Mixed ladder determinantal varieties from two-sided ladders

Gorla, E

Abstract: We study the family of ideals defined by mixed size minors of two-sided ladders of indeterminates. We compute their Gröbner bases with respect to a skew-diagonal monomial order, then we use them to compute the height of the ideals. We show that these ideals correspond to a family of irreducible projective varieties, that we call mixed ladder determinantal varieties. We show that these varieties are arithmetically Cohen–Macaulay, and we characterize the arithmetically Gorenstein ones. Our main result consists in proving that mixed ladder determinantal varieties belong to the same G-biliaison class of a linear variety.

DOI: https://doi.org/10.1016/j.jpaa.2007.01.016

Originally published at:
Gorla, E (2007). Mixed ladder determinantal varieties from two-sided ladders. Journal of Pure and Applied Algebra, 211(2):433-444.
DOI: https://doi.org/10.1016/j.jpaa.2007.01.016

MIXED LADDER DETERMINANTAL VARIETIES FROM TWO-SIDED LADDERS

ELISA GORLA

Abstract: We study the family of ideals defined by mixed size minors of two-sided ladders of indeterminates. We compute their Gröbner bases with respect to a skew-diagonal monomial order, then we use them to compute the height of the ideals. We show that these ideals correspond to a family of irreducible projective varieties, that we call mixed ladder determinantal varieties. We show that these varieties are arithmetically Cohen-Macaulay. We characterize the arithmetically Gorenstein ones, among those that satisfy a technical condition. Our main result consists in proving that mixed ladder determinantal varieties belong to the same G-biliaison class of a linear variety.

Introduction

In this paper we study the G-biliaison class of a family of varieties, whose defining ideals are generated by minors of a matrix of indeterminates. Other families of schemes defined by minors have been studied in the same contest. The results obtained in this paper are related in a very natural way to some of the results proven in [8], [6], and [5]. In [8] Kleppe, Migliore, Miró-Roig, Nagel, and Peterson proved that standard determinantal schemes are glicci, i.e. that they belong to the G-liaison class of a complete intersection. We refer to [9] for the definition of standard and good determinantal schemes, and to [10] for a comprehensive treatment of liaison theory. Hartshorne showed in [6] that the double G-links produced in [8] can indeed be regarded as G-biliaisons. Hence, standard determinantal schemes belong to the G-biliaison class of a complete intersection. In [5] we defined symmetric determinantal schemes as schemes whose saturated ideal is generated by the minors of size $t \times t$ of an $m \times m$ symmetric matrix with polynomial entries, and whose codimension is maximal for the given t and m. In the same paper we proved that these schemes belong to the G-biliaison class of a complete intersection. In this paper we prove that mixed ladder determinantal varieties belong to the G-biliaison class of a linear variety, hence they are glicci.

In Section 1 we define mixed ladder determinantal ideals and varieties. In order to prove irreducibility and Cohen-Macaulayness of these varieties, we use some standard Gröbner bases techniques that were used e.g. in [11], [7], [3], and [4]. In Theorem 1.10 we

1991 *Mathematics Subject Classification.* 14M06, 13C40, 14M12.

Key words and phrases. G-biliaison, G-liaison, ladder, minor, complete intersection, arithmetically Gorenstein scheme, arithmetically Cohen-Macaulay scheme.

The research in this paper was carried on while the author was a guest at the Max-Planck-Institut für Mathematik in Bonn. The author would like to thank the Max Planck Foundation for its support, and for providing an ideal working environment.

compute a Gröbner basis of a mixed ladder determinantal ideal with respect to a skew-diagonal monomial order. In Theorem 1.18 we show that mixed ladder determinantal ideals are irreducible. Hence they define irreducible varieties. In Theorem 1.21 we prove that mixed ladder determinantal varieties are arithmetically Cohen-Macaulay. In Theorem 1.25 we give a numerical characterization of the arithmetically Gorenstein mixed ladder determinantal varieties, among those that satisfy a technical assumption (stated in Property 2). In Theorems 1.14 and 1.15 we compute the codimension of a mixed ladder determinantal variety by producing subsets of the ladder, whose cardinality agrees with the codimension of the variety.

In Section 2 we use the results that we obtained in Section 1 in order to show that a mixed ladder determinantal variety can be obtained from a linear variety by a series of ascending G-biliaisons. The explicit biliaisons are produced in the proof of Theorem 2.1. They take place on suitable mixed ladder determinantal varieties. In Corollary 2.2 we conclude that mixed ladder determinantal varieties are glicci.

1. Mixed ladder determinantal ideals and varieties

Let V be a variety in $\mathbb{P}^r = \mathbb{P}^r_K$, where K is an algebraically closed field. By variety we mean a reduced (not necessarily irreducible) scheme. We denote by I_V the radical ideal associated to V in the coordinate ring of $\mathbb{P}^r$. Let $\mathcal{I}_V \subset \mathcal{O}_{\mathbb{P}^r}$ be the ideal sheaf of V. Let W be a variety containing V. We denote by $\mathcal{I}_{V|W}$ the ideal sheaf of V restricted to W, i.e. the quotient sheaf $\mathcal{I}_V/\mathcal{I}_W$. We let $H^0_*(\mathcal{I}) = \oplus_{s\in\mathbb{Z}} H^0(\mathbb{P}^r, \mathcal{I}(s))$ denote the 0-th cohomology module of the sheaf $\mathcal{I}$.

In this paper we deal with ideals generated by minors of a matrix of indeterminates, and with the varieties that they cut out.

Notation 1.1. Let X be an $m \times n$ matrix of indeterminates,

$$X = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{bmatrix}.$$

We assume that $m \leq n$.

Fix a choice of row indexes $1 \leq i_1 \leq i_2 \leq \ldots \leq i_t \leq m$ and of column indexes $1 \leq j_1 \leq j_2 \leq \ldots \leq j_t \leq n$. We denote by $X_{i_1,\ldots,i_t;j_1,\ldots,j_t}$ the determinant of the $t \times t$ submatrix of X consisting of the rows $i_1, \ldots, i_t$ and of the columns $j_1, \ldots, j_t$. If all the entries involved in the minor $X_{i_1,\ldots,i_t;j_1,\ldots,j_t}$ belong to a subset L of X, then we also write $L_{i_1,\ldots,i_t;j_1,\ldots,j_t}$ for $X_{i_1,\ldots,i_t;j_1,\ldots,j_t}$.

We are interested in the study of the ideals generated by minors that involve indeterminates contained in a subladder of X. We refer the reader to [2] for information about ladders and ladder determinantal ideals.

Definition/Notation 1.2. Let $L \subseteq X$ be a **ladder**. We say that L is a ladder if it has the following property: whenever $x_{i,j}, x_{k,l} \in L$ and $i \leq k, j \geq l$, then $x_{i,l}, x_{k,j} \in L$. In other words, a ladder has the property that whenever the skew diagonal of a submatrix is a subset of the ladder, then all the indeterminates in the submatrix belong to the ladder. Given $1 = b_1 < b_2 < \cdots < b_h \leq m$, $1 \leq a_1 < \cdots < a_h = n$, $1 \leq d_1 \leq \cdots \leq d_k = m$, and $1 = c_1 \leq c_2 \leq \cdots \leq c_k \leq n$, we let

$$L = \{x_{b,a} \mid b_i \leq b \leq m,\ 1 \leq a \leq a_i,\ 1 \leq b \leq d_j,\ c_j \leq a \leq n, \text{ for some } i, j\}.$$

We call $(b_1, a_1), \ldots, (b_h, a_h)$ the **upper outside corners**, and $(d_1, c_1), \ldots, (d_k, c_k)$ the **lower outside corners** of L. We allow to have more than one lower outside corner on the same row or column. However, we assume that all the lower outside corners are distinct. We assume that no two upper outside corners belong to the same row or to the same column. Since the only function of the upper outside corners is to delimit the ladder, the assumption that no two belong to the same row or column is not restrictive.

We include the possibility that $L = X$, i.e. **a matrix is a ladder**. We say that a ladder is **one-sided** if all its lower outside corners belong to the first column or to the last row. Alternatively, a ladder with arbitrary lower outside corners is one-sided if it has exactly one upper outside corner. We regard a matrix as a one-sided ladder.

We let
$$L_j = \{x_{b,a} \in L \mid 1 \leq b \leq d_j,\ c_j \leq a \leq n\}.$$
Then $L = \cup_{j=1}^k L_j$. Each L_j is a one-sided ladder, with unique lower outside corner (d_j, c_j). L_j has upper outside corners $(b_{l_j}, a_{l_j}), \ldots, (b_{h_j}, a_{h_j})$, where

$$l_j = \min\{i \mid x_{b_i,a_i} \in L_j\} \quad \text{and} \quad h_j = \max\{i \mid x_{b_i,a_i} \in L_j\}.$$

Notice that L_j is a subladder of

$$X_j = \{x_{b,a} \in X \mid b_{l_j} \leq b \leq d_j, c_j \leq a \leq a_{h_j}\}.$$

Moreover, X_j is the smallest submatrix of X containing L_j.

We study ideals generated by minors of a ladder of indeterminates. For a given matrix X and a ladder L, we denote by $I_t(L)$ the ideal generated by the $t \times t$ minors of X whose entries belong to L. We call $I_t(L)$ a **ladder determinantal ideal**. Let $K[L]$ denote the K-algebra generated by the entries of L. Then $I_t(L) \subseteq K[L]$. Narasimhan showed in [11] that these ideals are prime, hence they define irreducible projective varieties.

Definition 1.3. Let X be a matrix of indeterminates, let $L \subseteq X$ be a ladder. Fix a positive integer t. We call **ladder determinantal variety** a variety with ideal $I_t(L)$. We regard it as a subvariety of $\mathbb{P}^r = Proj(K[L])$.

As in [4], we extend the definition of ladder determinantal ideal to allow minors of different size in different regions of the ladder. In the paper of Gonciulea and Miller, they call mixed ladder determinantal varieties those corresponding to mixed ladder determinantal ideals coming from one-sided ladders. Moreover, they assume that no two consecutive outside corners belong to the same row or column (notice that their upper corners correspond to our lower corners, and viceversa). In this paper we drop both of those assumptions, so we deal with a larger family of varieties than in [4].

Definition 1.4. Let X be a matrix of indeterminates, let $L \subseteq X$ be a ladder as in Definition 1.2. Fix a vector of positive integers $t = (t_1, \dots, t_k)$. A **mixed ladder determinantal ideal** is an ideal of the form $I_t(L) := I_{t_1}(L_1) + \cdots + I_{t_k}(L_k)$. We call **mixed ladder determinantal variety** a variety defined by a mixed ladder determinantal ideal. As for ladder determinantal varieties, if $I_V = I_t(L)$ then V is a subvariety of $\mathbb{P}^r = Proj(K[L])$.

Remark 1.5. In Theorem 1.18 we show that every mixed ladder determinantal ideal is prime. Therefore every mixed ladder determinantal ideal is the ideal of a mixed ladder determinantal variety. Moreover, mixed ladder determinantal varieties are irreducible.

Remark 1.6. Every ladder determinantal variety is a mixed ladder determinantal variety, whose vector t has all equal entries.

We adopt the following conventions on the ladder L. All of the assumptions can be made without loss of generality.

Assumption 1. All the entries of L are involved in some t-minor. Otherwise we delete the superfluous entries. It is easy to check that the resulting set is again a ladder.

Assumption 2. No two consecutive corners coincide; no two upper outside corners belong to the same row or to the same column.

Assumption 3. $I_{t_i}(L_i) \not\subseteq I_{t_j}(L_j)$ for $i \neq j$. Therefore

$$d_{j+1} - d_j > t_{j+1} - t_j \quad \text{and} \quad c_{j+1} - c_j > t_j - t_{j+1}.$$

In particular, if $c_j = c_{j+1}$ then $t_j < t_{j+1}$. If $d_j = d_{j+1}$, then $t_j > t_{j+1}$.

We allow disconnected ladders, i.e. ladders for which $L_i \cap L_{i+1} = \emptyset$ for some i. We also allow ladders that are contained in a proper submatrix of X (i.e. we do not distinguish between mixed ladder determinantal varieties and cones over them). This is necessary in our context, since in some of the arguments we will need to delete some entries of a ladder (see e.g. Proposition 1.20 and Theorem 2.1). So even if we start with a ladder L that is connected and that is not contained in any proper submatrix of X, these properties will not always be preserved.

Notice that Assumption 1 is not necessarily verified by the subladders L_j, even if it is verified by L. Assumption 2 is automatically verified by the subladders L_j. Assumption 3 is trivially verified by every L_j, since they have exactly one lower outside corner (d_j, c_j) and all the minors have the same size t_j.

In Section 2 we show that mixed ladder determinantal varieties belong to the G-biliaison class of a complete intersection. Prior to that, we need to establish some properties of mixed ladder determinantal varieties, such as irreducibility and Cohen-Macaulayness. Irreducibility of ladder determinantal varieties was proven by Narasimhan in [11], while irreducibility of mixed ladder determinantal varieties associated to one-sided ladders was proven by Gonciulea and Miller in [4]. Cohen-Macaulayness of ladder determinantal varieties was established by Herzog and Trung in [7]. The analogous result for mixed ladder determinantal varieties associated to one-sided ladders was proven by Gonciulea and Miller in [4].

In this section we compute the codimension of mixed ladder determinantal varieties. Analogous results were proven by Herzog and Trung (for ladder determinantal varieties) and by Gonciulea and Miller (for mixed ladder determinantal varieties associated to one-sided ladders) in the papers mentioned above. In order to compute the codimension, we compute a Gröbner basis of mixed ladder determinantal ideals with respect to a skew-diagonal monomial order.

Notation 1.7. We consider the lexicographic order on the monomials of R, induced by the total order $<$ on the variables:

$$x_{d,c} < x_{b,a} \quad \text{if} \;\; b < d \quad \text{or} \;\; b = d, \; a > c.$$

The leading term of a minor with respect to this term order is the product of the elements on its skew-diagonal.

For $f \in K[L]$ we denote by $\mathrm{LT}(f)$ the leading term of f with respect to $<$. For an ideal $I \subset K[L]$ we denote $\mathrm{LT}(I)$ the leading term ideal of f with respect to $<$.

We now compute a Gröbner basis of $I_t(L)$ with respect to the monomial order $<$. We will use this Gröbner basis to compute the height of the mixed ladder determinantal ideal $I_t(L)$. The argument is a standard one, see for example [3] and [4]. We first recall two known results.

Lemma 1.8. *([2], Lemma 1.3.13) Let $I \subset K[X]$ be an ideal with Gröbner basis G. Let $L \subseteq X$ have the property that if $f \in G$ and $LT(f) \in K[L]$, then $f \in K[L]$. Then $G \cap K[L]$ is a Gröbner basis of $I \cap K[L]$.*

Lemma 1.9. *([2], Lemma 1.3.14) Let $I, J \subset K[x_1, \ldots, x_r]$ be homogeneous ideals. Let F be a Gröbner basis of I and G be a Gröbner basis of J. Then: $F \cup G$ is a Gröbner basis of $I + J$ if and only if for all $f \in F$ and $g \in G$ there is an $h \in I \cap J$ such that $LT(h) = lcm\,(LT(f), LT(g))$.*

In the next theorem we produce a Gröbner basis of $I_t(L)$ with respect to $<$.

Theorem 1.10. *Let $\mathcal{G}_j$ be the set of determinants of the $t_j \times t_j$ submatrices of L_j, such that at most $t_i - 1$ rows belong to L_i for $1 \leq i < j$ and at most $t_i - 1$ columns belong to L_i for $j \leq i < k$. Let $\mathcal{G} = \cup_{j=1}^k \mathcal{G}_j$. Then $\mathcal{G}$ is a Gröbner basis of $I_t(L)$ with respect to $<$.*

Proof. The $t_j \times t_j$ minors of L_j that do not belong to $\mathcal{G}_j$ can be written as a combination of the minors of $\mathcal{G}_i$ for some $i \neq j$. Therefore, $\mathcal{G}$ generates $I_t(L)$. Let $\mathcal{F}_j$ be the set of $t_j \times t_j$ minors in L_j, $\mathcal{F}_j \supseteq \mathcal{G}_j$. Moreover for each $f \in \mathcal{F}_j \setminus \mathcal{G}_j$, $\mathrm{LT}(f)$ is a multiple of $\mathrm{LT}(g)$ for some $g \in \mathcal{G}_i$, $i \neq j$. Let $\mathcal{F} = \cup_{j=1}^k \mathcal{F}_j$. From what we just observed, if $\mathcal{F}$ is a Gröbner basis of $I_t(L)$, then $\mathcal{G}$ is a Gröbner basis of $I_t(L)$ as well.

In order to prove that $\mathcal{F}$ is a Gröbner basis, we proceed by induction on k. If $k = 1$ then L is a one-sided ladder, $\mathcal{F} = \mathcal{G}$, and the thesis follows from Corollary 3.3 of [11].

If $k = 2$, then $L = L_1 \cup L_2$. Let $(d_1, 1), (m, c_2)$ be the lower outside corners of L. Let $(b_1, 1), (b_2, a_2), \ldots, (b_l, a_l)$ be the upper outside corners of L_1 and let $(b_g, a_g), \ldots, (b_{h-1}, a_{h-1}), (b_h, n)$ be the upper outside corners of L_2. Let X_1 and X_2 be as in Notation 1.2. Let $\mathcal{H}_i$ be the set of $t_i \times t_i$ minors contained in X_i, $i = 1, 2$. By Theorem 4.5.4

in [4], $\mathcal{H}_1 \cup \mathcal{H}_2$ is a Gröbner basis of $I = I_{t_1}(X_1) + I_{t_2}(X_2)$. By Lemma 1.8 $(\mathcal{H}_1 \cup \mathcal{H}_2) \cap K[L]$ is a Gröbner basis of $I \cap K[L]$. The assumption of the lemma holds because the leading term of each monomial is the product of the elements on the skew-diagonal, and since L is a ladder. We claim that $\mathcal{F}_1 \cup \mathcal{F}_2 = (\mathcal{H}_1 \cup \mathcal{H}_2) \cap K[L]$. In fact,

$$\mathcal{F}_i = \mathcal{H}_i \cap K[L_i] = \mathcal{H}_i \cap K[L]$$

for $i = 1, 2$. Since $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ generates $I_t(L)$, it is a Gröbner basis of it.

Assume that the thesis is true for $k-1$. Let $L' = \cup_{j=1}^{k-1} L_j$; L' has lower outside corners $(d_1, c_1), \dots, (d_{k-1}, c_{k-1})$. Let $t' = (t_1, \dots, t_{k-1})$. Then

$$L = L' \cup L_k \quad \text{and} \quad I_t(L) = I_{t'}(L') + I_{t_k}(L_k).$$

$\mathcal{F}_k$ is a Gröbner basis of $I_{t_k}(L_k)$ by Corollary 3.3 of [11]. Let $\mathcal{F}' = \cup_{j=1}^{k-1} \mathcal{F}_j$. By induction hypothesis $\mathcal{F}'$ is a Gröbner basis of $I_{t'}(L')$. Moreover, $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}_k$. Let $f \in \mathcal{F}'$ and $g \in \mathcal{F}_k$. Then $f \in \mathcal{F}_j$ for some $1 \leq j \leq k-1$. Let $H = L_j \cup L_k$. We have seen above that $\mathcal{F}_j \cup \mathcal{F}_k$ is a Gröbner basis of $I_{(t_j, t_k)}(H) = I_{t_j}(L_j) + I_{t_k}(L_k)$. Then by Lemma 1.9 there is an $h \in I_{t_j}(L_j) \cap I_{t_k}(L_k) \subseteq I_{t'}(L') \cap I_{t_k}(L_k)$ such that $\mathrm{LT}(h) = \mathrm{lcm}\,(\mathrm{LT}(f), \mathrm{LT}(g))$. Therefore, $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}_k$ is a Gröbner basis of $I_t(L)$, again by Lemma 1.9. □

We now use Theorem 1.10 to compute the height of mixed ladder determinantal ideals. In Theorem 1.15 we produce a subladder L' of L such that the number of entries in L' equals the height of $I_t(L)$. We start by fixing the notation.

Notation 1.11. For $1 \leq i \leq j \leq k$ we define

$$L_{i,j} = (L_i \setminus L_{i-1}) \cap (L_j \setminus L_{j+1}).$$

We adopt the convention that $L_{-1} = L_{k+1} = \emptyset$. Each $L_{i,j}$ is a one-sided ladder with exactly one lower outside corner. Some of the $L_{i,j}$'s may be empty. Set

$$t_{i,j} = \min\{t_l \mid L_{i,j} \subseteq L_l\}.$$

If $L_{i,j} = \emptyset$, we set $t_{i,j} = 1$.

Remark 1.12. It is easy to check that $\cup_{i,j} L_{i,j} = L$, and that $L_{i,j} \cap L_{p,q} \neq \emptyset$ if and only if $(i,j) = (p,q)$.

Moreover, one has the following inequalities between the $t_{i,j}$'s.

Lemma 1.13. *With the notation above, one has*

$$t_{i,j-1} \geq t_{i,j} \quad \textit{and} \quad t_{i,j} \geq t_{i-1,j}$$

whenever $L_{i,j}, L_{i,j-1}, L_{i-1,j} \neq \emptyset$. *For all* i, j *one has* $t_{i,j} \geq 1$.

Proof. $t_{i,j} \geq 1$ for all i, j since $t_l \geq 1$ for $l = 1, \dots, k$. If $L_{i,j} = \emptyset$, then $t_{i,j} = 1$ by definition. The other inequalities follow from the observation that

$$L_{i,j} = (L_i \setminus \bigcup_{l=1}^{i-1} L_l) \bigcap (L_j \setminus \bigcup_{l=j+1}^{k} L_l).$$

□

We are now ready to construct a subset $\mathcal{B} \subseteq L$, whose cardinality agrees with the dimension of the mixed ladder determinantal variety defined by $I_t(L) \subset K[L]$.

Theorem 1.14. *Let $\mathcal{B}_{i,j}$ be the subset of $L_{i,j}$ consisting of the elements that belong to the first $t_{i,j-1} - t_{i,j}$ columns or to the last $t_{i,j} - t_{i-1,j}$ rows. Let $\mathcal{B} = \cup_{i,j}\mathcal{B}_{i,j}$. Then*

$$ht\ I_t(L) = |L \setminus \mathcal{B}|.$$

Proof. It is well known that ht $I_t(L) =$ ht $\mathrm{LT}(I_t(L))$. Moreover ht $\mathrm{LT}(I_t(L)) = |\mathcal{S}|$, where $\mathcal{S}$ is a subset of the indeterminates in L of minimal cardinality among those with the

Property 1. Each monomial in a set of generators of $\mathrm{LT}(I_t(L))$ contains a variable from $\mathcal{S}$.

Geometrically, $\mathrm{LT}(I_t(L))$ corresponds to an intersection $\mathcal{U}$ of linear spaces, and the indeterminates in $\mathcal{S}$ cut out a linear space of minimal codimension among those which contain $\mathcal{U}$ (that therefore has the same codimension as $\mathcal{U}$). We claim that $L \setminus \mathcal{B}$ is a subset of L with the Property 1. In fact, we can choose as a set of generators for $\mathrm{LT}(I_t(L))$ the set of all skew-diagonal terms of $t_j \times t_j$ minor in L_j. Notice moreover that $t_j \geq t_{i,j}$ and that $L_{i,j} \setminus \mathcal{B}_{i,j} \neq \emptyset$ if $L_{i,j} \neq \emptyset$.

In order to complete the proof, we need to show that any $\mathcal{S}'$ with $|\mathcal{S}'| < |L \setminus \mathcal{B}|$ does not satisfy the Property 1. Let $\mathcal{S} = L \setminus \mathcal{B}$, and let

$$\delta_u = \{x_{b,a} \in L \mid b + a = u + 1\}, \quad u \geq 1.$$

Then $L = \cup_{u\geq 1}\delta_u$ and each two distinct elements in the union are disjoint. Since

$$\sum_{u\geq 1} |\mathcal{S} \cap \delta_u| = |\mathcal{S}| > |\mathcal{S}'| = \sum_{u\geq 1} |\mathcal{S}' \cap \delta_u|$$

then $|\mathcal{S} \cap \delta_u| > |\mathcal{S}' \cap \delta_u|$ for some $u \geq 1$. Notice that if $\delta_u \cap L_j \neq \emptyset$, then $\delta_u \subseteq L_j$. This follows from the definition of ladder. So let j be maximum with the property that $\mathcal{S} \cap \delta_u \subseteq L_j$. Then

$$|\mathcal{S}' \cap \delta_u \cap L_j| \leq |\mathcal{S}' \cap \delta_u| < |\mathcal{S} \cap \delta_u| = |L_j \cap \delta_u| - |(\mathcal{S} \setminus L_j) \cap \delta_u| = |L_j \cap \delta_u| - (t_j - 1).$$

Therefore there are t_j distinct variables in $(\delta_u \cap L_j) \setminus \mathcal{S}'$. Their product corresponds to a minimal generator of $\mathrm{LT}(I_t(L))$ that does not involve any element of $\mathcal{S}'$. □

We now construct a subladder L' of L such that the number of entries in L' equals the height of $I_t(L)$.

Theorem 1.15. *Let L be a ladder with upper outside corners and lower outside corners as in Notation 1.2. Let L' be the sublattice with the same upper outside corners as L, and lower outside corners $\{(d_j - t_j + 1, c_j + t_j - 1) \mid 1 \leq j \leq k\}$. Then*

$$ht\ I_t(L) = |L'|.$$

Proof. First notice that L' is a ladder, since the coordinates of its outside lower corners verify the inequalities

$$d_{j+1} - t_{j+1} + 1 > d_j - t_j + 1 \quad \text{and} \quad c_{j+1} + t_{j+1} - 1 > c_j + t_j - 1 \tag{1}$$

by Assumption 3. Notice that it follows from (1) that no two outside lower corners of L' belong to the same row or column. Let $\mathcal{M}$ be the set of minors in $\mathcal{G}$ that involve consecutive rows and columns (see the statement of Theorem 1.10 for the definition of $\mathcal{G}$).

Claim 1. $|\mathcal{M}| = |L \setminus \mathcal{B}|$.

There is a bijection between the two sets, given by the fact that every minor in $\mathcal{M}$ corresponds uniquely to its lower left corner.

Claim 2. $|\mathcal{M}| = |L'|$.

Again, there is a bijection between the two sets. Every minor in $\mathcal{M}$ corresponds uniquely to its upper right corner. The two claims together with Theorem 1.14 prove the thesis. □

Remark 1.16. Theorem 1.15 also follows from the observation that for each $1 \leq i \leq h$ and $1 \leq j \leq k$ such that $L_{i,j} \neq \emptyset$, we delete $r_{i,j} = t_{i,j} - t_{i-1,j}$ rows and $c_{i,j} = t_{i,j-1} - t_{i,j}$ columns from $L_{i,j}$. Therefore, the total number of entries deleted from a row that belongs to $L_i \setminus L_{i-1}$ is

$$\sum_{j:\, L_{i,j} \neq \emptyset} c_{i,j} = \sum_{j:\, L_{i,j} \neq \emptyset} (t_{i,j} - t_{i-1,j}) = t_i - 1.$$

Similarly, the total number of entries deleted from a column that belongs to $L_j \setminus L_{j+1}$ is

$$\sum_{i:\, L_{i,j} \neq \emptyset} r_{i,j} = \sum_{i:\, L_{i,j} \neq \emptyset} (t_{i,j-1} - t_{i,j}) = t_j - 1.$$

Remark 1.17. In Theorem 2.1 we show that if V is a mixed ladder determinantal variety with $I_V = I_t(L)$, then V is G-bilinked in $t_1 + \cdots + t_k - k$ steps to the linear variety associated to $I_1(L')$.

Following the proof of Theorem 4.1 in [11], we can show that mixed ladder determinantal ideals are prime. Hence they define irreducible projective varieties.

Theorem 1.18. *Let $X_i \subseteq X$ be as in Notation 1.2. Let $X' = X_1 \cup \ldots \cup X_k$ and let $I_t(X') = I_{t_1}(X_1) + \cdots + I_{t_k}(X_k) \subseteq K[X']$. Then*

$$I_t(X') \cap K[L] = I_t(L).$$

In particular, $I_t(L)$ is a prime ideal.

Proof. It is enough to show that $I_t(X') \cap K[L] \subseteq I_t(L)$. For $j = 1, \ldots, k$, let $\mathcal{G}_j$ be the set of determinants of the $t_j \times t_j$ submatrices of X_j, such that at most $t_i - 1$ rows belong to X_i for $1 \leq i < j$ and at most $t_i - 1$ columns belong to X_i for $j \leq i < k$. Let $\mathcal{G} = \cup_{j=1}^k \mathcal{G}_j$. By Theorem 1.10 $\mathcal{G}$ is a Gröbner basis of $I_t(X')$ with respect to $<$. Let $f \in I_t(X') \cap K[L]$ be a homogeneous form. Then

$$f = \mathrm{LT}(f) + g = \mathrm{LT}(m)h + g$$

where $g = f - \mathrm{LT}(f)$, $\mathrm{LT}(m)$ is the leading term of some element $m \in \mathcal{G}$, and $h \in K[L]$ is a monomial (possibly multiplied by a constant). Then

$$f = mh + f'$$

where $f' = f - mh \in I_t(X') \cap K[L]$, $deg(f) = deg(f')$ and $\mathrm{LT}(f') < \mathrm{LT}(f)$. The proof is complete by induction on $\mathrm{LT}(f)$. $\square$

Any mixed ladder determinantal variety has a nonempty open subset which is isomorphic to a nonempty open subset of a ladder determinantal variety. We prove this in Proposition 1.20, by repeatedly localizing the coordinate ring of the variety at a corner. We need the following lemma.

Lemma 1.19. *Let L be a ladder and let $I_t(L)$ be a mixed ladder determinantal ideal, as in Notation 1.2. Let $d_0 = 0$ and $c_{k+1} = n + 1$. Fix a $j \in \{1, \ldots, k\}$ such that*

$$t_j > 1, \quad d_{j-1} < d_j, \quad c_{j+1} > c_j.$$

Let M be the ladder obtained from L by removing the indeterminates

$$x_{d_{j-1}+1,c_j}, \ldots, x_{d_j,c_j}, x_{d_j,c_j+1}, \ldots, x_{d_j,c_{j+1}-1}.$$

The ladder M has lower outside corners

$$\{(d_1, 1), \ldots, (d_{j-1}, c_{j-1}), (d_j - 1, c_j + 1), (d_{j+1}, c_{j+1}), \ldots, (m, c_k)\}$$

and the same upper outside corners as L. Let $t' = (t_1, \ldots, t_{j-1}, t_j - 1, t_{j+1}, \ldots, t_k)$. Then there is an isomorphism

$$K[L]/I_t(L)[x^{-1}_{d_j,c_j}] \cong K[M]/I_{t'}(M)[x_{d_{j-1}+1,c_j}, \ldots, x_{d_j,c_j}, x_{d_j,c_j+1}, \ldots, x_{d_j,c_{j-1}-1}][x^{-1}_{d_j,c_j}].$$

Proof. Define

$$\varphi : K[L][x^{-1}_{d_j,c_j}] \longrightarrow K[M][x_{d_{j-1}+1,c_j}, \ldots, x_{d_j,c_j}, x_{d_j,c_j+1}, \ldots, x_{d_j,c_{j+1}-1}][x^{-1}_{d_j,c_j}]$$

by the assignment

$$\varphi(x_{u,v}) = \begin{cases} x_{u,v} + x_{u,c_j} x_{d_j,v} x^{-1}_{d_j,c_j} & \text{if } u \neq d_j, v \neq c_j \text{ and } x_{u,v} \in L_j, \\ x_{u,v} & \text{otherwise.} \end{cases}$$

The inverse of φ is

$$\psi : K[M][x_{d_{j-1}+1,c_j}, \ldots, x_{d_j,c_j}, x_{d_j,c_j+1}, \ldots, x_{d_j,c_{j+1}-1}][x^{-1}_{d_j,c_j}] \longrightarrow K[L][x^{-1}_{d_j,c_j}]$$

defined by the assignment

$$\psi(x_{u,v}) = \begin{cases} x_{u,v} - x_{u,c_j} x_{d_j,v} x^{-1}_{d_j,c_j} & \text{if } u \neq d_j, v \neq c_j \text{ and } x_{u,v} \in M_j, \\ x_{u,v} & \text{otherwise.} \end{cases}$$

Let

$$J_t(L) = I_t(L) K[L][x^{-1}_{d_j,c_j}]$$

and

$$J_{t'}(M) = I_{t'}(M) K[M][x_{d_{j-1}+1,c_j}, \ldots, x_{d_j,c_j}, x_{d_j,c_j+1}, \ldots, x_{d_j,c_{j+1}-1}][x^{-1}_{d_j,c_j}].$$

It suffices to show that $\varphi(J_t(L_j)) = J_{t'}(M_j)$ for $1 \leq j \leq k$, where $M_j = L_j \cap M$. Since L_j is a one-sided ladder, the thesis follows from Lemma 5.1.4 in [4]. $\square$

Proposition 1.20. *Let $I_t(L)$ be a mixed ladder determinantal ideal as in Notation 1.2. Let*

$$t_{\max} = \max\{t_1, \ldots, t_k\}, \qquad t_{\min} = \min\{t_1, \ldots, t_k\}.$$

Let $L_{\max}$ be the ladder obtained from L by enlarging each L_j along its lower border by a strip of $t_{\max} - t_j$ new indeterminates. In other words, $L_{\max}$ has the same upper outside corners as L, and lower outside corners $\{(d_j + (t_{\max} - t_j), c_j - (t_{\max} - t_j))\}$. Let $L_{\min}$ be the ladder obtained from L by removing from each L_j along its lower border a strip of $t_j - t_{\min}$ indeterminates. In other words, $L_{\min}$ has the same upper outside corners as L, and lower outside corners $\{(d_j - (t_j - t_{\min}), c_j + (t_j - t_{\min}))\}$. Let $I_{\max} = I_{t_{\max}}(L_{\max})$ and $I_{\min} = I_{t_{\min}}(L_{\min})$. Then there are $u \in K[L_{\max}]/I_{t_{\max}}(L_{\max})$, $v \in K[L]/I_t(L)$, and indeterminates $w_1, \ldots, w_i, z_1, \ldots, z_j$ such that

$$K[L_{\max}]/I_{\max}[u^{-1}] \cong K[L]/I_t(L)[w_1, \ldots, w_i][w_1^{-1}, \ldots, w_{i_0}^{-1}]$$

and

$$K[L_{\min}]/I_{\min}[z_1, \ldots, z_j][z_1^{-1}, \ldots, z_{j_0}^{-1}] \cong K[L]/I_t(L)[v^{-1}]$$

for some $i_0 \leq i$ and $j_0 \leq j$.

Proof. The thesis easily follows from Lemma 1.19. We sketch how it can be derived from it. In order to prove the first of the two ring isomorphisms, keep localizing the ring $K[L_{\max}]/I_{\max}$ at the indeterminates in the appropriate lower outside corners of $L_{\max}$. The size of the minors considered in each region decreases by 1 at each localization, and by Lemma 1.19 we have an isomorphism as claimed. A similar argument can be made starting from $K[L]/I_t(L)$. The products of the elements that were inverted give u and v, respectively. □

We now prove that $I_t(L)$ defines an arithmetically Cohen-Macaulay variety.

Theorem 1.21. *Mixed ladder determinantal ideals are Cohen-Macaulay.*

Proof. Let $I_t(L)$ be a mixed ladder determinantal ideal. We follow the Notation 1.2. Let $t_{\max}, t_{\min}, L_{\max}, L_{\min}$ be as in Proposition 1.20. Let $I_{\max} = I_{t_{\max}}(L_{\max})$ and $I_{\min} = I_{t_{\min}}(L_{\min})$. We saw that

$$K[L_{\max}]/I_{\max}[u^{-1}] \cong K[L]/I_t(L)[w_1, \ldots, w_i][w_1^{-1}, \ldots, w_{i_0}^{-1}],$$

and

$$K[L_{\min}]/I_{\min}[z_1, \ldots, z_j][z_1^{-1}, \ldots, z_{j_0}^{-1}] \cong K[L]/I_t(L)[v^{-1}].$$

The rings $K[L_{\max}]/I_{\max}$ and $K[L_{\min}]/I_{\min}$ are Cohen-Macaulay by Corollary 4.10 in [7]. So the thesis follows from the observation that a ring S is Cohen-Macaulay if and only if $S[x]$ is Cohen-Macaulay. □

In Theorem 5.2 of [3], Conca classifies the arithmetically Gorenstein ladder determinantal varieties associated to one-sided ladders, or to two-sided ladders satisfying a numerical condition. Using the result of Conca, Gonciulea and Miller classify the arithmetically Gorenstein mixed ladder determinantal varieties associated to one-sided ladders (see Theorem 6.3.1 of [4]). Using the same result of Conca together with Proposition 1.20, we can characterize the arithmetically Gorenstein mixed ladder determinantal varieties among those with the Property 2 below.

Notation 1.22. Let L be a ladder as in Notation 1.2. Let $t_{\max} = \max\{t_1, \ldots, t_k\}$. Let $\tau \geq 2$ be an integer. We denote by L_τ a ladder with the same upper outside corners as L, and lower outside corners $\{(d_j + (\tau - t_j), c_j - (\tau - t_j))\}$.

In other words, L_τ is the ladder obtained from L by deleting (resp. adding) $t_j - \tau$ (resp. $\tau - t_j$) rows and columns from the j-th lower step if $\tau \leq t_j$ (resp. $\tau \geq t_j$).

The same argument as in the proof of Proposition 1.20 shows the following.

Theorem 1.23. *Let $I_t(L)$ be a mixed ladder determinantal ideal as in Notation 1.2. Let $\tau \geq 2$ be an integer, and let L_τ be as in Notation 1.22. Then there are $u \in K[L_\tau]/I_\tau(L_\tau)$, $v \in K[L]/I_t(L)$, and indeterminates $w_1, \ldots, w_i, z_1, \ldots, z_j$ such that*

$$K[L_\tau]/I_\tau(L_\tau)[w_1, \ldots, w_i][w_1^{-1}, \ldots, w_{i_0}^{-1}, u^{-1}] \cong K[L]/I_t(L)[z_1, \ldots, z_i][z_1^{-1}, \ldots, z_{j_0}^{-1}, v^{-1}]$$

for some $i_0 \leq i$ and $j_0 \leq j$.

In Theorem 1.25 we characterize Gorensteinness for mixed ladder determinantal rings which satisfy the following condition.

Property 2. There exists $\tau \geq 2$ such that the ladder L_τ is connected and

$$\min_i\{a_{i+1} - a_i, b_{i+1} - b_i, c_{i+1} + t_{i+1} - c_i - t_i, d_{i+1} - t_{i+1} - d_i + t_i\} > \tau - 2.$$

Remarks 1.24.
- If the minimum condition is satisfied for τ, then it is satisfied for $\tau - 1$ as well. However, $L_{\tau-1}$ may be disconnected even when L_τ is connected.
  Equivalently, observe that for $\tau \gg 0$ the ladder L_τ is connected. However, the truth value of the minimum condition may change passing from a smaller value of τ to a larger one.
- If no two corners of L_2 belong to the same row or to the same column, then the minimum condition is automatically verified. If in addition L_2 is connected, then L satisfies Property 2.

We are now ready to give a numerical criterion to decide whether the ring $K[L]/I_t(L)$ is Gorenstein, for a ladder L that satisfies Property 2.

Theorem 1.25. *Let L be a ladder satisfying Property 2. Then $I_t(L)$ is Gorenstein if and only if all of the following hold:*

- $m - t_k = n - t_1$,
- $c_{j+1} - d_j = 2 + t_1 - t_j - t_{j+1}$ *for* $j = 1, \ldots, k-1$,
- $a_i - b_{i+1} = t_1 - 2$ *for* $i = 1, \ldots, h-1$.

Proof. Let $I_t(L)$ be a mixed ladder determinantal ideal. We follow the Notation 1.2. Let L_τ be as in Property 2, and $I = I_\tau(L_\tau)$. By Theorem 1.23

$$K[L_\tau]/I[w_1, \ldots, w_i][z_1^{-1}, \ldots, z_{j_0}^{-1}, u^{-1}] \cong K[L]/I_t(L)[z_1, \ldots, z_i][w_1^{-1}, \ldots, w_{i_0}^{-1}, v^{-1}].$$

By Theorem 5.2 of [3], the ring $K[L_\tau]/I$ is Gorenstein if and only if X_τ is a square matrix, and the inside lower corners (resp. inside upper corners) lie on certain diagonals. Here X_τ is the smallest matrix containing L_τ. The numerical conditions in [3] translate into the conditions in our thesis. Notice moreover that Assumption 3 guarantees that no two consecutive corners of L_τ belong to the same row or to the same column. So the thesis follows from the observation that a ring S is Gorenstein if and only if $S[x]$ is Gorenstein. □

Remark 1.26. It is natural that the conditions of Theorem 1.25 are independent of τ. The requirement that Property 2 holds for some τ is required in order to apply Theorem 5.2 of [3]. However, Gorensteinness is independent of τ. In fact, by Lemma 1.19 either the ideals $I_\tau(L_\tau)$ are Gorenstein for every choice of τ, or none of them is Gorenstein.

2. Linkage of mixed ladder determinantal varieties

In this section we prove that any mixed ladder determinantal variety can be obtained from a linear variety by a finite sequence of ascending elementary G-biliaisons. The biliaisons are performed on mixed ladder determinantal varieties, which are irreducible, hence generically Gorenstein. Therefore we conclude that mixed ladder determinantal varieties are glicci. In the end of the section, we sketch how one can show that the same conclusions hold for a mixed ladder determinantal variety associated to a matrix of generic forms.

The next theorem is the main result of this paper. It is analogous to Gaeta's Theorem, to Theorem 3.6 of [8], and to Theorem 4.1 of [6]. It is also analogous to Theorem 2.3 of [5] for a matrix that is not symmetric. The idea of the proof is as follows: starting from a mixed ladder determinantal variety V, we construct two more mixed ladder determinantal varieties V' and W such that V and V' are generalized divisors on W. We show that V' can be obtained from V by an elementary G-biliaison on W.

Theorem 2.1. *Any mixed ladder determinantal variety in $\mathbb{P}^r$ can be obtained from a linear variety by a finite sequence of ascending elementary biliaisons.*

Proof. Let $V \subset \mathbb{P}^r = Proj(K[L])$ be a mixed ladder determinantal variety. We follow the Notation 1.2. Let $L \subseteq X = (x_{i,j})$ be the ladder whose minors of size $t = (t_1, t_2, \ldots, t_k)$ define the variety V. In other words, $I_V = I_{t_1}(L_1) + \cdots + I_{t_k}(L_k) \subseteq R = K[L]$.

If $t_1 = \ldots = t_k = 1$ then V is a linear variety. Therefore, we concentrate on the case when $t_i \geq 2$ for some i.

Let L' be the subladder of L defined by the lower outside corners

$$\{(d_j - t_j + 1, c_j + t_j - 1) \mid j = 1, \ldots, k\}.$$

It follows from Theorem 1.15 that $I_t(L)$ defines a variety of codimension c equal to the number of entries in L'. Fix an i such that $t_i \geq 2$ and let (d_i, c_i) be the position of the i-th lower outside corner of L. Let

$$t' = (t_1, \ldots, t_{i-1}, t_i - 1, t_{i+1}, \ldots, t_k),$$

and let M be the subladder of L defined by the lower outside corners $\{(\delta_j, \gamma_j) \mid j = 1, \ldots, k\}$:

$$(\delta_j, \gamma_j) = \begin{cases} (d_i - 1, c_i + 1) & \text{if } j = i, \\ (d_j, c_j) & \text{if } j \neq i. \end{cases}$$

Informally, M is obtained from L by removing the last row and the first column from the i-th lower step of the ladder L (in the case of consecutive lower outside corners that lie on the same row or column however, one has to be careful about the corners while canceling). Notice that M is a ladder according to Definition 1.2, unless either $d_i = d_{i-1}$ or $c_i = c_{i+1}$. By Assumption 3, if $d_i = d_{i-1}$ then $t_{i-1} > t_i \geq 2$. Therefore we can replace i by $i-1$ and construct a new subset M of L, starting from the $(i-1)$-th lower outside corner of L. The process eventually ends, since the subset M constructed started from the leftmost lower outside corner of L on the row d_i is always a ladder. If $c_i = c_{i+1}$, then $t_{i+1} > t_i \geq 2$. In this case we can replace i by $i+1$ and construct a new subset M of L, starting from the $(i+1)$-th lower outside corner of L. The process eventually ends, since the subset M constructed started from the lowest outside lower corner of L on the column c_j is always a ladder. Summarizing, we have shown that starting from a ladder L with $t_j \geq 2$ for some j, we can construct a ladder $M \subset L$, with $t'_i = t_i - 1$ for some i and $t'_j = t_j$ for $j \neq i$.

Again, by Theorem 1.15 $I_{t'}(M) \subset K[M]$ defines a mixed ladder determinantal variety of codimension $c = \text{ht } I_1(L')$. Hence $I_{t'}(M) \subset K[L]$ defines a variety $V' \subset \mathbb{P}^r$ of codimension c, which is a cone over a mixed ladder determinantal variety. From now on, we will not distinguish between mixed ladder determinantal varieties and cones over them.

Let N be the ladder obtained from L by removing the entry x_{d_i,c_i}. N is a ladder, since both L and M are. N has the same upper outside corners as L, M, and it has lower outside corners

$$\{(d_1, c_1), \ldots, (d_{i-1}, c_{i-1}), (d_i - 1, c_i), (d_i, c_i + 1), (d_{i+1}, c_{i-1}), \ldots, (d_k, c_k)\}.$$

Let

$$\tau = (t_1, \ldots, t_{i-1}, t_i, t_i, t_{i+1}, \ldots, t_k)$$

be the vector obtained from t by repeating twice the entry t_i. Let W be the mixed ladder determinantal variety with associated ideal $I_\tau(N)$. It follows from Theorems 1.15, 1.18, and 1.21 that W has codimension $c - 1$, and is irreducible and arithmetically Cohen-Macaulay. So W is generically Gorenstein and V, V' are generalized divisor on it.

We denote by H a hyperplane section divisor on W. We claim that

$$V \sim V' + H,$$

where $\sim$ denotes linear equivalence of divisors on W. It follows that V is obtained by an ascending elementary biliaison from V'. Repeating this argument, after $t_1 + \cdots + t_k - k$ biliaisons we reduce to the case $t_1 = \ldots = t_k = 1$. Then the resulting variety is linear, hence we have bilinked V to a linear variety. Notice that the ideal of this linear variety is generated by the entries of L', where L' is the subladder of L defined in Theorem 1.15.

Let $\mathcal{I}_{V|W}, \mathcal{I}_{V'|W}$ be the ideal sheafs on W of V and V'. In order to prove the claim we must show that

$$\mathcal{I}_{V|W} \cong \mathcal{I}_{V'|W}(-1). \tag{2}$$

A minimal system of generators of $I_{V|W} = H^0_*(\mathcal{I}_{V|W}) = I_t(L)/I_\tau(N)$ is given by the images in the coordinate ring of W of the $t_i \times t_i$ minors of L that involve the entry x_{d_i,c_i}. To keep the notation simple, we denote both an element of R and its image in R/I_W with the same symbol, and we set $t = t_i$, $d = d_i$, $c = c_i$. We let a minor $L_{i_1,\ldots,i_{t-1},d;c,j_1,\ldots,j_{t-1}}$, resp. $L_{i_1,\ldots,i_{t-1};j_1,\ldots,j_{t-1}}$, be 0 if it involves one or more entries that do not belong to L. Hence we have

$$I_{V|W} = (L_{i_1,\ldots,i_{t-1},d;c,j_1,\ldots,j_{t-1}} \mid 1 \le i_1 < \ldots < i_{t-1} < d, c < j_1 < \ldots < j_{t-1} \le n).$$

A minimal system of generators of $I_{V'|W} = H^0_*(\mathcal{I}_{V'|W}) = I_{t-1}(M)/I_t(N)$ is given by the images in the coordinate ring of W of the minors of L_i of size $t_i - 1$ that do not involve row d and column c:

$$I_{V'|W} = (L_{i_1,\ldots,i_{t-1};j_1,\ldots,j_{t-1}} \mid 1 \le i_1 < \ldots < i_{t-1} < d, c < j_1 < \ldots < j_{t-1} \le n).$$

Notice that with our conventions we have

$$L_{i_1,\ldots,i_{t-1},d;c,j_1,\ldots,j_{t-1}} = 0 \quad \text{if and only if} \quad L_{i_1,\ldots,i_{t-1};j_1,\ldots,j_{t-1}} = 0$$

where $L_{i_1,\ldots,i_{t-1},d;c,j_1,\ldots,j_{t-1}}$ is a generator of $I_{V|W}$ and $L_{i_1,\ldots,i_{t-1};j_1,\ldots,j_{t-1}}$ is a generator of $I_{V'|W}$.

In order to produce an isomorphism as in (2), it suffices to observe that the ratios

$$\frac{L_{i_1,\ldots,i_{t-1},d;c,j_1,\ldots,j_{t-1}}}{L_{i_1,\ldots,i_{t-1};j_1,\ldots,j_{t-1}}} \tag{3}$$

are all equal as elements of $H^0(\mathcal{K}_W(1))$, where $\mathcal{K}_W$ is the sheaf of total quotient rings of W. Then the isomorphism (2) is given by multiplication by that element.

Equality of all the ratios in (3) follows from Lemmas 2.4 and 2.6 in [5], where we prove that

$$L_{i_1,\ldots,i_{t-1},d;c,j_1,\ldots,j_{t-1}} \cdot L_{k_1,\ldots,k_{t-1};l_1,\ldots,l_{t-1}} - L_{k_1,\ldots,k_{t-1},d;c,l_1,\ldots,l_{t-1}} \cdot L_{i_1,\ldots,i_{t-1};j_1,\ldots,j_{t-1}} \in I_W$$

for any choice of $i_1, \ldots, i_{t-1}$, $j_1, \ldots, j_{t-1}$, $k_1, \ldots, k_{t-1}$, $l_1, \ldots, l_{t-1}$. This completes the proof of the claim and of the theorem. □

Standard results in liaison theory give the following.

Corollary 2.2. *Every mixed ladder determinantal variety V can be bilinked in $t_1 + \ldots + t_k - k$ steps to a linear variety of the same codimension. In particular determinantal varieties associated to matrices of indeterminates, ladder determinantal varieties, and mixed ladder determinantal varieties are glicci.*

Remark 2.3. In Theorem 1.25 we classify the mixed ladder determinantal varieties which are arithmetically Gorenstein (among those that satisfy Property 2). The fact that arithmetically Gorenstein schemes are glicci follows from a result of Casanellas, Drozd, and Hartshorne (see [1]).

We conclude by noticing that by the upper-semicontinuity principle, one can argue that a generic mixed ladder determinantal scheme is glicci. The idea is as follows: fix m, n and upper and lower outside corners for a ladder L, and assume that all the ideals involved in the proof of Theorem 2.1 have maximal height (i.e. the height of the ideal that we would obtain if all the entries in L were distinct indeterminates). This is the case e.g. when the entries of L are generic polynomials in $S = K[x_0, \ldots, x_s]$ and $s \gg 0$. Let T be the scheme with saturated ideal $I_T = I_t(L) \subseteq S$. We call such a scheme a **generic mixed ladder determinantal scheme**. The proof of Theorem 2.1 can be repeated with no changes for T, under the assumption that T is a generic mixed ladder determinantal scheme. Therefore we have the following.

Theorem 2.4. *Any generic mixed ladder determinantal scheme can be obtained from a complete intersection by a finite sequence of ascending elementary G-biliaisons. In particular, a generic mixed ladder determinantal scheme is glicci.*

References

[1] M. Casanellas, E. Drozd, R. Hartshorne, Gorenstein liaison and ACM sheaves, J. Reine Angew. Math. **584** (2005), 149–171.

[2] A. Conca, Gröbner bases and determinantal rings, Ph.D. thesis, Universität-Gesamthochschule Essen (1993).

[3] A. Conca, Gorenstein ladder determinantal rings, J. London Math. Soc. (2) **54** (1996), no. 3, 453–474.

[4] N. Gonciulea, C. Miller, Mixed ladder determinantal varieties, J. Algebra **231** (2000), no. 1, 104–137.

[5] E. Gorla, The G-biliaison class of symmetric determinantal schemes, to appear in J. Algebra.

[6] R. Hartshorne, Generalized Divisors and Biliaison, preprint (2004).

[7] J. Herzog and N. V. Trung, Gröbner bases and multiplicity of determinantal and Pfaffian ideals, Adv. Math. **96** (1992), no. 1, 1–37.

[8] J. O. Kleppe, J. C. Migliore, R. M. Miró-Roig, U. Nagel, and C. Peterson, Gorenstein liaison, complete intersection liaison invariants and unobstructedness, Mem. Amer. Math. Soc. **154** (2001), no. 732.

[9] M. Kreuzer, J. C. Migliore, C. Peterson, and U. Nagel, Determinantal schemes and Buchsbaum-Rim sheaves, J. Pure Appl. Algebra **150** (2000), no. 2, 155–174.

[10] J. C. Migliore, Introduction to liaison theory and deficiency modules, Progress in Mathematics **165**, Birkhauser Boston, Inc., Boston, MA (1998)

[11] H. Narasimhan, The irreducibility of ladder determinantal varieties, J. Algebra **102** (1986), no. 1, 162–185.

Institut für Mathematik, Universität Zürich,
Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

Current address: Max Planck Institut für Mathematik,
Vivatsgasse 7, D-53111 Bonn, Germany

E-mail address: elisa.gorla@math.unizh.ch