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Abstract

In this article, I review two recent developments in the theory of computation of general equilibria. First, following Brown, DeMarzo and Eaves (1996) several papers have developed globally convergent algorithms for the computation of general equilibria in models with incomplete asset markets. I review some of the developments in that area. Second, new developments in computational algebraic geometry lead to algorithms to compute effectively all equilibria of systems of polynomial equations. I point out some applications of these algorithms to general equilibrium theory.

Computation of General Equilibria: New Developments

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1 Introduction

After Scarf [19] showed that there exist globally convergent (and effectively applicable) algorithms to compute economic equilibria, there is now a class of computable applied models which are routinely used to evaluate the economic consequences of different taxes and tariff structures (see for example Shoven and Whalley [22]). Research on efficient algorithms for the computation of general equilibria in these models largely took place outside of economics.

A large literature in numerical analysis has developed algorithms that are much faster than Scarf's original method and that can be used for large-scale applications. Efficient iterative schemes, mostly based on global Newton methods now allow applied researchers to solve for competitive equilibria in models with hundreds of commodities and agents (see e.g. Ferris and Pang [8]).

Recently, there has been substantial research in theoretical computer science on the development of polynomial time algorithms for the computation of general equilibria. For most existing methods, the number of operations needed to approximate equilibria within a fixed precision ϵ grows exponentially in $1/\epsilon$. Under restrictive assumptions on preferences, in models without production, researchers have developed algorithms to approximate equilibria 'in polynomial time', i.e. the running time of the algorithm increases polynomially in the input parameters and in the precision with which equilibria are computed. Codenotti et al. [4] give an overview on recent developments along this line.

In this article, I will not discuss any of these practical aspects of the solution of large-scale models. I will instead focus on the following two unrelated developments in the computation of general equilibria in economics.

1. The computation of equilibria in models with time, uncertainty and missing asset markets

2. The computation of all equilibria and the relationship between exact and approximate equilibria in the standard Arrow-Debreu model

2 Models with asset markets

Due to their essential static nature standard computable general equilibrium models suffer from an oversimplified treatment of uncertainty. Agents either solve a static problem or have myopic expectations and the model can therefore not explicitly incorporate investment and saving-decisions. The general equilibrium model with incomplete asset markets (GEI-model) provides a basic framework with several agents and several commodities to incorporate uncertainty and financial markets. See for example Magill and Quinzii [17] for an overview of the literature. The computation of equilibria in these models is challenging because in some specifications equilibria fail to exist while in others they are often numerically unstable.

Kehoe and Prescott [13] argue that real business cycle models provide an alternative way to extend computable general equilibrium to models with time and uncertainty. There is now a large literature on the computation of equilibria in dynamic stochastic economies. This is reviewed elsewhere in this dictionary (see Judd [12]), see also Judd's textbook [11].

In the standard GEI model there are two time periods[1] and S possible states of the world in the second period. There are L perishable commodities available for trade at each state. There are H agents with endowments $e^h \in \mathbb{R}_+^{(S+1)L}$ and utility functions $u^h : \mathbb{R}^{(S+1)L} \to \mathbb{R}$. It is assumed throughout this article that utility functions are smooth in the sense of Debreu[5] (i.e. utility is C^2, strictly increasing, strictly quasi-concave, exhibits non-zero Gaussian curvature and indifference curves do not cut the axes).

There are J assets available for trade. In each state s, asset j pays a bundle of commodities $a_j(s) \in \mathbb{R}^L$. It is without loss of generality to assume that the $LS \times J$ matrix

$$A = \begin{pmatrix} a_1(1) & \dots & a_J(1) \\ \vdots & \ddots & \vdots \\ a_1(S) & \dots & a_J(S) \end{pmatrix}$$

has full rank J. Allowing assets to pay in different commodities is crucial when one wants to extend the model to several time periods and long-lived securities.

In the following, it will be useful to write commodity prices as

$$p = (p(0), p(1), \dots, p(S)) \in \Delta^{(S+1)L-1} = \{p \in \mathbb{R}_+^{(S+1)L} : \sum_i p_i = 1\},$$

[1] Kubler and Schmedders (1999) show how the problem of computation of equilibria in multi-period finance models can be essentially reduced to the two period case.

and the $S \times J$ asset payoff matrix (as a function of spot prices $p(1) \dots p(S)$), $R(p)$, as

$$R(p) = \begin{pmatrix} p(1) \cdot a_1(1) & \dots & p(1) \cdot a_J(1) \\ \vdots & \ddots & \vdots \\ p(S) \cdot a_1(S) & \dots & p(S) \cdot a_J(S) \end{pmatrix}.$$

In part of the discussion we assume an exogenous short-sale constraint, i.e. there is a number $0 < K \leq \infty$ such that the 2-norm of an agent's portfolio must always be less than or equal to K. One can then write an agent's aggregate excess demand function as the solution of his maximization problem in the GEI economy.

$$(z^h(p), \phi^h(p)) = \arg \max_{z \in \mathbb{R}^{L(S+1)}, \phi \in \mathbb{R}^J} u(e^h + z) \qquad \text{s.t.}$$

$$p \cdot z = 0$$
$$((p(1) \cdot z(1), \dots, p(S) \cdot z(S))^T = R(p) \cdot \phi$$
$$\|\phi\| \leq K$$

A GEI equilibrium is a collection of prices, portfolios and a consumption allocation such that markets clear and each agent maximizes her utility, i.e. equilibrium prices p are characterized by $\sum_{h=1}^{H} z^h(p) = 0$.

In a slight idealization (see also the more precise definition in the next section), we assume that the maximization problem can be solved exactly and we define an ϵ-equilibrium as a price $\bar{p}$ such that

$$\| \sum_{h=1}^{H} z^h(\bar{p}) \| < \epsilon.$$

2.1 A general algorithm

Although generally $R(p)$ will have have full rank J, there will be so-called 'bad prices' at which the rank of $R(p)$ drops. When there are no short sale constraints, i.e. $K = \infty$ this leads to a discontinuity of excess demand. Scarf's algorithm fails: No matter how fine the simplicial sub-division, if the algorithm terminates at some $\bar{p}$, one cannot necessarily infer a bound on $\|z(\bar{p})\|$ and hence cannot find an ϵ-equilibrium.

Homotopy continuation methods (see Garcia and Zangwill [9] and Eaves[6]) turn out to be ideally suited for this numerical problem. In order to solve a system of equations $f(x) = 0$, $f : X \to Y$, the basic idea underlying homotopy methods is to find a smooth map $H : X \times [0, 1] \to Y$ with

$$H(x, 1) \equiv f(x) \text{ and } H(x, 0) \equiv g(x),$$

where $g : X \to Y$ has a known unique zero. The map H is called a smooth homotopy. In using homotopy methods it is crucial to set up the function, h, to ensure that there is a smooth path that connects $(x^s, 0)$ with $g(x^s) = 0$ to some $(\bar{x}, 1)$ with $f(\bar{x}) = 1$.

Brown, DeMarzo and Eaves [3] develop a homotopy algorithm which can be shown to be globally convergent in that it finds an ϵ-equilibrium for any $\epsilon > 0$ in a finite number of steps. Following the so-called Cass-trick, it is useful to introduce an unconstrained agent, i.e. to define the first agent maximization problem as

$$z^u(p) = \arg\max_z u^1(e+z) \text{ s.t. } p \cdot z = 0,$$

and aggregate demand as $z(p) = z^u(p) + \sum_{h=2}^H z^h(p)$. Note that $\bar{p}$ is a GEI equilibrium (given that $K = \infty$) if and only if $z(p) = 0$. An ϵ-equilibrium is characterized by $\|z(p)\| < \epsilon$.

Define the expenditure of the unconstrained agent y^u as

$$y^u = (p(1) \cdot z_1^u(p), \ldots, p(S) \cdot z_S^u(p))$$

Define an extended payoff matrix $R^*(p)$ by

$$R^*(p) = [R(p), y^u(p)]$$

and let $R^*_{-i}(p)$ be $R^*(p)$ with the i'th column deleted. For the constrained agents $h = 2, ..., H$ define

$$z^h(p, R^*_{-i}(p)) = \arg\max_{z,\phi} u^h(e^h + z) \text{ s.t. } p \cdot z = 0$$

$$(p(1) \cdot z(1), \ldots, p(S) \cdot z(S))^T = R^*_{-i}(p) \cdot \phi$$

Now consider a family of homotopies, indexed by i

$$H_i(p, t, \theta) = \begin{pmatrix} z^u(p) + t\sum_{h=2}^H z^h(p, R^*_{-i}(p)) \\ R^*(p)\theta \\ \theta \cdot \theta - 1 \end{pmatrix}$$

To prove existence of a homotopy path, Brown et al. [3] show that $\cup_{i=1}^{J+1} H_i^{-1}(0)$ contains a smooth path connecting the starting point to a solution at $t = 1$.

While generically in endowments, a homotopy path turns out to exist, the algorithm is hardly applicable in medium-sized problems, since the number of homotopies one has to consider can become quite large. An alternative is to focus on models with $K < \infty$ (or alternatively, models with transaction costs) or to consider algorithms which might fail in a small class of problems but which are generally more efficient.

2.2 Short-sale constraints

In the presence of short-sale constraints, the excess demand function is continuous and equilibrium existence can be proven with Brouwer's theorem. Therefore, one could presumably use a version of Scarf's algorithm to compute equilibria in this case. However, while there are no new mathematical problems to be solved, the fact that the rank of the asset-payoff matrix can still collapse in equilibrium poses difficult numerical problems. Simple Newton-method based algorithms often do not work (see Kubler and Schmedders [15]) unless one has a starting point very close to the actual solution. It turns out that just as in the problem without short-sale constraints, homotopy continuation methods can provide a basis for reliable algorithms.

Schmedders [20] develops a homotopy algorithm which can be used to solve models with a large number of heterogeneous households and goods. The basic idea of his algorithm is to modify the agents' problem by introducing a homotopy parameter $t \in [0,1]$ as follows.

$$\begin{aligned}(z^h(p,t), \phi^h(p,t)) = \quad & \arg\max_{z \in \mathbb{R}^{L(S+1)}, \phi \in \mathbb{R}^J} u(e^h + z) - (1-t)\tfrac{1}{2}\|\phi\|^2 \text{ s.t.} \\ & p \cdot z = 0 \\ & (p(1) \cdot z(1), \ldots, p(S) \cdot z(S)) = R(p) \cdot \phi \\ & \|\phi\| \leq K\end{aligned}$$

Under the assumptions on utilities this is still a convex problem and the first order Kuhn-Tucker conditions are necessary and sufficient. Schmedders provides various examples that show that even for $K = \infty$ his algorithm, although not guaranteed to converge, performs well in practice.

For $K < \infty$, the Kuhn-Tucker inequalities can be converted into a system of equalities via a change of variables, see Garcia and Zangwill [9] (Chapter 4). Kubler [14], Herings and Schmedders [10] and others subsequently used this idea to solve models with transaction costs, trading constraints and other market imperfections.

Of course, it is an important practical problem how to trace out a homotopy path numerically. See Watson [25] for a theoretical algorithm. For a practical description of numerical homotopy path-following methods see Schmedders [21].

3 Equilibria in semi-algebraic economies

While it is clear that sufficient assumptions for the global uniqueness of competitive equilibria are too restrictive to be applicable to models used in practice, it remains an open problem how serious a challenge the non-uniqueness of competitive equilibrium poses to applied equilibrium modeling. In the presence of multiple equilibria, comparative statics exercises become meaningless. Furthermore, even when for a given specification of the economy equilibria is globally unique, as Richter and Wong[18] point out, the possibility of multiple equilibria for close-by economies implies that it is generally impossible to compute prices and allocations that are close-by exact equilibrium prices and allocations (as opposed to computing prices at which aggregate excess demand is close to zero). In this section I argue that one can solve these problems by focusing on so-called "semi-algebraic" economies.

While the arguments are also applicable to the GEI model, for simplicity, consider a standard Arrow Debreu exchange economy, $(u^h, e^h)_{h=1}^H$. There are H agents trading L commodities. Each agent h has individual endowments $e^h \in \mathbb{R}^L_+$ and 'smooth preferences' characterized by an utility function $u^h : \mathbb{R}^L_+ \to \mathbb{R}$.

A Walrasian equilibrium is a collection of consumption vectors $(x^h)_{h=1}^H$ and prices $p \in \Delta^{L-1}$ such that

$$x^h \in \arg\max_{x \in \mathbb{R}^L_+} u^h(x) \text{ s.t. } p \cdot x \leq p \cdot e^h \tag{1}$$

$$\sum_{h=1}^H (x^h - e^h) = 0. \tag{2}$$

An approximate (ϵ-) equilibrium consists of an allocation an prices such that

$$\|u^h(x^h) - [\max_{x \in \mathbb{R}^L_+} u^h(x) \text{ s.t. } p \cdot x \leq p \cdot e^h]| < \epsilon \tag{3}$$

$$\|\sum_{h=1}^H (x^h - e^h)\| < \epsilon. \tag{4}$$

Given any $\epsilon > 0$, Scarf's algorithm (as well as the more efficient algorithms used in practice) finds a p,x^h which constitute an ϵ-equilibrium.

This leaves open two important theoretical questions.

1. Can one relate the approximate equilibrium prices and allocations, to exact equilibria, i.e. given a computed ϵ-equilibrium $(\bar{p}, (\bar{x}^h))$, does there exist a Walrasian equilibrium $\tilde{p}, (\tilde{x}^h)$ with $\|(\bar{p}, (\bar{x}^h)) - (\tilde{p}, (\tilde{x}^h))\|$ small? Can one find good bounds on this distance which tend to zero as $\epsilon \to 0$?

2. Given an economy $(u^h, e^h)_{h=1}^H$ with N Walrasian equilibria $(p^n, (x^h)^n)_{n=1}^N$ and any $\delta > 0$, is it possible to approximate all N equilibria, i.e. to find N ϵ-equilibria $(\tilde{p}^n, (\tilde{x}^h)^n)_{n=1}^N$ with $\|(p^n, (x^h)^n) - (\tilde{p}^n, (\tilde{x}^h)^n)\| < \delta$, for all $n = 1, ..., N$?

Clearly, the second problem is strictly more difficult to tackle than the first. Richter and Wong [18] show that for general economies even the answer to the first question is negative. In order to obtain positive answers to both questions, one needs to restrict possible preferences. One approach is to assume that better-sets are semi-algebraic sets. I will make the slightly more useful assumption that marginal utilities are semi-algebraic functions.

3.1 Semi-algebraic economies

We assume that for each h, $D_x u^h(x)$ is a semi-algebraic function, i.e. its graph $\{(x, y) \in \mathbb{R}^{2L}_+ : y = D_x u^h(x)\}$ is a finite union and intersection of sets of the form

$$\{(x, y) \in \mathbb{R}^{2L} : g(x, y) > 0\} \text{ or } \{(x, y) \in \mathbb{R}^{2L} : f(x, y) = 0\}$$

for polynomials with real coefficients, f and g.

For practical purposes, the focus on semi-algebraic preferences is quite general. First note, that Afriat's theorem implies that a finite set of observations on an individual's choices that can be rationalized by any utility function, can also be rationalized by semi-algebraic preferences (in fact, Afriat's construction is piece-wise linear). Furthermore, note that the constant elasticity of substitution utility function which is often used in applied work is semi-algebraic if the elasticities of substitution are rational numbers.

It follows from the Tarski-Seidenberg theorem, that for semi-algebraic economies, the answers to both questions above are positive, since the relevant statements can be written as first order sentences (see Basu et al. (2003)). However, algorithmic quantifier elimination which needs to be used to answer general questions in this framework is so computationally inefficient, that for practical purposes this does not help towards solving the above questions for interesting specifications of economies.

Nevertheless, given a semi-algebraic economy, it is possible to find a system of polynomial equations $f(x) = 0$, $f : \mathbb{R}^{H(L+1)+L-1} \to \mathbb{R}^{H(L+1)+L-1}$, and finitely many inequalities $g^i(x) \geq 0$, $g^i : \mathbb{R}^{H(L+1)+L-1} \to \mathbb{R}^M$,$i = 1, ..., N < \infty$ such that $p, (x^h)$ is a Walrasian equilibrium for the economy (u^h, e^h) if and only if there exist $\lambda^h \in \mathbb{R}_{++}$, $h = 1, ..., H$ such that for some $i = 1, ..., N$,

$$f(p, (x^h, \lambda^h)) = 0, \quad g^i(p, (x^h, \lambda^h)) \geq 0.$$

Therefore, the problem of finding Walrasian equilibria reduces to finding the real roots of polynomial systems of equations and verifying polynomial inequalities (see Kubler and Schmedders [16]).

Having reduced the problem of finding Walrasian equilibria to finding roots of a polynomial system of equations, one can then answer Questions 1 and 2 above affirmatively.

3.2 Question 1: Smale's alpha method

Smale's alpha method provides a simple sufficient conditions for approximate zeros to be close to exact zeros and can be viewed as an extension of the Newton-Kanterovich conditions. The following results are from Blum et al [2], Chapter 8.

Let $D \subset \mathbb{R}^n$ be open and let $f : D \to \mathbb{R}^n$ be analytic. For $z \in D$, define $f^{(k)}(z)$ to be the k'th derivative of f at z. This is a multi-linear operator which maps k-tuples of vectors in D into $\mathbb{R}^n$. Define the norm of an operator A to be

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

Suppose that the Jacobian of f at z, $f^{(1)}(z)$ is invertible and define

$$\gamma(z) = \sup_{k \geq 2} \left\| \frac{(f^{(1)}(z))^{-1} f^{(k)}(z)}{k!} \right\|^{\frac{1}{(k-1)}}$$

and

$$\beta(z) = \|(f^{(1)}(z))^{-1} f(z)\|.$$

Theorem 1 *Given a $\bar{z} \in D$, suppose the ball of radius $(1 - \frac{\sqrt{2}}{2})/\gamma(\bar{z})$ around $\bar{z}$ is contained in D and that*

$$\beta(\bar{z})\gamma(\bar{z}) < 0.157.$$

Then there exists a $\tilde{z} \in D$ with

$$f(\tilde{z}) = 0 \text{ and } \|\bar{z} - \tilde{z}\| \leq 2\beta(\bar{z}).$$

While the theorem applies to any locally analytic function, the bound $\gamma(z)$ can in general only be obtained if the system is in fact polynomial. For this case, the bound can be computed fairly easily. Given an ϵ-equilibrium the result gives an immediate bound on the distance between the approximation and an exact Walrasian equilibrium, hence answering Question 1 above.

3.3 Question 2: Polynomial system solving

In the following, I denote the collection of all polynomials in the variable $x_1, x_2, \ldots, x_n$ with coefficients in a field $\mathbb{K}$ by $\mathbb{K}[x_1, \ldots, x_n]$. The for this survey relevant examples of $\mathbb{K}$ are the field of rational numbers $\mathbb{Q}$, the field of real numbers $\mathbb{R}$, and the field of complex numbers $\mathbb{C}$. Polynomials over the field of rational numbers are computationally convenient since modern computer algebra systems perform exact computations over the field $\mathbb{Q}$. Economic parameters are typically real numbers, and equations characterizing equilibria lie in $\mathbb{R}[x]$. The algorithms to compute all solutions to polynomial systems always compute all solutions in an algebraically closed field, in this case $\mathbb{C}[x]$.

Given a polynomial system of equations $f : \mathbb{C}^M \to \mathbb{C}^M$ there are now a variety of algorithm to approximate numerically all complex and real zeros of f. Sturmfels' monograph [24] provides an excellent overview. In this survey, I briefly mention two possible approaches, homotopy continuation methods and solution methods based on Gröbner bases.

At the writing of this article, both approaches are too inefficient to be applicable to large economic models, but they can be used for models with 4-5 households and 4-5 commodities. To find all equilibria for a given economy, homotopy methods seem slightly more efficient, while Gröbner bases allow for statements about entire classes of economies.

3.3.1 All solution homotopies

Solving polynomial systems numerically means computing approximations to all isolated solutions. Homotopy continuation methods can provide paths to all approximate solutions. There are well known bounds on the maximal number of complex solutions of a polynomial system. The basic idea is to start at a generic polynomial system $g(x)$ whose number of roots is at least as large as the maximal number of solutions to $f(x) = 0$ and whose roots are all known. Then one needs to trace out all paths (in complex space) of the homotopy $H(x, t) = tg(x) + (1 - t)f(x)$, which do not diverge to infinity. Smale's alpha method can be applied along the path to ensure that the approximate solutions are close to real exact solutions (see Blum et al. [2]). It can be shown that all solutions to $f(x) = 0$ can be found in this manner.

Sommese and Wampler [23] provide a detailed overview. Applications of these methods in economics have so far been largely restricted to game theory, but the method is also applicable to Walrasian equilibria.

3.3.2 Gröbner basis

For given polynomials $f_1, \ldots, f_k$ in $\mathbb{Q}[x]$ the set

$$I = \{\sum_{i=1}^{k} h_i f_i : h_i \in \mathbb{Q}[x]\} = \langle f_1, \ldots, f_k \rangle,$$

is called the ideal generated by $f_1, \ldots, f_k$. It turns out that under conditions which can often be shown to hold in practice, the so-called 'reduced Gröbner basis' of this ideal,I, in the lexicographic term order has the shape

$$\mathcal{G} = \{x_1 - q_1(x_n), x_2 - q_2(x_n), \ldots, x_{n-1} - q_{n-1}(x_n), r(x_n)\}$$

where r is a polynomial of degree d and the q_i are polynomials of degree $d - 1$.

This basis can be computed exactly, using Buchberger's algorithm (recently, much more efficient versions of the basic algorithm have been developed, see e.g. Faugère [7]). The number of real solutions to the original system then equals the number of real solutions of the univariate polynomial $r(.)$ which can be determined exactly by Sturm's method (see Sturmfels [24]) for details. The roots of $r(.)$ can be approximated numerically with standard methods and the remaining solution to the original system is linear in these roots.

Kubler and Schmedder [16] use the method to test for uniqueness of equilibria in semi-algebraic classes of economies.

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