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Abstract

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Dans cette Note on étudie la caractérisation ponctuelle du jacobien des applications BnV au sens des distributions. On étend un résultat bien connu de Müller à une classe plus naturelle de fonctions, en utilisant le théorème de la divergence pour écrire le jacobien comme une intégrale de contour.
Partial Differential Equations

An extension of the identity \( \text{Det} = \text{det} \)

**Une extension de l'identité \( \text{Det} = \text{det} \)**

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**A B S T R A C T**

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**R É S U M É**

Dans cette Note on étudie la caractérisation ponctuelle du jacobien des applications \( BnV \) au sens des distributions. On étend un résultat bien connu de Müller à une classe plus naturelle de fonctions, en utilisant le théorème de la divergence pour écrire le jacobien comme une intégrale de contour.

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1. Introduction

We first define the notion of distributional Jacobian and of \( BnV \) function:

**Definition 1.1.** Let \( \Omega \subset \mathbb{R}^m \) be an open set, assume \( p \) and \( q \) satisfy:

\[
P \geq n - 1, \quad \frac{1}{q} + \frac{n - 1}{p} \leq 1.
\]

For \( u \in L^q \cap W^{1,p}(\Omega, \mathbb{R}^n) \) with \( m \geq n \), we let \( j(u) \) be the \((m - n + 1)-current\) given by the action \( (j(u), \omega) := (-1)^n \int_\Omega u^1 du^2 \wedge \cdots \wedge du^n \wedge \omega \) on forms \( \omega \) in \( C_0^\infty(\Omega) \). The distributional Jacobian of \( u \) is the \((m - n)-current\) \( [Ju] := \partial j(u) \).

We say that a map \( u \in W^{1,p} \cap L^q \) belongs to \( BnV \) if its distributional Jacobian \( [Ju] \) has finite mass (and hence it can be represented by a Radon Measure).

If \( m = n \), \( [Ju] \) is a distribution and a simple calculation gives that \( [Ju] = \frac{1}{m} \text{div}(\text{Cof}(\nabla u)u) \), where \( \text{Cof}(\nabla u) \) is the matrix of cofactors of \( \nabla u \). This case of Definition 1.1 was first introduced by Ball in [2]. Subsequent works by Šverák [17] and Müller and Spector [15] were devoted to analyze the regularity properties of such maps and their applications to problems in elasticity. A powerful theory for these variational problems has been developed by Giaquinta, Modica and Souček (see [9])...
Lemma 2.3.

Let $\nu(\cdot) \in \mathcal{M}$ be a BnV map. Let $\nu$ be the density of the absolutely continuous part of the distributional Jacobian $\nu$ with respect to the Lebesgue measure: $\nu = \nu \mathcal{L}^m + |\nu|^1 = |\nu|^m + |\nu|^1$. Then $\nu(x) = \det \nabla \nu(x)$ for $\mathcal{L}^m$-almost every $x \in \Omega$.

Theorem 1.3. If $u \in L^1 \cap W^{1,1}(\Omega, \mathbb{R}^n)$ is a BnV map, then $\nu(x) = (e_1 \wedge \cdots \wedge e_m) \cdot \nabla^1 u(x) \wedge \cdots \wedge \nabla^n u(x)$ for $\mathcal{L}^m$-a.e. $x \in \Omega$ (see [8], 1.5.2 for the definition of $\nu \mathcal{L}^\omega$).

Theorem 1.2 was originally proved by Müller in [14] assuming $u \in W^{1,p} \cap BnV$ with $p \geq n^2/(n + 1)$. Müller's result was first conjectured by Ball in [2]. Note that, by Sobolev's embedding, $p \geq n^2/(n + 1)$ implies that $u \in L^1$ for some $q$ satisfying (1). Theorem 1.3 was claimed by the first author in [5]. Indeed, the arguments of [5] show Theorem 1.3 assuming Theorem 1.2 and are outlined here in Section 3 for completeness. However, in the aforementioned paper, the first author overlooked that Müller's proof is not valid in the full range of exponents (1).

2. Proof of Theorem 1.2

Similarly to [14], Theorem 1.2 will be proved using a blow up procedure, which needs two lemmas.

Lemma 2.21. If $u \in BnV(B_R, \mathbb{R}^n)$ then for $\mathcal{L}^1$-a.e. $\rho \in (0, R)$:

$$[Ju](B_{\rho}) = \int_{\partial B_{\rho}} u^1 d\tau \cdots \wedge d\tau^n = \int_{\partial B_{\rho}} u^1 d\tau \cdots \wedge d\tau^n, (\tau) d\mathcal{T}^{n-1}.$$  \hspace{1cm} (2)

where $\tau$ is the simple $(n-1)$-vector orienting $\partial B_{\rho}$ as the boundary of $B_{\rho}$.

Proof. Let $\varphi_{\delta, r}$ be a standard Lipschitz cut-off, taking the value 1 for $|x| \leq r - \delta$ and 0 for $|x| \geq r$, with $\varphi_{\delta, r}(x) = (r - |x|)/\delta$ for $r - \delta \leq |x| \leq r$. Let $f(r) := \int_{\partial B_{\rho}} u^1 d\tau \cdots \wedge d\tau^n$: then $f \in L^1([0, 1])$ because of (1) and Fubini's Theorem. This implies that $\mathcal{L}^1$-a.e. $r$ is a Lebesgue point, that is: $\int_{r - \delta}^{r + \delta} |f(s) - f(r)| ds = o(\delta)$. Moreover, $f \phi_{\delta, r}(x) = \int \phi r \phi_{\delta, r}(x) \wedge d\tau \cdots \wedge d\tau^n = \int_{\partial B_{\rho}} u^1 d\tau \cdots \wedge d\tau^n$. Hence at every Lebesgue point $\langle Ju, \varphi_{\delta, r}\rangle \rightarrow \int_{\partial B_{\rho}} u^1 d\tau \cdots \wedge d\tau^n$; on the other hand, by dominated convergence, $\langle Ju, \varphi_{\delta, r}\rangle \rightarrow \langle Ju, \phi_{\delta, r}\rangle$.

Definition 2.2. Let $u \in BnV(\Omega, \mathbb{R}^n)$ and let $x_0 \in B_R \subset \Omega$. We define $u_{\varepsilon}(y) := (u(x_0 + \varepsilon y) - u(x_0))/\varepsilon$.

Lemma 2.3. Let $u$ be as above and set $\delta_0(x) := \alpha(x - x_0)$. Then $[Ju_{\varepsilon}] = \frac{1}{\varepsilon} \delta_{\frac{x_0}{\varepsilon}}[Ju]$.

Proof. Let $\phi \in C_c^\infty(B_1)$ be a test function. Since $\langle Ju_{\varepsilon}, \phi \rangle = \langle Ju_{\varepsilon}, d\phi \rangle$ we have:

$$\langle Ju_{\varepsilon}, \phi \rangle = \frac{1}{\varepsilon} \int_{\partial B_1} u^1(x_0 + \varepsilon y) - u^1(x_0)/\varepsilon \det(\nabla u^2(x_0 + \varepsilon y), \ldots, \nabla u^n(x_0 + \varepsilon y), \nabla \phi(y)) dy$$

$$= \frac{1}{\varepsilon} \int_{\partial B_1} u^1(x) - u^1(x_0)/\varepsilon \det(\nabla u^2(x), \ldots, \nabla u^n(x), \nabla \phi(x - x_0)/\varepsilon) dx$$

$$= \frac{1}{\varepsilon^n} \int_{\Omega} [Ju_{\varepsilon}, d\phi \left( \frac{x - x_0}{\varepsilon} \right)] = \frac{1}{\varepsilon^n} \langle Ju_{\varepsilon}, \phi \left( \frac{x - x_0}{\varepsilon} \right) \rangle. \hspace{1cm} \Box$$

Taking the supremum over $\phi \in C_c^\infty(B_1)$: $\|\phi\|_\infty \leq 1$ we conclude $\|Ju_{\varepsilon}\| = \frac{1}{\varepsilon^n} \|Ju\|$. Since the Radon–Nikodym decomposition commutes with the push forward, $[Ju_{\varepsilon}] = \frac{1}{\varepsilon^n} \delta_{\frac{x_0}{\varepsilon}}[Ju]^1$ and $[Ju_{\varepsilon}] = \frac{1}{\varepsilon^n} \delta_{\frac{x_0}{\varepsilon}}[Ju]^1$, which allows to conclude

$$\|Ju_{\varepsilon}\| = \frac{1}{\varepsilon^n} \langle Ju, \phi \left( \frac{x - x_0}{\varepsilon} \right) \rangle \quad \forall r > 0.$$  \hspace{1cm} (3)
Proof of Theorem 1.2. To simplify the notation we use $u_h$ for the function $u_{h^{-1}}$ given by Definition 2.2. We use formula (2) to the blow-up sequence $(u_h)$ around a “good” point $x_0$ to get $[Ju_h](B_ρ(x_0)) = ∫_{∂B_ρ(x_0)} u^1_h \wedge \cdots \wedge du^n_h$, and hence we let $h \uparrow \infty$ to obtain

$$
ν(χ_0)|B_ρ| = ∫_{∂B_ρ(x_0)} (L \cdot x)^1 L^2 \wedge \cdots \wedge L^n = ∫_{B_ρ(x_0)} (L \cdot x)^1 \cof(L)^1_k \cdot η = \det(L)|B_ρ|,
$$

where $L := \nabla u(χ_0)$ and $η$ is the exterior unit normal to $∂B_ρ$.

Step 1: By the standard theory of Sobolev functions (see [7]), a.e. $x_0 ∈ Ω$ satisfies the following properties:

(a) $\lim_{r \downarrow 0} \frac{1}{r^n} \left[ ∫_{B_r(x_0)} |Ju|^p \right]^{\frac{1}{p}} = o(1) + \frac{h^n}{L} \int_{B_r(0)} v(x) - ν(x_0) \, dx = 0$;

(b) $∇u$ is approximately continuous at $x_0$ and in particular $∫_{B_r(x_0)} |∇u(x) - ∇u(x_0)|^p \, dx = o(r^m)$.

From now on we fix $x_0$ satisfying (a) and (b) and, without loss of generality, we assume $x_0 = 0$. Observe first of all that condition (a) and Eq. (3) imply:

$$
[J_u_h](B_r(0)) = h^n \left[ [Ju]^p \right]^{\frac{1}{p}}(B_r(0)) + ∫_{B_r(0)} |v(x) - ν(x_0)| \, dx = 0 \quad ∀ r > 0.
$$

Step 2: We observe that, being $(u_h)$ a sequence, there is a set of radii $ρ ∈ (0, 1)$ of full measure such that (2) holds for every $h$. Moreover by (b), using Fubini’s and Fatou’s Theorems, for a.e. $ρ$ there exists a subsequence (not relabeled and possibly depending on $ρ$) such that $\nabla u_h → L := \nabla u(0)$ in $L^p(∂B_ρ)$. We fix now a radius $ρ$ with all the properties above and we do not relabel the relevant subsequence. Hence $du^1_h \wedge \cdots \wedge du^n_h → L^2 \wedge \cdots \wedge L^n$ in $L^{p-1}(∂B_ρ)$, since

$$
du^1_h \wedge \cdots \wedge du^n_h \rightarrow L^2 \wedge \cdots \wedge L^n = \sum_1 \left( L^2 \wedge \cdots \wedge (du^1_h - L^1) \wedge \cdots \wedge du^n_h.\right.
$$

In the borderline case $p = (n - 1)$, the convergence is improved to the first Hardy space $H^1(∂B_ρ)$ because of the Coifman–Lions–Meyer–Semmes estimate (see [3]):

$$
\|dv\|_{H^1(∂B_ρ)} \leq C \|dv\|_{L^{n-1}(∂B_ρ)} \leq C \|dv\|_{L^{n-1}(∂B_ρ)}.
$$

Suppose first of all that $p > n - 1$. Then by the Poincaré’s inequality and the Sobolev embedding theorem, the sequence $(u_h)$ is equicontinuous, with the estimate $\|u_h - L \cdot x - C_h\|_{L^{n-1}(∂B_ρ)} \leq C \|∇u_h - L\|_{L^p(∂B_ρ)} → 0$. Here $C_h$ is the average of $u_h$ on $∂B_ρ$. Since $∫_{∂B_ρ} du^1_h \wedge \cdots \wedge du^n_h = 0$, we conclude,

$$
[J_u_h](B_ρ) = ∫_{∂B_ρ} (u^1_h - C_h^1) \, du^2_h \wedge \cdots \wedge du^n_h \rightarrow ∫_{∂B_ρ} (L \cdot x)^1 L^2 \wedge \cdots \wedge L^n = \det(L)|B_ρ|.
$$

Finally if $p = n - 1$ we use the John–Nirenberg embedding and Poincaré’s inequality to get $[u_h - C_h - L \cdot x]|_{BMO} + \|u_h - C_h - L \cdot x\|_{L^1} \leq C \|∇u_h - L\|_{L^n(∂B_ρ)} → 0$. Recall that, by Fefferman’s Theorem, $BMO$ is the dual space of $H^1$ and thus $∫ f g \leq C([f]_{BMO} + \|f\|_{L^1})\|g\|_{H^1}$ whenever $fg$ is integrable (see [16], Chapter IV; take into account that the original Theorem of Fefferman, proved in $\mathbb{R}^n$, must be suitably modified to our situation where the domain is a compact manifold, see [10]). We thus infer that $∫_{∂B_ρ} (u^1_h - C_h^1) du^2_h \wedge \cdots \wedge du^n_h → ∫_{∂B_ρ} (L \cdot x)^1 L^2 \wedge \cdots \wedge L^n = \det(L)|B_ρ|.$

3. Proof of Theorem 1.3

Given a normal current $T ∈ N_k(\mathbb{R}^m)$ and a Lipschitz map $π : \mathbb{R}^m → \mathbb{R}^l$ with $k ≥ l$, we can define a weakly*-measurable map $x → (T, π, x) ∈ N_{k-1}(\mathbb{R}^m)$, uniquely characterized by the validity of the identity $∫_{\mathbb{R}^m} (T, π, x) \, dx = T_L(π(ψ)) \, dπ$ for every $ψ ∈ C^1_c(\mathbb{R}^l)$ (this is the so-called “slicing of the current”, see for instance [8]). In [5], the first author proved a slicing theorem for Jacobians, namely:

**Theorem 3.1.** Let $ι^k : \mathbb{R}^k → \{x\} × \mathbb{R}^k$ be the natural injection of $\mathbb{R}^k$ into $\mathbb{R}^m$, and let $π : \mathbb{R}^{m-k} × \mathbb{R}^k → \mathbb{R}^{m-k}$ a projection, with $k ≥ m$. Denote by $u^i$ the trace $u(x, ·) = u_0(x)$. Then $[Ju], π, x = (-1)^{(m-k)n}ι^k[Ju^i]$. Moreover this property holds separately for the absolutely continuous part and the singular part of $[Ju]$. This theorem allows us to pass from Theorem 1.2 to Theorem 1.3.
Proof of Theorem 1.3. Set $\pi(x) = (x^1, \ldots, x^{m-n})$, and $y = (x^{m-n+1}, \ldots, x^n)$. By Theorem 3.1, $\langle [Ju]^a, f \, d\pi \rangle = \langle [Ju]^a \, L \, d\pi, f \rangle = \int_{\mathbb{R}^{m-n}} \langle [Ju]^a \, L, \pi \rangle (f) \, d\mathcal{L}^{m-n}(x)$. Thus, using Theorem 1.2, we conclude

$$\langle [Ju]^a, f \, d\pi \rangle = \int_{\mathbb{R}^{m-n}} \left( \int_{\mathbb{R}^n} (-1)^{(m-n)n} \det(\nabla_y u(x, y)) f(x, y) \, d\mathcal{L}^n(y) \right) \, d\mathcal{L}^{m-n}(x)$$

$$= \int_{\mathbb{R}^n} \det(\nabla_y u(x, y)) f(x, y) \, dy \wedge d\pi = \int_{\mathbb{R}^m} f(\epsilon_1 \wedge \cdots \wedge e_m \, L \, du^1 \wedge \cdots \wedge du^n, d\pi) \, d\mathcal{L}^m.$$ 

It is easy to show that, for every $A \in GL(n, \mathbb{R})$, the identity $\langle [Ju \circ A], f \, d\pi \rangle = \deg(A) \cdot (A^{-1})^\ast [Ju]$ holds, where $\deg(A)$ is the sign of the determinant of $A$. If then $I$ is a multiindex of length $m - n$, and $\pi^I(x) = (x^1, \ldots, x^{m-n})$, we let $A$ be a permutation matrix satisfying $\pi = \pi^I \circ A$. Then

$$\langle [Ju]^a, f_I \, d\pi^I \rangle = \deg(A) \int_{\mathbb{R}^m} f_I \circ A(\epsilon_1 \wedge \cdots \wedge e_m \, L \, du^1 \circ A) \wedge \cdots \wedge du^n \circ A, d(\pi^I \circ A) \rangle \, d\mathcal{L}^m$$

$$= \deg(A) \int_{\mathbb{R}^m} A^\ast (f_I \, du^1 \wedge \cdots \wedge du^n \wedge d\pi^I) = \int_{\mathbb{R}^m} f_I \, du^1 \wedge \cdots \wedge du^n \wedge d\pi^I.$$ 

It is then sufficient to write a generic form as $\omega = \sum_I f_I \, dx^I$ to conclude the proof. \qedsymbol

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