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Multiperiod Mean-Variance Efficient Portfolios with Endogenous Liabilities

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Abstract

We study the optimal policies and mean-variance frontiers (MVF) of a multiperiod mean-variance optimization of assets and liabilities (AL). This makes the analysis more challenging than for a setting based on purely exogenous liabilities, in which the optimization is only performed on the assets while keeping liabilities fixed. We show that under general conditions for the joint AL dynamics, the optimal policies and the MVF can be decomposed into an orthogonal set of basis returns using exterior algebra. This formalism, novel to financial applications, allows us to study analytically the structure of optimal policies and MVF representations under endogenous liabilities in a multidimensional and multiperiod setting. Using a numerical example, we illustrate our methodology by analyzing the impact of the rebalancing frequency on the MVF and by highlighting the main differences between exogenous and endogenous liabilities.

JEL Classification Codes: C0, C61, G11.

Key Words: Assets and Liabilities, Mean-Variance Frontiers, Markowitz Model, Endogenous Liabilities.
In this paper we extend previous research on multiperiod mean-variance AL optimization by allowing for a simultaneous optimization of the balance sheet as a whole. Using a geometric approach, we study the optimal policies and mean-variance frontiers (MVF) of a multiperiod mean-variance optimization of assets and liabilities (AL). Our model is based on a separate specification of the assets’ and liabilities’ dynamics, where the surplus results as the final aggregated difference between assets and liabilities. Such a modeling approach has several advantages. Most importantly, it can account for different distributional properties of each AL component separately. In practice, these differences do arise, since each component of the balance sheet can differ with respect to, e.g., duration, liquidity, and embedded optionalities. A direct specification of the surplus dynamics as the only relevant state variable would make it impossible to account for such differences.

Most of the literature on AL optimization starts from the optimization of the aggregate surplus, keeping liabilities fixed and optimizing the assets mix only. Assuming exogenously fixed liabilities might be realistic for some types of applications, as for instance those related to the AL optimization of pension funds. However, such an assumption is too restrictive for many financial institutions like banks. Indeed, these institutions typically do optimize not only their assets mix, but also their liabilities mix. The bank’s liability management objectives are to maintain liquidity, provide and enhance access to a diverse and stable source of funds, provide competitively priced and attractive products to customers, conduct funding at a low cost relative to current market conditions, and engage in sound balance sheet management strategies. Therefore, banks offer, e.g., several types of deposit accounts, which differ in terms of the services provided by the bank, the rate of return earned by the depositor, and the risk incurred by the depositor. These products provide a lot of flexibility to effectively comply with the bank’s AL objectives. Hence, given the practical relevance of the liabilities mix, we do treat them endogenously and not as exogenously fixed.

The contemporaneous optimization of the balance sheet as a whole makes the analysis of AL optimal portfolios and MVF more challenging than in a setting based on purely exogenous liabilities. Indeed, increasing the number of state variables and controls in the model may have

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1 See Hilli, Koivu, Pennanen and Ranne (2007), among others, who present a stochastic programming model for the asset liability management of a Finnish pension company.
the drawback of making the optimization problem more difficult to solve. Furthermore, the antisymmetric structure of assets and liabilities make it impossible to apply, e.g., the geometric structure developed in Leippold, Trojani and Vanini (2004). Under general conditions on the joint AL dynamics, we show that the optimal policies and MVF can be decomposed into an orthogonal set of basis returns by using a formalism based on Grassmann algebra. To our knowledge, our paper is the first that uses this formalism to study a financial application.

Our effort to remain as analytical as possible has several advantages. For instance, the analytical solution of the multiperiod problem allows the risk manager to directly quantify the impact and gains from changing the rebalancing frequency. Another aspect that can be addressed is whether one should manage the liabilities actively or passively. Using the formalism presented in this paper, both the exogenous and endogenous case can be solved and compared against each other. Furthermore, since the number of assets and liabilities does not alter the geometrical structure of the optimal surplus decomposition, we can directly analyze the effects of adding or removing AL components on the efficient set and the optimal policies.

The paper is structured as follows. The next section briefly reviews the literature on mean-variance portfolio selection and AL optimization. Section 2 introduces our model and formulates a general orthogonal decomposition theorem for the optimal final surplus in a multiperiod mean-variance optimization of AL. Section 3 proves the general theorem and elaborates on the geometric distinction between AL optimizations under exogenous and endogenous liabilities. Section 4 illustrates the methodology by studying the impact of the rebalancing frequency on the MVF and by highlighting in some more detailed numerical examples the main differences between exogenous and endogenous liabilities. Section 5 concludes.

1 Background

Our model extends the basic intuition behind the static Markowitz (1952) model to a dynamic framework where portfolios of assets and liabilities are optimized jointly in a multiperiod setting. Owing to its practical value, several authors proposed some generalization of Markowitz’s
seminal work to a multiperiod context using different modelling techniques\footnote{See also Steinbach (2001) for a review of the literature on mean-variance models in financial portfolio analysis.}. Early essays in this direction were for instance proposed in Smith (1967), Mossin (1968), Merton (1969), Samuelson (1969), and more recently Duffie and Richardson (1991), Grauer and Hakansson (1993), and Schweizer (1995). Bajeux-Besnainou and Portait (1998) obtained an explicit dynamic asset allocation in a continuous-time mean-variance framework using martingale techniques. Li, Zhou and Lim (2002) study the solutions of a continuous time mean-variance portfolio optimization problem under short-selling constraints on stocks. Their approach adopts the embedding technique first applied by Li and Ng (2000) in a discrete-time mean-variance model\footnote{Such a technique has been extended to a unified framework in Zhou and Li (2000).}.

Models that include explicitly liabilities in a static Markowitz-type optimization were proposed already in the early nineties; see for instance Sharpe and Tint (1990), Elton and Gruber (1992), Leibowitz, Kogelman and Bader (1992), and later Keel and Müller (1995). In a static context, such an inclusion of liabilities is mathematically a simple extension of the standard Markowitz model. However, in a multiperiod model, incorporating liabilities in a portfolio selection problem is more challenging, because the relevant Hamilton Jacobi Bellman equations depend on multivariate state dynamics. Leippold et al. (2004) propose a geometric approach to multiperiod mean-variance AL optimization that largely simplifies the mathematical analysis and the economic interpretation of such model settings. However, their formalism is not directly applicable to studying the joint optimization of assets and liabilities due to the inherent antisymmetric structure. Therefore, it is an open question whether there exists a geometric structure to our optimization problem that allows for a similar analytical tractability as in Leippold et al. (2004).

In this paper, we contribute to previous research on multiperiod mean-variance AL optimization by allowing for a simultaneous optimization of the balance sheet as a whole. This renders the AL dynamics endogenous and makes the analysis more complex than under purely exogenous liabilities. Building on the geometric formalism of exterior algebra, we can study the model with endogenous liabilities from an analytical perspective. As we show below, this
approach can be applied also in the presence of short-selling constraints on the assets and/or the liabilities side.

An alternative route to AL optimization is based on approaches that put AL management at the core of enterprise-wide risk management for financial institutions. This often requires models that can handle at the same time a large number and variety of constraints and rich state variable dynamics. Multi-stage stochastic programming approaches to solve multiperiod investment problems have been proved very useful in such settings. These approaches build on the generation of scenario trees with the objective to minimize the expected costs along the scenario paths. The cost functions are tailored to the circumstances and goals of the financial entity under consideration. Recent endeavors to such multiperiod models, some of them implemented and tested in the industry, include Carinò et. al (1994), Consigli and Dempster (1998), Mulvey (1999), Pflug (2000), Zaremba (2000), Siegmann and Lucas (2001), and Hilli et al. (2007). However, in this paper we aim at isolating and studying the effect of endogenous liabilities in contrast to the case with exogenous liabilities. Therefore, for the sake of analytical tractability, we stick to a mean-variance framework and we leave possible extensions to future research.

2 Model and Main Results

Let the aggregate value of assets at time $t$ be $A(t)$ and the aggregate value of liabilities at time $t$ be $L(t)$. The firm’s aggregate surplus is defined as $S(t) = A(t) - L(t)$ and the joint aggregate assets and liabilities state vector is $s(t) = (A(t), L(t))^\top$. Without loss of generality but for simplicity of exposition, we consider two assets and two liabilities in the balance sheet with gross returns $r_{A,1}(t), r_{A,2}(t)$ and $r_{L,1}(t), r_{L,2}(t)$, respectively. We consider the first asset and the first liability as the reference benchmark for the assets and the liabilities side. We define the joint excess returns vector for the second asset and the second liability relative to the benchmark as $\varphi(t) = (\varphi_A(t), \varphi_L(t))^\top$, where $\varphi_A(t) = r_{A,2}(t) - r_{A,1}(t)$ and $\varphi_L(t) = r_{L,2}(t) - r_{L,1}(t)$. To

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4Early contributions include Cohen and Thore (1970), Both (1972), Both and Dash (1979), Brodt (1979), Brodt (1984), Kallberg, White and Ziemba (1982), and Kusy and Ziemba (1986). See also Ziemba and Mulvey (1998), Zenios and Ziemba (2003), and the overview in Kouwenberg and Zenios (2003) for a complete exposition of these approaches to multiperiod AL optimization.
shorten notation, we write \( r_{A,1}(t) = r_A(t) \) and \( r_{L,1}(t) = r_L(t) \) and \( \mathbf{r}(t) = (r_A(t), r_L(t))^\top \). Similarly, \( \mathbf{R}(t) = \text{Diag}[\mathbf{r}(t)] \) and \( \Phi(t) = \text{Diag}[\varphi(t)] \) are diagonal matrices of benchmark and excess AL returns.

All return variables in the model are assumed to be elements of the space \( L_2 \) (on \( \mathbb{R} \)) of random variables with finite second moments. The scalar product in \( L_2 \) is denoted by \( \langle \cdot, \cdot \rangle \) and we write \( \mathbf{I} \) for the unit vector in \( L_2 \). Information is generated by \( \mathbf{s}(t), t = 0, \ldots, T - 1 \). Conditional expectations at time \( t \) are denoted by \( \mathbb{E}_t(\cdot) \) while unconditional expectations are denoted by \( \mathbb{E}(\cdot) \). Collecting all return processes in the vector \( \mathbf{\mu}_t = (r_A(t), \varphi_A(t), r_L(t), \varphi_L(t))^\top \), we assume the matrices \( \mathbb{E}_t(\mathbf{\mu}_t \mathbf{\mu}_t^\top) \) of conditional second moments at time \( t = 0, \ldots, T - 1 \) to be positive definite.

Consider an investor at time \( t = 0 \) with initial assets \( A(0) \) and initial liabilities \( L(0) \), starting to invest dynamically over a time horizon of length \( T \). Transactions can take place at discrete times \( t = 0, \ldots, T - 1 \). For a given value of aggregate assets \( A(t) \) at time \( t \), we denote by \( W_A(t) \geq 0 \) the amount of wealth invested in the excess return \( \varphi_A(t) \). The amount \( A(t) - W_A(t) \geq 0 \) is invested at time \( t \) in the asset benchmark return \( r_A(t) \). Similarly, \( W_L(t) \geq 0 \) is the amount invested at time \( t \) in the excess liability return \( \varphi_L(t) \), while \( L(t) - W_L(t) \geq 0 \) is the amount invested in the liability benchmark return \( r_L(t) \). Finally, \( \mathbf{w}(t) = (W_A(t), W_L(t))^\top \) is the control vector of amounts invested in the assets and liabilities excess returns. Based on the above notations, the balance sheet at time \( t \) takes the stylized form:

<table>
<thead>
<tr>
<th>Balance Sheet</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Benchmark Asset:</strong></td>
</tr>
<tr>
<td><strong>Other Assets:</strong></td>
</tr>
<tr>
<td><strong>Benchmark Liability:</strong></td>
</tr>
<tr>
<td><strong>Other Liabilities:</strong></td>
</tr>
<tr>
<td><strong>Surplus:</strong></td>
</tr>
</tbody>
</table>

The joint dynamics for assets and liabilities are directly implied by the self financing condition on a candidate AL portfolio policy:

\[
A(t + 1) = r_A(t)A(t) + \varphi_A(t)W_A(t), \quad L(t + 1) = r_L(t)L(t) + \varphi_L(t)W_L(t).
\]

(2.1)
Therefore, the vector dynamics for the bivariate state process of assets and liabilities is

\[ s(t + 1) = R(t)s(t) + \Phi(t)w(t). \quad (2.2) \]

The dynamics (2.2) cannot be reduced to a single surplus dynamics unless \( r_A = r_L \). We assume that the investor maximizes the expected value of the final surplus \( S(T) = A(T) - L(T) \) subject to some given variance penalization:

\[
P(1) : \quad \max_w \left[ \mathbb{E}(S(T)) - \theta \text{VAR}(S(T)) \right],
\]

subject to (2.2),

where \( \theta > 0 \). Problem (1) can be substantially simplified by considering the problem:

\[
P(2) : \quad \max_w \mathbb{E} \left[ (\psi S(T) - \theta S(T)^2) \right],
\]

subject to (2.2),

where \( \psi > 0 \) is an auxiliary parameter. In particular, following Li and Ng (2000), Leippold et al. (2004) prove that (i) any solution of \( P(1) \) is also a solution of \( P(2) \) and (ii) if \( w^* \) is a solution of \( P(2) \) for given \((\psi^*, \theta)\), then it is also a solution for \( P(1) \), if the condition

\[
\psi^* = 1 + 2\theta \mathbb{E}(S(T) | w^*),
\]

(2.3)
is satisfied. Although \( P(1) \) and \( P(2) \) are equivalent problems under condition (2.3), problem \( P(2) \) is separable in the sense of dynamic programming. Therefore, problem \( P(2) \) can be solved by means of standard dynamic programming techniques while problem \( P(1) \) cannot.

We further extend problem \( P(2) \) by introducing a set of short-selling constraints on assets and liabilities. More specifically, \( A(\cdot) \) is the state dependent set of constraints defined by

\[
A(s) = \{ w \mid 0_{2 \times 1} \leq w \leq s \},
\]

(2.4)
and we can formulate the relevant optimization problem as

$$
P(3) \begin{cases} 
\max_{w \in A(s)} \mathbb{E} [\gamma S(T) - \frac{1}{2} S(T)^2] 
\text{subject to (2.2), } w(t) \in A(s(t)); \ t = 0, ..., T - 1, 
\end{cases}$$

where $\gamma = \frac{\psi}{2\theta}$. To solve this problem, we follow a geometric approach and decompose the multiperiod optimal policies in a linear combination of policies with orthogonal terminal date pay-offs in $L_2$. In the theorem below, the main result of our paper, we state this orthogonal decomposition result for the optimal surplus $S^*(T)$ for a fixed time horizon $T - t$.\footnote{More precisely, we would have to write $S^*_t(T)$ to indicate the dependency of the optimal surplus in (2.5) on the investment horizon $T - t$. For notational convenience, we set $S^*(T) = S^*_t(T)$.}

**Theorem 1.** Consider the mean-variance AL optimization problem $P(3)$. If the joint gross benchmark $R(t)$ and the joint excess return $\Phi(t)$ vectors are in $L_2$ for all $t = 0, 1, \ldots, T - 1$, then there exist $t + 2$ random variables $r_s, r_\gamma \in L_2$ such that the optimal final surplus $S^*(T)$ can be decomposed as:

$$S^*(T) = r_s(T - t)^\top s(T - t) + \gamma \sum_{i=1}^{t} r_\gamma(T - i), \ \forall t = 0, \ldots, T - 1. \quad (2.5)$$

The set

$$\{r_s(T - t)^\top s(T - t), r_\gamma(T - t), r_\gamma(T - t + 1), \ldots, r_\gamma(T - 1)\} \quad (2.6)$$

is a $(t + 1)$-dimensional orthogonal subset of $L_2$.

The above theorem allows us to decompose the multiperiod optimal surplus as a linear combination of $L_2$-orthogonal payoffs. The first portfolio component in (2.5), $r_s(T - t)^\top s(T - t)$, depends on the current state of the assets and liabilities and corresponds to the minimum-second-moment (MSM) portfolio (see below). The second component is state-independent and is weighted by the risk aversion term $\gamma$. Section 3 provides the proof of the above theorem. There, we also give an explicit construction of the returns $r_s$ and $r_\gamma$.\footnote{More precisely, we would have to write $S^*_t(T)$ to indicate the dependency of the optimal surplus in (2.5) on the investment horizon $T - t$. For notational convenience, we set $S^*(T) = S^*_t(T)$.}
3  Derivation of the Main Results

3.1  Single-Period Optimization

Consider first the solution of the optimization problem \( P(3) \) at time \( T - 1 \), where we drop all time indexes \( T - 1 \) for brevity. Below, we write \( d = (1, -1)^\top \) for the “difference vector,” i.e., the vector which translates assets and liabilities into a surplus. Then, subject to the state dynamics in equation (2.2), the value function \( J(s) \) of problem \( P(3) \) is defined by:

\[
J(s) = \max_{w \in A(s)} \mathbb{E} \left[ \gamma S(T) - \frac{1}{2} S(T)^2 \right]
\]

\[
= \max_{w \in A(s)} \mathbb{E} \left[ \gamma d \top s(T) - \frac{1}{2} (d \top s(T))^2 \right]
\]

\[
= \max_{w \in A(s)} \left[ \gamma d \top \mathbb{E}(\Phi) w - s \top \mathbb{E} \left( R d d \top \Phi \right) w - \frac{1}{2} w \top \mathbb{E} \left( \Phi d d \top \Phi \right) w \right] + C , \tag{3.7}
\]

using the state dynamics (2.2), where \( C \) is some constant independent of \( w \). The first-order condition implies

\[
w^* = \left[ \mathbb{E} \left( \Phi d d \top \Phi \right) \right]^{-1} \left[ \gamma \mathbb{E}(\Phi) d - \mathbb{E} \left( \Phi d d \top R \right) s + \begin{pmatrix} \alpha_1 - \alpha_2 \\ \alpha_3 - \alpha_4 \end{pmatrix} \right],
\]

where \( \alpha_i \leq 0, i = 1, \ldots, 4 \), are the Kuhn-Tucker multipliers for the constraints in the set \( A(s) \). More precisely, \( \alpha_1 \) and \( \alpha_2 \) are the multipliers for the short-selling assets constraints while \( \alpha_3 \) and \( \alpha_4 \) are the multipliers for the liabilities constraints. If no constraint is binding, then the optimal policy is given by

\[
w_0^* = \left[ \mathbb{E} \left( \Phi d d \top \Phi \right) \right]^{-1} \left[ \gamma \mathbb{E}(\Phi) d - \mathbb{E} \left( \Phi d d \top R \right) s \right] , \tag{3.8}
\]

and the optimal surplus \( S^*(T) \) is

\[
S^*(T) = d \top (Rs + \Phi w_0^*) = r_s \top s + \gamma r, \tag{3.9}
\]
where

\[
\begin{align*}
\mathbf{r}_s^\top &= \mathbf{d}^\top \left( \mathbf{R} - \Phi \left[ \mathbf{E} \left( \Phi \mathbf{d} \mathbf{d}^\top \Phi \right) \right]^{-1} \mathbf{E} \left( \Phi \mathbf{d} \mathbf{d}^\top \mathbf{R} \right) \right), \\
\mathbf{r}_\gamma &= \mathbf{d}^\top \Phi \left[ \mathbf{E} \left( \Phi \mathbf{d} \mathbf{d}^\top \Phi \right) \right]^{-1} \mathbf{E} (\Phi) \mathbf{d}.
\end{align*}
\]

(3.10) (3.11)

We see from (3.9) that the optimal surplus $S^\ast(T)$ has been decomposed into a linear combination of two “returns” in $\mathbf{r}_s$ and an “excess returns” $\mathbf{r}_\gamma$. The weights in the linear combinations are the single components of the state variable $\mathbf{s}$ and the risk aversion related coefficient $\gamma$. The square integrability of the joint gross benchmark and the joint excess return vectors imply the square integrability of $\mathbf{r}_s^\top$ and $\mathbf{r}_\gamma$. Now, our goal is to identify $\mathbf{r}_s^\top \mathbf{s}$ and $\mathbf{r}_\gamma$ as two particular $L^2$ projections in some orthogonal subspaces of $L^2$. This orthogonal decomposition is general and holds irrespectively of the fact that some constraints on assets or liabilities are binding. The presence of binding constraints only changes the spaces on which the relevant projections are defined. The situation in which constraints are binding is easier to analyze and can be handled using a standard projection formalism in $L^2$, as in Leippold et al. (2004). Therefore, in the next sections we discuss first this latter case. In a second step, we study the unconstrained case. To simplify our exposition, we make a separate discussion for each case, but we will use the same notation $\mathbf{r}_s$ and $\mathbf{r}_\gamma$ for the surplus returns. No confusion should occur.

### 3.1.1 Binding Short-Selling Constraints on Liabilities

It is sufficient to study two cases. In the first case, the constraint $W_L \geq 0$ binds while the constraint $L - W_L \geq 0$ is slack. Then $W_L = 0$ follows. In the second case, the constraint $L - W_L \geq 0$ binds while the constraint $W_L \geq 0$ is slack, implying $W_L = L$. Hence, the optimal policies for the two cases are

\[
\mathbf{w}^\ast = (W_A^\ast, 0)^\top, \quad \text{or} \quad \mathbf{w}^\ast = (W_A^\ast, L)^\top.
\]

(3.12)

As a consequence, we can interpret the solution for the first and the second case as the solution of a problem with exogenous liabilities. In the first case, the liabilities amount $L$ is invested exclusively in the gross benchmark liability return $r_L$. In the second case, it is invested exclud-
sively in the gross liability return \( r_L + \varphi_L \). In both cases, \( \mathbf{w}^* \) can be written as the solution to an exogenous benchmark liability optimization of the form studied in Leippold et al. (2004). Using the scalar product in \( L_2 \) the optimal policy can be then written as:

\[
W_A^* = \arg\max_{W_A} \mathbb{E} \left[ \gamma S(T) - \frac{1}{2} \left( S(T) \right)^2 \right] \bigg| \begin{array}{ll}
W_L &= 0 \text{ or } W_L = L \\
\end{array}
\]

\[
= \begin{cases}
\frac{1}{\langle \varphi_A, \varphi_A \rangle} (\langle 1, \varphi_A \rangle - \langle r_A, \varphi_A \rangle A + \langle r_L, \varphi_A \rangle L) & \text{if } W_L = 0 \\
\frac{1}{\langle \varphi_A, \varphi_A \rangle} (\langle 1, \varphi_A \rangle - \langle r_A, \varphi_A \rangle A + \langle r_L + \varphi_L, \varphi_A \rangle L) & \text{if } W_L = L
\end{cases}
\]

The optimal surplus \( S^*(T) \) becomes

\[
S^*(T) = r_s^\top s + \gamma r_\gamma,
\]

where

\[
r_\gamma = \frac{\langle 1, \varphi_A \rangle}{\langle \varphi_A, \varphi_A \rangle} \varphi_A,
\]

\[
r_s^\top = \begin{cases}
\left( r_A - \frac{\langle r_A, \varphi_A \rangle}{\langle \varphi_A, \varphi_A \rangle} \varphi_A, - \left( r_L - \frac{\langle r_L, \varphi_A \rangle}{\langle \varphi_A, \varphi_A \rangle} \varphi_A \right) \right) & \text{if } W_L = 0 \\
\left( r_A - \frac{\langle r_A, \varphi_A \rangle}{\langle \varphi_A, \varphi_A \rangle} \varphi_A, - \left( r_L + \varphi_L - \frac{\langle r_L + \varphi_L, \varphi_A \rangle}{\langle \varphi_A, \varphi_A \rangle} \varphi_A \right) \right) & \text{if } W_L = L
\end{cases}
\]

Using an orthogonal projection notation, the above expressions can be written more compactly as

\[
r_\gamma = \frac{\langle 1, \varphi_A \rangle}{\langle \varphi_A, \varphi_A \rangle} \varphi_A = \mathbb{P}_{\varphi_A}(1),
\]

where \( \mathbb{P}_x(y) \) is the orthogonal projection of \( y \in L_2 \) on the subspace spanned by \( x \in L_2 \). Similarly, we can write \( r_s \) as

\[
r_s^\top = \begin{cases}
\left( \mathbb{P}_{\varphi_A}^\perp(r_A), -\mathbb{P}_{\varphi_A}^\perp(r_L) \right) & \text{if } W_L = 0 \\
\left( \mathbb{P}_{\varphi_A}^\perp(r_A), -\mathbb{P}_{\varphi_A}^\perp(r_L + \varphi_L) \right) & \text{if } W_L = L
\end{cases}
\]

where \( \mathbb{P}_x(y) \) is the orthogonal projection of \( y \in L_2 \) on the orthogonal subspace to the subspace spanned by \( x \in L_2 \). Since \( r_\gamma \) and the components of \( r_s \) are projections on orthogonal subspaces of \( L_2 \), it follows that the surplus \( S^*(T) \) has been decomposed as an orthogonal sum in \( L_2 \). We summarize this result in the next corollary.
Corollary 2. When the short-selling constraints on liabilities are binding, the optimal surplus $S^*(T)$ allows an orthogonal decomposition into two $L_2$-orthogonal returns $r_\gamma$ and $r_\top s$. These returns are projections on the excess asset return and its orthogonal complement.

Hence, the optimal surplus $S^*(T)$ in (3.13) consists of two components, which have the following interpretation. The pay-off difference

$$r_\top s = A\mathbb{P}_{\varphi_A}^+(r_A) - L\mathbb{P}_{\varphi_A}^+(r_L)$$

(3.18)
is the single-period global minimum-second-moment (MSM) final surplus, which is obtained as the difference between a single-period assets-only and a single-period liabilities-only MSM pay-off (corresponding to $\gamma = 0$, $L = 0$, and $\gamma = 0$, $A = 0$, respectively). On the other hand, the pay-off $r_\gamma = \mathbb{P}_{\varphi_A}(l_l)$ is the asset excess return which is closest to the fictive risk-free pay-off $l_l$ in the $L^2$-norm. While the global MSM pay-off in equation (3.18) is the final surplus of a portfolio with a generally non-zero initial position in both assets and liabilities, the pay-off $\mathbb{P}_{\varphi_A}(l_l)$ is an asset excess return and can be generated by a zero initial cost portfolio investing only in the available asset returns. A similar type of decomposition and interpretation as the one obtained in Corollary 2 arises when the short-selling constraints on assets are binding. The next section gives the projection operators that make such a representation explicit in the case of binding constraints on assets.

3.1.2 Binding Short-Selling Constraints on Assets

When constraints on assets are binding, we again have to study two cases. In the first case, the constraint $W_A \geq 0$ is binding and $W_A = 0$. In the second case, the constraint $A - W_A \geq 0$ is binding and $W_A = A$. Hence, the optimal policies for the two cases satisfy

$$w^* = (0, W_L^*)^\top, \text{ or } w^* = (A, W_L^*)^\top.$$

11
Such an optimization problem can be handled analogously to the one in the previous section. In particular, we can use a notation based on projections and write the optimal surplus $S^*(T)$ as

$$S^*(T) = r_s^T s + \gamma r_\gamma,$$

(3.19)

where

$$r_\gamma = \mathbb{P}_{\varphi_L}(1),$$

(3.20)

and

$$r_s^T = \begin{cases} 
(\mathbb{P}_{\varphi_L}^\perp (r_A), -\mathbb{P}_{\varphi_L}^\perp (r_L)) & \text{if } W_A = 0 \\
(\mathbb{P}_{\varphi_L}^\perp (r_A + \varphi_A), -\mathbb{P}_{\varphi_L}^\perp (r_L)) & \text{if } W_A = A.
\end{cases}$$

(3.21)

Since $r_\gamma$ and the components of $r_s$ are projections on orthogonal subspaces of $\mathcal{L}_2$, the surplus $S^*(T)$ is again decomposed as an orthogonal sum in $\mathcal{L}_2$. By contrast with the previous section, projections are now on subspaces related to the excess liability return rather than the excess asset return.

**Corollary 3.** When the short-selling constraints on assets are binding, the optimal surplus $S^*(T)$ allows an orthogonal decomposition into two $\mathcal{L}_2$-orthogonal returns $r_\gamma$ and $r_s$. These returns are projections on the excess liability return and its orthogonal complement.

The pay-off difference

$$r_s^T s = A\mathbb{P}_{\varphi_A}^\perp (r_A) - L\mathbb{P}_{\varphi_A}^\perp (r_L)$$

(3.22)

is again a single-period global MSM final surplus, which is obtained as the difference between a single-period assets-only and a single-period liabilities-only MSM pay-off. The pay-off $\mathbb{P}_{\varphi_L}(1)$ is now the liabilities excess return which is closest to the fictive risk-free pay-off $1$ in the $\mathcal{L}_2$-norm.

12
3.1.3 No Binding Constraints

We now consider the situation in which no constraint is binding. The returns $r_s$ and $r_\gamma$ in (3.10) and (3.11) for the single-period problem can be rewritten explicitly as:

$$
\begin{align*}
\mathbf{r}_s &= \left( r_A - \frac{(\langle r_A, \varphi_A \rangle - \langle r_A, \varphi_A \rangle \langle \varphi_A, \varphi_L \rangle \varphi_A - \langle r_A, \varphi_A \rangle \langle \varphi_A, \varphi_L \rangle}{(\varphi_A, \varphi_A) - \langle \varphi_A, \varphi_A \rangle} \varphi_L - \frac{(\langle r_A, \varphi_A \rangle - \langle r_A, \varphi_A \rangle \langle \varphi_A, \varphi_L \rangle \varphi_A - \langle r_A, \varphi_A \rangle \langle \varphi_A, \varphi_L \rangle}{(\varphi_A, \varphi_A) - \langle \varphi_A, \varphi_A \rangle} \right),
\end{align*}
$$

(3.23)

and

$$
\begin{align*}
\mathbf{r}_\gamma &= \left( \frac{\langle \varphi_L, 1 \rangle \langle \varphi_A, \varphi_L \rangle}{\langle \varphi_A, \varphi_A \rangle} - 2 \langle \varphi_A, 1 \rangle \langle \varphi_A, \varphi_L \rangle \varphi_A + \langle \varphi_A, 1 \rangle \langle \varphi_A, \varphi_L \rangle - \langle \varphi_L, 1 \rangle \langle \varphi_A, \varphi_L \rangle \varphi_L \left( \frac{(\langle \varphi_A, \varphi_L \rangle - \langle \varphi_A, \varphi_L \rangle \langle \varphi_A, \varphi_A \rangle}{(\varphi_A, \varphi_A) - \langle \varphi_A, \varphi_A \rangle} \varphi_L - \frac{(\langle \varphi_A, \varphi_L \rangle - \langle \varphi_A, \varphi_L \rangle \langle \varphi_A, \varphi_A \rangle \varphi_A - \langle \varphi_A, \varphi_L \rangle \langle \varphi_A, \varphi_L \rangle}{(\varphi_A, \varphi_A) - \langle \varphi_A, \varphi_A \rangle} \right) \right).
\end{align*}
$$

(3.24)

Apparently, (3.23) and (3.24) are more complex expressions than those obtained when some constraint is binding or when liabilities are treated exogenously as in Leippold et al. (2004). In their model, the orthogonal returns $r_\gamma$ and $r_s$ are simply given by:

$$
\begin{align*}
\mathbf{r}_s &= \left( r_A - \frac{(\langle r_A, \varphi_A \rangle - \langle r_A, \varphi_A \rangle \langle \varphi_A, \varphi_A \rangle)}{\langle \varphi_A, \varphi_A \rangle - \langle \varphi_A, \varphi_A \rangle} \varphi_A, \mathbf{r}_s^T = \left( r_A - \frac{\langle r_A, \varphi_A \rangle}{\langle \varphi_A, \varphi_A \rangle} \varphi_A, - \left( r_L - \frac{\langle r_L, \varphi_A \rangle}{\langle \varphi_A, \varphi_A \rangle} \varphi_A \right) \right).
\end{align*}
$$

(3.25)

Therefore, it is not obvious how the geometric structure, which makes the problem of optimizing the surplus with exogenous liability tractable, will translate also into the endogenous liability problem. Moreover, in a true multi-period setting and when several assets and liabilities are present, the expressions in (3.23) and (3.24) will become even more complicated and difficult to interpret and to handle. In the case of orthogonal assets and liabilities excess returns, i.e., $\langle \varphi_A, \varphi_L \rangle = 0$, equations (3.23) and (3.24) imply that $r_s$ and $r_\gamma$ are again projections on a space of excess returns and its orthogonal complement. However, in the non-orthogonal case the complexity of the arising expressions makes the structure of the optimal surplus less transparent. Therefore, to get more insight into the financial structure of the optimal surplus, we propose a notation based on exterior algebra which renders the relevant expressions manageable also in a multi-period context and in the presence of several assets and liabilities.

Let $\Lambda^n(\mathcal{L}_2)$ be the $n$-fold antisymmetric tensor product space of $\mathcal{L}_2$ and set $\Lambda^0(\mathcal{L}_2) = \mathbb{R}$. Taking the infinite direct sum over $n$ defines the infinite dimensional Grassmann algebra, if
the appropriate topology is added. For details on the general construction, see Berezin (1966). In the sequel, it will be sufficient to work with finite dimensional subsets of the Grassmann algebra, and specifically with the 2-fold antisymmetric tensor product space $\Lambda^2(L_2)$ of $L_2$. However, the methodology is in principle applicable also to an infinite dimensional setting, i.e., a setting with an infinite number of assets or liabilities.

Let us denote by ‘∧’ the wedge product and the inner product on $\Lambda^2(L_2)$ by $\langle \cdot, \cdot \rangle_2$. This product is defined by

$$\langle w \wedge x, y \wedge z \rangle_2 = \begin{vmatrix} \langle w, y \rangle & \langle w, z \rangle \\ \langle x, y \rangle & \langle x, z \rangle \end{vmatrix}; \quad w, x, y, z \in L_2,$$

where $| \cdot |$ is the determinant of a $2 \times 2$ matrix. In particular, the scalar product in $L_2$ is an inner product of elements in $\Lambda^1(L_2) = L_2$. This feature allows us to interpret the AL solutions when some constraint is binding as a geometric special case of the solutions when no constraint is binding. To write compactly the returns (3.23) and (3.24), we define the following operator:

$$[x, y] = \frac{\langle x \wedge y, \varphi_A \wedge \varphi_L \rangle_2}{\langle \varphi_A \wedge \varphi_L, \varphi_A \wedge \varphi_L \rangle_2}; \quad x, y, \varphi_A, \varphi_L \in L_2.$$

Using this notation, the returns (3.23) and (3.24) are given by the expressions:

$$r_s = \left( r_A - (r_A, \varphi_L) \varphi_A - [r_A, \varphi_A] \varphi_L \\ - (r_L - ([r_L, \varphi_L] \varphi_A - [r_L, \varphi_A] \varphi_L)) \right), \quad r_\gamma = [1, \varphi_L] \varphi_A - [1, \varphi_A] \varphi_L . \quad (3.27)$$

We next show that also in the case of no binding constraints the components of $r_s$ and the pay-off $r_\gamma$ are orthogonal elements of $L_2$. Similarly to the previous sections, we introduce some appropriate projection operators that highlight this point. To this end, we define for any $x = (x_1, x_2)^T \in L_2^2$ the operator $P_{[x]}$ by:

$$P_{[x]}(y) = \begin{pmatrix} [y, x_2]^T \\ [x_1, y] \end{pmatrix} x . \quad (3.28)$$
With this last definition we obtain for any $y \in \mathcal{L}_2$:

$$
\mathbb{P}_{[\phi]}(\mathbb{P}_{[\phi]}(y)) = [[y, \varphi_L]\varphi_A - [y, \varphi_A]\varphi_L, \varphi_L] \varphi_A - [[y, \varphi_L]\varphi_A - [y, \varphi_A]\varphi_L, \varphi_A]\varphi_L.
$$

Using the bilinearity of the operator $[x,y]$ and since $[\varphi_A, \varphi_A] = [\varphi_L, \varphi_L] = 0$, $[\varphi_A, \varphi_L] = -[\varphi_L, \varphi_A] = 1$, we get

$$
\mathbb{P}_{[\phi]}(\mathbb{P}_{[\phi]}(y)) = \mathbb{P}_{[\phi]}(y).
$$

Similar calculations show that $\mathbb{P}_{[\phi]}$ is a self-adjoint operator in $\mathcal{L}_2$. Therefore, $\mathbb{P}_{[\phi]}(\cdot)$ is an orthogonal projection on a finite dimensional subspace $M$ of $\mathcal{L}_2$. In our case, $M$ is the linear space generated by $\varphi_A, \varphi_L$. Based on these last results, the orthogonality of the returns $r_\gamma$ and $r_s^\top$ in (3.27) becomes apparent. In fact, from (3.27) we have:

$$
\begin{align*}
  r_s &= \begin{pmatrix}
    \mathbb{P}_{[\phi]}(r_A) \\
    -\mathbb{P}_{[\phi]}(r_L)
  \end{pmatrix}, \quad r_\gamma = \mathbb{P}_{[\phi]}(\mathbb{I}).
\end{align*}
$$

We summarize the relevant findings in the next corollary.

**Corollary 4.** When no short-selling constraint is binding, the optimal surplus $S^*(T)$ allows an orthogonal decomposition into two $\mathcal{L}_2$-orthogonal returns $r_\gamma$ and $r_s^\top$. These returns are projections on the space generated by the excess assets and liabilities returns and its orthogonal complement.

The pay-off difference

$$
  r_s^\top s = A\mathbb{P}_{[\phi]}(r_A) - L\mathbb{P}_{[\phi]}(r_L)
$$

is also in this general setting a single-period global MSM final surplus, which is obtained as the difference between a single-period assets-only and a single-period liabilities-only MSM pay-off. Similarly, the pay-off $\mathbb{P}_{[\phi]}(\mathbb{I})$ is now the asset and liability excess return which is closest to the fictive risk-free pay-off $\mathbb{I}$ in the $\mathcal{L}_2$-norm. The exact contribution of the excess returns $\varphi_A, \varphi_L$ and the benchmark returns $r_A, r_L$, to $r_s$ and $r_\gamma$ is explicitly given by the coefficients $[r_A, \varphi_A]$, $[r_L, \varphi_A]$, $[r_A, \varphi_L]$, $[r_L, \varphi_L]$, $[\mathbb{I}, \varphi_A]$, and $[\mathbb{I}, \varphi_L]$. 

15
The orthogonal decomposition results implied by (3.27) and for the shortselling constraints (3.14), (3.15) and (3.20), (3.21), respectively, can be used to rewrite the value function as

\[
J(s) = E \left[ \gamma S^*(T) - \frac{1}{2} S^*(T)^2 \right] \\
= E \left[ \gamma \left( r_s^T s + \gamma r_\gamma \right) - \frac{1}{2} \left( r_s^T s + \gamma r_\gamma \right)^2 \right] \\
= \gamma^2 E \left[ r_\gamma - \frac{1}{2} r_\gamma^2 \right] + \gamma E \left[ (1 - r_\gamma) r_s^T \right] s - \frac{1}{2} s^T E \left[ r_s r_s^T \right] s.
\]

Further, using the properties

\[
\langle r_s, r_\gamma \rangle = 0, \quad \langle r_\gamma, r_\gamma \rangle = \langle r_\gamma, 1 \rangle,
\]

the value function simplifies to:

\[
J(s) = \frac{\gamma^2}{2} E \left[ r_\gamma \right] + \gamma E \left[ r_s^T \right] s - \frac{1}{2} s^T E \left[ r_s r_s^T \right] s. \quad (3.32)
\]

### 3.2 Multiperiod Optimization

To clarify the differences between single-period and multiperiod optimizations, we consider first a two-period problem. For brevity, we drop the time indices \( T - 2 \). Moreover, since under binding constraints we can treat the optimization problem as one with exogenous assets or liabilities, we focus only on the situation in which no short-selling constraint is binding. Given the state dynamics (2.2) and using (3.32), the value function at time \( T - 2 \) is

\[
J(s) = \max_{w \in A(s)} E \left[ J(s(T - 1)) \right] \\
= \max_{w \in A(s)} E \left[ \gamma r_s(T - 1)^T s(T - 1) - \frac{1}{2} s(T - 1)^T (r_s(T - 1) r_s(T - 1)^T) s(T - 1) + \frac{\gamma^2}{2} r_\gamma \right] \\
= \max_{w \in A(s)} \left[ \left( \gamma d^T - s^T R d d^T \right) \Phi w - \frac{1}{2} w^T \Phi d d^T \Phi w \right] + C.
\]
where $C$ collects all terms that do not depend on $w$ and $\tilde{R}, \tilde{\Phi}$ are defined by $\tilde{R} = RR_s(T - 1)$, $\tilde{\Phi} = \Phi R_s(T - 1)$ with:

$$R_s(T - 1) = \text{Diag} \left( \begin{array}{c} P_{[\varphi(T-1)]^\perp} (r_A(T - 1)) \\ P_{[\varphi(T-1)]^\perp} (r_L(T - 1)) \end{array} \right).$$

(3.33)

Therefore, the functional form of the value function at time $T - 2$ is the same as the one at time $T - 1$ and the optimal policy at time $T - 2$ is given by

$$w^* = \left[ \mathbb{E} \left( \tilde{\Phi} dd^\top \tilde{\Phi} \right) \right]^{-1} \left[ \gamma \mathbb{E} (\tilde{\Phi} d) - \mathbb{E} (\tilde{\Phi} dd^\top \tilde{R})s \right].$$

(3.34)

Then, the optimal surplus becomes

$$S^*(T) = r_s(T - 1)^\top s(T - 1) + \gamma r_\gamma(T - 1)$$
$$= d^\top R_s(T - 1)(Rs + \Phi w^*) + \gamma r_\gamma(T - 1)$$
$$= r_s(T - 2)^\top s + \gamma (r_\gamma(T - 2) + r_\gamma(T - 1)),$$

(3.35)

where

$$r_s^\top(T - 2) = d^\top \left( \tilde{R} - d^\top \Phi \left[ \mathbb{E} \left( \tilde{\Phi} dd^\top \tilde{\Phi} \right) \right]^{-1} \mathbb{E} \left( \tilde{\Phi} dd^\top \tilde{R} \right) \right),$$

(3.36)

$$r_\gamma(T - 2) = d^\top \Phi \left[ \mathbb{E} \left( \tilde{\Phi} dd^\top \tilde{\Phi} \right) \right]^{-1} \mathbb{E} \Phi d.$$  

(3.37)

In particular, defining $\tilde{\varphi} = (\tilde{\varphi}_A, \tilde{\varphi}_L)^\top$ and $\tilde{r} = (\tilde{r}_A, \tilde{r}_L)^\top$ for the vectors of the diagonal elements of $\tilde{\Phi}$, it follows:

$$r_s^\top(T - 2) = \left( \tilde{r}_A - [\tilde{r}_A, \tilde{\varphi}_L]\tilde{\varphi}_A + [\tilde{r}_A, \tilde{\varphi}_A]\tilde{\varphi}_L, (\tilde{r}_L - [\tilde{r}_L, \tilde{\varphi}_A]\tilde{\varphi}_A + [\tilde{r}_L, \tilde{\varphi}_A]\tilde{\varphi}_L) \right)$$

$$= \left( P_{[\varphi]^\perp} (\tilde{r}_A), -P_{[\varphi]^\perp} (\tilde{r}_L) \right),$$

(3.38)

$$r_\gamma(T - 2) = [1, \tilde{\varphi}_L]\tilde{\varphi}_A - [1, \tilde{\varphi}_A]\tilde{\varphi}_L = P_{[\varphi]} (1).$$

(3.39)

Hence, $r_s(T - 2)$ and $r_\gamma(T - 2)$ are again projections on orthogonal subspaces of $\mathcal{L}_2$. Moreover, since by construction $\varphi_A(T - 1)$ and $\varphi_L(T - 1)$ are orthogonal to $\tilde{\varphi}_A, \tilde{\varphi}_L$ and $\tilde{r}_A, \tilde{r}_L$, the single-
period excess return \( r_\gamma(T - 1) \) is also orthogonal to \( r_s(T - 2) \) and \( r_\gamma(T - 2) \). As a consequence, we can write the two-period optimal surplus \( S^*(T) \) in (3.35) as a linear combination of the orthogonal pay-offs \( r_s(T - 2)^\top s, r_\gamma(T - 2), \) and \( r_\gamma(T - 1) \).

**Corollary 5.** The two-period optimal surplus \( S^*(T) \) allows an orthogonal decomposition into three \( L_2 \)-orthogonal pay-offs \( r_s(T - 2) \), \( r_\gamma(T - 2) \) and \( r_\gamma(T - 1) \).

We can now proceed to derive the optimal surplus for an arbitrary time \( T - t \). It follows that the value function at time \( T - t \) is of the same functional form as in the two-period model, with return matrices \( \tilde{R}(T - t) \) and \( \tilde{\Phi}(T - t) \) given by:

\[
\tilde{R}(T - t) = R(T - t)R_s(T - t + 1) \quad , \quad \tilde{\Phi}(T - t) = \Phi(T - t)R_s(T - t + 1) ,
\]

where

\[
R_s(T - t + 1) = \text{Diag} \left[ \begin{pmatrix} P[\tilde{\varphi}(T - t + 1)] (\tilde{r}_A(T - t + 1)) \\ P[\tilde{\varphi}(T - t + 1)] (\tilde{r}_L(T - t + 1)) \end{pmatrix} \right]. \tag{3.40}
\]

The resulting orthogonal final surplus components are implied by the pay-offs

\[
\begin{align*}
\mathbf{r}_s(T - t)^\top &= \begin{pmatrix} P[\tilde{\varphi}(T - t)] (\tilde{r}_A(T - t)) \\ -P[\tilde{\varphi}(T - t)] (\tilde{r}_L(T - t)) \end{pmatrix} \tag{3.41}
\end{align*}
\]
\[
\begin{align*}
r_\gamma(T - t + i) &= P[\tilde{\varphi}(T - t + i)] (L), \quad i = 0, ..., T - t - 1 \tag{3.42}
\end{align*}
\]

and we obtain the optimal surplus \( S^*(T) \) as

\[
S^*(T) = \mathbf{s}(T - t)^\top \mathbf{r}_s(T - t) + \gamma \sum_{i=1}^{t} r_\gamma(T - i). \tag{3.43}
\]

Therefore, the final surplus \( S^*(T) \) is decomposed into an orthogonal sum of a \((T - t)\)-period MSM AL surplus and a set of \( T - t \) “local” AL excess returns. For states where short-selling constraints are binding, the relevant projections map returns on some lower dimensional spaces
than in the unconstrained case. If for example the constraint $W_L = 0$ binds at time $T - t$, the relevant return components degenerate to

\[
\begin{align*}
    \mathbf{r}^T_s(T - t) &= \langle P_{\varphi_1(T-t)} \rho_A(T - t), - P_{\varphi_1(T-t)} \rho_L(T - t) \rangle, \\
    r_\gamma(T - t) &= P_{\varphi_1(T-t)}(\mathbf{1}), \\
\end{align*}
\]  

(3.43) (3.44)

where $\varphi_1$ is the first component of $\varphi$. Finally, when short-selling constraints are simultaneously binding on assets and liabilities, no optimization has to be performed and the resulting orthogonal projections collapse to the identity operator. Hence, we have proven Theorem 1.

3.3 Multi-Asset and Multi-Liabilities Optimization

We conclude Section 3 with a remark on the multidimensional extension of our geometrical model, where we have $n$ assets $A_1, \cdots, A_n$ and $m$ liabilities $L_1, \cdots, L_m$. For each asset and liability class, we define the respective gross benchmark-returns, $r_A(t) \in \mathbb{R}^{1 \times n}$ and $r_L(t) \in \mathbb{R}^{1 \times m}$, together with $n$ excess returns on assets, $\rho^A(t) \in \mathbb{R}^{1 \times n}$, and $m$ excess returns on liabilities, $\rho^L(t) \in \mathbb{R}^{1 \times m}$. To adopt our geometrical AL framework for the multidimensional setting, we have to consider the inner product $\langle \cdot, \cdot \rangle_{n+m}$ on the space $\Lambda^{n+m}(L_2)$. Then, the definition (3.26) will be based on $(n + m)$-forms. So, for instance, the normalizing two-form in definition (3.26) will become

\[
\langle \varphi_{A_1} \wedge \cdots \wedge \varphi_{A_n} \wedge \varphi_{L_1} \wedge \cdots \wedge \varphi_{L_m}, \varphi_{A_1} \wedge \cdots \wedge \varphi_{A_n} \wedge \varphi_{L_1} \wedge \cdots \wedge \varphi_{L_m} \rangle_{n+m} = \\
\begin{bmatrix}
\langle \rho_{A_1}, \rho_{A_1} \rangle & \cdots & \langle \rho_{A_1}, \rho_{A_n} \rangle \\
\vdots & \ddots & \vdots \\
\langle \rho_{A_n}, \rho_{A_1} \rangle & \cdots & \langle \rho_{A_n}, \rho_{A_n} \rangle \\
\langle \rho_{L_1}, \rho_{A_1} \rangle & \cdots & \langle \rho_{L_1}, \rho_{A_n} \rangle \\
\vdots & \ddots & \vdots \\
\langle \rho_{L_m}, \rho_{A_1} \rangle & \cdots & \langle \rho_{L_m}, \rho_{A_n} \rangle \\
\end{bmatrix}_{(n \times n)} \\
\begin{bmatrix}
\langle \rho_{A_1}, \rho_{L_1} \rangle & \cdots & \langle \rho_{A_1}, \rho_{L_m} \rangle \\
\vdots & \ddots & \vdots \\
\langle \rho_{A_n}, \rho_{L_1} \rangle & \cdots & \langle \rho_{A_n}, \rho_{L_m} \rangle \\
\langle \rho_{L_1}, \rho_{L_1} \rangle & \cdots & \langle \rho_{L_1}, \rho_{L_m} \rangle \\
\vdots & \ddots & \vdots \\
\langle \rho_{L_m}, \rho_{L_1} \rangle & \cdots & \langle \rho_{L_m}, \rho_{L_m} \rangle \\
\end{bmatrix}_{(m \times m)}.
\]
When there are no binding constraints, we can still represent the returns (3.23) and (3.24) as in equation (3.27). In particular, we have

\[
\mathbf{r}_s = \begin{pmatrix}
    r_{A_1} - \sum_{i=1}^{n} \rho_{A_i} [\mathbf{r}_{A_1}, \mathbf{\rho}_L] + \sum_{j=1}^{m} \rho_{L_j} [\mathbf{r}_{A_1}, \mathbf{\rho}_A] \\
    \vdots \\
    r_{A_n} - \sum_{i=1}^{n} \rho_{A_i} [\mathbf{r}_{A_n}, \mathbf{\rho}_L] + \sum_{j=1}^{m} \rho_{L_j} [\mathbf{r}_{A_n}, \mathbf{\rho}_A] \\
    -r_{L_1} + \sum_{i=1}^{n} \rho_{A_i} [\mathbf{r}_{L_1}, \mathbf{\rho}_L] - \sum_{j=1}^{m} \rho_{L_j} [\mathbf{r}_{L_1}, \mathbf{\rho}_A] \\
    \vdots \\
    -r_{L_m} + \sum_{i=1}^{n} \rho_{A_i} [\mathbf{r}_{L_m}, \mathbf{\rho}_L] - \sum_{j=1}^{m} \rho_{L_j} [\mathbf{r}_{L_m}, \mathbf{\rho}_A]
\end{pmatrix}
\]  

(3.45)

and

\[
\mathbf{r}_\gamma = \sum_{i=1}^{n} \varphi_{A_i} [\mathbf{1}^{(i)}, \varphi_L] - \sum_{j=1}^{m} \varphi_{L_j} [\mathbf{1}^{(j)}, \varphi_A],
\]

(3.46)

where, e.g., \([\mathbf{r}_{A_1}, \mathbf{\rho}_L]\) is defined as

\[
[\mathbf{r}_{A_1}, \mathbf{\rho}_L] = \frac{\langle \varphi_{A_1} \land \cdots \land \varphi_{A_{i-1}} \land r_{A_1} \land \varphi_{A_i+1} \land \cdots \land \varphi_{A_n} \land \varphi_{L_1} \land \cdots \land \varphi_{L_m}, \varphi_{A_1} \land \cdots \land \varphi_{A_n} \land \varphi_{L_1} \land \cdots \land \varphi_{L_m} \rangle_{n+m}}{\langle \varphi_{A_1} \land \cdots \land \varphi_{A_n} \land \varphi_{L_1} \land \cdots \land \varphi_{L_m}, \varphi_{A_1} \land \cdots \land \varphi_{A_n} \land \varphi_{L_1} \land \cdots \land \varphi_{L_m} \rangle_{n+m}}.
\]

Accordingly, we have

\[
[\mathbf{r}_{L_1}, \mathbf{\rho}_A] = \frac{\langle \varphi_{A_1} \land \cdots \land \varphi_{L_m}, \varphi_{A_1} \land \cdots \land \varphi_{A_n} \land \varphi_{L_1} \land \cdots \land \varphi_{L_{i-1}} \land r_{L_1} \land \varphi_{L_{i+1}} \land \cdots \land \varphi_{L_m}, \varphi_{A_1} \land \cdots \land \varphi_{A_n} \land \varphi_{L_1} \land \cdots \land \varphi_{L_m} \rangle_{n+m}}{\langle \varphi_{A_1} \land \cdots \land \varphi_{A_n} \land \varphi_{L_1} \land \cdots \land \varphi_{L_m}, \varphi_{A_1} \land \cdots \land \varphi_{A_n} \land \varphi_{L_1} \land \cdots \land \varphi_{L_m} \rangle_{n+m}}.
\]

In exactly the same way, we can define the terms \([\mathbf{1}^{(i)}, \varphi_L]\) and \([\mathbf{1}^{(j)}, \varphi_A]\). Instead of replacing the excess returns with the corresponding benchmark returns in the \((n + m)\)-forms, we replace them with the fictive risk-free payoffs \(\mathbf{1}\). We note that for both the non-binding case as well as for the binding cases, the geometrical structure presented in the previous sections will be preserved when generalized to a multi-assets and multi-liabilities. Hence, the advantages of using the compact notation of exterior algebra, as in (3.45) and (3.46), becomes even more apparent in such a setup.
4 Application: Exogenous vs. Endogenous Liabilities

The orthogonal decomposition in Theorem 1 can be used to analyze several issues related to AL portfolio optimization such as, e.g., the determination of the optimal funding ratio or the analysis of the impact of the rebalancing frequency. This section discusses some of these topics with a particular emphasis on the distinction between AL optimizations under exogenous or endogenous liabilities.

4.1 Geometric structure

In both the endogenous and exogenous liability cases, the final surplus can be decomposed into a sum of two return components. These components contain both projection operators \( r_s \) and \( r_\gamma \), respectively. The projection property describes the formal similarities of the two models. But significant differences hold between the geometrical structure of these operators in the endogenous and exogenous case. One issue is the relation between MVF’s under exogenous and endogenous liabilities. Based on the orthogonal decomposition in Theorem 1 we can focus without loss of generality on the single-period case. Any MVF can be represented in the \((\text{VAR}(S), \mathbb{E}(S))\)-space by a curvature parameter \( a \), a vertical shift parameter \( b \) and a horizontal shift parameter \( c \).

**Corollary 6.** The single-period MVF is a quadratic form

\[
\text{VAR}(S^*(T)) = a\mathbb{E}(S^*(T))^2 + 2b\mathbb{E}(S^*(T)) + c,
\]

\[\text{To clarify our point, we denote the linear span of the four vectors } r_A, \varphi_A, r_L, \varphi_L \text{ by } W. \text{ From the results derived in Section 3 it follows that the returns } r_s \text{ and } r_\gamma \text{ are linear combinations or projections from } W \text{ into the reals. The difference is given by the Fourier coefficients of the linear combinations of the AL returns. In the exogenous case, the Fourier coefficients are given by the inner product on } \Lambda^1(W) \cong W. \text{ In the endogenous case, by the inner product on } \Lambda^2(W).\]

\[\text{In particular, in the multiperiod setting the moments of the sum } \sum_{i=1}^t r_\gamma(T - i), \text{ instead of those of } r_\gamma(T - 1) \text{ (see the corollary below), will affect the quadratic form describing the MVF’s. Similarly, the relevant MSM returns } r_s \text{’s and their moments will also be different.} \]
where under endogenous liabilities the coefficients \( \{a, b, c\} \) are

\[
a = \frac{1}{\langle P_{\varphi}(l), l \rangle} - 1,
b = -\frac{\langle P_{\varphi}^\perp (r) s, l \rangle}{\langle P_{\varphi}(l), l \rangle},
c = \langle P_{\varphi}^\perp (r) s, P_{\varphi}^\perp (r) s \rangle + \frac{\langle P_{\varphi}^\perp (r) s, l \rangle^2}{\langle P_{\varphi}(l), l \rangle}.
\]

Under exogenous liabilities the coefficients \( \{\tilde{a}, \tilde{b}, \tilde{c}\} \) of the quadratic form are

\[
\tilde{a} = \frac{1}{\langle P_{\varphi}(l), l \rangle} - 1,
\tilde{b} = -\frac{\langle P_{\varphi}^\perp (r) s, l \rangle}{\langle P_{\varphi}(l), l \rangle},
\tilde{c} = \langle P_{\varphi}^\perp (r) s, P_{\varphi}^\perp (r) s \rangle + \frac{\langle P_{\varphi}^\perp (r) s, l \rangle^2}{\langle P_{\varphi}(l), l \rangle}.
\]

**Proof.** The first and second moment of the optimal surplus are given by

\[
E[S^*(T)] = E[r_s^T s] + \gamma E[r_\gamma],
E[(S^*(T))^2] = E\left[ (r_s^T s + \gamma r_\gamma)^2 \right].
\]

Solving for \( \gamma \) and plugging in the arising expressions into the definition of the variance, we obtain the coefficients \( \{a, b, c\} \) using (3.31). The coefficients \( \{\tilde{a}, \tilde{b}, \tilde{c}\} \) are obtained by exploiting the projection properties \( \langle P_{\varphi}(l), P_{\varphi}(l) \rangle = \langle P_{\varphi}(l), l \rangle \) and \( \langle P_{\varphi}(l), P_{\varphi}^\perp (r) s \rangle = 0 \).

The corollary states that in an AL optimization under exogenous liabilities the implied MVF is affected by liabilities in only two ways. First, through a vertical shift caused by the parameter \( \tilde{c} \) and, second, by a sidewise shift caused by the parameter \( \tilde{b} \). Therefore, the introduction of liabilities induces a pure translation of the MVF in the mean-variance space, caused by a modified global MSM surplus, relative to a pure assets optimization. Under endogenous liabilities, the curvature parameter \( a \) depends on the structure of the liabilities returns. Hence, in this case the AL optimization also produces a change in the curvature of
the relevant MVF. As expected, the curvature under endogenous liabilities is lower. This is proved in the next proposition.

**Proposition 7.** The curvature of the MVF under endogenous liabilities is always less or equal to the curvature of the MVF under exogenous liabilities. This leads to a better risk-return tradeoff in the \((VAR(S), E(S))\)-space.

**Proof.** We have to show that

\[
\frac{1}{\langle P\varphi_A(\mathbb{I}), \mathbb{I} \rangle} \geq \frac{1}{\langle P\varphi_L(\mathbb{I}), \mathbb{I} \rangle},
\]

which, after some algebraic manipulation, is equivalent to

\[
\frac{\langle \varphi_A, \varphi_A \rangle}{\langle \mathbb{I}, \varphi_A \rangle^2} \geq \frac{\langle \varphi_A, \varphi_A \rangle \langle \varphi_L, \varphi_L \rangle - \langle \varphi_A, \varphi_L \rangle^2}{\langle \mathbb{I}, \varphi_A \rangle^2 \langle \varphi_L, \varphi_L \rangle - 2\langle \mathbb{I}, \varphi_A \rangle \langle \mathbb{I}, \varphi_L \rangle \langle \varphi_A, \varphi_L \rangle + \langle \varphi_A, \varphi_A \rangle \langle \mathbb{I}, \varphi_L \rangle^2}. \tag{4.47}
\]

But, this last inequality is equivalent to

\[
(\langle \varphi_A, \varphi_L \rangle \langle \mathbb{I}, \varphi_A \rangle - \langle \mathbb{I}, \varphi_L \rangle \langle \varphi_A, \varphi_A \rangle)^2 \geq 0, \tag{4.48}
\]

which completes the proof. \(\square\)

The change in the MVF due to the presence of endogenous liabilities is not arbitrary. In fact, there is another interesting geometrical property, which relates the endogenous AL problem with the exogenous one, namely that the MVFs under exogenous and endogenous liabilities have to share one tangential point.

**Proposition 8.** The MVF under endogenous and exogenous liability share a tangential point in \((E(S), VAR(S))\)-space.

**Proof.** The claim is proved, if the discriminant of the parabola vanishes, i.e.,

\[
(b - \tilde{b})^2 = (a - \tilde{a})(c - \tilde{c}). \tag{4.49}
\]
This can be shown after some lengthy but straightforward calculations.  

From Proposition 7 and 8, we can expect that neglecting the potential endogeneity of liabilities may lead to substantial efficiency losses for the optimal portfolio mix.

4.2 Numerical Example

Next, we analyze in a numerical example the differences between the unconstrained MVF’s under exogenous and endogenous liabilities. For these two cases, the MVF can be calculated analytically. Since the MVF with exogenous liabilities can be interpreted as a constrained MVF with binding constraints on endogenous liabilities, we have two extreme situations: One situation in which the constraints never bind and one in which constraints always bind. Hence, studying these two cases provides insights about the maximal additional mean-variance utility that we can obtain from an AL optimization. For our numerical exercise, we assume $A(0) = 1.05$ and $L(0) = 1$. The vector of the mean returns and the vector of volatilities are set equal to

$$(\mu(r_A), \mu(\varphi_A), \mu(r_L), \mu(\varphi_L)) = (1.04, 0.20, 1.06, 0.12),$$

$$(\sigma(r_A), \sigma(\varphi_A), \sigma(r_L), \sigma(\varphi_L)) = (0.085, 0.272, 0.083, 0.195).$$

For the correlation matrix we assume:

\[
\begin{array}{cccc}
  & r_A & \varphi_A & r_L & \varphi_L \\
 r_A & 1 & 0.178 & 0.445 & -0.010 \\
 \varphi_A & 1 & 0.446 & -0.157 & \\
r_L & 1 & 0.270 & \\
\varphi_L & 1 & \\
\end{array}
\]

These calculations can be obtained from the authors.
The upper panel of Figure 1 plots single-period MVFs for a one-year investment horizon. As expected, the MVF under endogenous liabilities (the solid curve) is an envelope of the MVF under exogenous liabilities (the dotted curve). Under the given model parameters, the tangential point of the two curves lies on the inefficient part of the endogenous liabilities MVF. This needs not be the case for other model inputs.

The lower panel of Figure 1 plots multiperiod MVFs. To this end, we assume iid returns, which allows us to express the multiperiod surplus components in Theorem 1 in closed-form. We assume an investment horizon of one year. Dotted curves represent the MVFs under exogenous liabilities when (from right to left) the rebalancing frequency is 1, 2, 6, 12 (monthly), and 360 (daily). Solid curves represent MVFs under endogenous liabilities. We
<table>
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<tr>
<th>Freq</th>
<th>Surplus Volatility 10%</th>
<th>Surplus Volatility 14%</th>
<th>Surplus Volatility 22%</th>
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<td>Ex</td>
<td>En</td>
<td>Diff</td>
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<td>0.97</td>
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<td>39.18%</td>
</tr>
<tr>
<td>1d</td>
<td>0.82</td>
<td>1.07</td>
<td>30.48%</td>
</tr>
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</table>

Table 1: We report the Sharpe ratios for exogenous (Ex) and endogenous (En) surplus strategies at given levels of surplus volatility and at different rebalancing frequencies (Freq). For the calculation of the Sharpe ratio, we fix the riskless interest rate at 4%. We also report the percentage difference (Diff) in the Sharpe ratios between the strategy with endogenous liability and endogenous liability.

Observe that, when the rebalancing frequency increases, the MVFs are shifted to the right. This rightward shift results from intertemporal diversification opportunities. The benefit from intertemporal diversification is most pronounced at low frequencies. Switching from a static to a semi-annual optimization shifts the frontiers substantially into a more favorable risk-return tradeoff. However, a switch from a monthly to a daily rebalancing produces only moderate changes in the MVF. Furthermore, Figure [x] shows that for any given surplus variance the benefits from intertemporal diversification are more pronounced under endogenous liabilities.

In Table 1, we report the Sharpe ratios for different surplus maximization strategies and rebalancing frequencies. To get an idea about the quantitative impact of the different strategies in our numerical study, we fix the level of the surplus volatility and calculate the Sharpe ratios for the case of endogenous and exogenous liability and their relative differences. For example, given a surplus volatility of 22%, we can increase the Sharpe ratio by more than 42%, if we treat liabilities as endogenous in a static optimization problem. If we increase the rebalancing frequency to daily, the advantage of treating liabilities as endogenous is still above a 26% increase in the Sharpe ratio. If we decrease the surplus volatility, the relative advantage increases to 87.50% for the static problem and to 30.48% for a daily rebalancing strategy. Since the rebalancing frequency is typically low in AL management and surplus volatility should also be kept low, we conclude that treating liabilities as endogenous is even more important.
5 Conclusions

We propose a geometric approach for the joint multiperiod mean-variance optimization of assets and liabilities. We show that the optimal surpluses and minimum variance frontiers can be decomposed into an orthogonal set of basis returns. This decomposition holds both under binding and non-binding short selling constraints and for multiple assets and liabilities. We obtain the orthogonal basis returns by applying projection operators, which can be represented by means of inner products on exterior algebras. This analysis highlights the geometric difference to the asset-only and exogenous liability case. If short-selling constraints are binding, the optimization problem reduces to an optimization problem in which, depending on the constraint, some assets or liabilities are binding. In this case, the effect of a binding constraint on the optimal surplus is reflected by a simpler structure of the projection operators arising in the orthogonal surplus decomposition (as in Leippold et al. (2004)). Finally, we highlight some properties of the minimum variance frontiers under endogenous liabilities and we quantify in a numerical example the impact of the rebalancing frequency on the prevailing dynamic risk and return tradeoff. Although we concentrate only on analytical expressions, our approach may serve as a basis for model extensions including more complex constraints. To this end, our approach can provide valuable guidance in efficiently formulating numerical computations. However, we leave these issues open for future research.
References


