On a class of integro-differential problems

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Abstract. The paper is concerned with the existence of solutions to an integro-differential problem arising in the neutron transport theory. By an anisotropic singular perturbations method we show that solutions of such a problem exist and are close to those of some nonlocal elliptic problem. The existence of the solutions of the nonlocal elliptic problem with bounded data is ensured by the Schauder fixed point theorem. Then an asymptotic method is applied in the general case. The limits of the solutions of the nonlocal elliptic problems are solutions of our integro-differential problem.

1. Introduction and motivation. The Boltzmann transport equation, governing the neutron distribution in a nuclear reactor leads, by using the Vladimirov method given in [12], to the even-parity second order transport equation. So, let us consider the within-group transport equation for the neutron angular flux $\psi(r, \hat{\Omega})$ with the simplifying restriction that scattering be isotropic

$$\hat{\Omega} \cdot \nabla \psi(r, \hat{\Omega}) + \sigma(r) \psi(r, \hat{\Omega}) - \sigma_s(r) \phi(r) = s(r, \hat{\Omega})$$

coupled with some boundary conditions. The unit vector $\hat{\Omega}$ represents the traveling direction of a neutron, the gradient operator $\nabla$ acts on the spatial variable $r$ only, $\sigma(r)$ is the total macroscopic cross section, $\sigma_s(r)$ is the macroscopic scattering cross section and $s(r, \hat{\Omega})$ is the source term. The scalar flux $\phi(r)$ is given by

$$\phi(r) = \int_{\hat{\Omega}} \psi(r, \hat{\Omega}) d\hat{\Omega}.$$  

We used the physical notation where $\int_{\hat{\Omega}}$ denotes the integration on the set representing the traveling directions. To derive the even-parity transport equation let us decompose $\psi(r, \hat{\Omega})$ into the sum of even and odd angular-parity components

$$\psi^+(r, \hat{\Omega}) = \frac{1}{2} (\psi(r, \hat{\Omega}) + \psi(r, -\hat{\Omega}))$$

$$\psi^-(r, \hat{\Omega}) = \frac{1}{2} (\psi(r, \hat{\Omega}) - \psi(r, -\hat{\Omega}))$$

i.e.

$$\psi(r, \hat{\Omega}) = \psi^+(r, \hat{\Omega}) + \psi^-(r, \hat{\Omega}).$$

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Then the scalar flux is only written in terms of \( \psi^+ \) by
\[
\phi(r) = \int_{\hat{\Omega}} \psi(r, \hat{\Omega}) d\hat{\Omega} = \int_{\hat{\Omega}} \psi^+(r, \hat{\Omega}) d\hat{\Omega}.
\]

On the other hand, rewriting the transport equation (1) for \(-\hat{\Omega}\)
\[
-\hat{\Omega} \cdot \nabla \psi(r, -\hat{\Omega}) + \sigma(r) \psi(r, -\hat{\Omega}) - \sigma_s(r) \phi(r) = s(r, -\hat{\Omega})
\]
and summing (1) and (2) term by term, we get
\[
\hat{\Omega} \cdot \nabla \psi^+(r, \hat{\Omega}) + \sigma(r) \psi^+(r, \hat{\Omega}) - \sigma_s(r) \phi(r) = s^+(r, \hat{\Omega}),
\]
then subtracting them leads to
\[
\hat{\Omega} \cdot \nabla \psi^+(r, \hat{\Omega}) + \sigma(r) \psi^-(r, \hat{\Omega}) = s^-(r, \hat{\Omega}),
\]
whence
\[
\psi^-(r, \hat{\Omega}) = \frac{1}{\sigma(r)} [s^-(r, \hat{\Omega}) - \hat{\Omega} \cdot \nabla \psi^+(r, \hat{\Omega})],
\]
where \( \sigma(r) \) is assumed different from 0. Replacing the term \( \psi^-(r, \hat{\Omega}) \) in (3) using the above identity yields the second order form of the transport equation
\[
-\hat{\Omega} \cdot \nabla \left( \frac{1}{\sigma(r)} \hat{\Omega} \cdot \nabla \psi^+(r, \hat{\Omega}) \right) + \sigma(r) \psi^+(r, \hat{\Omega})
= \sigma_s(r) \int_{\hat{\Omega}} \psi^+(r, \hat{\Omega}) d\hat{\Omega} + s^+(r, \hat{\Omega}) - \hat{\Omega} \cdot \nabla \frac{s^-(r, \hat{\Omega})}{\sigma(r)}.
\]
This is a second order partial differential equation in \( r \) with a nonlocal term given by a partial integral on the angular domain. For more details we refer the reader to [10, 12].

Motivated by the above model we consider in the following some integro-differential problems.

First let \( \omega_1 \) (resp. \( \omega_2 \)) be a bounded open subset of \( \mathbb{R}^m \) (resp. \( \mathbb{R}^n \)) where \( m \) and \( n \) are positive integers. We split the components of a point in \( x \in \mathbb{R}^{m+n} \) into the \( m \) first components and the \( n \) last ones i.e.
\[
X_1 = (x_1, \ldots, x_m) \quad \text{and} \quad X_2 = (x'_1, \ldots, x'_n).
\]

With this notation we set
\[
\nabla u = \left( \begin{array}{c} \nabla X_1 u \\ \nabla X_2 u \end{array} \right) = \left( \begin{array}{c} (\partial_{x_1} u, \ldots, \partial_{x_m} u)^T \\ (\partial_{x'_1} u, \ldots, \partial_{x'_n} u)^T \end{array} \right).
\]

We set
\[
\Omega = \omega_1 \times \omega_2.
\]
Let us denote by \( A = (a_{ij}(x)) \) a \( n \times n \) matrix such that
\[
a_{ij} \in L^\infty(\Omega) \quad \forall i, j = 1, \ldots, n,
\]
and for some \( \lambda > 0 \) we have the ellipticity hypothesis
\[
A \xi \cdot \xi \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \quad \text{a.e. } x \in \Omega.
\]
Let \( a \in C(\mathbb{R}) \) be a continuous function satisfying
\[
a(r) = O(r) \quad \text{when } |r| \to \infty
\]
and \( l \) be the nonlocal term defined as
\[
l(u) = \int_{\omega_1} h(X_1, X'_1, X_2) u(X'_1, X_2) dX'_1, \quad \forall u \in L^2(\Omega),
\]
for some \( h(X_1, X'_1, X_2) \) and \( h \in L^1(\omega_1) \).
where $h$ is a measurable function satisfying
\[ h \in L^\infty(\omega_1 \times \Omega). \]  
Let us consider the integro-differential problem to find $u$ such that
\[
\begin{cases}
-\nabla X_1 (A \nabla X_2 u) + \chi u = a(l(u)) & \text{in } \Omega, \\
u(X_1, \cdot) = 0 & \text{on } \partial \omega_2 \text{ a.e. } X_1 \in \omega_1.
\end{cases}
\]  
where $\chi$ is nonnegative constant. We say that a function $u_0 \in L^2(\Omega)$ such that $u_0(X_1, \cdot) \in H^1_0(\omega_2)$ for a.e. $X_1 \in \omega_1$, is a weak solution of the problem (10) if the integral identity
\[
\int_{\omega_2} A \nabla X_2 u_0 \cdot \nabla X_2 v \, dx = \int_{\omega_2} a(l(u_0)) v \, dx,
\]  
holds for a.e. $X_1 \in \omega_1$ and $\forall v \in H^1_0(\omega_2)$. In order to show the existence of such a solution, we perturb the first equation in (10). This is done by introducing the following anisotropic singular perturbations problem
\[
\begin{cases}
-\varepsilon^2 \Delta X_1 u_\varepsilon - \nabla X_2 (A \nabla X_2 u_\varepsilon) + \chi u_\varepsilon = a(l(u_\varepsilon)) & \text{in } \Omega, \\
u_\varepsilon = 0 & \text{on } \partial \Omega,
\end{cases}
\]  
where $\varepsilon > 0$ will go to 0, $\Delta X_1 = \sum_{i=1}^m \partial_{x_i}^2$ is the usual Laplace operator in $X_1$. It is clear that the problem (12) is a nonlocal semilinear elliptic problem, subject to homogeneous Dirichlet boundary conditions. The theory of the anisotropic singular perturbations is developed in [1]-[5], [8, 9, 11] where different convergence results are shown for linear boundary value problems. In these references we can see that the limit problem (when $\varepsilon = 0$) is a partial differential equation only in $X_2$ possibly parameterized by $X_1$. This is what inspired us to use this asymptotic technique, to show the existence of a solution of the problem (10) as a limit of $u_\varepsilon$ when $\varepsilon \to 0$.

The existence of solutions of the nonlocal elliptic problem (12) is shown in two steps in the following section. In the first one we assume that $a$ is bounded which allows the application of the Schauder fixed point theorem, then when $a$ satisfies the initial hypothesis (7) we establish the existence of a solution $u_\varepsilon$ by using an asymptotic method. In the last section we deal with the asymptotic behaviour of $u_\varepsilon$ solution to (12) when $\varepsilon \to 0$ in order to show that the only possible limits are the solutions of problem (10).

2. Nonlocal elliptic problems. Through all this section we assume that $\varepsilon$ is fixed.

2.1. Bounded data. In this subsection we suppose that $a$ is bounded i.e.
\[ |a(s)| \leq \alpha \quad \forall s \in \mathbb{R}, \]  
where $\alpha$ is a positive constant. Under this assumption we can take $\chi = 0$ (instead of $\chi \geq 0$) since it does not play any role. Our aim here is to study the existence of the solution of the nonlocal elliptic problem (12), using a fixed point argument. This is the subject of the following theorem.

**Theorem 2.1.** Under the assumptions (5), (6), (9) and (13), there exists at least one weak solution to problem (12).
Proof. We use the Schauder fixed point theorem. For \( w \in L^2(\Omega) \), let \( u \in H^1_0(\Omega) \) be the solution to the following linear elliptic problem (see [6, 7])

\[
\begin{aligned}
-\varepsilon^2 \Delta X_1 u - \nabla X_2 (A \nabla X_2 u) &= a(l(w)) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(14)

Define the mapping \( T \) from \( L^2(\Omega) \) into itself by

\[
w \mapsto u = T(w).
\]

(15)

Thus, the pair \((w, u)\) \(\in L^2(\Omega) \times H^1_0(\Omega)\) satisfies

\[
\int_\Omega \varepsilon^2 \nabla X_1 u \cdot \nabla X_1 v + A \nabla X_2 u \cdot \nabla X_2 v \, dx = \int_\Omega a(l(w)) v \, dx, \quad \forall v \in H^1_0(\Omega).
\]

(16)

Taking \( v = u \) we get

\[
\int_\Omega \varepsilon^2 |\nabla X_1 u|^2 + A |\nabla X_2 u|^2 \, dx = \int_\Omega a(l(w)) u \, dx.
\]

The coerciveness assumption (6), (13) and the Cauchy-Schwarz inequality lead to

\[
\int_\Omega \varepsilon^2 |\nabla X_1 u|^2 + \lambda |\nabla X_2 u|^2 \, dx \leq (\int_\Omega [a(l(w))]^2 \, dx)^{1/2} (\int_\Omega u^2 \, dx)^{1/2} \leq \alpha |\Omega|^{1/2} (\int_\Omega u^2 \, dx)^{1/2},
\]

where \(|\Omega|\) denotes the measure of \( \Omega \). Applying the Poincaré inequality in the \( X_2 \) directions and the Young inequality we derive

\[
\int_\Omega \varepsilon^2 |\nabla X_1 u|^2 + \lambda |\nabla X_2 u|^2 \, dx \leq \frac{1}{2\lambda} \alpha^2 C_{\omega_2}^2 |\Omega| + \frac{\lambda}{2} (\int_\Omega |\nabla X_2 u|^2 \, dx),
\]

whence

\[
\int_\Omega \varepsilon^2 |\nabla X_1 u|^2 + \frac{\lambda}{2} |\nabla X_2 u|^2 \, dx \leq \frac{1}{2\lambda} \alpha^2 C_{\omega_2}^2 |\Omega|,
\]

where \( C_{\omega_2} \) is the Poincaré constant in \( \omega_2 \). This means that

\( u \) is bounded in \( H^1(\Omega) \),

(17)

this of course independently of \( w \). In particular one has

\[
|u|_{L^2(\Omega)} \leq C,
\]

where \( C \) is a constant independent of \( w \). Let \( B \) be the closed ball in \( L^2(\Omega) \) centered at the origin with radius \( C \). To apply the Schauder fixed point theorem it only remains to show that \( T \) is continuous on \( B \) for the \( L^2(\Omega) \) topology. Let \( w_n \in B \) be a converging sequence such that

\[
w_n \to w \quad \text{in } L^2(\Omega).
\]

(18)

We set \( u_n = T(w_n) \in B \). Then there exist a subsequence \( n' \) and \( u \in H^1_0(\Omega) \) such that

\[
u_n' \to u \quad \text{in } H^1(\Omega),
\]

(19)

\[
u_n' \to u \quad \text{in } L^2(\Omega),
\]

(20)

\( w_n' \to w \) a.e. in \( \Omega \).

(21)
The pair \((w_{n'}, u_{n'})\) satisfies (16) i.e.
\[
\int_\Omega \varepsilon^2 \nabla_{X_1} u_{n'} \cdot \nabla_{X_1} v + A \nabla_{X_2} u_{n'} \cdot \nabla_{X_2} v \, dx = \int_\Omega a(l(w_{n'})) v \, dx.
\]
Passing to the limit in \(n'\) we derive
\[
\int_\Omega \varepsilon^2 \nabla_{X_1} u \cdot \nabla_{X_1} v + A \nabla_{X_2} u \cdot \nabla_{X_2} v \, dx dt = \lim_{n' \to \infty} \int_\Omega a(l(w_{n'})) v \, dx.
\]
Let us compute the last limit. Thanks to the Lebesgue theorem with (18), (21) we deduce
\[l(w_{n'}) \to l(w) \text{ a.e. in } \Omega.\]
Applying again Lebesgue’s theorem, taking into account the above limit with the continuity of \(a\), leads to
\[
\lim_{n' \to \infty} \int_\Omega a(l(w_{n'})) v \, dx = \int_\Omega a(l(w)) v \, dx, \quad \forall v \in H^1_0(\Omega).
\]
Thus
\[
-\varepsilon^2 \Delta_{X_1} u - \nabla_{X_2} (A \nabla_{X_2} u) = a(l(w)) \text{ in } D'(\Omega).
\]
Since \(u \in H^1_0(\Omega)\) we deduce that
\[u = T(w).\]
We can easily infer that the convergences (19) and (20) hold for the whole sequence \(n\) since problem (16) has a unique solution. Then \(T\) is continuous and by the Schauder fixed point theorem we get the existence of a solution to (12). \(\Box\)

In the following we drop the hypothesis (13) and we show the existence theorem in more general case.

2.2. More general assumptions. In this section we keep the assumption (7) of the introduction for \(a\) and we assume that \(\chi\) is large enough. For instance we suppose
\[
\chi > |\omega_1| |h|_{L^\infty(\omega_1 \times \Omega)} \lim_{|r| \to \infty} \sup_{|r| < \infty} a(r). \tag{22}
\]
Let us introduce a sequence of piecewise linear functions \(\theta_n : \mathbb{R} \to \mathbb{R}\) defined as
\[
\theta_n(r) = \begin{cases} r & \text{if } |r| \leq n, \\ \text{sign}(r)n & \text{if } |r| \geq n,
\end{cases}
\]
and construct continuous functions \(a_n\) defined as
\[a_n = a \circ \theta_n.\]
Then we have

**Theorem 2.2.** Under the assumptions (5)-(7), (9) and (22), there exists at least one weak solution to problem (12).

**Proof.** According to the previous subsection and since \(a_n\) is a bounded function there exists \(u^n \in H^1_0(\Omega)\) solution to
\[
\begin{cases}
-\varepsilon^2 \Delta_{X_1} u^n - \nabla_{X_2} (A \nabla_{X_2} u^n) + \chi u^n = a_n(l(u^n)) \text{ in } \Omega, \\
u^n = 0 \text{ on } \partial \Omega,
\end{cases}
\]
in the weak sense i.e.

\[
\int_{\Omega} \varepsilon^2 \nabla X_1 u^n \cdot \nabla X_1 v + A \nabla X_2 u^n \cdot \nabla X_2 v + \chi u^n v \, dx = \int_{\Omega} a_n(l(u^n)) v \, dx \quad \forall v \in H_0^1(\Omega). \tag{23}
\]

By testing with \( u^n \) and using the ellipticity assumption we derive

\[
\int_{\Omega} \varepsilon^2 |\nabla X_1 u^n|^2 + \lambda |\nabla X_2 u^n|^2 + \chi |u^n|^2 \, dx \leq \int_{\Omega} |a_n(l(u^n)) u^n| \, dx. \tag{24}
\]

Let us first estimate the last term in the above identity. Since \( a \) satisfies (7) and \( \chi \) is large enough (it satisfies (22)) there exists \( r_0 \) such that \( \forall r, |r| \geq r_0 \) we have

\[
|a(r)| \leq \frac{\chi}{|\omega_1| |h|_{L^\infty(\omega_1 \times \Omega)}} |r|. \tag{25}
\]

Thus we set

\[
\Omega_{r_0} = \{ x \in \Omega \mid |\theta_n(l(u^n))| \geq r_0 \}
\]

and rewrite the last term in (24) as

\[
\int_{\Omega} |a_n(l(u^n)) u^n| \, dx = \int_{\Omega/\Omega_{r_0}} |a_n(l(u^n)) u^n| \, dx + \int_{\Omega_{r_0}} |a_n(l(u^n)) u^n| \, dx. \tag{26}
\]

Then by (25) we get

\[
\int_{\Omega} |a_n(l(u^n)) u^n| \, dx \leq |\Omega|^{1/2} \max_{|r| \leq r_0} |a(r)||u^n|_{L^2(\Omega/\Omega_{r_0})} + \frac{\chi}{|\omega_1| |h|_{L^\infty(\omega_1 \times \Omega)}} \int_{\Omega_{r_0}} |u^n| |l(u^n)| \, dx, \tag{27}
\]

since \( |\theta_n(r)| \leq |r|, \forall r \in \mathbb{R} \) and \( \forall n \in \mathbb{N} \). The last integral in the above inequality can be estimated as follows

\[
\int_{\Omega_{r_0}} |u^n| |l(u^n)| \, dx \\
\leq \int_{\Omega_{r_0}} |u^n| \int_{\omega_i} |h(X_1, X_1', X_2)||u^n(X_1', X_2)| \, dX_i' \, dx \\
\leq |h|_{L^\infty(\omega_1 \times \Omega)} \left( \int_{\Omega_{r_0}} |u^n|^2 \, dx \right)^{1/2} \left( \int_{\omega_i} \left( \int_{\omega_i} |u^n(X_1', X_2)|^2 \, dX_i' \right)^2 \, dx \right)^{1/2}.
\]

Applying the Young inequality we derive

\[
\int_{\Omega_{r_0}} |u^n| |l(u^n)| \, dx \\
\leq |h|_{L^\infty(\omega_1 \times \Omega)} \left( \int_{\Omega_{r_0}} |u^n|^2 \, dx \right)^{1/2} \left( \int_{\omega_i} |u^n(X_1', X_2)|^2 \, dX_i' \, dx \right)^{1/2} \\
\leq |\omega_1| |h|_{L^\infty(\omega_1 \times \Omega)} \left( \int_{\Omega_{r_0}} |u^n|^2 \, dx \right)^{1/2} \left( \int_{\Omega} |u^n|^2 \, dx \right)^{1/2}.
\]
Then using this and Young’s inequality in (27) we deduce that
\[
\int_\Omega |a_n(l(u^n)) u^n| dx \\
\leq |\Omega|^{1/2} \max_{|r| \leq r_0} |a(r)||u^n|_{L^2(\Omega/\Omega_{r_0})} + \chi |u^n|_{L^2(\Omega)} |u^n|_{L^2(\Omega_{r_0})}
\]
\[
\leq \frac{1}{2\chi} |\Omega| (\max_{|r| \leq r_0} |a(r)|)^2 + \frac{\chi}{2} |u^n|_{L^2(\Omega/\Omega_{r_0})}^2 + \frac{\chi}{2} |u^n|_{L^2(\Omega)}^2
\]
\[
= C + \chi |u^n|_{L^2(\Omega)}^2,
\]
where \(C = \frac{1}{2\chi} |\Omega| (\max_{|r| \leq r_0} |a(r)|)^2\). Going back to (24) and using the last estimate we obtain
\[
\varepsilon^2 |\nabla X_1 u^n|_{L^2(\Omega)}^2 + \lambda |\nabla X_2 u^n|_{L^2(\Omega)}^2 \leq C, \quad \forall n \in \mathbb{N},
\]
which means that

\[
|\nabla u_n|_{L^2(\Omega)}^2 \text{ is bounded},
\]
this of course independently of \(n\). Then there exist a subsequence \(n'\), \(g \in L^2(\Omega)\) and \(u \in H_0^1(\Omega)\) such that
\[
|u|_{L^2(\Omega)} \leq g \quad \text{a.e. in } \Omega. \tag{31}
\]

Passing to the limit in (23) we get
\[
\int_\Omega \varepsilon^2 \nabla X_1 u \cdot \nabla X_2 v + A \nabla X_2 u \cdot \nabla X_2 v + \chi uv \, dx = \lim_{n' \to \infty} \int_\Omega a_n'(l(u^{n'}))v \, dx. \tag{32}
\]
We can compute the last limit using Lebesgue’s theorem. Indeed, by (29), (30), (31) we get
\[
|l(u^{n'})|(x) \leq \int_{\omega_1} |h(X_1, X_1', X_2)| g(X_1', X_2) dX_1' \quad \text{a.e. in } \Omega, \tag{33}
\]
\[
l(u^{n'}) \to l(u) \quad \text{a.e. in } \Omega. \tag{34}
\]

Then
\[
\theta_{n'}(l(u^{n'})) \to l(u) \quad \text{a.e. in } \Omega.
\]
Moreover the right hand side of (33) is a function in \(L^2(\Omega)\) which we will denote by \(H\). Then it is easy to remark (cf. (25)) that for some constant \(C\) one has
\[
|a_n l(u^{n'})| \leq \max_{|r| \leq r_0} |a(r)| + CH.
\]

Thanks to the continuity of \(a\) we deduce also
\[
a_{n'}(l(u^{n'})) \to a(l(u)) \quad \text{a.e. in } \Omega.
\]

Thus by the Lebesgue theorem it follows that
\[
\int_\Omega \varepsilon^2 \nabla X_1 u \cdot \nabla X_1 v + A \nabla X_2 u \cdot \nabla X_2 v + \chi uv \, dx = \int_\Omega a(l(u))v \, dx. \tag{35}
\]
This completes the proof since we already have \(u \in H_0^1(\Omega)\). \(\square\)

Note that this technique of bounded and a.e. converging subsequence as in (30), (31) will be used subsequently in this paper.
3. **Anisotropic singular perturbations method.** Before stating our asymptotic behaviour result, let us introduce the functional space \( V \) defined as

\[
V := \{ v \in L^2(\Omega) \mid \partial_{x_i}v \in L^2(\Omega), \ i = 1, \cdots, n \ \text{and} \ v(X_1, \cdot) \in H^1_0(\omega_2) \ \text{a.e.} \ X_1 \in \omega_1 \}
\]

equipped with the norm

\[
|u|^2_V = |\nabla_{X_2} u|^2_{L^2(\Omega)} + |u|^2_{L^2(\Omega)}.
\]  

(36)

It is clear that \( V \) is a Hilbert space since if \( u_n \) is a Cauchy sequence in \( V \), there exist \( u \in L^2(\Omega) \) with \( \partial_{x_i}u \in L^2(\Omega), \ i = 1, \cdots, n \) such that \( u_n \to u \) with respect to the norm (36), and for a.e. \( X_1 \in \omega_1 \) -up to a subsequence-

\[
u_n(X_1, \cdot) \to u(X_1, \cdot) \ \text{in} \ H^1(\omega_2).
\]

In this section we also assume that

\[\nabla h \in L^{\infty}(\omega_2; L^2(\omega_1 \times \omega_1)).\]  

(37)

Then let us show the following lemma that plays a principal role in this study.

**Lemma 3.1.** Let \( w_n \in V \) be a sequence converging weakly toward \( w \) in \( V \). Then we have

\[l(w_n) \to l(w) \ \text{in} \ H^1(\Omega)\]

and

\[l(w_n) \to l(w) \ \text{in} \ L^2(\Omega).
\]

**Proof.** Using (37) we have

\[
\int_{\Omega} |\nabla_{X_1} l(w_n)|^2 dx
\]

\[
= \int_{\Omega} \int_{\omega_1} |\nabla_{X_1} h(X_1, X_1', X_2)w_n(X_1', X_2)|^2 dX_1' dx
\]

\[
\leq \int_{\omega_1 \times \omega_2} \int_{\omega_1} |\nabla_{X_1} h(X_1, X_1', X_2)|^2 |w_n(X_1', X_2)|^2 dX_1' dx
\]

\[
\leq \int_{\omega_2} \int_{\omega_1} (|\nabla_{X_1} h(X_1, X_1', X_2)|^2 dX_1') (\int_{\omega_1} |w_n(X_1', X_2)|^2 dX_1') dx
\]

\[
= \int_{\omega_2} (\int_{\omega_1} |w_n(X_1', X_2)|^2 dX_1') (\int_{\omega_1} |\nabla_{X_1} h(X_1, X_1', X_2)|^2 dX_1' dX_1) dx
\]

\[
\leq \left( \text{ess sup}_{\omega_2} \int_{\omega_1 \times \omega_1} |\nabla_{X_1} h(X_1, X_1', X_2)|^2 dX_1' dX_1 \right) \int_{\Omega} w_n^2(x) dx.
\]
Similarly
\[ \int_{\Omega} |\nabla X_2 l(w_n)|^2 \, dx \]
\[ = \int_{\Omega} \left| \int_{\omega_1} \nabla X_2 h(X_1, X_1', X_2) w_n(X_1', X_2) + h \nabla X_2 w_n(X_1', X_2) \, dX_1' \right|^2 \, dx \]
\[ \leq \int_{\omega_1 \times \omega_2} \left( \int_{\omega_1} |\nabla X_2 h||w_n(X_1', X_2)| + |h| \nabla X_2 w_n(X_1', X_2) \, dX_1' \right)^2 \, dx \]
\[ \leq 2 \int_{\omega_1 \times \omega_2} \left( \int_{\omega_1} |\nabla X_2 h(X_1, X_1', X_2)||w_n(X_1', X_2)| \, dX_1' \right)^2 \, dx \]
\[ + 2 \int_{\omega_1 \times \omega_2} \left( \int_{\omega_1} |h(X_1, X_1', X_2)||\nabla X_2 w_n(X_1', X_2) \, dX_1' \right)^2 \, dx \]
\[ \leq 2(ess \sup_{\omega_2} \int_{\omega_1 \times \omega_2} |\nabla X_2 h(X_1, X_1', X_2)|^2 \, dX_1' \, dX_1) \int_{\Omega} (w_n)^2 \, dx \]
\[ + 2(ess \sup_{\omega_2} \int_{\omega_1 \times \omega_2} |h(X_1, X_1', X_2)|^2 \, dX_1' \, dX_1) \int_{\Omega} |\nabla X_2 w_n|^2 \, dx. \]

Then since
\[ w_n \rightharpoonup w \ \text{in} \ V, \]
it follows that \( w_n \) is bounded in \( V \) and by the above estimates we deduce that the sequence \( l(w_n) \) is bounded in \( H^1(\Omega) \). Thus, there exist a subsequence \( n' \) and \( W \in H^1(\Omega) \), such that
\[ l(w_{n'}) \rightarrow W \ \text{in} \ H^1(\Omega), \quad \text{(38)} \]
\[ l(w') \rightarrow W \ \text{in} \ L^2(\Omega). \quad \text{(39)} \]

On the other hand we have for every \( v \in D(\Omega) \)
\[ \int_{\Omega} l(w_{n'}) \, v \, dx = \int_{\Omega} v(X_1, X_2) \left( \int_{\omega_1} h(X_1, X_1', X_2) w_{n'}(X_1', X_2) \, dX_1' \right) \, dx \]
\[ = \int_{\omega_1} \left( \int_{\Omega} h(X_1, X_1', X_2) v(X_1, X_2) w_{n'}(X_1', X_2) \, dX_1' \, dX_2 \right) \, dX_1. \]

Since \( w_n \rightharpoonup w \) in \( V \) it follows that for a.e. \( X_1 \in \omega_1 \)
\[ \int_{\Omega} h(X_1, X_1', X_2) v(X_1, X_2) w_{n'}(X_1', X_2) dX_1' dX_2 \]
\[ \rightarrow \int_{\Omega} h(X_1, X_1', X_2) v(X_1, X_2) w(X_1', X_2) dX_1' dX_2. \]

Using Lebesgue’s theorem we derive
\[ \int_{\omega_1} \left( \int_{\Omega} h(X_1, X_1', X_2) v(X_1, X_2) w_{n'}(X_1', X_2) dX_1' dX_2 \right) \, dX_1 \]
\[ \rightarrow \int_{\omega_1} \left( \int_{\Omega} h(X_1, X_1', X_2) v(X_1, X_2) w(X_1', X_2) dX_1' dX_2 \right) \, dX_1, \]
i.e.
\[ \int_{\Omega} l(w_{n'}) \, v \, dx \rightarrow \int_{\Omega} l(w) \, v \, dx \ \forall v \in D(\Omega). \]

Thus we have
\[ l(w_{n'}) \rightarrow l(w) \ \text{in} \ D'(\Omega). \]
Combining this with (38) and (39) we deduce

\[ W = l(w). \]

Since \( w \) is the unique limit of \( w_n \), the whole sequence \( l(w_n) \) converges to \( l(w) \) i.e.

\[
\begin{align*}
  l(w_n) &\to l(w) \text{ in } H^1(\Omega), \\
  l(w_n) &\to l(w) \text{ in } L^2(\Omega).
\end{align*}
\]

This completes the proof of the lemma.

Next, let us recall the definition of \( \varepsilon - \) nets in metric spaces. This definition is useful when the problem (10) has more than one solution.

**Definition 3.2 (\( \varepsilon - \) nets).** Given a metric space \((X; d)\), a subset \( Y \) of \( X \) is said to be an \( \varepsilon - \) net of another subset \( Y' \), if for all \( x \in Y' \), there exists an \( a \in Y \) such that

\[ d(x, a) < \varepsilon. \]

Now, we are ready to state the main result.

**Theorem 3.3.** Under the assumptions of Theorem 2.2 and if (37) holds, then the problem (11) is equivalent to finding \( u_0 \in V \) such that

\[
\int_{\Omega} A \nabla X_1 u_0 \cdot \nabla X_1 v + \nabla X_2 u_0 \cdot \nabla X_2 v + \chi u_0 v dx = \int_{\Omega} a(l(u_0)) v dx,
\]

\( \forall v \in V, \)

and the set of solutions of (10) is not empty. Moreover if we consider the metric structure of \( V \) corresponding to the norm (36), then for every \( r > 0 \), there exists \( \varepsilon_0 > 0 \) such that the set of the solutions of (10) consists a \( r - \) net of the set

\[ A_{\varepsilon_0} = \{ u_\varepsilon \text{ solution to (12) for } \varepsilon < \varepsilon_0 \}, \]

and we also have

\[ \varepsilon \nabla X_1 u_\varepsilon \longrightarrow 0 \text{ in } L^2(\Omega). \]

This theorem has the following immediate corollary that gives the convergence of \( u_\varepsilon \).

**Corollary 1.** Under the assumptions of Theorem 3.3 and if problem (10) has only one solution \( u_0 \) then we have

\[ u_\varepsilon \longrightarrow u_0, \quad \nabla X_2 u_\varepsilon \longrightarrow \nabla X_2 u_0 \quad \text{and} \quad \varepsilon \nabla X_1 u_\varepsilon \longrightarrow 0 \text{ in } L^2(\Omega). \]

**Remark 1.** We will also see in the proof below that from every subsequence of \( (u_\varepsilon)_\varepsilon \) there exists another converging subsequence to a solution of (10) in the sense of Corollary 1.

**Proof of Theorem 3.3.** Let us take \( v = u_\varepsilon \) in the weak formulation

\[
\int_{\Omega} \varepsilon^2 \nabla X_1 u_\varepsilon \cdot \nabla X_1 v + A \nabla X_2 u_\varepsilon \cdot \nabla X_2 v + \chi u_\varepsilon v dx = \int_{\Omega} a(l(u_\varepsilon)) v dx. \quad (40)
\]

By the ellipticity assumption we derive

\[
\int_{\Omega} \varepsilon^2 |\nabla X_1 u_\varepsilon|^2 + \lambda |\nabla X_2 u_\varepsilon|^2 + \chi(u_\varepsilon)^2 dx \leq \int_{\Omega} |a(l(u_\varepsilon))| u_\varepsilon dx. \quad (41)
\]

As similarly done before we set

\[ \Omega_{r_0} = \{ x \in \Omega \mid |l(u_\varepsilon)| \geq r_0 \}. \]
Then the right hand side of (41) can be written as
\[\int_\Omega |a(l(u_\varepsilon))u_\varepsilon|\,dx = \int_{\Omega/\Omega_{\rho_0}} |a(l(u_\varepsilon))u_\varepsilon|\,dx + \int_{\Omega_{\rho_0}} |a(l(u_\varepsilon))u_\varepsilon|\,dx.\] (42)

By (25) we get
\[\int_\Omega |a(l(u_\varepsilon))u_\varepsilon|\,dx \leq |\Omega|^{1/2} \max_{|r| \leq \rho_0} |a(r)||u_\varepsilon|_{L^2(\Omega/\Omega_{\rho_0})}
+ \frac{\chi}{|\omega_1|\,|h|_{L^\infty(\omega_1 \times \Omega)}} \int_{\Omega_{\rho_0}} |u_\varepsilon||l(u_\varepsilon)|\,dx.
\]

Then using the same argument as in the proof of Theorem 2.2 we deduce that
\[\int_\Omega |a(l(u_\varepsilon))u_\varepsilon|\,dx \leq |\Omega|^{1/2} \max_{|r| \leq \rho_0} |a(r)||u_\varepsilon|_{L^2(\Omega/\Omega_{\rho_0})}
+ \frac{\chi}{|\omega_1|\,|h|_{L^\infty(\omega_1 \times \Omega)}} \int_{\Omega_{\rho_0}} |u_\varepsilon||l(u_\varepsilon)|\,dx.
\]

Then the right hand side of (41) can be written as
\[\int_\Omega |a(l(u_\varepsilon))u_\varepsilon|\,dx \leq \frac{1}{2\chi} |\Omega|\,(\max_{|r| \leq \rho_0} |a(r)|)^2 + \frac{\chi}{2}|u_\varepsilon|_{L^2(\Omega/\Omega_{\rho_0})}^2 + \frac{\chi}{2}|u_\varepsilon|_{L^2(\Omega)}^2 + \frac{\chi}{2}|u_\varepsilon|_{L^2(\Omega)}^2
\]
\[\leq C + \chi|u_\varepsilon|_{L^2(\Omega)},\]
where \(C = \frac{1}{2\chi}|\Omega|\,(\max_{|r| \leq \rho_0} |a(r)|)^2\). Going back to (41) and using the last estimate we obtain
\[\varepsilon^2|\nabla X_1u_\varepsilon|_{L^2(\Omega)}^2 + \lambda|\nabla X_2u_\varepsilon|_{L^2(\Omega)}^2 \leq C,
\]
i.e.
\[u_\varepsilon, \quad |\varepsilon\nabla X_1u_\varepsilon|, \quad |\nabla X_2u_\varepsilon| \quad \text{are bounded in } L^2(\Omega),\] (43)
this of course independently of \(\varepsilon\). It follows that there exist \(u_0 \in L^2(\Omega), u_1 \in (L^2(\Omega))^n\) such that - up to a subsequence
\[u_\varepsilon \rightharpoonup u_0, \quad \nabla X_1u_\varepsilon \rightharpoonup u_1 \quad \text{in } L^2(\Omega).
\]
(The convergence is meant component by component). Of course the convergence in \(L^2(\Omega)\) –weak – implies the convergence in \(D'(\Omega)\) and by the continuity of the derivation in \(D'(\Omega)\) we deduce that
\[u_\varepsilon \rightharpoonup u_0, \quad \nabla X_2u_\varepsilon \rightharpoonup \nabla X_2u_0 \quad \text{in } L^2(\Omega).\] (44)

Also, since \(u_\varepsilon\) is bounded in \(L^2(\Omega)\) we have
\[\int_\Omega \varepsilon \nabla X_1u_\varepsilon \cdot v\,dx = -\int_\Omega \varepsilon u_\varepsilon \text{div}_X v\,dx \rightharpoonup 0, \quad \forall v \in [D(\Omega)]^m.
\]
It follows since \(|\varepsilon\nabla X_1u_\varepsilon|\) is bounded in \(L^2(\Omega)\) that
\[\varepsilon \nabla X_1u_\varepsilon \rightharpoonup 0 \quad \text{in } L^2(\Omega).
\]
We then go back to the equation satisfied by \(u_\varepsilon\) and compute the limit of the right hand side, using Lemma 3.1, we obtain - up to a subsequence -
\[l(u_\varepsilon) \rightharpoonup l(u_0) \quad \text{a.e. in } \Omega.
\]
The continuity of \(a\) gives
\[a(l(u_\varepsilon)) \rightharpoonup a(l(u_0)) \quad \text{a.e. in } \Omega.\] (45)
Together with Lebesgue’s theorem and (25) we obtain
\[\int_\Omega a(l(u_\varepsilon))v\,dx \rightharpoonup \int_\Omega a(l(u_0))v\,dx.\] (46)
Now we also have
\[ \int_{\Omega} A \nabla X_2 u_0 \cdot \nabla X_2 v + \chi u_0 v \, dx = \int_{\Omega} a(l(u_0)) v \, dx. \]
Taking \( v = u_\varepsilon \) in (47) and passing to the limit we get
\[ \int_{\Omega} A \nabla X_2 u_0 \cdot \nabla X_2 u_\varepsilon + \chi u_0 u_\varepsilon \, dx = \int_{\Omega} a(l(u_0)) u_0 \, dx. \]  
Next we expend
\[ I_\varepsilon := \int_{\Omega} \varepsilon^2 \nabla X_1 u_\varepsilon \cdot \nabla X_1 u_\varepsilon \, dx + \int_{\Omega} A \nabla X_2 (u_\varepsilon - u_0) \cdot \nabla X_2 (u_\varepsilon - u_0) \, dx + \chi \int_{\Omega} (u_\varepsilon - u_0)^2 \, dx, \]
to get
\[ I_\varepsilon = \int_{\Omega} \varepsilon^2 \nabla X_1 u_\varepsilon \cdot \nabla X_1 u_\varepsilon + A \nabla X_2 u_\varepsilon \cdot \nabla X_2 u_\varepsilon \, dx + \chi \int_{\Omega} u_\varepsilon^2 \, dx. \]
Using (40) we derive
\[ I_\varepsilon = \int_{\Omega} a(l(u_\varepsilon)) u_\varepsilon \, dx \]
\[ - \int_{\Omega} A \nabla X_2 u_\varepsilon \cdot \nabla X_2 u_\varepsilon \, dx - \int_{\Omega} A \nabla X_2 u_\varepsilon \cdot \nabla X_2 u_0 \, dx \]
\[ + \int_{\Omega} A \nabla X_2 u_0 \cdot \nabla X_2 u_0 \, dx - 2\chi \int_{\Omega} u_\varepsilon u_0 \, dx + \chi \int_{\Omega} u_0^2 \, dx. \]
Thanks to Lemma 3.1, (45) and applying the Lebesgue theorem we deduce that
\[ a(l(u_\varepsilon)) \to a(l(u_0)) \quad \text{in} \quad L^2(\Omega). \]
This with the weak convergence of \( u_\varepsilon \) leads to
\[ \int_{\Omega} a(l(u_\varepsilon)) u_\varepsilon \, dx \to \int_{\Omega} a(l(u_0)) u_0 \, dx. \]
Using this to pass to the limit in \( I_\varepsilon \) we get
\[ \lim_{\varepsilon \to 0} I_\varepsilon = \int_{\Omega} a(l(u_0)) u_0 \, dx - \int_{\Omega} A \nabla X_2 u_0 \cdot \nabla X_2 u_0 \, dx - \chi \int_{\Omega} u_0^2 \, dx = 0, \]
since we have (48). Using the ellipticity assumption we derive
\[ \int_{\Omega} \varepsilon^2 |\nabla X_1 u_\varepsilon|^2 + \lambda |\nabla X_2 (u_\varepsilon - u_0)|^2 \, dx + \chi \int_{\Omega} (u_\varepsilon - u_0)^2 \, dx \leq I_\varepsilon. \]
It follows that
\[ \varepsilon^2 |\nabla X_1 u_\varepsilon| \to 0, \quad \nabla X_2 u_\varepsilon \to \nabla X_2 u_0, \quad u_\varepsilon \to u_0 \quad \text{in} \quad L^2(\Omega). \]
Now we also have
\[ \int_{\omega_1} \int_{\omega_2} |\nabla X_2 (u_\varepsilon - u_0)|^2 \, dX_2 \, dX_1 \to 0. \]
From this it follows that for a.e. $X_1$

$$\int_{\omega_2} |\nabla_{X_2} (u_\varepsilon - u_0)|^2 dX_2 \longrightarrow 0.$$ 

Since

$$\left\{ \int_{\omega_2} |\nabla_{X_2} v|^2 dX_2 \right\}^{\frac{1}{2}}$$

is a norm on $H_0^1(\omega_2)$ and we can easily show that $u_\varepsilon(X_1, \cdot) \in H_0^1(\omega_2)$ (see [2]) we have

$$u_0(X_1, \cdot) \in H_0^1(\omega_2)$$

for a.e. $X_1$. As a consequence we have

$$u_0 \in V.$$ 

Using the same argument as in [4] and taking into account (47) we show that $u_0$ satisfies (11).

Now we have to show that the only possible limits are solutions of (10). For that we suppose that there exist a subsequence of $u_\varepsilon$ still labeled by $u_\varepsilon$ and a constant $c > 0$ such that

$$|u_\varepsilon - u_0|_V > c,$$

for any solution $u_0$ to (10). Of course, from the above proof we can extract again a subsequence that converges to a solution of (10), which contradicts (50). By the same argument we deduce that

$$\varepsilon \nabla_{X_1} u_\varepsilon \longrightarrow 0 \quad \text{in} \ L^2(\Omega).$$

This completes the proof.

**Remark 2.** Thanks to the hypothesis (22), we can also consider the Neumann or the mixed boundary conditions.

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