ON SIMPLIFICATION OF THE COMBINATORIAL LINK FLOER HOMOLOGY

ANNA BELIAKOV

ABSTRACT. We give a new combinatorial construction of the hat version of the link Floer homology over \( \mathbb{Z}/2\mathbb{Z} \) and verify that in many examples, our complex is smaller than Manolescu–Ozsváth–Sarkar one.

INTRODUCTION

Knot Floer homology is a powerful knot invariant constructed by Ozsváth–Szabo \([11]\) and Rasmussen \([14]\). In its basic form, the knot Floer homology \( \widehat{HF}(K) \) of a knot \( K \in S^3 \) is a finite–dimensional bigraded vector space over \( \mathbb{F} = \mathbb{Z}/2\mathbb{Z} \)

\[ \widehat{HF}(K) = \bigoplus_{d \in \mathbb{Z}, i \in \mathbb{Z}} \widehat{HF}_d(K, i), \]

where \( d \) is the Maslov and \( i \) is the Alexander grading. Its graded Euler characteristic

\[ \sum_{d, i} (-1)^d \text{rank} \, \widehat{HF}_d(K, i) t^i = \Delta_K(t) \]

is equal to the symmetrized Alexander polynomial \( \Delta_K(t) \). The knot Floer homology enjoys the following symmetry extending that of the Alexander polynomial.

\[ \widehat{HF}_d(K, i) = \widehat{HF}_{d-2i}(K, -i) \tag{1} \]

By the result of Ozsváth–Szabo \([11]\), the maximal Alexander grading \( i \), such that \( \widehat{HF}_i(K, i) \neq 0 \) is the Seifert genus \( g(K) \) of \( K \). Moreover, Yi Ni showed \([7]\), that the knot is fibered if and only if \( \text{rank} \, \widehat{HF}_i(K, g(K)) = 1 \). A concordance invariant bounding from below the slice genus of the knot can also be extracted from knot Floer homology \([9]\). For torus knots the bound is sharp, providing the new proof of Milnor conjecture. A purely combinatorial proof of the Milnor conjecture was given by Rasmussen in \([15]\) by using Khovanov homology \([3]\).

The knot Floer homology was extended to links in \([13]\). The first combinatorial construction of the link Floer Homology was given in \([5]\) over \( \mathbb{F} \) and then in \([6]\) over \( \mathbb{Z} \). Both constructions use grid diagrams of links.

Key words and phrases. Heegaard Floer homology, Maslov index.
Figure 1. A rectangular diagram for $5_2$ knot. The number associated to a domain is minus the winding number for its points. The sets $X$ and $O$ consist of black and white points, respectively.

A grid diagram is a square grid on the plane with $n \times n$ squares. Each square is decorated either with an $X$, an $O$, or nothing. Moreover, every row and every column contains exactly one $X$ and one $O$. The number $n$ is called complexity of the diagram. Following [6], we denote the set of all $O$’s and $X$’s by $O$ and $X$, respectively.

Given a grid diagram, we construct an oriented, planar link projection by drawing horizontal segments from the $O$’s to the $X$’s in each row, and vertical segments from the $X$’s to the $O$’s in each column. We assume that at every intersection point the vertical segment overpasses the horizontal one. This produces a planar rectangular diagram $G$ for an oriented link $L$ in $S^3$. Any link in $S^3$ admits a rectangular diagram (see e.g. [2]). An example is shown in Figure 1.

In [5], [6] the grid lies on the torus, obtained by gluing the top most segment of the grid to the bottommost one and the leftmost segment to the right most one. In the torus, the horizontal and vertical segments of the grid become circles. The MOS complex is then generated by $n$–tuples of intersection points between horizontal and vertical circles, such that exactly one point belongs to each horizontal (or vertical) circle. Two generators $x$ and $y$ are connected by a differential if there exists a rectangle with vertices among $x$ and $y$ and edges lying alternatively on horizontal and vertical circles, which does not contain $X$’s and $O$’s and points among $x$ and $y$. The Alexander grading is given by formula (3) below, and the Maslov grading by (4) plus one. The MOS complex has $n!$ generators. This number greatly succeeds
Figure 2. **Admissible collection of ovals for 5_2 knot.** The dots show a generator with Alexander grading 1.

the rank of its homology. For the trefoil, for example, the number of generators is 120, where the rank of $\hat{\text{HFK}}(3_1)$ is 3.

In this paper, we construct another combinatorial complex computing link Floer homology, which has significantly less generators. All knots with less than 6 crossings admit rectangular diagrams where all differentials in our complex are zero, and the rank of the homology group is equal to the number of generators.

**Main results.** Our construction also uses rectangular diagrams. Given an oriented link $L$ in $S^3$, let $G$ be its rectangular diagram in $\mathbb{R}^2$. Let us draw $2n - 2$ narrow ovals around all but one horizontal and all but one vertical segments of the rectangular diagram $G$. We denote by $S$ the set of unordered $(n-1)$–tuples of intersection points between the horizontal and vertical ovals, such that exactly one point appears on each horizontal (or vertical) oval. We assume throughout this paper that the ovals intersect transversally. An example is shown in Figure 2.

Given $x, y \in S$, it is easy to find an oriented closed curve $\gamma_{x,y}$ composed of arcs belonging to horizontal and vertical ovals, where each piece of a horizontal oval connects a point in $x$ to a point in $y$ (and hence each piece of the vertical one goes from a point in $y$ to a point in $x$). In $S^2 = \mathbb{R}^2 \cup \infty$, there exists a domain $D$ bounded by $\gamma_{x,y}$. Suppose that the orientation of $D$ is induced by the orientation of $S^2$. If $\gamma_{x,y}$ is oriented as the boundary of $D$, then we denote $D$ by $D_{x,y}$ and say that it connects $x$ to $y$; otherwise, $D = D_{y,x}$ and it connects $y$ to $x$. The points $x$ and $y$ are called *corners* of the domain.
Figure 3. The complex \((C(G), \partial)\) with long ovals.

Let \(D_i\) be the closures of the connected components of the complement of ovals in \(S^2\). Let \(D_0\) be the connected component which is unbounded in \(\mathbb{R}^2\). Any domain connecting two generators is of the form \(\sum_i n_i D_i\) with \(n_i \geq 0\). Suppose \(D_{x,y}\) connects \(x\) to \(y\). In [4], Lipshitz defined its Maslov index \(M(D_{x,y})\) as follows. Let \(k\) be the number of the acute corners in \(D_{x,y}\), and \(l\) be the number of the obtuse corners. Then

\[
M(D_{x,y}) = \chi(D_{x,y}) - k/4 + l/4 + n_x + n_y
\]

where \(\chi(D)\) is the Euler characteristic of the domain \(D\) considered as a 2–chain, and \(n_x = \sum_{x_i \in x} n_{x_i}\), with \(n_{x_i} \in \{0, 1/4, 1/2, 3/4\}\) for an isolated, acute, straight or obtuse corner, respectively.

We call a domain \(D\) decomposable if it can be represented as a union of two domains of Maslov index zero and one; otherwise, the domain is indecomposable.

A collection of ovals is called admissible, if all Maslov index one domains without \(X\)’s and \(O\)’s inside, connecting two elements of \(S\), are indecomposable. An example of the admissible collection is given by long ovals, where all ovals are as long as the \(n \times n\) grid and \(D_0\) contains a point among \(X\)’s and \(O\)’s (compare Figure 3).

A chain complex \((C(G), \partial)\) computing the hat version of the link Floer homology of \(L\) over \(\mathbb{F} = \mathbb{Z}/2\mathbb{Z}\) can be defined as follows. The generators are elements of \(S\) for
an admissible collection of ovals. The differential of \( x \in S \) is a sum over all \( y \in S \), such that \( x \) and \( y \) can be connected by a Maslov index one domain, which does not contain \( X \)'s and \( O \)'s. Note that this domain is indecomposable by definition.

The gradings in this complex can be constructed analogously to those in [5]. Suppose \( \ell \) is the number of components of \( L \). Let us introduce an Alexander–grading on the complex \((C(G), \partial)\) as a function \( A : S \to \mathbb{Z}^\ell \), defined as follows.

First, we define a function \( a : S \to \mathbb{Z}^\ell \). For a point \( p \), the \( i \)-th component of \( a \) is minus the winding number of the projection of the \( i \)-th component of the oriented link around \( p \). In the grid diagram, we have \( 2n \) distinguished squares containing \( X \)'s or \( O \)'s. Let \( \{ c_{i,j} \}, i \in \{ 1, \ldots, 2n \}, j \in \{ 1, \ldots, 4 \}, \) be the vertices of these squares. Given \( x \in S \), we set

\[
(3) \quad A(x) = \sum_{x \in x} a(x) - \frac{1}{8} \left( \sum_{i,j} a(c_{i,j}) \right) - \left( \frac{n_1 - 1}{2}, \ldots, \frac{n_\ell - 1}{2} \right),
\]

where here \( n_i \) is the complexity of the \( i \)-th component of \( L \), i.e. the number of horizontal segments on this component.

The homological or Maslov grading is given by a function \( M : S \to \mathbb{Z} \), which is defined as follows. Given two collections \( A, B \) of finitely many points in the plane, let \( I(A, B) \) be the number of pairs \((a_1, a_2) \in A \) and \((b_1, b_2) \in B \) with \( a_1 < b_1 \) and \( a_2 < b_2 \). Define

\[
(4) \quad M(x) = I(x, x) - I(x, \emptyset) - I(\emptyset, x) + I(\emptyset, \emptyset).
\]

It is easy to see that the differential preserves the Alexander–grading and drops the Maslov–grading by one (compare Proposition 2.6 in [6]). Let \( V_i \) be the two–dimensional bigraded vector space spanned by one generator in Alexander and Maslov grading zero and another one in Maslov grading \(-1\) and Alexander grading minus the \( i \)-th basis vector.

**Theorem 1.** Suppose \( G \) is a rectangular diagram of an oriented \( \ell \)–component link \( L \), where the \( i \)-th component of \( L \) has complexity \( n_i \). Then the homology \( H_*(C(G), \partial) \) is equal to \( \widehat{HFK}(L) \otimes \bigotimes_{i=1}^{\ell} V_{n_i}^{-1} \).

In the case, when the ovals used to define \((C(G), \partial)\) are sufficiently short, the complex is simpler than the MOS complex (see Section 3). We prove Theorem 1 by showing that \((C(G), \partial)\) is homotopy equivalent to the Ozsváth–Szabo complex associated to a suitable chosen \( 2n \)–pointed Heegaard diagram of \( S^3 \) compatible with \( L \). The main point is to show that any indecomposable domain of Maslov index one contributes to the differential.
Our complex can easily be generalized to provide other variants of link Floer homology over $\mathbb{F}$. For the simplicity of exposition, we do not discuss these generalizations.

The paper is organized as follows. We recall the construction of the link Floer homology in Section 1. Theorem 1 is proven in Section 2. In Section 3 we discuss computations made by Droz [1].

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1. **Link Floer Homology with Multiple Basepoints**

For the readers convenience, we review here the construction of knot and link Floer homology, considering the case where the link meets the Heegaard surface in extra intersection points. Our exposition follows closely [5, Section 2].

Let $(\Sigma, \alpha, \beta, w, z)$ be a Heegaard diagram for $S^3$, where $\Sigma$ is a surface of genus $g$, $k$ is some positive integer, $\alpha = \{\alpha_1, ..., \alpha_{g+k-1}\}$ are pairwise disjoint, embedded curves in $\Sigma$ which span a half–dimensional subspace of $H_1(\Sigma; \mathbb{Z})$ (and hence specify a handlebody $U_\alpha$ with boundary equal to $\Sigma$), $\beta = \{\beta_1, ..., \beta_{g+k-1}\}$ is another collection of attaching circles specifying $U_\beta$, and $w = \{w_1, ..., w_k\}$ and $z = \{z_1, ..., z_k\}$ are distinct marked points with

$$w, z \subset \Sigma - \alpha - \beta.$$

Let $\{A_i\}_{i=1}^k$ resp. $\{B_i\}_{i=1}^k$ be the connected components of $\Sigma - \alpha$ resp. $\Sigma - \beta$.

We suppose that our basepoints are placed in such a manner that each component $A_i$ or $B_i$ contains exactly one basepoint among the $w$ and exactly one basepoint among the $z$. We can label our basepoints so that $A_i$ contains $z_i$ and $w_i$, and then $B_i$ contains $w_i$ and $z_{\nu(i)}$, for some permutation $\nu$ of $\{1, ..., k\}$.

In this case, the basepoints uniquely specify an oriented link $L$ in $S^3 = U_\alpha \cup U_\beta$, by the following conventions. For each $i = 1, ..., k$, let $\xi_i$ denote the arc in $A_i$ from $z_i$ to $w_i$ and let $\eta_i$ denote the arc in $B_i$ from $w_i$ to $z_{\nu(i)}$. Let $\tilde{\xi}_i \subset U_\alpha$ be an arc obtained by pushing the interior of $\xi_i$ into $U_\alpha$, and $\tilde{\eta}_i$ be the arc obtained by pushing the
interior of $\eta_i$ into $U_\beta$. Now, we can let $L$ be the oriented link obtained as the union
$$\bigcup_{i=1}^{k} \left( \tilde{\xi}_i + \tilde{\eta}_i \right).$$

**Definition 1.1.** In the above case, we say that $(\Sigma, \alpha, \beta, w, z)$ is a $2k$–pointed Heegaard diagram compatible with the oriented link $L$ in $S^3$.

Let $\ell$ denote the number of components of $L$. Clearly, $k \geq \ell$. In the case where $k = \ell$, these are the Heegaard diagrams used in the definition of link Floer homology [13], see also [11], [14]. In the case where $k > \ell$, these Heegaard diagrams can still be used to calculate link Floer homology, in a suitable sense.

**Definition 1.2.** A periodic domain is a two–chain of the form
$$P = \sum_{i=1}^{k} (a_i \cdot A_i + b_i \cdot B_i)$$
which has zero local multiplicity at all of the $\{w_i\}_{i=1}^{k}$. A Heegaard diagram is said to be admissible if every non–trivial periodic domain has some positive and negative local multiplicities at some other points.

For simplicity, we consider now the case where $L$ is a knot. The case of links can be handled with a few minor notational changes, as outlined in Subsection 1.1 below.

Let $(\Sigma, \alpha, \beta, w, z)$ be a Heegaard diagram compatible with an oriented knot $K$. Let us consider the complex $C(\Sigma, \alpha, \beta, w, z)$ generated over $\mathbb{F}$ by intersection points between tori $T_\alpha = \alpha_1 \times \cdots \times \alpha_{g+k-1}$ and $T_\beta = \beta_1 \times \cdots \times \beta_{g+k-1}$ in $\text{Sym}^{g+k-1}(\Sigma)$ endowed with the differential

$$\partial x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\phi \in \pi_2(x,y) \mid M(\phi) = 1, n_{w_i}(\phi) = n_{z_i}(\phi) = 0 \forall i = 1, \ldots, n} \# \left( \frac{M(\phi)}{\mathbb{R}} \right) \cdot y.$$  

where $\pi_2(x,y)$ denotes the space of homology classes of Whitney disks (domains) connecting $x$ to $y$, $M(\phi)$ is the Maslov index of $\phi$, $n_p(\phi)$ denotes the local multiplicity of $\phi$ at the reference point $p$ (i.e. the algebraic intersection number of $\phi$ with the subvariety $\{p\} \times \text{Sym}^{g+k-2}(\Sigma)$), $M(\phi)$ is the moduli space of pseudo–holomorphic representatives of $\phi$, and $\#()$ denotes a count modulo two. In the case when the Heegaard diagram is admissible, the sum in Equation (5) is finite. We refer to [12] for further details.
The relative Alexander grading of two intersection points \( x \) and \( y \) is defined by the formula

\[
A(x) - A(y) = \left( \sum_{i=1}^{n} n_{z_i}(\phi) \right) - \left( \sum_{i=1}^{n} n_{w_i}(\phi) \right).
\]

The absolute \( A \)-grading can be fixed by requiring

\[
\sum_{x \in T_{\alpha} \cap T_{\beta}} t^{A(x)} \equiv \Delta_K(t) \cdot (1 - t^{-1})^{n-1} \pmod{2},
\]

where \( \Delta_K(t) \) is the symmetrized Alexander polynomial which could be made to work over \( \mathbb{Z} \) by introducing signs.

Moreover, there is a relative Maslov grading, defined by

\[
M(x) - M(y) = M(\phi) - 2 \sum_{i=1}^{n} n_{w_i}(\phi).
\]

The relative Maslov grading can also be lifted to an absolute grading as explained in e.g. [5].

For this complex, the function \( A \) defines an Alexander grading which is preserved by the differential. The following proposition shows how to extract the usual knot Floer homology from the above variants using multiple basepoints. The proof is given in [5].

**Proposition 1.1.** Let \((\Sigma, \alpha, \beta, w, z)\) be a \(2k\)-pointed admissible Heegaard diagram compatible with a knot \( K \). Then, we have an identification

\[
H_*(C(\Sigma, \alpha, \beta, w, z), \partial) \cong \widehat{HFK}(K) \otimes V^{\otimes (k-1)},
\]

where \( V \) is the two-dimensional vector space spanned by two generators, one in bigrading \((-1, -1)\), another in bigrading \((0, 0)\).

### 1.1. Modifications for links.

Recall that knot Floer homology has a generalization to the case of oriented links \( L \). For an \( \ell \) component oriented link \( L \) in \( S^3 \), this takes the form of a multi-graded theory

\[
\widehat{HFK}(L) = \bigoplus_{d \in \mathbb{Z}, h \in \mathbb{H}} \widehat{HFK}_d(L, h),
\]

where \( \mathbb{H} \cong H_1(S^3 - L) \cong \mathbb{Z}^\ell \), with the latter isomorphism induced by an ordering of the link components. We sketch now the changes to be made to the above discussion to define link Floer homology for Heegaard diagrams with extra basepoints.

Suppose now that \((\Sigma, \alpha, \beta, w, z)\) is a Heegaard diagram compatible with an oriented link \( L \) in the sense of Definition [5].
Let us label the basepoints keeping track of which link component they belong to. Specifically, suppose \( L \) is a link with \( \ell \) components, and for \( i = 1, \ldots, \ell \), we choose \( n_i \) basepoints to lie on the \( i \)th component. Letting \( S \) be the index set of pairs \( (i, j) \) with \( i = 1, \ldots, \ell \) and \( j = 1, \ldots, n_i \). We now have basepoints \( \{z_{i,j}\}_{(i,j) \in S} \) and \( \{w_{i,j}\}_{(i,j) \in S} \).

We can now form the chain complex \( C(\Sigma, \alpha, \beta, w, z) \) over \( F \) analogous to the version before, generated by intersection points of \( T_{\alpha} \cap T_{\beta} \). This complex has a relative Maslov grading as before. It also has an Alexander grading which in this case is an \( \ell \)–tuple of integers, \( A : T_{\alpha} \cap T_{\beta} \to \mathbb{Z}^\ell \), determined up to an overall additive constant by the formula

\[
A(x) - A(y) = \left( \sum_{j=1}^{n_1} (n_{z_{1,j}}(\phi) - n_{w_{1,j}}(\phi)), \ldots, \sum_{j=1}^{n_\ell} (n_{z_{\ell,j}}(\phi) - n_{w_{\ell,j}}(\phi)) \right).
\]

The indeterminacy in this case can be fixed with the help of Proposition 1.2.

The differential drops Maslov grading by one and preserves the Alexander multi–grading, and hence the homology groups \( H_*(C(\Sigma, \alpha, w, z)) \) inherit a Maslov grading and an Alexander multi–grading.

**Proposition 1.2.** Let \( (\Sigma, \alpha, \beta, w, z) \) be a \( 2k \)–pointed admissible Heegaard diagram compatible with an oriented link \( L \), with \( n_i \) pairs of basepoints corresponding to the \( i \)th component of \( L \). Then, there are multi–graded identifications

\[
H_*(C(\Sigma, \alpha, \beta, w, z), \partial) \cong \widehat{\text{HFK}}(L) \otimes \bigotimes_{i=1}^\ell V_i^{\otimes(n_i-1)},
\]

where \( V_i \) is the two–dimensional vector space spanned by one generator in Maslov and Alexander gradings zero, and another in Maslov grading \(-1\) and Alexander grading corresponding to minus the \( i \)th basis vector.

By the result of \([13]\), the bigraded groups \( \widehat{\text{HFK}}(L) \) are link invariants, i.e. they do not change if

- the complex structure on \( \Sigma \) is varied;
- the \( \alpha \)– and \( \beta \)–curves are moved by isotopies (in the complement of the basepoints);
- the \( \alpha \)– and \( \beta \)–curves are moved by handle–slides (in the complement of the basepoints);
- the Heegaard diagram is stabilized.
1.2. MOS complex. It was shown in [5], that for a link of complexity \( n \) the MOS complex, described in Introduction, coincides with the complex \((C(\Sigma, \alpha, \beta, w, z), \partial)\), where \( \Sigma \) is a torus, \( \alpha \) are \( n \) parallel meridians on \( \Sigma \), \( \beta \) are \( n \) parallel longitudes (both providing \( n \times n \) grid on the torus), and where the basepoints \( w \) and \( z \) are identified with \( X \) and \( O \). Hence, by Proposition 1.2 the homology of this complex is equal to

\[
\hat{HFK}(L) \otimes \bigotimes_{i=1}^{\ell} V_i^{\otimes (n_i-1)}.
\]

2. Proofs

2.1. Shortening of ovals. The next lemma allows to “short” an oval by removing any bigon without basepoints (see Figure 4).

**Lemma 2.1.** Assume that the complex \((C, \partial)\) contains a bigon with corners \( x \) and \( y \), counted for the differential. Then there exists a homotopy equivalent complex \((C', \partial')\) with the set of generators \( S(C') \) obtained from \( S(C) \) by removing all generators containing \( x \) or \( y \).

**Proof.** Let us define the homotopy equivalence explicitly.

Let \( I : S(C') \to S(C) \) and \( P : S(C) \to S(C') \) be the obvious inclusion and projection. For any \( x \in S(C) \), let \( h(x) \) be zero if \( y \notin x \), otherwise \( h(x) \) is obtained from \( x \) by replacing \( y \) by \( x \). We define \( F : S(C) \to S(C') \) and \( G : S(C') \to S(C) \) as follows.

\[
F = P(\text{Id} + \partial h) \quad G = (\text{Id} + h \partial) I
\]

Here \( \text{Id} \) is the identity map. Let us endow the complex \( C' \) with the differential \( \partial' = F \partial G \). It is easy to check that \( \partial'^2 = 0 \) and that \( F \) and \( G \) are chain maps. Indeed, \( \partial' F = F \partial IP \) and \( IP \partial G = G \partial' \), or \( \partial' F = F \partial \) and \( \partial G = G \partial' \) over \( \mathbb{F} \).

Moreover, we have \( GF = \text{Id} = \partial h + h \partial \), i.e. \( GF \) is homotop to the identity on \( C \). On the other hand, \( FG \) is the identity map on \( C' \).

Assume \( \partial(x) = y \) and \( x, y \) do not contain the corners of the bigon \( x \) and \( y \). Then the new differential \( \partial' = P(\partial + \partial h \partial) I \) coincides with \( \partial \), i.e. for all such \( x \) and \( y \), we have \( \partial'(x) = y \). In addition, \( \partial' \) is nonzero the following case. Assume \( a, b \in S(C) \) do not contain \( x \) and \( y \), and \( x, y \in S(C) \) contain these points. Moreover, \( \partial(a) = y \), \( h(y) = x \), and \( \partial(x) = b \), then \( \partial'(a) = b \). \( \square \)
2.2. Counting of indecomposable domains. Assume $G$ is a rectangular diagram of complexity $n$ for an oriented link $L$. Let us denote by $\alpha = \{\alpha_1, \ldots, \alpha_{n-1}\}$ the horizontal ovals and by $\beta = \{\beta_1, \ldots, \beta_{n-1}\}$ the vertical ones, encircling all but one horizontal and vertical segments of $G$, respectively. Then $(S^2, \alpha, \beta, X, O)$ is an admissible 2n–pointed Heegaard diagram compatible with $L$. Indeed, any periodic domain has positive and negative coefficients. For simplicity, we denote by $(F, \partial)$ the complex $(C(S^2, \alpha, \beta, X, O), \partial)$, defined in Section 1. It is generated by $(n - 1)$–tuples of intersection points between ovals (one point on each oval). The differential $\partial$ is given by counting of all Maslov index one domains connecting two generators, which do not contain $X$’s and $O$’s and admit holomorphic representatives.

**Theorem 2.2.** The complex $(F, \partial)$ is homotopy equivalent to a complex with the same generators, where any indecomposable Maslov index one domain connecting two generators and not containing $X$’s and $O$’s, counts for the differential.

*Proof.* Let $(F', \partial')$ be the complex $(C(S^2, \alpha, \beta, X, O), \partial)$ defined with long ovals. By the result of Ozsváth and Szabo, $(F', \partial')$ is homotopy equivalent to $(F, \partial)$. Let $(C, \partial)$ be obtained from $(F', \partial')$ by applying Lemma 2.1 several times. Assume that $(C, \partial)$ has the same set of generators as $(F, \partial)$. We prove the claim by showing that in $(C, \partial)$ any indecomposable Maslov index one domain, connecting two generators and not containing $X$’s and $O$’s, counts for the differential. For domains without straight corners, the proof by induction on the number of boundary components $c$ in the domain.

Let $c = 1$. Any bigon counts by the Riemann mapping theorem. Assume that any indecomposable immersed polygon with $2n$ corners counts. Suppose that our complex has an immersed polygon $D$ with $2(n + 1)$ corners without basepoints inside, connecting two generators. Then by prolongating one oval in the complement of basepoints, $D$ can be split into two polygons, say $P$ and $P'$, with less than $2(n + 1)$ corners (compare Figure 5). Let us denote by $C'$ the complex obtained after prolongating the oval. By Lemma 2.1 $(C, \partial)$ and $(C', \partial')$ are homotopy equivalent. By assumption, $P$ and $P'$ count for the differential in $C'$. Moreover, their corners (after removing the corners of the bigon) define $a$ and $b$ as in the proof of Lemma 2.1 such that $\partial(a) = b$ is given by counting $D = D_{a, b}$.

Assume that for $c = n - 1$, the claim holds. Suppose the complex $C$ has an indecomposable domain $D$ with $c = n$ boundary components and without basepoints inside. Since $M(D) = 1$ and there are no straight corners, $D$ has $n - 1$ obtuse angles by (2). An example with $c = 2$ is drawn in Figure 6.

Let us stretch one oval in $D$, in such a way that $D$ gives rise to a domain $D'$ with $c = n - 1$ boundary components and $n - 1$ obtuse angles, i.e. $D'$ has Maslov
Figure 5. Immersed polygon realizing a differential from black to white points. The prolongated oval is shown by dashed line.

Figure 6. An indecomposable annulus realizing a differential from black to white points. The prolongated oval is shown by dashed line.

index two. Let $x$ and $y$ be the corners of the bigon, obtained after stretching. The stretched oval connects $y$ to some boundary component, say $B$. By Proposition 2.4 below, $y$ can also be connected to $B$ by a unique path inside $D$. Hence, $D'$ can be represented as a union of two domains connecting some generators. The unique path connecting $y$ to $B$ leaves any boundary component along a cut at an obtuse angle. Moreover, if the path goes along the boundary of a component without obtuse angles, then it also meets a component with two obtuse angles (compare Proof of Theorem 2.4). Therefore, the two domains obtained after splitting of $D'$ have Maslov index one. Removing $x$ and $y$ and also the end points of cuts from the corners of
these domains, we obtain $a$ and $b$ as in the proof of Lemma 2.1. This proves that
the domain $D$ counts for the differential $\partial$.

Assume that we have an indecomposable Maslov index one domain with a straight
corner (i.e. one of the boundary components is an oval with a common corner of
two generators). Such domain can be obtained by shortening an oval as shown in
Figure 7. Here again the white corners show the generator $a$ and the black corners
the generator $b$, such that $\partial(a) = b$ is given by counting the domain. A similar
argument works for any indecomposable Maslov index one domain with straight
corners.

2.3. Proof of Theorem 1. By Proposition 1.2 the homology of the complex $(F, \partial)$
is given by the following formula.

$$ H(F, \partial) \cong \widehat{\text{HFK}}(L) \otimes \bigotimes_{i=1}^{\ell} V_{i}^{\otimes(n_{i}-1)}, $$

By Theorem 2.2 $(F, \partial)$ is homotopy equivalent to a complex with the same genera-
tors, where all indecomposable domains count for the differential. In the case where
there are no decomposable domains, the last complex coincides with $(C(G), \partial)$.

2.4. Structure of indecomposable domains. Let us denote by $D_{x,y}$ a Maslov
index one domain connecting $x$ to $y$. Let $c$ be the number of boundary components
of $D_{x,y}$. We call a boundary component bad, if it has no obtuse angles. Let $b$ be the
number of bad components in the domain. The path in $D_{x,y}$ starting at an obtuse
angle and following the horizontal or vertical oval until the boundary of $D_{x,y}$ will
be called a cut at the obtuse angle. There are two cuts at any obtuse angle.
Figure 8. Case $b = 0, c = 2$. a) Decomposable domain. The cuts are shown by dashed lines. b) Indecomposable domain. The corners from $x$ are marked by $x$ and the points from $\tilde{y}$ by $y$.

Definition 2.3. A boundary component $C_1 \subset \partial D_{x,y}$ is called $y$–connected to $C_2 \subset \partial D_{x,y}$ if for any point $y \in C_1$ disjoint from the corners, there exists a unique path starting at $y$ and ending in $C_2$, such that

1) the path goes along cuts or $\partial D_{x,y}$, where the segments of horizontal and vertical ovals alternate along the path;

2) the corners of the path come alternatively from $x$ and $\tilde{y}$, where $\tilde{y}$ contains $y$ and intersection points of cuts with $\partial D_{x,y}$;

3) the first corner belongs to $x$.

Proposition 2.4. If $D_{x,y}$ is indecomposable and has no straight corners, then any two connected components of its boundary are $y$–connected.

Proof. The proof is by induction on $c$ and $b$. Assume $b = 0, c = 2$. If one of the cuts goes from the inner boundary component to itself, the domain is decomposable (see Figure 8 (a)). If it is not the case, then an easy check verifies the claim (compare Figure 8 (b)).

Assume the claim holds for $b = 0, c = n - 1$. Let us add an $n$-th good component $A$ to the domain. Then if there are no cuts meeting $A$, we are done, since two cuts from the obtuse angle $y$–connect $A$ to any other component by the induction hypothesis. If $C \subset \partial D_{x,y}$ is connected to $A$ by a cut, then no cut from $A$ meets $C$, otherwise the domain is decomposable (compare Figure 9 (a)). It remains two possibilities shown in Figure 10. It is easy to check that in both cases $C$ and $A$ are $y$–connected to all other components. For this, it is sufficient to find a required path for two choices of $y$ (before and after one corner) on each component. An example is shown in Figure 11. Note that a particular form of the domain does not matter for the argument.

Assume $b = 1, c = n$. Let us denote by $B$ the bad component. Then there exists a component $E \subset \partial D_{x,y}$ with two obtuse angles. If two cuts from different
Figure 9. Case $b = 0$, $c = n$. Decomposable domain. All angles without cuts are assumed to be acute.

Figure 10. $C$ is connected to $A$ by one or two cuts. All angles without cuts are assumed to be acute.

Figure 11. $C$ is $y$–connected to all other components. The two choices of $y$ are shown by red and blue dots. The connected paths have the corresponding colors.
Figure 12. Case $b = 1$, $c = n$. All drawn components are $y$–connected. All angles without cuts are assumed to be acute.

obtuse angles connect $E$ to $B$ or to $\partial D_{x,y} - B$, then the domain is decomposable. Indeed, in these cases, $E$ is connected to $B$ or to the other components (which are $y$–connected by the induction hypothesis) by two paths, which allow to split the domain into Maslov index one and zero domains, since Maslov index is additive with respect to the union of domains. Therefore, two cuts from one obtuse angle should connect $E$ to $B$ and two cuts from the other obtuse angle to $\partial D_{x,y} - B$. If a cut connecting $E$ to $B$ goes through other boundary components, then no of them can be connected to $\partial D_{x,y} - B$ or to each other; otherwise the domain is decomposable by the same arguments as before. Hence, the situation looks like in Figure 12 where all components are $y$–connected.

If there are components connected to $B$ by one or two cuts, they are also $y$–connected to the remaining components, since $B$ is $y$–connected to them.

In the case $b = k$, $c = n$, we assume that the domain is obtained from $b = k - 1$, $c = n - 1$ domain by adding one bad component $B$ and making one angle obtuse. All other components are $y$–connected to each other by the induction hypothesis. The new obtuse angle should be connected to $B$ by two cuts as in Figure 12. The rest is similar to the previous case. □

3. Computations

3.1. 5_2 knot. Figure 13 shows a rectangular diagram for 5_2 knot of complexity $n = 7$ obtained from the original diagram in Figure 1 by cyclic permutations (compare [2]). The main advantage of this diagram is that there are no regions counted for the differential.
The Alexander grading of a generator is given by the formula $A(x) = \sum_{x \in X} a(x) - 2$. The maximal Alexander grading is equal to one. There are two generators in this grading shown by colored dots in Figure 13. Both of them have Maslov grading 2.

In Alexander grading zero, the homology of our complex is isomorphic to $\hat{\text{HFK}}(5_2, 0) \otimes 6 \hat{\text{HFK}}(5_2, 1)$.

Note that we can obtain 12 generators in Alexander grading zero, just by moving one point of a generator in Alexander grading one to the other side of the oval. In three cases, depicted by white dots there are two possibilities to move a point. This gives 3 another generators. As a result, in Alexander grading zero our complex has 15 generators. Note that all movings drop Maslov index by one.

The homology of our complex is $\hat{\text{HFK}}(K) \otimes V^{n-1}$. To compute $\hat{\text{HFK}}$ it is sufficient to know this homology in all positive Alexander gradings. The negative gradings can be reconstructed by using the symmetry $(1)$.

Using this symmetry, we deduce that $\hat{\text{HFK}}(5_2)$ has rank two in the Alexander–Maslov bigrading $(1, 2)$, rank three in $(0, 1)$, and rank two in the bigrading $(-1, 0)$. To compare, the Alexander polynomial is $\Delta_{5_2}(t) = 2(t + t^{-1}) - 3$. The knot $5_2$ is not fibered and its Seifert genus is one.

**Figure 13.** A complex for $5_2$ knot. The colored dots show generators in the maximal Alexander grading equal to 1. A number assigned to a region is minus the winding number for its points.
3.2. Further examples. Jean–Marie Droz wrote a computer program calculating the homology of our complex [1]. As a by product, his program generates rectangular diagrams of knots and allows to change them by Cromwell–Dynnikov moves.

According to his computations, for all knots with less than 9 crossings and some knots with less than 15 crossings, one can find rectangular diagrams, such that the maximally short ovals are admissible. In all these cases, the number of generators in our complex is significantly smaller than that in the MOS complex.

Indeed, for all knots admitting rectangular diagrams of complexity 10, the number of generators in the MOS complex is $10! = 3'628'800$. Our complex has on average about 50'000 generators. The knot $8n15$ has the maximal number $92'672$ generators, among them only $1'249$ in the positive Alexander gradings.

The knot $12n2000$ admits a rectangular diagram of complexity 12, where $12! = 479'001'600$. Our complex has $1'411'072$ generators with $16'065$ in the positive Alexander gradings.

References


Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
E-mail address: anna@math.unizh.ch