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Abstract

We review the recent literature in macroeconomics that analyses Markov equilibria in dynamic general equilibrium model. After defining the Markov equilibrium concept we first summarize what is known about the existence and uniqueness of such equilibria in models where sequential equilibria can be obtained by solving a suitable social planner problem. We then discuss the existence problems of Markov equilibria in models where equivalence of equilibrium allocations and solutions to social planner problems cannot be established and review techniques the literature has developed to deal with the existence problem, as well as recent applications of these techniques in macroeconomics.

Markov Equilibria in Macroeconomics

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1 Introduction

We say that a dynamic economy has a Markovian structure (or is Markovian, for short), if the stochastic processes that specify the fundamentals of the economy (such as endowments, preferences and technologies) are Markov processes.[1] In many applications attention is restricted to first order Markov processes in which the probability distributions over fundamentals today are functions exclusively of their values yesterday.

In dynamic economies sequential equilibria are sequences of functions mapping histories of realizations of the stochastic process of the fundamentals into allocations and prices such that all agents in the economy maximize their objectives, given prices, and all markets clear. Under fairly mild conditions (i.e. convexity and continuity assumptions on the primitives) such equilibria exist. However, in order to characterize and compute equilibria it is often useful to look for equilibria of a different form.

Recursive Markov equilibria can be characterized by a state space, a policy-function and a transition functions. The policy function maps the state today into current endogenous choices and prices, and the transition function maps the state today into a probability distribution over states tomorrow (see e.g. the definition in Ljunqvist and Sargent's (2000) textbook). In most of this survey we will use the terms 'Markov equilibria' and 'recursive Markov equilibria' interchangeably, however, below we will also consider Markov-equilibria which are not recursive and will refer to these as *generalized Markov equilibria*. This characterization leaves open of course what are the appropriate state variables that constitute the state space.

Most simply, the state space would consist of the set of possible exogenous shocks governing endowments, preferences and technologies. But other than in exceptional cases (see, e.g. Lucas' (1978) asset pricing application where asset prices are solely functions of the underlying shocks to technology), such a strongly stationary Markov equilibrium does not exist.

[1] Deterministic economies are special cases in which the stochastic processes for the fundamentals have degenerate distributions.

In addition to the exogenous shocks endogenous variables have to be included in the state space to assure existence of a Markov equilibrium. We define as the minimal state space the space of all exogenous shocks and endogenous variables that are payoff relevant today, in that they affect current production or consumption sets or preferences (see Maskin and Tirole, 2001).

We call Markov equilibria with this minimal state space 'simple Markov equilibria'. In the remainder of this essay we want to discuss what we know about the existence and uniqueness of such Markov equilibria, both in general and for important specific example. As it turns out, when equilibria are Pareto efficient and thus equilibrium allocation can be determined by solving a suitable social planner problem simple Markov equilibria can be shown to exist under fairly mild conditions. We therefore discuss this case first. On the other hand, when equilibria are not Pareto efficient, e.g. when markets are incomplete or economic agents behave strategically, forward-looking variables often have to be included for a Markov equilibrium to exist; therefore simple Markov equilibria in the sense defined above do not exist in general. We discuss this case in section 3.

2 Markov Equilibria in Economies where Equilibria are Pareto Optimal

In this section we discuss the existence and uniqueness of simple Markov equilibria in economies whose sequential market equilibrium allocations can be determined as solutions to a suitable social planner problem. In these economies the problem of proving the existence of a Markov equilibrium reduces to showing that the solution of the social planner can be written as a time-invariant optimal policy function of the minimal set of state variables, as defined above.

This is commonly done by re-formulating the optimization problem of the social planner as a functional equation and showing that the optimal Markov policy function generates a sequential allocation which solves the original social planner problem; this is what Bellman (1957) called the principle of optimality. This principle can be established under weak conditions, see Stokey et al. (1989). Equipped with this result the existence of a Markov equilibrium then follows from the existence of a solution to the functional equation associated with the social planner problem.

If the functional equation can be shown to be a contraction mapping (sufficient conditions for this were provided by Blackwell, 1965), then it follows that there exists a unique value function solving the functional equation and an optimal policy correspondence. In addition, the contraction mapping theorem also gives an iterative procedure to find the solution to the functional equation from any starting guess, which is helpful for numerical work.

Under weaker conditions other fixed point theorems may be employed to argue at least for the existence (if not uniqueness) of a solution to the functional equation, with associated optimal Markov policy correspondence. In order to establish that the policy correspondence is actually a function (and thus the Markov equilibrium is unique) in general strict concavity of the return function needs to be assumed. Stokey et al. (1989) provide a summary of the main results in the general theory of dynamic programming.

This technique of analyzing and computing dynamic equilibria in Pareto optimal economies is now widely used in macroeconomics. Its first application can be found in Lucas and Prescott (1971) in their study of optimal investment behavior under uncertainty. Lucas (1978) used recursive techniques to study asset prices in an endowment economy and showed that the Markov equilibrium has a particularly simple from. Kydland and Prescott (1982) showed how powerful these techniques are for a quantitative study of the business cycle implications of the neoclassical growth model with technology shocks to production. The volume by Cooley (1995) provides a comprehensive overview over this line of research.

3 Generalized Markov Equilibria

In models where the first welfare theorem is not applicable (for example models with incomplete financial markets or with distorting taxes), in models where there are infinitely many agents (e.g. OLG models) or in models with strategic interaction existence of simple Markov equilibria (i.e. Markov equilibria with minimal state space) cannot be guaranteed[2] (see Santos (2002), Krebs (2004), Kubler and Schmedders (2002) and Kubler and Polemarchakis (2004) for simple counter-examples). The functional equations characterizing equilibrium have no contraction properties and more general fixed-point theorems than the contraction mapping theorem, such as Schauder's fixed point theorem cannot be applied because it is difficult to guarantee compactness of the space of admissible functions. Coleman (1991) is an important example where existence can be shown. However, his results rely on monotonicity conditions on the equilibrium dynamics which are not satisfied in general models.

In the applied literature a solution to this problem was suggested early on. For example Kydland and Prescott (1980) analyze a Ramsey dynamic optimal taxation problem. To make the problem recursive they add as a state variable last period's marginal utility.

On the theoretical side Duffie et al. (1994) were the first to rigorously analyze situations where recursive equilibria may fail to exist in general equilibrium models. Kubler and Schmedders (2003) and Miao and Santos (2005) refine their approach and make it applicable for computations.[3] We now present their basic idea.

Consider a Markovian economy where a date-event (or node) can be associated with a finite history of shocks, $s^t = (s_0, \dots, s_t)$. The shocks follow a

[2] An important exception are Bewley-style models with incomplete markets where simple recursive Markov equilibria exist, see e.g. Krebs (2005).

[3] Miao and Santos (2005) also give a clear explanation of how this approach relates to the work by Abreu, Pearce and Stacchetti (1990).

Markov chain with support $\mathcal{S} = \{1, ..., S\}$. Denote by $z(s^t)$ the vector of all endogenous variables at node s^t. Typically this would include the vector of household asset holdings across individuals and the capital stock at the beginning of the period, but also prices and endogenous choices at note s^t, as well as shadow variables such as Lagrange multipliers. A competitive equilibrium is a process of endogenous variables $\{z(s^t)\}$ with $z(s^t) \in \mathcal{Z} \subset \mathbb{R}^M$, which solve the optimization problems of all agents in the economy, and clear markets. The set $\mathcal{Z}$ denotes the set of all possible values of the endogenous variables.

We focus on dynamic economic models where an equilibrium can be characterized by a set of equations[4] relating current-period exogenous and endogenous variables to endogenous and exogenous variables next period. Examples of such equations are the Euler equations of individual households, first order conditions of firms, as well as market-clearing conditions for all markets. We assume that such a set of equations characterizing equilibrium is given and denote it by

$$h(\hat{s}, \hat{z}, z_1, \ldots, z_S) = 0.$$

The arguments $(\hat{s}, \hat{z})$ denote the exogenous state variables and endogenous variables for the current period. Note that the endogenous variables might contain variables which were determined in the previous period, such as the capital stock, individuals' assets etc. The variables $(z_s)_{s=1}^S$ denote endogenous variables in the subsequent period, in states $s = 1, \ldots, S$, respectively. We refer to $h(\cdot) = 0$ as the set of "equilibrium equations".

As explained above, to analyze Markov equilibria, one needs to specify an appropriate state space. We assume that the equilibrium set $\mathcal{Z}$ can be written as the product $\mathcal{Y} \times \hat{\mathcal{Z}}$, where $\mathcal{Y}$ denotes the set into which the endogenous state variables fall. In the neoclassical growth model, $\mathcal{Y}$ would consist of the set of possible values of the capital stock, in models with heterogeneous agents one would need to add the set of possible wealth distributions across agents. Unfortunately, as the references cited above show a recursive Markov equilibrium with this state space may not exist. We therefore require a more general notion of Markov equilibrium for these types of economies.

A *generalized Markov equilibrium* consists of a (non-empty valued) "policy correspondence", P that maps the state today into possible endogenous variables today and a "transition function" F that maps the state and endogenous variables today into endogenous variables next period. Formally, the maps

$$P : \mathcal{S} \times \mathcal{Y} \rightrightarrows \hat{\mathcal{Z}} \text{ and } F : \text{ graph}(P) \to \mathcal{Z}^S$$

should satisfy that for all shocks and endogenous variables in the current period, $(\hat{s}, \hat{z}) \in$ graph(P), the transition function prescribes values next period that are consistent with the equilibrium equations, i.e.

$$h(\hat{s}, \hat{z}, F(\hat{s}, \hat{z})) = 0,$$

[4] It is straightforward to incorporate inequality constraints into this framework. For expositional purposes we focus on equations.

and lie in the policy correspondence, i.e.

$$(s, F_s(\hat{s}, \hat{z})) \in \text{ graph}(P) \text{ for all } s \in \mathcal{S}.$$

It follows that a generalized Markov equilibrium is recursive according to our earlier definition, if the associated policy correspondence is single valued. It is simple if the state space is the natural minimal state space.

It is easy to see that Markov equilibria are in fact competitive equilibria in the usual sense. Duffie et al. (1994) show that under mild assumptions on the primitives of the model generalized Markov equilibria exist whenever competitive equilibria exist. The basic idea of their approach is very similar to backward induction, using critically a natural monotonicity property of the inverse of the equilibrium equations.[5]

For practical purposes it is of course crucial that the chosen state space is relatively small and that the Markov equilibrium is recursive. In an asset pricing model with heterogeneous agents, Kubler and Schmedders choose the state space to consist of the beginning-of-period wealth distribution, but can only show the existence of a generalized Markov equilibrium. One cannot rule out that the equilibrium is not recursive, the same value of the state-variables might occur with different values of the endogenous variables. The counter-examples to existence mentioned above show that this is precisely the problem: If for given initial conditions there exist multiple competitive equilibria the one that realizes is pinned down by lagged variables. Without ruling out multiplicity of equilibria, it does not seem possible to prove the existence of recursive equilibria with the natural state space.

Miao and Santos (2005) enlarge the state space with the shadow values of investment of all agents and prove that with this larger state space a recursive Markov equilibrium exists. The basic insight of their approach is that one needs to add variables to the natural state space that uniquely select one out of several possible endogenous variables.

The main practical problem with the approach originated by Duffie et al. (1994) and refined by Miao and Santos (2005) is that it provides a method to construct all Markov equilibria. There might exist some recursive equilibria for the natural (minimal) state space, but this approach naturally solves for all other recursive Markov equilibria as well.[6]

In many recent applications of recursive methods to macroeconomics the focus of researchers studying nonoptimal economies is to find a recursive equilibrium with minimal state space. Notable examples in which even this natural state space is large are Rios-Rull (1996), Heaton and Lucas (1996) and Krusell and Smith (1998). They mark the boundary of economies that currently can be analyzed with recursive techniques.

In dynamic endowment economies with either informational frictions or limited enforceability of contracts constrained-efficient (efficient, subject to the in-

[5] See their original paper, Kubler and Schmedders (2003) or Miao and Santos (2005) for details.

[6] Datta et al. (2005) provide ideas how to solve for the one Markov equilibrium with minimal state space.

formational or enforcement constraints) consumption allocations usually display a high degree of dependence on past endowment shocks, even though the natural state space only contains the current endowment shock. Therefore Markov equilibria with minimal state space do not exist. However, using ideas by Spear and Srivastava (1987) and Abreu, Pearce and Stacchetti (1990) the papers by Atkeson and Lucas (1992) and Thomas and Worrall (1988) demonstrate that nevertheless the constrained social planner problem has a convenient recursive structure if one includes promised lifetime utility as a state variable into the recursive problem. This approach or its close alternative, namely to introduce as additional state variable Lagrange multiplies on the incentive or enforcement constraints (as in Marcet and Marimon, 1999) has seen many applications in macroeconomics, since it allows to make a large class of dynamic models with informational of enforcement frictions recursive and hence tractable. Miao and Santos (2005) show how such problems with strategic interactions can be incorporated into the framework above.

In optimal policy problems in which the government has no access to a commitment technology a recent discussion about the desirability of a restriction to Markov policies with minimal state space has emerged. Such restrictions rule out reputation if one confines attention to smooth policies. See Phelan and Stacchetti (2001) and Klein and Rios-Rull (2003) for examples of the two opposing views on this issue. However, as Krusell and Smith (2003) argue if one allows discontinuous policy functions reputation effects can be generated even with Markov policies.[7]

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