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On a hyperbolic-parabolic system arising in magnetoelasticity

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Abstract

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1 Introduction

The evolution equation of spin fields in ferromagnets introduced by Landau and Lifschitz [8], and derived in an equivalent form by Gilbert [6], reads

\[ \gamma^{-1} \mathbf{m}_t = - \mathbf{m} \times (\mathbf{H}_{\text{eff}} + \mathbf{m}_t). \]

(1.1)

The unknown \( \mathbf{m} \), the magnetization vector, is a map from \( \Omega \) (a bounded open set of \( \mathbb{R}^d \), \( d \geq 1 \)) to \( S^2 \) (the unit sphere of \( \mathbb{R}^3 \)) and \( \gamma \) is a positive constant which represents the damping factor introduced to describe dissipative local phenomena. The magnetization distribution is well described by a free energy functional which we assume to be composed of three terms, namely the exchange energy \( E_{\text{ex}} \), the elastic energy \( E_{\text{el}} \) and the elastic-magnetic energy \( E_{\text{em}} \). We neglect other contributions to the free energy due, for example, to anisotropy and demagnetization terms.

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Let $u$ be the displacement vector, then the total free energy $E$ for a deformable ferromagnet is given by

$$E(m, u) = E_{ex}(m) + E_{em}(m, u) + E_{el}(u).$$

The effective field $H_{eff}$ is obtained by the first variational derivative with respect to $m$ of the total free energy $E(m, u)$ that is formally

$$H_{eff} = \delta_m E(m, u). \quad (1.2)$$

To the equation (1.1) we associate the evolution equation for the displacement $u$ which we formally write as

$$\rho u_{tt} = -\delta_u E(m, u), \quad (1.3)$$

where $\rho$ is a positive parameter. Qualitative and numerical results concerning the evolution models for ferromagnets mechanically at rest have been obtained by several authors. We quote here the first existence theorem due to Visintin [14] and the next results, concerning also systems with further dissipation terms in [1], [2], [10], [15]. Nonuniqueness and singularities are established in [1], [11], [4], [12] in a single theoretical framework which includes also other applications as the heat flow of harmonic maps.

In [13] the 3D-dimensional model (1.1)-(1.3) for magnetoelastic materials has been studied. The existence of weak solutions of the proposed nonlinear hyperbolic-parabolic differential system has been proved combining the Faedo-Galerkin approximations and the penalty method, moreover some asymptotic behaviors have been deduced from compactness properties.

For obtaining uniqueness results one has to look at simplified models. It is a common practice (see for example [2], [5], [7], [11], [13]) to replace equation (1.1) by the quasilinear parabolic equation (Ginzburg-Landau type equation)

$$m_t + \gamma^{-1} m_t \times m = -H_{eff} - \frac{|m|^2 - 1}{\varepsilon} m, \quad (1.4)$$

where $\varepsilon$ is a small positive parameter and $m : \Omega \rightarrow \mathbb{R}^3$. Indeed, the last term of (1.4) has been introduced in order to represent the constraint $|m| = 1$ in the limit $\varepsilon \to 0$. We focus our attention to the hyperbolic-parabolic system (1.3)-(1.4). In the next section we detail the three energetic terms and propose a simplified one-dimensional dynamical model.

Although the proposed simplified model does not take into account some specific aspects of the magnetoelastic materials, some interesting features arise in the study of the equations. We report, in section 3, the results obtained in [5] by the variational analysis of the associated static problem where a bifurcation phenomenon (see also [3] for a similar result) appears in the minimization of the energy functional. The proof of the existence of a unique solution to the proposed simplified dynamical problem is given in section 4.
2 The model

We start with detailing the terms of the energy $E(m, u)$ in the general 3D case. Let $x_i, i = 1, 2, 3$ be the position of a point $x$ of $\Omega$ and denote by

$$u_i = u_i(x), \quad i = 1, 2, 3$$

the components of the displacement vector $u$ and by

$$\epsilon_{kl}(u) = \frac{1}{2}(u_{k,l} + u_{l,k}), \quad i, j = 1, 2, 3$$

the deformation tensor where, as a common praxis, $u_{k,l}$ stands for $\partial u_k / \partial x_l$.

Moreover we denote by

$$m_j = m_j(x), \quad j = 1, 2, 3$$

the component of the unit magnetization vector $m$. In the sequel, where not specified, the Latin indices vary in the set $\{1, 2, 3\}$ and the summation of the repeated indices is assumed. We define

$$E_{ex}(m) = \frac{1}{2} \int_\Omega a_{ij} m_{k,i} m_{k,j} d\Omega, \quad (2.1)$$

where $(a_{ij})$ is a symmetric positive definite matrix which is supposed to be diagonal for most materials with all diagonal elements equal to a positive number $a$ (in the sequel we assume $a \equiv 1$). The magneto-elastic energy for cubic crystals is assumed, that implies

$$E_{em}(m, u) = \frac{1}{2} \int_\Omega \lambda_{ijkl} m_i m_j \epsilon_{kl}(u) d\Omega, \quad (2.2)$$

where $\lambda_{ijkl} = \lambda_1 \delta_{ijkl} + \lambda_2 \delta_{ij} \delta_{kl} + \lambda_3 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$ with $\delta_{ijkl} = 1$ if $i = j = k = l$ and $\delta_{ijkl} = 0$ otherwise. Finally we introduce the elastic energy

$$E_{el}(u) = \frac{1}{2} \int_\Omega \sigma_{klmn} \epsilon_{kl}(u) \epsilon_{mn}(u) d\Omega, \quad (2.3)$$

where $\sigma_{klmn}$ is the elasticity tensor satisfying the following symmetry property

$$\sigma_{klmn} = \sigma_{mnkl} = \sigma_{lkmn}$$

and moreover the inequality

$$\sigma_{klmn} \epsilon_{kl} \epsilon_{mn} \geq \beta \epsilon_{kl} \epsilon_{kl}$$

holds for some $\beta > 0$.

A simplified energy functional and hence a simplified dynamical model can be obtained assuming that $\Omega$ is a subset of $\mathbb{R}$ and neglecting some components of the unknowns $u$ and $m$. More precisely we consider the single space variable $x$ and assume $\Omega = (0, 1), u = (0, w, 0)$ and $m = (m_1, m_2, 0)$. Then one has

$$\epsilon_{kl}(w) = \epsilon_{12}(w) = \epsilon_{21}(w) = \frac{1}{2} w_x, \quad (2.4)$$

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\[ \lambda_{ijkl} = \lambda_{ij12} = \lambda_3 (\delta_{i1} \delta_{j2} + \delta_{i2} \delta_{j1}) = \lambda_{ij21}, \quad (2.5) \]

and the different energies are now

\[ E_{ex}(m) = \frac{1}{2} \int_0^1 |m|_x^2 dx, \quad ((a_{ij}) = a \cdot Id = Id), \quad (2.6) \]

\[ E_{em}(m, u) = \frac{\lambda}{2} \int_0^1 (m_1 m_2 + m_2 m_1) w_x dx \quad (\lambda_3 = \lambda), \quad (2.7) \]

\[ E_{el}(u) = \frac{1}{2} \int_0^1 w_x^2 dx \quad (\sigma_{1221} = 1). \]

In order to deal with the constraint \(|m| = 1\), we introduce the penalization term

\[ \frac{1}{4\epsilon} \int_0^1 (|m|^2 - 1)^2 dx. \]

If for \(m = (m_1, m_2)\) we define the linear operator \(\Lambda\) by \(\Lambda(m) = (m_2, m_1)\) and neglect the cross term \(\gamma^{-1} m_t \times m\), the system (1.3)-(1.4) reduces to

\[
\begin{align*}
\{ & m_t + m \frac{|m|^2 - 1}{\epsilon} + \lambda \Lambda(m) w_x - m_{xx} = 0, \\
& w_{tt} - w_{xx} - \frac{\lambda}{2} (\Lambda(m) \cdot m)_x = 0,
\}
\end{align*}
\quad (2.10)
\]

in \(Q = \Omega \times (0, T]\).

The following initial and boundary conditions are assumed,

\[ w(\cdot, 0) = w_0, \quad w_t(\cdot, 0) = w_1, \quad m(\cdot, 0) = m_0, \quad |m_0| = 1 \quad \text{in} \ \Omega, \quad (2.11) \]

\[ w = 0, \quad \frac{\partial m}{\partial \nu} = 0 \quad \text{on} \ \Sigma = \partial \Omega \times (0, T] , \quad (2.12) \]

where \(\nu\) is the outer unit normal at the boundary \(\partial \Omega\).

We shall prove the following existence and uniqueness result. We assume that

\[ w_0 \in H^1_0(\Omega), \quad w_1 \in L^2(\Omega), \quad m_0 \in H^1(\Omega), \quad (2.13) \]

then we have

**Theorem 2.1** Given \(T > 0\) and \(\epsilon^{-1} > 2\lambda^2\) there exists a unique solution to (2.10), (2.11), (2.12), (2.13), with \(w \in C^0([0, T]; H^1_0(\Omega)) \cap C^1([0, T]; L^2(\Omega)), \quad m \in C^0([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)) \) and \(m_t \in L^2(0, T; L^2(\Omega)) \).
The total energy $E(m, w)$ of the system (2.10), which accounts also for the kinetic energy term is defined as

$$E(m, w) = \frac{1}{2} \int_\Omega |m_x|^2 dx + \frac{1}{2} \int_\Omega w_x^2 dx + \frac{1}{2} \int_\Omega w_t^2 dx + \frac{1}{2} \int_\Omega \Lambda(m) \cdot m w_x dx + \frac{1}{4} \int_\Omega (|m|^2 - 1)^2 dx.$$  \hspace{1cm} (2.14)

Setting

$$E_0 = \frac{1}{2} \int_\Omega |m_0 x|^2 dx + \frac{1}{2} \int_\Omega (w_0 x)^2 dx + \frac{1}{2} \int_\Omega (w_1 x)^2 dx + \frac{1}{2} \int_\Omega \Lambda(m_0) \cdot m_0 w_0 x dx,$$

we show the dissipative behaviour $E(t) \leq E_0$ for the solution to (2.10)-(2.13).

### 3 The variational analysis of the steady problem

In this section we summarize some results from [5] on the stationary problem associated with the simplified energy functional in one dimension. The minimization of the nonlocal functional presents some interesting features. It is proved in [5] that there exists a critical value $\lambda^*$, such that: for $\lambda < \lambda^*$ and $\epsilon$ small enough, the absolute minimum of the functional is zero and it is achieved only by constants of modulus one (trivial solutions) while for $\lambda > \lambda^*$ the minimum is negative and it is achieved by non trivial functions, for every $\epsilon > 0$. A similar bifurcation phenomenon was observed by Bethuel, Brezis, Coleman and Hélein in [3] in their study of nematics between cylinders.

The stationary problem considered in [5] is described by the nonlocal system

$$\begin{cases}
  m_{xx} - \epsilon^{-1}(|m|^2 - 1)m + \mu \Lambda(m) \cdot m - \int_0^1 \Lambda(m) \cdot m dx = 0, \\
  m_x(0, t) = m_x(1, t) = 0,
\end{cases} \quad (3.1)
$$

where $\mu = \lambda^2/2$ and the functional studied reduces to

$$F(m) = F_{\mu, \epsilon}(m) = \frac{1}{2} \int_0^1 |m_x|^2 dx + \frac{\epsilon^{-1}}{4} \int_0^1 (|m|^2 - 1)^2 dx$$

$$- \frac{\mu}{4} \left[ \int_0^1 (\Lambda(m) \cdot m)^2 dx - (\int_0^1 \Lambda(m) \cdot m dx)^2 \right]. \quad (3.2)$$

For the minimization problem

$$\mathcal{F}_{\mu, \epsilon} = \inf_{m \in H^1(0,1)} F(m). \quad (3.3)$$

the following results have been obtained.

**Theorem 3.1** For each $\mu$ and for each positive $\epsilon$ small enough, i.e., such that $\epsilon^{-1} - \mu > 0$, the minimum of the functional $F(m)$ is achieved by a function $m^* \in H^1(0,1)$. Furthermore, $m^*$ is a solution to (3.1) and is therefore of class $C^\infty$.  

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The functional $F(m)$ has some obvious symmetry properties. One has clearly $F(S_i(m)) = F(m)$ for each $S_i$ in the group

$$\mathcal{G} = \{S_0, \ldots, S_7\}$$ (3.4)

generated by the rotation by $\pi/2$ and the complex conjugation.

Denoted by $\lambda_2$ the first nontrivial eigenvalue for the Neumann problem:

\begin{align*}
-f_{xx} &= \lambda f \quad \text{in } (0,1), \\
f_x(0) &= f_x(1) = 0,
\end{align*} (3.5)

one has

**Proposition 3.2** Put

$$I(\mu) = \inf_{m \in H^1((0,1); S^1)} F(m).$$ (3.6)

Then:

(i) For $\mu \leq \lambda_2/2$ we have $I(\mu) = 0$ and the minimum is attained only by the constant functions, $m \equiv \alpha \in S^1$.

(ii) For $\mu > \lambda_2/2$ we have $I(\mu) < 0$ and the minimum is attained by $m^0 = e^{i\phi^0}$ where $\phi^0$ is a nontrivial solution of the problem

\begin{align*}
-\phi_{xx}^0 &= \mu \left( \sin 2\phi^0 - \int_0^1 \sin 2\phi^0 \, dt \right) \cos 2\phi^0 \quad \text{in } (0,1), \\
\phi_x^0(0) &= \phi_x^0(1) = 0.
\end{align*} (3.7)

We have also the convergence result:

**Proposition 3.3** For each $\mu > 0$, any sequence of minimizers $\{m_{\varepsilon_n}\}$, with $\varepsilon_n \to 0$, has a subsequence which converges in $H^1(0,1)$ and in $C[0,1]$ to $m^0 \in C^\infty([0,1]; S^1)$ which is a minimizer for $I(\mu)$.

The main theorem states

**Theorem 3.4**

(i) For each $\mu < \lambda_2/2$ there exists $\varepsilon_0(\mu) > 0$ such that for $\varepsilon \leq \varepsilon_0(\mu)$ we have $I_{\mu,\varepsilon} = 0$ and the only minimizers for (3.3) are constant functions $m^\varepsilon \equiv \alpha \in S^1$.

(ii) For $\mu > \lambda_2/2$ we have $I_{\mu,\varepsilon} < 0$ for every $\varepsilon > 0$. For each $\varepsilon > 0$ we may choose a representative for the minimizer $m^\varepsilon$ (by replacing $m^\varepsilon$ with $S_i(m^\varepsilon)$, see (3.4)) such that $\lim_{\varepsilon \to 0} m^\varepsilon = m^0$ in $H^1(0,1)$ and in $C[0,1]$, where $m^0 \in C^\infty([0,1]; S^1)$ is a non-trivial minimizer for $I(\mu)$.

(iii) In the limiting case $\mu = \lambda_2/2$, we have for a subsequence, $\lim_{\varepsilon \to 0} m^\varepsilon = \alpha$ in $H^1(0,1)$ and in $C[0,1]$, for some constant $\alpha \in S^1$. 

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In [5] the associated gradient flow problem is also studied. For any \( t > 0 \) there exists a unique classical solution \( u(t) \) to the problem

\[
\frac{du}{dt} = -\text{grad} \, F(u), \quad u(x, 0) = u_0.
\] (3.8)

Moreover, \( \lim_{t \to \infty} u(t) = u_\infty \) exists and the function \( u_\infty \) is a stationary point of the energy functional (3.2).

4 Proof of Theorem 2.1

We give the proof of Theorem 2.1 in the more general case in which mechanical and magnetic external forces act on the system. That is, we consider

\[
\begin{cases}
  u_{tt} = u_{xx} + \frac{1}{2} \lambda (\Lambda(m) \cdot m)_x + f,
  \\
  m_t = m_{xx} - \lambda \Lambda(m) u_x - \varepsilon^{-1}(|m|^2 - 1)m + g,
\end{cases}
\] (4.1)

with initial and boundary conditions

\[
\begin{cases}
  u|_{t=0} = u_0(x), \quad u_t|_{t=0} = u_1(x), \quad u|_{\partial \Omega \times (0,T)} = 0, \\
  m|_{t=0} = m_0(x), \quad m_x|_{\partial \Omega \times (0,T)} = 0.
\end{cases}
\] (4.2)

We assume that

\[
\begin{cases}
  u_0 \in H^1_0(\Omega), \quad u_1 \in L^2(\Omega), \quad m_0 \in H^1(\Omega), \\
  f \in L^2(0,T;L^2(\Omega)), \quad g \in L^2(0,T;L^2(\Omega)).
\end{cases}
\] (4.3)

For the proof we need some preliminary results (see the lemmas below). Arguing as in [16] where several hyperbolic-parabolic systems are considered, first we prove a local existence and uniqueness result. Then, we establish a uniform a priori estimate for the solution which allows to get a global result by using the continuation method. For any pair of positive constants \( M_1, M_2 \), we define the following convex set \( B^h \):

\[
B^h = \left\{ (u, u_t) \in B^h_1 \equiv C^0([0,h]; H^1_0(\Omega)) \times C^0([0,h]; L^2(\Omega)), \right. \\
\left. (m, m_t) \in B^h_2 \equiv C^0([0,h]; H^1(\Omega)) \cap L^2(0,h; H^2(\Omega)) \times L^2(0,h; L^2(\Omega)), \right. \\
\left. u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \quad m|_{t=0} = m_0, \quad m_x|_{\partial \Omega \times (0,h)} = 0, \right. \\
\left. \sup_{0 \leq t \leq h} \|u_t\|^2_{L^2(\Omega)} + \sup_{0 \leq t \leq h} \|u_t\|^2 \leq M_1, \right. \\
\left. \sup_{0 \leq t \leq h} \|m_t\|^2_{L^2(\Omega)} + \int_0^h \|m_t\|^2 dt \leq M_2. \right. 
\]
Hereafter we denote by $\| \cdot \|$ the $L^2(\Omega)$--norm and by $\| \cdot \|_\infty$ the $L^\infty(\Omega)$--norm.

**Remark 4.1** If $\phi \in C^0([0, h]; H^1_0(\Omega))$ and $\psi \in C^0([0, h]; H^1(\Omega))$ we have

$$\sup_{0 \leq t \leq h} \| \phi_t \|_\infty^2 \leq \sup_{0 \leq t \leq h} \| \phi_x \|^2$$  \hspace{1cm} (4.4)

and

$$\sup_{0 \leq t \leq h} \| \psi_t \|_\infty^2 \leq 2 \sup_{0 \leq t \leq h} \| \psi \|_{H^1(\Omega)}^2.$$  \hspace{1cm} (4.5)

The inequalities (4.4), (4.5) can be easily checked. In particular, the inequality (4.5) follows from $|\psi(\xi, \cdot)| \leq |\psi(s, \cdot)| + \int_\xi^s \psi_x(x, \cdot) \, dx$, by integration over $s \in (0, 1)$.

**Lemma 4.2** There exists a positive time $t^* \in (0, T]$, depending on the data $u_0, u_1, m_0, f, g$, such that the problem (4.1), (4.2), (4.3) admits a unique solution $(u, m)$ in $\Omega \times [0, t^*]$. Moreover, we have $u \in C^2([0, t^*]; H^1_0(\Omega)) \cap C^1([0, t^*]; L^2(\Omega))$ and $m \in C^0([0, t^*]; H^1(\Omega)) \cap L^2(0, t^*; H^2(\Omega))$, $m_t \in L^2(0, t^*; L^2(\Omega))$.

**Proof.** Let $h \in (0, T]$ and $(\bar{u}, \bar{m}) \in B^h$ we consider the following linear problem

$$\begin{cases}
\begin{align*}
u_{tt} &= u_{xx} + \frac{1}{2} \lambda(\Lambda(\bar{m}) \cdot \bar{m})_x + f, \\
m_t &= m_{xx} - \lambda \Lambda(\bar{m}) \bar{u}_x - \epsilon^{-1}(|\bar{m}|^2 - 1) \bar{m} + g,
\end{align*}
\end{cases}$$  \hspace{1cm} (4.6)

with initial and boundary conditions (4.2).

We start by giving an outline of the proof. First of all we observe that for each fixed $(\bar{u}, \bar{m}) \in B^h$ there exists a unique solution $(u, m)$ to the linear problem (4.6), (4.2), (4.3) with $u \in B^1_{h}$ and $m \in B^2_{h}$ (see [9], [12]). Then we get the following estimates for the solution to the problem (4.6), (4.2), (4.3)

(i) $\| u_t \|^2 + \| u_x \|^2 \leq (C_1 + t K_1(M_2)) e^{2t}$, \hspace{1cm} $t \in [0, h]$

(ii) $\int_0^t \| m_t \|^2 \, d\tau + \| m \|_{H^1(\Omega)}^2 \leq (C_2 + t K_2(M_1, M_2)) e^t$, \hspace{1cm} $t \in [0, h]$

where $C_1$ is a positive constant depending on $\| u_{0, x} \|, \| u_1 \|$ and $\| f \|_{L^2(0, T; L^2(\Omega))}, C_2$ is a positive constant depending on $\| m_0 \|_{H^1(\Omega)}$ and $\| g \|_{L^2(0, T; L^2(\Omega))}$, and $K_l (l = 1, 2)$ are positive constants depending on $M_1, M_2$.

From (i) and (ii) it follows that if we choose $M_1 = 2C_1$ and $M_2 = 2C_2$, then for $t$ small enough the solution $(u, m) \in B^h$, and hence the mapping $(\bar{u}, \bar{m}) \mapsto (u, m)$ maps $B^h$ into itself. Existence of the solution would then follow from Banach fixed point theorem once we establish that the mapping $(\bar{u}, \bar{m}) \mapsto (u, m)$ is a contraction. Indeed, let $(\bar{u}, \bar{m}) \in B^h$ and $(\tilde{u}, \tilde{m}) \in B^h$ be fixed and denote by $(u, m)$ and $(\bar{u}, \bar{m})$ the corresponding solutions of the linearized problem. We shall prove the following inequality for the difference functions $(z, q) = (u - \bar{u}, m - \bar{m})$ and $(\bar{z}, \bar{q}) = (\bar{u} - \bar{u}, \bar{m} - \bar{m})$, we have

$$\| z \|^2 + \| q \|^2 \leq (C_3 + t K_3(M_3, M_4)) e^t,$$  \hspace{1cm} (4.7)

where $C_3$ is a positive constant depending on $\| \bar{u}_{0, x} \|, \| \bar{u}_1 \|$ and $\| \bar{f} \|_{L^2(0, T; L^2(\Omega))}$, $C_4$ is a positive constant depending on $\| \bar{m}_0 \|_{H^1(\Omega)}$ and $\| \bar{g} \|_{L^2(0, T; L^2(\Omega))}$, and $K_4 (l = 1, 2)$ are positive constants depending on $M_3, M_4$.
(iii) \[ \| z_t \|^2 + \| z_x \|^2 + \int_0^t \| q_t \|^2 dt + \| q \|_{H^1(\Omega)}^2 \leq \]
\[ \leq t K_3(M_1, M_2) e^t \quad \sup_{0 \leq t \leq h} (\| z_t \|^2 + \| z_x \|^2 + \int_0^t \| q_t \|^2 dt + \| q \|_{H^1(\Omega)}^2), \]
with \( K_3 \) a positive constant depending on \( M_1 \) and \( M_2 \). The contraction property for \( t \) small enough follows immediately. Therefore, in order to complete the proof, we only need to establish the estimates (i)-(iii).

We will use repeatedly (see (4.5)) that for \( m \in B^h \)
\[ \sup_{0 \leq t \leq h} \| m \|_\infty < \sqrt{2} \sqrt{M_2}. \tag{4.7} \]

The estimate (i). Multiplying the first equation of (4.6) by \( u_t \) and integrating on \( \Omega \) we have
\[ \frac{1}{2} \frac{d}{dt} \| u_t \|^2 + \frac{1}{2} \frac{d}{dt} \| u_x \|^2 = \lambda \int \Omega \Lambda(m) \cdot m_x u_t dx + \int \Omega f u_t dx \leq \]
\[ \leq \lambda \sup_{0 \leq t \leq h} \| \Lambda(m) \|_\infty \int \Omega m_x u_t dx + \int \Omega f u_t dx \]
\[ \leq \lambda \sqrt{2} \sqrt{M_2} \| m \|_\infty \| u_t \| + \| f \| \| u_t \|, \quad \text{by (4.7)} \]
and hence by the Young inequality
\[ \frac{1}{2} \frac{d}{dt} \| u_t \|^2 + \frac{1}{2} \frac{d}{dt} \| u_x \|^2 \leq \lambda^2 M_2^2 + \frac{1}{2} \| f \|^2 + \| u_t \|^2. \]
Setting \( y = \| u_t \|^2 + \| u_x \|^2 \) we derive
\[ y' \leq 2 \lambda^2 M_2^2 + \| f \|^2 + 2 y, \]
which implies
\[ (e^{-2t} y)' \leq (2 \lambda^2 M_2^2 + \| f \|^2) e^{-2t} \leq 2 \lambda^2 M_2^2 + \| f \|^2. \]
Integrating between 0 and \( t \) we derive
\[ e^{-2t} y \leq y(0) + 2 \lambda^2 M_2^2 t + \int_0^T \| f \|_\infty^2 dt \]
\[ = \| u_1 \|^2 + \| u_{0x} \|^2 + 2 \lambda^2 M_2^2 t + \int_0^T \| f \|_\infty^2 dt. \]

Estimate (i) follows with \( C_1 = \| u_1 \|^2 + \| u_{0x} \|^2 + \int_0^T \| f \|_\infty^2, K_1(M_2) = 2 \lambda^2 M_2^2. \)

The estimate (ii). As in the proof of the previous estimate, we multiply the second equation of (4.6) by \( m_t \) and integrate on \( \Omega = (0,1) \) we get
\[ \| m_t \|^2 + \frac{1}{2} \frac{d}{dt} \| m_x \|^2 = -\lambda \int \Omega \Lambda(m) \cdot m_t u_t dx - \varepsilon^{-1} \int \Omega (|m|^2 - 1) \tilde{m} \cdot m_t dx + \int \Omega g \cdot m_t dx \]
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\[ \leq \lambda \sup_{0 \leq t \leq h} \| \bar{m} \|_\infty \| m_t \|_\infty \| \bar{u}_x \| + \varepsilon^{-1} \sup_{0 \leq t \leq h} \| | \bar{m} |^2 - 1 \|_\infty \| \bar{m} \| + \| g \| \| m_t \| \]
\[ \leq \{ \lambda \sqrt{2M_1 M_2} + \varepsilon^{-1}(2M_2 + 1)\sqrt{2M_2} + \| g \| \} \| m_t \| \]
\[ \leq \frac{1}{2} \| m_t \|^2 + \varepsilon \| m_t \|^2 + \varepsilon^{-1} \| m_t \|^2 + 2 \| g \|, \]
where we have set
\[ K = \lambda \sqrt{2M_1 M_2} + \varepsilon^{-1}(2M_2 + 1)\sqrt{2M_2}. \]

Thus we have
\[ \| m_t \|^2 + \frac{d}{dt} \| m_x \|^2 \leq 2K^2 + 2\| g \|^2, \]
and integrating between 0, t we obtain
\[ \int_0^t \| m_t \|^2 dt + \| m_x \|^2 \leq \| m_0 \|_0^2 + 2 \int_0^t \| g \|^2 dt + 2K^2 t. \quad (4.8) \]

Now multiplying the second equation of (4.6) by \( m \) and integrating on \( \Omega \) leads to
\[ \frac{1}{2} \frac{d}{dt} \| m \|^2 + \| m_x \|^2 = -\lambda \int_\Omega \Lambda(m) \cdot \bar{m} \bar{u}_x dx - \varepsilon^{-1} \int_\Omega (| \bar{m} |^2 - 1)m \cdot m dx + \int_\Omega g \cdot m dx \]
\[ \leq \lambda \sup_{0 \leq t \leq h} \| m \|_\infty \| \bar{m} \|_\infty \| \bar{u}_x \| + \varepsilon^{-1} \sup_{0 \leq t \leq h} \| | \bar{m} |^2 - 1 \|_\infty \| \bar{m} \| + \| g \| \| m \| \]
\[ \leq \{ \lambda \sqrt{2M_1 M_2} + \varepsilon^{-1}(2M_2 + 1)\sqrt{2M_2} + \| g \| \} \| m \|. \]

Hence by Young’s inequality
\[ \frac{1}{2} \frac{d}{dt} \| m \|^2 \leq \frac{1}{2} \| m \|^2 + \frac{1}{2} \{ K + \| g \| \}^2, \]
which implies
\[ \frac{d}{dt} \left( \| m \|^2 e^{-t} \right) \leq \{ K + \| g \| \}^2 e^{-t} \leq 2K^2 + 2\| g \|^2. \]

Integrating between 0 and t we obtain
\[ \| m \|^2 \leq (\| m_0 \|^2 + 2 \int_0^t \| g \|^2 dt + 2K^2 t) e^t. \]

Combining with (4.8) we obtain (ii) with
\[ C_2 = \| m_0 \|^2_{H^1(\Omega)} + 4 \int_0^T \| g \|^2 dt, \quad K_2(M_1, M_2) = 2K^2. \]

The estimate (iii). The proof of the inequality (iii) can be carried out in an analogous way to the estimates established above. First we consider the equation for \( z \), that is
\[ z_{tt} = z_{xx} + \lambda \Lambda(m) \cdot \bar{m}_x - \lambda \Lambda(\bar{m}) \cdot \bar{m}_x. \]
Multiplying by $z_t$ and integrating on $\Omega$ we get
\[
\frac{1}{2} \frac{d}{dt} (\|z_t\|^2 + \|z_x\|^2) = \lambda \int_{\Omega} \{\Lambda(\bar{m}) \cdot m_x - \Lambda(m) \cdot \bar{m}_x\} z_t \, dx
\]
\[
= \lambda \int_{\Omega} \Lambda(m) \cdot (m_x - \bar{m}_x) z_t \, dx + \lambda \int_{\Omega} \Lambda(m - \bar{m}) \cdot \bar{m}_x z_t \, dx
\]
\[
\leq \lambda \sup_{0 \leq t \leq h} \|\Lambda(m)\|_{\infty} \|m_x - \bar{m}_x\| \|z_t\| + \lambda \sup_{0 \leq t \leq h} \|\Lambda(m - \bar{m})\|_{\infty} \|\bar{m}_x\| \|z_t\|
\]
\[
\leq \{\lambda \sqrt{2M_2} \|\bar{m}_x - \bar{m}_x\| + \lambda \sqrt{2M_2} \|m_x - \bar{m}_x\|\} \|z_t\|, \quad \text{(see (4.7))}
\]
\[
\leq \frac{1}{2} \|z_t\|^2 + \frac{1}{2} \{\lambda \sqrt{2M_2} \|m_x - \bar{m}_x\| + \lambda \sqrt{2M_2} \|m_x - \bar{m}_x\|\}^2
\]
\[
\leq \frac{1}{2} \{\|z_t\|^2 + \|z_x\|^2\} + 2\lambda M_2 \|\bar{q}\|_{H^1(\Omega)}^2.
\]
Setting $y = \|z_t\|^2 + \|z_x\|^2$ and noting that $y(0) = 0$ we derive
\[
(e^{-t} y)' \leq 4\lambda M_2 \|\bar{q}\|_{H^1(\Omega)}^2 e^{-t} \leq 4\lambda M_2 \sup_{0 \leq t \leq h} \|\bar{q}\|_{H^1(\Omega)}^2,
\]
which implies
\[
\|z_t\|^2 + \|z_x\|^2 \leq (4\lambda M_2 t)e^t \sup_{0 \leq t \leq h} \|\bar{q}\|_{H^1(\Omega)}^2. \quad (4.9)
\]
From the second equation (4.6) we have
\[
q_t = q_{xx} - \Lambda(\bar{m}) \bar{u}_x + \lambda \Lambda(\bar{m}) \bar{u}_x - \varepsilon^{-1}(|\bar{m}|^2 - 1) \bar{m} + \varepsilon^{-1}(|\bar{m}|^2 - 1) \bar{m}
\]
\[
= q_{xx} - \Lambda(\bar{m})(\bar{u}_x - \bar{u}_x) + \lambda \Lambda(\bar{m} - \bar{m}) \bar{u}_x - \varepsilon^{-1}(|\bar{m}|^2 - 1)(\bar{m} - \bar{m}) + \varepsilon^{-1}(|\bar{m}|^2 - |\bar{m}|^2) \bar{m}.
\]
Multiplying this equation by $q_t$ and integrating on $\Omega$ we get
\[
\|q_t\|^2 + \frac{1}{2} \frac{d}{dt} \|q_x\|^2 = -\lambda \int_{\Omega} \Lambda(\bar{m}) \cdot q_t(\bar{u}_x - \bar{u}_x) \, dx + \lambda \int_{\Omega} \Lambda(\bar{m} - \bar{m}) \cdot q_t \bar{u}_x \, dx
\]
\[
- \varepsilon^{-1} \int_{\Omega} (|\bar{m}|^2 - 1)(\bar{m} - \bar{m}) \cdot q_t \, dx + \varepsilon^{-1} \int_{\Omega} ((\bar{m} - \bar{m}) \cdot (\bar{m} + \bar{m}) \bar{m} \cdot q_t) \, dx
\]
\[
\leq \lambda \sqrt{2M_2} \|q_t\| \|\bar{u}_x - \bar{u}_x\| + \lambda \sup_{0 \leq t \leq h} \|\bar{m} - \bar{m}\| \|q_t\| \|\bar{u}_x\|
\]
\[ + \epsilon^{-1}(2M_2 + 1)\|\dot{\bar{m}} - \bar{m}\|_2 + \epsilon^{-1}4M_2\|\dot{m} - \bar{m}\|_2 \]

\[ \leq \{\lambda \sqrt{2M_2}\|\dot{z}_x\| + [\lambda \sqrt{2M_1} + \epsilon^{-1}(2M_2 + 1) + \epsilon^{-1}4M_2]\|q\|_H^1(\Omega)\}q_t \]

\[ \leq 2\lambda^2 M_2\|\dot{z}_x\|^2 + [\lambda \sqrt{2M_1} + \epsilon^{-1}(2M_2 + 1) + \epsilon^{-1}4M_2]^2\|q\|^2_{H^1(\Omega)} + \frac{1}{2}\|q_t\|^2. \]

It follows that

\[ \|q_t\|^2 + \frac{d}{dt}\|q_x\|^2 \leq K(M_1, M_2, \lambda, \epsilon) \sup_{0 \leq t \leq h} (\|\dot{z}_x\|^2 + \|q\|^2_{H^1(\Omega)}), \]

where

\[ K(M_1, M_2, \lambda, \epsilon) = \{4\lambda^2 M_2 + 2[\lambda \sqrt{2M_1} + \epsilon^{-1}(2M_2 + 1) + \epsilon^{-1}4M_2]^2\}. \]

Integrating between 0 and t we derive

\[ \int_0^t\|q_t\|^2 dt + \|q_x\|^2 \leq (K(M_1, M_2, \lambda, \epsilon)t)e^t \sup_{0 \leq t \leq h} (\|\dot{z}_x\|^2 + \|q\|^2_{H^1(\Omega)}). \]

Next, we multiply the equation by q to get

\[ \frac{1}{2} \frac{d}{dt}\|q\|^2 + \|q_x\|^2 = -\lambda \int_\Omega \Lambda(\bar{m}) \cdot q(\bar{u}_x - \bar{u}_x) dx + \lambda \int_\Omega \Lambda(\bar{m} - \bar{m}) \cdot q \bar{u}_x dx - \]

\[ -\epsilon^{-1} \int_\Omega (|\bar{m}|^2 - 1)(\bar{m} - \bar{m}) \cdot q dx + \epsilon^{-1} \int_\Omega (\bar{m} - \bar{m})(\bar{m} + \bar{m}) \bar{m} \cdot q dx. \]

The estimate of the right hand side is the same as above, except that q, is now replaced by q. Therefore, we obtain

\[ \frac{1}{2} \frac{d}{dt}\|q\|^2 + \|q_x\|^2 \leq K(M_1, M_2, \lambda, \epsilon) \sup_{0 \leq t \leq h} (\|\dot{z}_x\|^2 + \|q\|^2_{H^1(\Omega)}) + \frac{1}{2}\|q\|^2, \]

with

\[ K(M_1, M_2, \lambda, \epsilon) = \{2\lambda^2 M_2 + [\lambda \sqrt{2M_1} + \epsilon^{-1}(2M_2 + 1) + \epsilon^{-1}4M_2]^2\}. \]

It follows that

\[ \frac{d}{dt}(\|q\|^2e^{-t}) \leq K(M_1, M_2, \lambda, \epsilon) \sup_{0 \leq t \leq h} (\|\dot{z}_x\|^2 + \|q\|^2_{H^1(\Omega)}). \]

Integrating this inequality leads to

\[ \|q\|^2 \leq K(M_1, M_2, \lambda, \epsilon)te^{t} \sup_{0 \leq t \leq h} (\|\dot{z}_x\|^2 + \|q\|^2_{H^1(\Omega)}), \]

and the estimate (iii) follows. \[ \square \]
Lemma 4.3 (Uniform a priori estimate.) Let \( \varepsilon^{-1} > 2 \lambda^2 \). Then, there exists a positive constant \( C_T = C(u_0, u_1, m_0, f, g, \varepsilon, \lambda, T) \) such that the solution of (4.1), (4.2), (4.3) verifies the following estimate,

\[
\sup_{\tau \in [0,T]} \{ \| u_t(\cdot, \tau) \|^2 + \| u_x(\cdot, \tau) \|^2 + \| m(\cdot, \tau) \|^2_{H^1(\Omega)} + \int_0^T \| m_t(\cdot, t) \|^2 dt \} \leq C_T. \tag{4.10}
\]

**Proof.** Multiplying the first equation of (4.1) by \( u_t \) and the second by \( m_t \) and integrating on \( \Omega \), we get

\[
\frac{1}{2} \frac{d}{dt} \| u_t \|^2 + \frac{1}{2} \frac{d}{dt} \| u_x \|^2 + \frac{1}{2} \frac{d}{dt} \| m_x \|^2 + \| m_t \|^2 + \varepsilon^{-1} \frac{d}{dt} \| m \|^2 - 1 \|^2 =
\]

\[
= \frac{\lambda}{2} \int_{\Omega} (\Lambda(m) \cdot m)_x u_t dx + \int_{\Omega} f u_t dx - \lambda \int_{\Omega} \Lambda(m) \cdot m_t u_x + \int_{\Omega} g \cdot m_t dx.
\]

Remark that

\[
- \lambda \int_{\Omega} \Lambda(m) \cdot m_t u_x dx = - \frac{\lambda}{2} \int_{\Omega} (\Lambda(m) \cdot m)_x u_t dx = \frac{\lambda}{2} \int_{\Omega} (\Lambda(m) \cdot m u_x) dx - \frac{\lambda}{2} \int_{\Omega} (\Lambda(m) \cdot m)_x u_t dx.
\]

Then from above we obtain

\[
\frac{1}{2} \frac{d}{dt} \{ \| u_t \|^2 + \| u_x \|^2 + \| m_x \|^2 + \lambda \int_{\Omega} \Lambda(m) \cdot m u_x dx + \frac{\varepsilon^{-1}}{2} \| m \|^2 - 1 \|^2 \} \leq
\]

\[
\leq - \| m_t \|^2 + \frac{1}{2} \| f \|^2 + \frac{1}{2} \| u_x \|^2 + \frac{1}{2} \| g \|^2 + \frac{1}{2} \| m_t \|^2. \tag{4.11}
\]

Now one has

\[
| \frac{\lambda}{2} \int_{\Omega} \Lambda(m) \cdot m u_t dx | \leq \int \frac{\lambda}{2} |m|^2 |u_x| dx \leq \frac{1}{4} \| u_x \|^2 + \frac{\lambda^2}{4} \int_{\Omega} |m|^4 dx \leq
\]

\[
\leq \frac{1}{4} \| u_x \|^2 + \frac{\lambda^2}{4} \int_{\Omega} (|m|^2 - 1)^2 dx \leq \frac{1}{4} \| u_x \|^2 + \frac{\lambda^2}{2} \int_{\Omega} (|m|^2 - 1)^2 dx + \frac{\lambda^2}{2}.
\]

This implies

\[
\| m_x \|^2 + \| u_x \|^2 + \lambda \int_{\Omega} \Lambda(m) \cdot m u_x dx + \frac{\varepsilon^{-1}}{2} \| m \|^2 - 1 \|^2 + \lambda^2 \geq
\]

\[
\geq \| m_x \|^2 + \frac{1}{2} \| u_x \|^2 + (\frac{\varepsilon^{-1}}{2} - \lambda^2) \| m \|^2 - 1 \|^2 \geq 0. \tag{4.12}
\]
Multiplying (4.11) by 2 and adding to the right the left hand side of (4.12) we get
\[ \frac{d}{dt} K(t) \leq -\|m_t\|^2 + \|f\|^2 + \|g\|^2 + K(t) \]
where
\[ K(t) = \|u_t\|^2 + \|u_x\|^2 + \|m_x\|^2 + \lambda \int_\Omega \Lambda(m) \cdot m u_x dx + \frac{\varepsilon^{-1}}{2} \|m\|^2 - 1 + \lambda^2. \]
From the above it follows easily that
\[ \frac{d}{dt} \{ e^{-t} K(t) \} + \|m_t\|^2 e^{-t} \leq \|f\|^2 + \|g\|^2. \]
Integration for \( t \in (0, \tau) \), for any \( \tau \in (0, T] \), yields the following inequality at time \( \tau \) (using (4.12) again),
\[ \|u_t\|^2 + \|u_x\|^2 + \|m_x\|^2 + \frac{\varepsilon^{-1}}{2} - \lambda^2 \|m\|^2 - 1 + \int_0^\tau \|m_t\|^2 dt \leq \]
\[ \leq \left( K(0) + \int_0^\tau (\|f\|^2 + \|g\|^2) dt \right) e^\tau. \]
This leads to (4.10) since
\[ \|m\|^2 = \int_\Omega |m|^2 dx \leq \left( \int_\Omega |m|^4 dx \right)^{1/2} = \left( \int_\Omega (|m|^2 - 1 + 1)^2 dx \right)^{1/2} \leq \]
\[ \leq \{ 2\|m\|^2 - 1 \}^{1/2}. \]

Once the uniform estimate has been proved, the local solution can be extended step by step until the given fixed time \( T \). More precisely we extend the solution on a sequence of intervals \( (0, t_n] \) such that \( t_n \to t^* \). Then considering the initial problem starting from \( t^* \) one can extend the solution thanks to (4.10).

**Remark 4.4** In the homogeneous case \( f = g = 0 \), the constant \( C_T \) in (4.10) is independent of the time \( T \); moreover from equation (4.11) we get
\[ \begin{align*}
\frac{1}{2} \frac{d}{dt} \|u_t\|^2 + \frac{1}{2} \frac{d}{dt} \|u_x\|^2 + \frac{1}{2} \frac{d}{dt} \|m_x\|^2 + \frac{\varepsilon^{-1}}{4} \frac{d}{dt} \|m\|^2 - 1 + \frac{\lambda}{2} \int_0^1 \Lambda(m) \cdot m u_x dx &= -\|m_t\|^2 \tag{4.13}
\end{align*} \]
and hence the dissipative energy behaviour easily follows.
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References


