Some remarks on Liouville type theorems

Brezis, H; Chipot, M; Xie, Y

Posted at the Zurich Open Repository and Archive, University of Zurich
ZORA URL: https://doi.org/10.5167/uzh-12642

Originally published at:
The authors present here elementary proofs of statements related to the Liouville theorem for the equation

$$-\nabla \cdot (A(x)\nabla u(x)) + a(x)u = 0$$

in $\mathcal{D}'(\mathbb{R}^k)$, where $a \in L^\infty_{\text{loc}}(\mathbb{R}^k)$, $a(x) \geq 0$ and $A(x) = (a_{ij}(x))$ is a $k \times k$ uniformly elliptic matrix of bounded measurable coefficients. When $A(x) \equiv \delta_{ij}$ the equation is the so-called stationary Schrödinger equation [see Y. Pinchover, in *Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon’s 60th birthday*, 329–355, Proc. Sympos. Pure Math., 76, Part 1, Amer. Math. Soc., Providence, RI, 2007; MR2310209 (2008e:35002); B. Simon, Bull. Amer. Math. Soc. (N.S.) 7 (1982), no. 3, 447–526; MR0670130 (86b:81001a)]. When $a \equiv 0$ it is well known that every bounded solution has to be constant (see [L. C. Evans, *Partial differential equations*, Amer. Math. Soc., Providence, RI, 1998; MR1625845 (99e:35001); M. Meier, Manuscripta Math. 29 (1979), no. 2-4, 207–228; MR0545042 (80m:35024); J. Moser, Comm. Pure Appl. Math. 14 (1961), 577–591; MR0159138 (28 #2356) and also [H. Berestycki, I. Capuzzo Dolcetta and L. Nirenberg, Topol. Methods Nonlinear Anal. 4 (1994), no. 1, 59–78; MR1321809 (96d:35041); M. Rigoli and A. G. Setti, NoDEA Nonlinear Differential Equations Appl. 9 (2002), no. 1, 15–36; MR1891293 (2002k:35096) for some nonlinear versions]. The case where $a \neq 0$ and $k \geq 3$ is very different and in this case nontrivial bounded solutions might exist. Many of the results in this paper are known in one form or another [see S. Agmon, in *Differential equations (Birmingham, Ala., 1983)*, 7–17, North-Holland, Amsterdam, 1984; MR0799327 (87a:35060); C. J. K. Batty, Math. Ann. 292 (1992), no. 3, 457–492; MR1152946 (93g:47050); W. Arendt, C. J. K. Batty and P. Bénilan, Math. Z. 209 (1992), no. 4, 511–518; MR1156433 (93i:47057); A. A. Grigor’yan, Trudy Sem. Petrovsk. No. 14 (1989), 66–77, 265–266; MR1001354 (90m:35050); Bull. Amer. Math. Soc. (N.S.) 36 (1999), no. 2, 135–249; MR1659871 (99k:58195); A. A. Grigor’yan and W. Hansen, Math. Ann. 312 (1998), no. 4, 659–716; MR1660247 (2000a:58092); Y. Pinchover, Differential Integral Equations 5 (1992), no. 3, 481–493; MR1157482 (93b:35035); R. G. Pinsky, *Positive harmonic functions and diffusion*, Cambridge Univ. Press, Cambridge, 1995; MR1326606 (96m:60179); Trans. Amer. Math. Soc. 360 (2008), no. 12, 6545–6554; MR2434298 (2009i:35058)], but the proofs presented here are based on simple self-contained PDE techniques. For instance, Liouville-type results are proved in the case where the growth of $u$ is controlled (i.e., for $r$ large $r^{-2} \int r^{-\beta} \int r^{-\beta} \Omega u^2dx \leq C$, with $\Omega$ being a bounded domain containing the origin), in the case of decay of $a(x)$ (i.e., for $|x|$ large $a(x) \geq \frac{c}{|x|^\beta}$, $\beta < 2$ or $\beta = 2$ and in such a case $a_{ij} \in C^1(\mathbb{R}^k \setminus B(0, R))$ with $|\partial_i(a_{ij}(x)) \cdot x_j | \leq D$ for $|x| > R$ and some $R > 0$) and in the case where at infinity $a$ has enough mass locally (i.e., $a(x) \geq a_0 > 0$ at infinity or $a(x) \geq a_p$ where $a_p$ is a periodic function). When $A_{ij} \equiv \delta_{ij}$ bounded nontrivial solutions exist if $a \neq 0$ and $\int_{|x| > 1} a(x)|x|^{-k+2}dx < \infty$ for $k \geq 3$, and do not exist if $a(x) \geq \pi(|x|)$ for $|x|$ large with $\int \infty r\pi(r)dr = +\infty$. 

---

**Some remarks on Liouville type theorems. (English summary)**

{For the entire collection see MR2416199 (2009f:35002)}

Luisa Moschini

© Copyright American Mathematical Society 2010, 2015