $e^{\mathcal{H}}$, we can apply Lemma 10, the parameterized Shape Lemma, to the entire system (12) – (14). The set of solutions to this system is identical to the solution set of a system with the shape $\mathcal{G}$ (omitting the variable t and the accompanying polynomial). Finally, by construction all Walrasian equilibria of the economy satisfy equations (12) and (13). Proposition 3 implies that for generic endowments all Walrasian equilibria also satisfy equation (14). □

Observe from this proof that the UPR is not unique. Different choices of the random weights μ_i in Equation (13) lead to different UPRs. Also, the generic set of endowments for which the theorem holds depends on μ. For large (parameterized) classes of economies there exists a UPR for which the rational function determining the L'th price is simply $p_L = y$, i.e. μ is taken to be $\mu = (0, \ldots, 0, 1)$. Clearly such a UPR is preferred since it enables us to avoid the additional variable y. Such a UPR exists if for each economy in the class all solutions to the polynomial equilibrium system, $F(c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) = 0$, have distinct values for the L'th price. As Paolo Siconolfi (private communication, 2007) pointed out to us, all competitive equilibria have distinct L'th price for an open and dense set of (semi-algebraic) preferences. We do not attempt to derive this result here in terms of generic sets of preference parameters. In our examples below we can always use the last price as the last variable and omit the new variable for generic values of the preference parameters.

From a practical perspective it is important to note that for the same equilibrium correspondence some UPRs, using the SINGULAR implementation of Buchberger's algorithm, are substantially easier to compute than others. The choice of linear form or last variable does matter for running times.

In the non-generic case, when the Shape Lemma fails to hold, Buchberger's algorithm still produces a Gröbner basis. Thus an examination of the SINGULAR output always allows us to diagnose whether the input system yields a basis in shape form. Furthermore, under weaker conditions than those required for the Shape Lemma, the Gröbner basis will be in a triangular form that still facilitates a simple computation of all complex solutions.

3.4 Arbitrary Parameters

As we mentioned in the introduction, modelers are often interested in characterizing the map from exogenous parameters to competitive equilibria where the parameters are not restricted to consist of profiles of individual endowments but instead may contain preference parameters, tax rates, or other economically interesting exogenous parameters. In the following we explain how we can generalize Theorem 1 to such situations.

There are two occasions in the proof of Theorem 1 where we restrict the set of parameters, agents' individual endowments, to a generic subset of the entire set of parameters. We need to invoke this restriction both for the application of Lemma 10, the parameterized Shape Lemma, and for the application of Proposition 3 which states an important full rank condition for Walrasian equilibria. It is worthwhile to emphasize the fundamental difference between these two restrictions on the parameter set.

Proposition 3 states a semi-algebraic version of the well-known restriction on endowments that ensures that all equilibria satisfy the standard regularity condition. Some economies do not satisfy this condition and thus need to be excluded. So this condition on agents' individual endowments rests on the underlying economic model. On the contrary, the condition in the parameterized Shape Lemma, which reads rather similar, does not hinge directly on the economic model. Instead it is a consequence of the application of Buchberger's algorithm for the computation of the Gröbner basis in shape form. This result holds for arbitrary parameters.

For an illustration of the described distinction let us consider Arrow-Debreu exchange economies that are parameterized by both profiles of endowments as well as preference parameters. We assume that for each agent h, utility u^h is parameterized by some $\zeta \in \mathbb{R}^K_{++}$ and we assume that $\partial_{c_l} u^h(\zeta; c)$ is semi-algebraic in both ζ and c. Lemma 3 immediately implies that there exist polynomials $m^h_l(\zeta; c, y)$ such that for every $c \in \mathbb{R}^L_+$,

$$m^h_l(\zeta; c, \partial_{c_l} u^h(\zeta; c)) = 0.$$

We now consider an arbitrary (possibly lower dimensional) set of parameters $\Xi \subset \mathbb{R}^{HL}_{++} \times \mathbb{R}^{HK}_{++}$ that contains both some individuals' endowments and some preference parameters. To indicate the dependence on these parameters, we rewrite the polynomial equilibrium system from above as

$$F(\xi; c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) = \begin{cases} m^h(\zeta^h; c^h, \lambda^h p), & h \in \mathcal{H} \\ p \cdot (c^h - e^h), & h \in \mathcal{H} \\ \sum_{h \in \mathcal{H}} (c^h_l - e^h_l), & l = 1, \ldots, L-1 \\ \sum_l p_l - 1 \end{cases}$$

In this formulation, it should be clear that some of the e^h_l and some of the ζ^h are constant while others can be part of the vector ξ. We can now state a different version of Theorem 1 which perhaps is more appealing to applied general equilibrium modelers.

Theorem 2 *Suppose that for generic parameters $\xi \in \Xi$, every Walrasian equilibrium satisfies*

$$\det \left[D_{c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p} F(\xi; c^{\mathcal{H}}, \lambda^{\mathcal{H}}, p) \right] \neq 0.$$

Then for generic $\xi \in \Xi$, every Walrasian equilibrium x^* of the economy along with an accompanying positive number y^* is among the finitely many common zeros of the polynomials in a set $\mathcal{G}$ of the shape

$$\mathcal{G} = \{x_1 - v_1(\xi; y), \ \ldots \ , \ x_M - v_M(\xi; y), \ r(\xi; y)\}. \tag{15}$$

The non-zero polynomial $r \in \mathbb{R}[\xi; y]$ is not constant in y. Moreover, each v_i, $i = 1, \ldots, M$, is a polynomial in y of degree less than the degree of r. The coefficients of this polynomial are rational terms in ξ.

The proof of this theorem closely follows the proof of Theorem 1, except that we assume generic regularity instead of proving it. Therefore, we do not need equation (14) and do not invoke the regularity result of Proposition 3 but can entirely rely on the parameterized Shape Lemma and on Lemma 7. This lemma continues to hold since it only relies on finitely many solutions which again is guaranteed by the assumption of regularity.

The new theorem raises the question, of course, if there are other sets of parameters which ensure generic regularity. Observe that the proofs of Propositions 2 and 3 carry over to other parameter sets as long as these include the individual endowments of at least one agent. Let $\xi = (e^1, \xi_{\backslash e^1})$ consist of the parameters of interest and define $d^h(\xi; p, \tau)$ to be agent h's individual demand function as a (semi-algebraic) function of the parameters, prices and income (of course d^h will only depend on some of the elements of ξ or might not depend on ξ at all).

We redefine the function H from equation (10) in the proof of Proposition 2 as follows,

$$H(p, \tau, \xi_{\backslash e^1}) = \begin{cases} d^1(\xi; p, \tau) + \sum_{h=2}^{H} (d^h(\xi; p \cdot e^h) - e^h) \\ \xi_{\backslash e^1} \end{cases}$$

Since Walras' law implies that if $H(p, \tau, \xi_{\backslash e^1}) = \xi$ then $\tau = p \cdot e^1$, the vector p is a competitive equilibrium price for the economy with parameters ξ if and only if there is a τ such that $H(p, \tau, \xi_{\backslash e^1}) = \xi$.

In the following corollary we summarize our discussion on more general parameter sets in the spirit of the discussion in the introduction.

COROLLARY 1 *Suppose* Ξ *is a semi-algebraic set that contains individual endowments of at least one agent. Then there exists a semi-algebraic correspondence* $\rho : \Xi \to \mathbb{R}_+$ *and rational functions* $v_i : \Xi \times \mathbb{R}_{++} \to \mathbb{R}$, $i = 1, \ldots, M$, *such that for a generic vector of parameters* $\xi \in \Xi$ *the vector* x^* *is a competitive equilibrium if and only if there is a* $y \in \rho(\xi)$ *such that* $x_i^* = v_i(\xi; y)$.

The corollary follows from Theorem 2 in conjunction with the observation that endowments of one individual suffice to guarantee generic regularity and the fact that the set

$$\{(\xi, y) : \; r(\xi; y) = 0 \;\&\; v_i(\xi; y) > 0, i = 1, \ldots, M \;\&\; (v_1(\xi; y), \ldots, v_M(\xi; y)) \text{ solve } (2) - (5)\}$$

is a semi-algebraic set and the graph of the correspondence ρ.

3.5 Practical Implications: Equilibrium Multiplicity

Theorems 1 and 2 and Corollary 1 describe the structure of the equilibrium correspondence for semi-algebraic economies. As such this result is theoretical in nature. But the rather simple description of the equilibrium correspondence has significant practical implications, too. Here we emphasize the perhaps most obvious one, namely the analysis of equilibrium multiplicity.

Theorem 1 reduces the problem of solving the system of equilibrium equations essentially to solving a single univariate polynomial equation. This equivalence enables us to employ bounds on the number of zeros of univariate polynomials to derive bounds on the number of solutions to the equilibrium system; that is, we can obtain bounds on the number of equilibria. In some applications this property enables us to prove uniqueness of equilibrium.

The Fundamental Theorem of Algebra, see e.g. Sturmfels (2002), states that a univariate polynomial, $f(x) = \sum_{i=0}^{d} a_i x^i$, with rational, real or complex coefficients a_i, $i = 0, 1, \ldots, d$, has d zeros, counting multiple roots, in the field $\mathbb{C}$ of complex numbers. That is, the degree d of the polynomial f is an upper bound on the number of complex zeros. More importantly for our economic analysis even better bounds are available for the number or real zeros. For a finite sequence $a_0, \ldots, a_k$ of real numbers the number of sign changes is the number of products $a_i a_{i+l} < 0$, where $a_i \neq 0$ and a_{i+l} is the next non-zero element of the sequence. Zero elements are ignored in the calculation of the number of sign changes. The classical Descartes's Rule of Signs, see Sturmfels (2002), states that the number of positive real zeros of f does not exceed the number of sign changes in the sequence of the coefficients of f. This bound is remarkable because it bounds the number of (positive) real zeros. It is possible that a polynomial system is of very high degree and has many solutions but the Descartes bound on the number of positive real zeros of the representing polynomial r in the Shape Lemma proves that the system has a single real positive solution.

The Descartes bound is not tight and overstates the true number of positive real solutions for many polynomials. Sturm's Theorem, see Sturmfels (2002) or Bochnak et al. (1998), yields an exact bound on the number of positive real solutions of a univariate polynomial. For a univariate polynomial f, the Sturm sequence of $f(x)$ is a sequence of polynomials $f_0, \ldots, f_k$ defined as follows,

$$f_0 = f, \ f_1 = f', \ f_i = f_{i-1} q_i - f_{i-2} \ \text{ for } 2 \leq i \leq k$$

where f_i is the negative of the remainder on division of f_{i-2} by f_{i-1}, so q_i is a polynomial and the degree of f_i is less than the degree of f_{i-1}. The sequence stops with the last nonzero remainder f_k. Sturm's Theorem provides an exact root count, see e.g. Bochnak et al. (1998) for a proof.

Lemma 11 (Sturm's Theorem)
Let f be a polynomial with Sturm sequence $f_0, \ldots, f_k$ and let $a < b \in \mathbb{R}$ with neither a nor b a root of f. Then the number of roots of f in the interval $[a, b]$ is equal to the number of sign changes of $f_0(a), \ldots, f_k(a)$ minus the number of sign changes of $f_0(b), \ldots, f_k(b)$.

Buchberger's algorithm computes Gröbner bases exactly for the case of rational coefficients, that is, the set of polynomials $\mathcal{G}$ can be computed exactly whenever marginal utility can be written as a polynomial with rational coefficients. Once the UPR for an economy (or a class of economies parameterized by endowments or preference parameters) is known, we can use the univariate polynomial to determine the number of real zeros of the system and the number of competitive equilibria. For fixed values of all parameters, Sturm's algorithm provides an exact method to determine the number of solutions to a univariate polynomial in the interval $[0, \infty)$. Therefore, we can determine the exact number of solutions of the univariate polynomial. Using simple bracketing, we can then approximate all solutions numerically, up to arbitrary precision. Given the solutions to the univariate representation, the other solutions can then be computed by evaluating polynomials up to arbitrary precision. This is the only point in the procedure where the computation is not exact. Even this last computational step could be performed exactly by resorting to methods from computational algebraic number theory, see Cohen (1996). While the theory for the representation of algebraic numbers in the reals is sound we suspect that any implementation of these methods would be too slow to be of interest.

4 Applications

In this section we apply our tools to some parameterized economies. A simple class of semi-algebraic utility can be obtained by assuming that utility is separable, i.e. $u^h(c) = \sum_{l=1}^{L} u_{hl}(c_l)$ with each u'_{hl} being a semi-algebraic function. We first consider the case of quadratic utility because the resulting equations are simple and provide a nice illustration of our tools. We then move to the case of utility exhibiting constant elasticity of substitution (CES). This latter case is prevalent in economic applications.

4.1 Quadratic Utility

There are two agents and two commodities, utility functions for agent h and good l are

$$u_{hl}(c) = a_{hl}c - \frac{1}{2}b_{hl}c^2.$$

For the case where utility is symmetric across goods, i.e. $u_{h1} = u_{h2}$, there always exists a unique Walrasian equilibrium. The following polynomial system of equations is solved by any interior Walrasian equilibrium. (We write b_h for $b_{h1} = b_{h2}$ and normalize utility so that $a_{hl} = 1$ for $h = 1, 2$ and $l = 1, 2$. To avoid confusion between exponents and the traditional agent superscript h we write c_{hl} instead of c_l^h, e_{hl} instead of e_l^h, and λ_h instead of λ^h.)

$$\begin{aligned}
1 - b_1 c_{11} - \lambda_1 p_1 &= 0\\
1 - b_1 c_{12} - \lambda_1 p_2 &= 0\\
1 - b_2 c_{21} - \lambda_2 p_1 &= 0\\
1 - b_2 c_{22} - \lambda_2 p_2 &= 0\\
p_1(c_{11} - e_{11}) + p_2(c_{12} - e_{12}) &= 0\\
p_1(c_{21} - e_{21}) + p_2(c_{22} - e_{22}) &= 0\\
c_{11} + c_{21} - e_{11} - e_{21} &= 0\\
p_1 + p_2 - 1 &= 0
\end{aligned}$$

Note that as discussed we can parameterize the economies not just by agents' endowments but in addition by other parameters, here the utility parameters b_1 and b_2.

Observe that for a specific value of p_2 the last equation fixes the value of p_1. The remaining equations are then linear in the remaining variables. Therefore, the system cannot have two solutions with the same value for p_2. Thus, if we use p_2 as the last variable then our results holds for this system without an additional linear form. Implementing this system in SINGULAR yields an equivalent system with the shape $\mathcal{G}$. The last equation in the variable p_2 is of the form

$$r(e_{\mathcal{H}}, b_1, b_2; p_2) = C_2 p_2^2 + C_1 p_2$$

with the coefficients

$$\begin{aligned} C_2 &= b_1b_2e_{11} + b_1b_2e_{12} + b_1b_2e_{21} + b_1b_2e_{22} - 2b_1 - 2b_2, \\ C_1 &= -b_1b_2e_{12} - b_1b_2e_{22} + b_1 + b_2. \end{aligned}$$

We observe that the univariate equation in $\mathcal{G}$ depends on all six parameters of the model. Obviously, the equation $r(\cdot; p_2) = 0$ has two solutions. One solution is $p_2 = 0$, which is not a Walrasian equilibrium. It is easy to check that for economically meaningful values of the parameters b_h and endowments $e^{\mathcal{H}}$ it holds that $C_2 < 0$ and $C_1 > 0$ and so $p_2^* = -C_1/C_2 \in (0,1)$ is a Walrasian equilibrium price. The remaining equations (which we do not report here) then yield all remaining variable values. The UPR now asserts that the interior Walrasian equilibrium (if there is one) is unique.

Next we allow utility to differ across agents and goods. For this general case the univariate polynomial r has the form

$$r(e_{\mathcal{H}}, (a_{hl}, b_{hl})_{h=1,2,l=1,2}; p_2) = C_4p_2^4 + C_3p_2^3 + C_2p_2^2 + C_1p_2,$$

where C_1, C_2, C_3 and C_4 are polynomials in the parameters. All four polynomials contain positive and negative monomials in the parameters and so their respective signs depend on the actual parameter values.

Again $p_2 = 0$ is a solution to this equation which does not correspond to a Walrasian equilibrium. Thus, there can be at most 3 Walrasian equilibria. For many parameter values only exactly one of the solutions to $r = 0$ corresponds to a Walrasian equilibrium. However, it is easy to "reverse-engineer" parameter values to obtain an economy with 3 equilibria. For example, suppose $e^1 = (10, 0)$, $e^2 = (0, 10)$ and

$$u'_{11}(c) = 9 - c, \quad u'_{12}(c) = 29/4 - 7/8c, \quad u'_{21}(c) = 116 - 26c, \quad u'_{22}(c) = 24 - 4c.$$

It is easy to verify that this economy has at 3 equilibria with prices (p_1, p_2) being $(4/5, 1/5)$, $(3/5, 2/5)$ and $(1/2, 1/2)$, respectively. In fact, the representing polynomial from the Gröbner basis for the equilibrium system is $r(p_2) = 50p_2^4 - 55p_2^3 + 19p_2^2 - 2p_2$. By Descartes' bound this system has at most three positive solutions. For them to be equilibrium prices they must lie in $(0,1)$. We can apply Sturm's theorem and use SINGULAR to compute the number of sign changes of the Sturm sequence at 0 and the number of sign changes of the Sturm sequence at 1. It turns out that there are exactly 3 solutions in $(0,1)$.

4.2 CES Utility

We consider economies with $H = 2$ agents and $L \geq 2$ commodities. Suppose agents have CES utility functions with marginal utility of the form

$$u'_{hl}(c) = (\alpha_{hl}\ c_l)^{-\sigma}. \tag{16}$$

For simplicity we assume that elasticities of substitution are identical across agents and that σ is an integer. After transforming agents' first-order conditions into polynomial expressions we obtain the specific form of Equations (11) for our CES-framework (using the same notation as in the previous example).

$$\begin{aligned} \alpha_{hl}^{\sigma} c_{hl}^{\sigma} \lambda_h p_l - 1 &= 0, \quad h \in \mathcal{H}, l = 1, \ldots, L, \\ \sum_{l=1}^{L} p_l (c_{hl} - e_{hl}) &= 0, \quad h = 1, \ldots, H, \\ \sum_{h=1}^{H} c_{hl} - e_{hl} &= 0, \quad l = 1, \ldots, L-1, \\ \sum_{l=1}^{L} p_l - 1 &= 0. \end{aligned}$$

We can greatly reduce running times of SINGULAR if we write the equilibrium equations slightly differently. In particular, we normalize $p_1 = 1$ and eliminate all Lagrange multipliers. Defining $q_l = p_l^{1/\sigma}$, $l = 2, \ldots, L$, we obtain a system of equations that is equivalent as far as the economic model is concerned.

$$\alpha_{h1} c_1^h - \alpha_{hl} c_{hl} q_l = 0, \quad h \in \mathcal{H}, l = 2, \ldots, L, \tag{17}$$

$$c_{h1} - e_{h1} + \sum_{l=2}^{L} q_l^{\sigma} (c_{hl} - e_{hl}) = 0, \quad h = 1, \ldots, H, \tag{18}$$

$$\sum_{h=1}^{H} c_{hl} - e_{hl} = 0, \quad l = 1, \ldots, L-1. \tag{19}$$

Any positive real solution of this system is in fact a Walrasian equilibrium. For almost all parameters all equilibria have a distinct last price p_L.

The resulting UPR is now as follows.

$$\begin{aligned} r(e_{\mathcal{H}}, \alpha_{\mathcal{H}}; y) &= \textstyle\sum_{i=0}^{\sigma} v_i^r(e_{\mathcal{H}}, \alpha_{\mathcal{H}})\, y^i \\ c_{hl} &= \textstyle\sum_{i=0}^{\sigma-1} v_i^{c_{hl}}(e_{\mathcal{H}}, \alpha_{\mathcal{H}})\, y^i, \qquad h \in \mathcal{H}, l = 1, \ldots, L \\ q_l &= v_1^{q_l}(e_{\mathcal{H}}, \alpha_{\mathcal{H}})\, y + v_2^{q_l}(e_{\mathcal{H}}, \alpha_{\mathcal{H}}), \quad l = 2, \ldots, L-1 \\ q_L &= y, \end{aligned}$$

where the v^r are polynomials, the v^c and v^q are rational functions. (The only exponent is i.) Note that all equilibria are uniquely described by $r(e_{\mathcal{H}}, \alpha_{\mathcal{H}}, y) = 0$ and $y > 0$. Note also that prices are linear in y, independent of the number of goods and of σ. Allocations are polynomials in y of degree $\sigma - 1$. We can use this representation to bound the maximal number of equilibria and for given endowments and preference parameters compute the exact number of equilibria by solving a single univariate polynomial.

Descartes' bound implies that there can be at most σ real zeros to the polynomial system, and so, independently of L, we can bound the number of equilibria by the elasticity of substitution σ. But since for $\sigma \to \infty$ the number of equilibria remains finite, this bound cannot be tight for sufficiently large σ.

For the case of only $L = 2$ commodities the expressions simplify. Without loss of generality normalize $\alpha_{h2} = 1 - \alpha_{h1}$ for both agents $h = 1, 2$. To simplify the notation further denote q_2 simply by q and α_{h1} by α_h. The resulting polynomial r in the UPR is then

$$\begin{aligned} r(e_{\mathcal{H}}, \alpha_{\mathcal{H}}; y) \quad = \quad & (\alpha_1 e_{22} + \alpha_2 e_{12} - \alpha_1 \alpha_2 (e_{12} + e_{22}))\, y^{\sigma} - \alpha_1 \alpha_2 (e_{11} + e_{21})\ y^{\sigma - 1} + \\ & (1 - \alpha_1)(1 - \alpha_2)(e_{12} + e_{22})\ y + (\alpha_1 \alpha_2 (e_{11} + e_{21}) - \alpha_1 e_{11} - \alpha_2 e_{21}) \end{aligned}$$

The univariate polynomial r has exactly four terms for $\sigma \geq 3$. Since $0 < \alpha^1, \alpha^2 < 1$ the polynomial has always exactly three sign changes. Descartes's Rule of Signs implies that there can be at most 3 real positive solutions.

The bound of three equilibria is tight, as the following simple case illustrates. Suppose $\sigma = 3$, $\alpha_1 = 1/5, \alpha_2 = 4/5$ and $e_{12} = e_{21} = 1$. If $e_{11} = e_{22} = f > 44$ the economy has three equilibria – with these parameters the univariate representation above becomes

$$r(y) = (f + 16)y^3 - (4f + 4)y^2 + (4f + 4)y - f - 16$$

whose 3 positive real roots for $f > 44$ correspond to 3 Walrasian equilibria.

The rather small upper bound on the number of equilibria is no longer valid once we consider economies with more than two agents. While we detect still a lot of structure in the equations, we are unable to derive general bounds on the number of equilibria. One drawback of using SINGULAR for our computations is that with the current state of technology we can only solve models of moderate size, say of about 20 – 25 polynomial equations of small or moderate degree. While our paper builds the theoretical foundation for computing all equilibria in general equilibrium models, we currently cannot solve applied models that often have hundreds or thousands of equations. We expect that the development of ever faster computers and more efficient or perhaps even parallelizable algorithms will allow for the computation of Gröbner bases for larger and larger systems. For recent advances see, for example, Faugère (1999).

5 Incomplete Financial Markets

In this paper we have shown how we can use tools from algebraic geometry to analyze the equilibrium correspondence of static finite Arrow-Debreu economies if equilibria are described by polynomial equations and inequalities. In order to convince the reader that the described tools are not restricted to such economies but instead are applicable to many other economic models we now briefly demonstrate an application of these tools to models with incomplete financial markets. Such models are well known to be much more complicated than the standard Arrow Debreu model. We first present a short description of an incomplete markets model (see Magill and Quinzii (1996) for a general discussion) and then discuss results for some examples.

5.1 Models with Incomplete Financial Markets

We consider an exchange economy under uncertainty. As before there are H individuals, $h \in \mathcal{H} = \{1, 2, \ldots, H\}$, and L physical commodities, $l = 1, 2, \ldots, L$. Uncertainty is modeled through a set of $S+1$ states of nature, $s \in \mathcal{S} = \{0, 1, \ldots, S\}$. Commodities cannot be transferred across states. Consumption sets are $\mathbb{R}_+^{L(S+1)}$ and prices are denoted by $p \in \mathbb{R}_+^{L(S+1)}$. We write p_{sl} for the price of commodity l in state s and define $p_s = (p_{s1}, \ldots, p_{sL})$. Each individual h is characterized by endowments, $e^h \in \mathbb{R}_+^{L(S+1)}$, and a utility function, $u^h : \mathbb{R}_+^{L(S+1)} \to \mathbb{R}$. As before we assume that for each agent $h \in \mathcal{H}$, u^h is C^1 on $\mathbb{R}_{++}^{L(S+1)}$, strictly increasing and strictly concave and that the gradient $\partial_c u^h(c) \gg 0$ is a semi-algebraic function.

To transfer wealth across states of nature agents must trade financial securities. There are J securities. Asset j can be traded at state $s = 0$ at a price q_j and its payoff in each state $s = 1, \ldots, S$ is assumed to be a polynomial function of prices in that state, $A_s^j \in \mathbb{R}[p_s]$. Agents portfolios are denoted by $\theta \in \mathbb{R}^J$.

A competitive equilibrium consists of prices p, q, allocations $c^{\mathcal{H}}$ and portfolios $\theta^{\mathcal{H}}$ such that for each agent $h \in \mathcal{H}$,

$$(c^h, \theta^h) \in \arg\max_{(c,\theta) \in \mathbb{R}_+^{L(S+1)} \times \mathbb{R}^J} u^h(c) \quad \text{s.t.} \quad p_0 \cdot (c_0 - e_0^h) + q \cdot \theta \leq 0$$

$$p_s \cdot (c_s - e_s^h) \leq \textstyle\sum_j \theta_j A_s^j(p_s), \text{ for } s = 1, \ldots, S,$$

and

$$\sum_{h \in \mathcal{H}} (c^h - e^h) = 0.$$

Using the 'Cass-Trick' (see Cass (2006) for an overview of equilibrium theory with incomplete markets), we take agent 1 to be 'unconstrained' and define an interior Walrasian equilibrium to be a solution $\big(c^{\mathcal{H}}, (\theta^h, \lambda^h)_{h=2,\ldots,H}, \mu, p, q\big)$, with $p, q, c^{\mathcal{H}}$ positive, to the following system of equations.

$$\partial_c u^1(c^1) - \mu p = 0 \tag{20}$$

$$p \cdot (c^1 - e^1) = 0 \tag{21}$$

$$\partial_{c_s} u^h(c^h) - \lambda_s^h p_s = 0, \quad \forall h = 2, \ldots, H, s \in \mathcal{S} \tag{22}$$

$$-q\lambda_0^h + \sum_{s=1}^{S} \lambda_s^h A_s(p_s) = 0, \quad \forall h = 2, \ldots, H, s \in \mathcal{S} \tag{23}$$

$$p_0 \cdot (c_0 - e_0^h) + q \cdot \theta = 0, \quad \forall h = 2, \ldots, H \tag{24}$$

$$p_s \cdot (c_s^h - e_s^h) - \theta^h \cdot A_s(p) = 0, \quad \forall h = 2, \ldots, H, s = 1, \ldots, S \tag{25}$$

$$\sum_{h \in \mathcal{H}} (c_{sl}^h - e_{sl}^h) = 0, \quad \forall s \in \mathcal{S}, l = 1, \ldots, L, (s,l) \neq (S, L) \tag{26}$$

$$\sum_{s \in \mathcal{S}} \sum_{l=1}^{L} p_{sl} - 1 = 0 \tag{27}$$

As before, we can rewrite marginal utility in terms of polynomials and replace the non-polynomial Equations (20) and (22) by $m^1(c^1, \mu p) = 0$ and $m^h_s(c^h_s, \lambda_s p_s) = 0$, respectively. In order for Theorem 1 to carry over to this setting, we need to show that for almost all endowments, this polynomial system of equations is regular, that is, we have to show analogues of Propositions 1 – 3 above.

We define for $h > 1$,

$$d^h(p; e^h) = \arg\max_{c \in \mathbb{R}^{LS}_+} u^h(c) \quad \text{s.t.} \quad p_0 \cdot (c_0 - e^h_0) + q \cdot \theta \leq 0$$

$$p_s \cdot (c_s - e^h_s) \leq \textstyle\sum_j \theta_j A^j_s(p), \text{ for } s = 1, \ldots, S,$$

We assume that the payoff matrix $A(p) = \left(A^1(p), \ldots, A^J(p)\right)$ has rank J for generic $p \in \Delta^{SL}_+$ and assume that for each agent h the function d^h is continuous for prices in this generic set. The unconstrained agent's demand,$d^1(p, \tau)$ is defined as above $d^1(p, \tau) = \arg\max_{c \in \mathbb{R}^{SL}_+} u^1(c)$ s.t. $p \cdot c \leq \tau$.

There are now two crucial insights: First, the unconstrained agent can be chosen arbitrarily among the set of agents, this ensures that there cannot be robust equilibria for which individual consumption lies in the 'bad set' B^h and the proof of Proposition 1 goes through as before. The other insight is that the set of prices for which $A(p)$ has full rank J is a semi-algebraic set (see also Anderson and Raimondo (2007) for a thorough analysis of this set). Therefore, we can perform our analysis on this set and Propositions 2 and 3 carry through. It remains to be shown that this restriction to prices does not robustly wipe out equilibria for which $A(p)$ has rank less than J. But this again follows from Hardt triviality; given any economy with some of the original assets taken out, generically there will be no equilibrium for which prices lie in the lower-dimensional set for which $A(p)$ has rank less than J.

Note that this does not imply that equilibrium necessarily exists. But when it exists, it can be described by Theorem 1.

5.2 Example

We consider the simplest possible example to illustrate how the possibility of non-existence of equilibrium is irrelevant for our method. We assume that there is no uncertainty in the second period, i.e. suppose that $S = 1$ so that there are two time periods with one state in the second period, $s = 0, 1$. There are two agents and two goods in the second period, for simplicity there is only one good in the first period. Agents' utility functions are

$$u^h(c) = u^h_0(c_{01}) + u^h_1(c_{11}, c_{12}) \quad \text{where} \quad u^h_0(c) = c - 1/200c^2, u^h_1(c_1, c_2) = u^h_0(c_1) + u^h_0(c_2).$$

There is a single asset paying $p_{s2} - p_{s1}$ in the second period at $s = 1$. Clearly, equilibrium does not need to exist since for $p_{12} = p_{11}$ the rank of the payoff matrix drops from one to zero. Suppose agent 2 has constant endowments of one, $e^2_{01} = e^2_{11} = e^2_{12} = 1$ while agent 1's endowments are $e^1_{01} = 2$, $e^1_1 = (e, 1)$ with $99 > e > 0$ being a parameter (we only consider $e < 99$ to rule out that one agent could be satiated in equilibrium). Note that for $e = 1$ we must have $p_{12} = p_{11}$ in any competitive equilibrium.

For this simple case of identical utility across commodities, the univariate polynomial r in the UPR is linear in y and all variables are directly rational functions of the parameter. So the UPR directly computes the equilibrium of the parameterized economy. All coefficients are rational numbers and the solution is exact. We normalize $p_{01} = p_{11} = 1$. The competitive equilibrium prices and portfolios are given by

$$q = \frac{1-e}{197}, \quad p_{12} = \frac{198}{199-e}, \quad \theta^1 = \frac{-19503e^2 + 3920103e - 11623788}{e^3 - 399e^2 + 40394e - 39996}.$$

At $e = 1$ the price of the asset becomes zero and the denominator in the expression of θ^1 is zero. The portfolio holding is not defined. At this point the rank of the payoff matrix drops to zero. Equilibrium nevertheless exists, there is no trade. At the non-generic point $e = 1$, the UPR does not give the correct solution to the system, since Lemma 10 only holds for generic parameters. For all other (real) values of e the denominator of the asset position never vanishes. For $1 \neq e < 99$ an equilibrium always exist with the payoff matrix having full rank 1.

We modify the example so that in the second period agent 2 does not have identical utility across the two goods. Suppose

$$u_1^2(c_1, c_2) = c_1 - 1/200c_1^2 + c_2 - 1/160c_2^2.$$

We make no other modifications to the economy. The analysis is now more complicated. The equilibrium prices in state 1 depend on the wealth distribution across the two agents.

$$\begin{aligned}
r(y,e) &= 3058149y^3 + (46572e - 6197797)y^2 + (335e^2 - 100175e + 3233873)y \\
&\quad +(e^3 - 359e^2 + 18625e - 24176) \\
q &= y \\
p_{12} &= \frac{197y + e - 199}{e - 199} \\
\theta^1 &= \frac{-27125783404e^2 + 5434208802433e - 7199403102164}{9e^5 - 7175e^4 + 1623387e^3 - 77212929e^2 + 243727714e - 191476296}y^2 + \\
&\quad \frac{-137849568e^3 - 27849610063e^2 + 5427223492904e - 7197195219345}{9e^5 - 7175e^4 + 1623387e^3 - 77212929e^2 + 243727714e - 191476296}y + \\
&\quad \frac{176315e^4 - 88351742e^3 + 37963058360e^2 - 117616735896e + 97640477377}{9e^5 - 7175e^4 + 1623387e^3 - 77212929e^2 + 243727714e - 191476296}
\end{aligned}$$

The expression for θ^1 is not well-defined if the denominator becomes zero, i.e. if

$$9e^5 - 7175e^4 + 1623387e^3 - 77212929e^2 + 243727714e - 191476296 = 0.$$

This polynomial has three real zeros which are (approximately) 1.332, 2 and 59.6. As $e \to 1.332$, $p_{12} \to p_{11}$, but at $p_{11} = p_{12}$, the asset pays zero in both states. In the equilibrium without the asset spot prices in state 1 are no longer equal, therefore no equilibrium can exist for this endowment value. For $e = 59.6$ this situation is similar and there is no equilibrium either. For $e = 2$ the situation is different, just like in the first example, equilibrium does exist (and there is no trade in the asset), but for this non-generic value of e the UPR does not represent the competitive equilibrium. For all other values of $e < 99$ equilibrium does exist and is described by our UPR.

6 Conclusion

This paper has developed a method to characterize and to compute the equilibrium correspondence for exchange economies with semi-algebraic preferences. We first have shown how equilibria in these economies can be characterized as particular solutions to square polynomial systems of equations. Subsequently we have applied powerful methods from computational algebraic geometry to obtain an equivalent system of equations that has a very simple structure, the univariate polynomial representation. The computer algebra system SINGULAR enables us to compute the UPR explicitly. In particular, if all coefficients in the polynomial equilibrium system are parameters and rational numbers then the computation is exact, that is, without rounding errors.

We have presented the development of the new methods in the context of Arrow-Debreu exchange economies. But clearly the results apply to many different models. To illustrate this generality we have shown an application to a model with incomplete financial markets, the well-known GEI model. For brevity we have not presented further applications in this paper. We just mention that we have also successfully applied our tools to general equilibrium models with production, the Lucas asset pricing model with heterogeneous agents and complete markets, OLG models, and strategic market games.

The nature of the analysis in this paper is rather technical. We emphasize once again, however, that the results have important implications for applied general equilibrium modeling. In this paper we have illustrated how the UPR can be used for an analysis of equilibrium multiplicity. Using the UPR we can determine an upper bound on the number of equilibria and approximate all equilibria numerically. Thus this paper has laid the theoretical foundation for the analysis of equilibrium multiplicity in general equilibrium models. Another potentially interesting application of our results is the reverse-engineering of economies from "observables." Modelers can use the UPR to determine whole sets of model parameters so that the resulting equilibrium values match observed values of endogenous variables.

References

[1] Anderson, R.M. and R.C. Raimondo, 2007, Incomplete Markets with no Hart Points, *Theoretical Economics*, 2, 115 – 133.

[2] Basu, S., R. Pollack and M.-F. Roy, 2003, *Algorithms in Real Algebraic Geometry*, Springer Verlag, Berlin.

[3] Bates, D.J., J.D. Hauenstein, A.J. Sommese, and C.W. Wampler, 2008, Bertini: Software for Numerical Algebraic Geometry, available at `http://www.nd.edu/~sommese/bertini/`.

[4] Becker, E., M.G. Marianari, T. Mora and C. Treverso, 1994, The Shape of the Shape Lemma, *International Symposium on Symbolic and Algebraic Computation*, Proceedings, 129 – 133.

[5] Blume, L.E. and W.R. Zame, 1992, The Algebraic Geometry of Competitive Equilibrium, in *Economic Theory and International Trade; Essays in Memoriam J. Trout Rader*, ed. by W. Neuefeind and R. Reizman, Springer-Verlag, Berlin.

[6] Bochnak, J., M. Coste and M.-F. Roy, 1998, *Real Algebraic Geometry*, Springer-Verlag, Berlin.

[7] Brown, D.J., P. DeMarzo and C. Eaves, 1996, Computing Equilibra when Asset Markets are Incomplete, *Econometrica* 64, 1 – 27.

[8] Brown, D.J. and F. Kubler, 2008, *Computational Aspects of General Equilibrium Theory* Springer-Verlag, Berlin.

[9] Brown, D.J. and R.L. Matzkin, 1996, Testable Restrictions on the Equilibrium Manifold, *Econometrica* 64, 1249 – 1262.

[10] Cass, D., 2006, Musings on the Cass trick, *Journal of Mathematical Economics*, 42, 374 – 383.

[11] Chiappori, P.-A. and J.-C. Rochet, 1987, Revealed Preferences and Differentiable Demand, *Econometrica* 55, 687 – 691.

[12] Cohen, H., 1996, *A Course in Computational Algebraic Number Theory*, Graduate Texts in Mathematics, Springer-Verlag, Berlin.

[13] Cox, D.A., J.B. Little and D.B. O'Shea, 1997, *Ideals, Varieties and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Undergraduate Texts in Mathematics, Springer-Verlag, New York.

[14] Cox, D.A., J.B. Little and D.B. O'Shea, 1998, *Using Algebraic Geometry*, Graduate Texts in Mathematics, Springer-Verlag, New York.

[15] Faugère, J.C., 1999, A New Efficient Algorithm for Computing Gröbner Bases (f4), *Journal of Pure and Applied Algebra*, 139, 61 – 88.

[16] Greuel G.-M., G. Pfister, and H. Schönemann, 2005, SINGULAR 3.0. A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern. `http://www.singular.uni-kl.de`

[17] Kubler, F. and K. Schmedders, 2000, Computing Equilibria in Stochastic Finance Economies, *Computational Economics* 15, 145 – 172.

[18] Magill, M. and M. Quinzii, 1996, *Theory of Incomplete Markets*, MIT Press, Cambridge, MA.

[19] Mas-Colell, A., 1977, On the Equilibrium Price Set of an Exchange Economy, *Journal of Mathematical Economics* 4, 117 – 126.

[20] Mas-Colell, A., 1985, *The Theory of General Economic Equilibrium: A Differentiable Approach*, Cambridge University Press, Cambridge.

[21] Smale, S., 1974, Global Analysis and Economics IIA, *Journal of Mathematical Economics*, 1, 1 – 14.

[22] Sommese, A.J. and C.W. Wampler, II, 2005, *The Numerical Solution of Systems of Polynomials Arising in Engineering and Science*, World Scientific Publishing Co., Singapore.

[23] Sturmfels, B., 2002, *Solving Systems of Polynomial Equations*, CBMS Regional Conference Series in Mathematics No. 97.

[24] Verschelde, J., 1999, Algorithm 795: PHCpack: A General-purpose Solver for Polynomial Systems by Homotopy Continuation, *ACM Transactions on Mathematical Software*, 25, 251 – 276. Available for download at `http://www.math.uic.edu/∼jan/`.