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Abstract

This paper performs specification analysis on the term structure of variance swap rates on the S&P 500 index and studies the optimal investment decision on the variance swaps and the stock index. The analysis identifies two stochastic variance risk factors that govern the short and long end of the variance swap term structure variation, respectively. Furthermore, the highly negative estimate for the market price of variance risk makes it optimal for an investor to take short positions in a short-term variance swap contract, long positions in a long-term variance swap contract, and short positions in the stock index.

JEL Classification: G12, G13, C52.

Keywords: Variance swap; term structure; variance risk dynamics; variance risk premium; asset allocation.
I. Introduction

The financial market is becoming increasingly aware of the fact that return variance on stock indexes is stochastic (Engle (2004)) and the variance risk is heavily priced (?). Associated with this recognition is the development of a large number of variance-related derivative products. The most actively traded is the variance swap contract. The contract has zero value at inception. At maturity, the long side of the variance swap contract receives the difference between a standard measure of the realized variance and a fixed rate, called the *variance swap rate*, determined at the inception of the contract. Traditional derivative contracts such as calls, puts, and straddles also have variance risk exposure, but entering a variance swap contract represents the most direct way of achieving exposure to or hedging against variance risk.

Variance swap contracts on major equity indexes are actively traded over the counter. Variance swap rate quotes on such indexes are now readily available from several broker dealers. In this paper, we obtain more than a decade worth of variance swap rate quotes from a major investment bank on the S&P 500 index at five fixed time-to-maturities from two months to two years. With the data, we perform specification analysis on the variance risk dynamics and the term structure of variance swap rates. We propose a class of models on the variance risk dynamics and derive their pricing implications on the term structure of variance swap rates. We estimate the model specifications by exploiting the rich information embedded in both the time series and the term structure of the variance swap rate quotes. Based on the estimated variance risk dynamics, we study both theoretically and empirically how investors can use the term structure of variance swap contracts to span the variance risk and to revise their dynamic asset allocation decisions.

Our specification analysis and model estimation show that two stochastic variance risk factors are needed to explain the term structure variation of the variance swap rates, with one factor controlling the instantaneous variance rate variation and the other controlling the central tendency variation of the return variance. The instantaneous variance rate is much more transient than the central tendency factor under both the risk-neutral and the statistical measures. Thus, the two factors generate different loading patterns across the term structure of the variance swap rates. The instantaneous variance rate factor dominates the short-term variance swap rate dynamics whereas the central tendency factor dominates the long-term variance swap rate dynamics.

With the estimated variance risk dynamics, we study how the presence of variance swap contracts across several maturities alters an investor’s optimal asset allocation decision. We consider a dynamic asset allocation problem, where an investor equipped with a constant relative risk aversion utility function allocates her initial
wealth among the money market account, the S&P 500 stock index, and variance swap contracts at different maturities on the index to maximize her utility of terminal wealth. We derive the optimal allocation decisions in analytical forms for cases with and without the availability of the variance swap contracts in the investment.

In the absence of the variance swap contracts, the presence of stochastic variance and its correlation with the index return creates an intertemporal hedging demand for the stock index investment. However, with the variance swap contracts directly spanning the variance risk, there no longer exists an intertemporal hedging demand for the stock index investment. Under a two-factor variance risk dynamics, we can use two variance swap contracts at two distinct maturities to span the variance risk. The optimal investments in the stock index and the two variance swap contracts depend on the market prices of the three sources of risks: the return risk, the instantaneous variance rate risk, and the variance central tendency risk. The highly negative estimate for the market price of the instantaneous variance risk makes it optimal to take short positions in a short-term variance swap contract, long positions in a long-term variance swap contract, and short positions in the stock index. When we perform a historical analysis on different investment strategies, we find that incorporating variance swap contracts into the portfolio mix significantly increases the historical performance of the investment.

The remainder of the paper is organized as follows. The next section discusses the literature that forms the background of our study. Section III introduces a class of affine stochastic variance models, under which we value the term structure of variance swap rates. Section IV discusses the data, estimation strategy, and the estimation results on variance risk dynamics, variance risk premium, and their effects on the variance swap term structure. Section V derives the optimal asset allocation decisions with and without the variance swap contracts. Section VI calibrates the allocation decisions to the estimated risk dynamics and risk premiuma, and studies the historical performance of different investment strategies. Section VII concludes.

II. Background

Our study is related to several strands of literature. The first strand includes all traditional studies that estimate the variance dynamics joint with the return dynamics. See Engle (2004) for a review. More recently, a rapidly growing literature infers the variance dynamics based on realized variance estimators constructed from high-frequency returns. Important contributions include Andersen, Bollerslev, Diebold, and Ebens (2001), Andreou and Ghysels (2002), Andersen, Bollerslev, Diebold, and Labys (2003), Barndorff-Nielsen and Shep-
hard (2004a,b), A¨ıt-Sahalia, Mykland, and Zhang (2005), Andersen, Bollerslev, and Meddahi (2005), Oomen (2005), Zhang, Mykland, and A¨ıt-Sahalia (2005), Bandi and Russell (2006), Hansen and Lunde (2006), and A¨ıt-Sahalia and Mancini (2008). The realized variance estimators make return variance almost an observable quantity and hence sharpen the identification of variance dynamics. In this paper, we also use realized variance in combination with our variance swap quotes to enhance the identification of the variance risk dynamics and variance risk premium. Nevertheless, we do not construct the realized variance estimators using high-frequency returns, but using daily returns in accordance with the specification of a typical variance swap contract.

The second strand of literature combines information in time series returns and option prices to infer the variance dynamics and variance risk premium together with the return dynamics and return risk premium. Prominent examples include Bates (1996, 2000, 2003), Chernov and Ghysels (2000), Jackwerth (2000), Pan (2002), Jones (2003), Eraker (2004), A¨ıt-Sahalia and Kimmel (2007), Broadie, Chernov, and Johannes (2007), and Carr and Wu (2008a). The joint inference of return and variance dynamics dictates that misspecification on one leads to erroneous conclusions on the other. Bergomi (2004) highlights how traditional stochastic volatility and Lévy jump models impose structural constraints on the relations between the Black and Scholes (1973) implied volatility skew along the strike dimension, the spot-volatility correlation, and the term structure of the volatility of volatility. With variance swap rate quotes, we show that we can directly study and estimate the variance dynamics and variance risk premium without specifying the underlying return dynamics, and hence without interference from the potential misspecifications of the return dynamics.

Several studies also form option portfolios to separate the variance risk exposure from the return risk exposure. For example, Bakshi and Kapadia (2003a,b) consider the profit and loss arising from delta-hedging a long position in a call option. They use a portfolio of vanilla options to approximate the value of the 30-day variance swap rate on five stock indexes and 35 individual stocks. They compare the synthetic 30-day variance swap rates to the ex post realized variance to determine the variance risk premium. Bondarenko (2004) uses a similar procedure to synthesize variance swap rates on the S&P 500 index and link the variance risk premium to hedge fund behavior. Wu (2005) estimates the variance risk dynamics and variance risk premium by combining the information in realized variance estimators from high-frequency returns and the VIX, a volatility index on the S&P 500 index constructed by the Chicago Board of Options Exchange to approximate the 30-day variance swap rate (Carr and Wu (2006)). Different from these studies, we obtain direct variance swap rate quotes and thus avoid the approximation errors inherent in the procedure of synthesizing variance swaps from vanilla options (Carr and Wu (2008b)). Furthermore, by having variance swap rates across several
maturities, we can more effectively identify the multi-dimensional structure of the variance risk dynamics and their impacts on the variance swap term structure and variance swap investment.

Our investment analysis is related to a strand of literature that studies asset allocation problems in the presence of derivative securities. Carr and Madan (2001) and Carr, Jin, and Madan (2001) study how to use vanilla options across different strikes to span the jump risk with random jump size, in the absence of stochastic variance. Complementary to their study, we focus on how to use variance swap contracts of different maturities to span the multi-dimensional stochastic variance risk. Liu and Pan (2003) analyze investments in vanilla options in the presence of both jumps and stochastic variance. In this case, the allocation to a vanilla option at a given strike and maturity is a result of mixed effects from spanning the jump risk and the stochastic variance risk. To disentangle the effects, they assume a constant jump size and hence effectively seclude themselves from the strike dimension analyzed in Carr and Madan (2001) and Carr, Jin, and Madan (2001). Compared to the options contracts, variance swap contracts provide a more direct way of spanning the variance risk. Since the variance swap is a linear contract in variance risk, by trading these contracts, the investor does not create additional delta exposures to the underlying stock index, as would be the case for strategies involving vanilla options. Furthermore, if the purpose is to gain exposure to variance risk and benefit from variance risk premium, using variance swap contracts also alleviates the constraints imposed by margin requirements. Santa-Clara and Saretto (2007) show that margin requirements often limit the notional amount of capital that can be invested in option-based strategies and frequently force investors to close down positions and realize losses. The margin requirements for variance swap investments are much smaller than a corresponding investment strategy in options. Option strategies to short variance require as much as 20 times more margin than a corresponding variance swap strategy (Jung (2006)).

Finally, recognizing the virtue of the variance swap contract as a traded asset with the most direct and simple linkage to variance risk, several researchers propose to directly model the dynamics of the variance swap rate (Duanmu (1993) and Carr and Sun (2007)) or the forward variance swap rate (Dupire (1993), Bergomi (2005, 2008), and Buehler (2006)). In particular, Bergomi takes the initial forward variance swap rate curve as given and models the dynamics of the logarithm of forward variance rate as controlled by multiple factors. By contrast, we start with the instantaneous variance rate as an affine function of a finite number of factors and derive the fair value for the variance swap term structure. While Bergomi’s direct modeling of the forward variance swap rate curve can prove useful for pricing volatility derivatives and exotic structures, our finite-dimensional factor model provides fair valuation on the variance swap term structure.
III. Affine Models of Variance Swap Term Structure

We start with a complete stochastic basis \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\), with \(\mathbb{P}\) being the statistical probability measure. We use \(\mathbb{Q}\) to denote a risk-neutral measure that is absolutely continuous with respect to \(\mathbb{P}\). No arbitrage guarantees that there exists at least one such measure that prices all traded securities (Duffie (1992)).

We use \(v_t\) to denote the time-\(t\) instantaneous variance rate, and use \(V_{t,T} \equiv \int_t^T v_s ds\) to denote the aggregate return variance during the period \([t, T]\) with \(\tau = T - t\) being the length of the horizon. We assume that the dynamics of the instantaneous variance rate \(v_t\) is controlled by a \(k\)-dimensional Markov process \(X\), which starts at \(X_0\) and satisfies the following stochastic differential equation under the risk-neutral measure \(\mathbb{Q}\):

\[
dX_t = \mu(X_t) dt + \Sigma^X(X) dB^X_t + (q dN^X(\lambda(X_t)) - \bar{q} \lambda(X_t) dt),
\]

where \(\mu(X_t) \in \mathbb{R}^k\) denotes the instantaneous drift function, \(B^X\) denotes a \(k\)-dimensional independent Brownian motion with \(\Sigma^X(X)\Sigma^X(X)^\top \in \mathbb{R}^{k \times k}\) being the symmetric and positive definite instantaneous covariance matrix, and \(N^X\) denotes \(k\) independent Cox processes with intensities \(\lambda(X_t) \in \mathbb{R}^k\) and with the random jump magnitudes \(q\) being a diagonal \((k \times k)\) matrix, characterized by its two-sided Laplace transform \(L_q(\cdot)\) and with \(\bar{q} = \mathbb{E}_\mathbb{Q}[q]\). The last two terms in equation (1) form a \(k\)-dimensional jump martingale.

To analyze the variance risk dynamics using variance swap rates, we adopt the affine framework of Duffie, Pan, and Singleton (2000) and model the term structure of variance swaps within the affine class.

**Definition 1** In affine stochastic variance models, the Laplace transform of the aggregate return variance \(V_{t,T}\) under the risk-neutral measure \(\mathbb{Q}\) is an exponential-affine function of the state vector \(X_t\):

\[
L_V(u) \equiv \mathbb{E}_\mathbb{Q}\left[ e^{-uV_{t,T}} \bigg| \mathcal{F}_t \right] = \exp \left( -b(\tau)^\top X_t - c(\tau) \right),
\]

where \(b(\tau) \in \mathbb{R}^k\) and \(c(\tau)\) is a scalar.

We confine our attention to time-homogeneous models by allowing the coefficients depending only on the horizon \(\tau = T - t\), but not on the calendar time \(t\). The following proposition presents a set of sufficient conditions for the affine definition in equation (2) to hold.

**Proposition 1** If the instantaneous variance rate \(v_t\), the drift vector \(\mu(X)\), the diffusion covariance matrix
$\Sigma^X(X)\Sigma^X(X)^\top$, and the jump arrival rate $\lambda(X)$ of the Markov process $X$ are all affine in $X$ under the risk-neutral measure $Q$, the Laplace transform $L_V(u)$ is exponential-affine in $X_t$.

The conditions are directly adopted from Duffie, Pan, and Singleton (2000) on general asset pricing modeling. We specify the $Q$-dynamics as follows,

\begin{align}
    v_t &= b^\top v_t + c_v, \quad b_v \in \mathbb{R}^k, c_v \in \mathbb{R}, \\
    \mu(X_t) &= \kappa(\theta - X_t), \quad \kappa \in \mathbb{R}^{k \times k}, \theta \in \mathbb{R}^k, \\
    \Sigma^X(X)\Sigma^X(X)^\top &= \text{diag} [\alpha + \beta X_t], \quad \alpha \in \mathbb{R}^k, \beta \in \mathbb{R}^{k \times k}, \\
    \lambda(X_t) &= \alpha_\lambda + \beta_\lambda X_t, \quad \alpha_\lambda \in \mathbb{R}^k, \beta_\lambda \in \mathbb{R}^{k \times k},
\end{align}

where we adopt the convention that $\text{diag} [v]$ maps the vector $v$ onto a diagonal matrix and $\text{diag} [M]$ maps the diagonal elements of the matrix $M$ onto a vector. We further constrain $\beta$ and $\beta_\lambda$ to be diagonal matrices. Then, the coefficients $\{b(\tau), c(\tau)\}$ for the Laplace transform in equation (3) are determined by the following ordinary differential equations:

\begin{align}
    b'(\tau) &= ub_v - (\kappa + \bar{q}\beta_\lambda)^\top b(\tau) - \frac{1}{2} \beta \text{diag} \left[ b(\tau)b(\tau)^\top \right] - \beta_\lambda \left( L_q(b(\tau)) - 1 \right), \\
    c'(\tau) &= uc_v + (\kappa\theta - \bar{q}\alpha_\lambda)^\top b(\tau) - \frac{1}{2} \alpha^\top \text{diag} \left[ b(\tau)b(\tau)^\top \right] - \alpha_\lambda^\top \left( L_q(b(\tau)) - 1 \right),
\end{align}

with the boundary conditions $b(0) = 0$ and $c(0) = 0$.

A. The term structure of variance swap rates

The terminal payoff of a variance swap contract is the difference between the realized variance over a certain time period and a fixed variance swap rate, determined at the inception of the contract. The variance difference is multiplied by a notional amount that converts the difference into dollars. Since the contract is worth zero at inception, no-arbitrage dictates that the time-$t$ variance swap rate with expiry date $T$, $VS_t(T)$, is equal to the risk-neutral expected value of the aggregate return variance over the horizon $[t, T]$,

\begin{equation}
    VS_t(T) = \frac{1}{\tau} \mathbb{E}_t^Q [V_t], \quad \tau = T - t,
\end{equation}

where the $1/\tau$ scaling represents an annualization in the variance swap rate quote.
Under the affine stochastic variance framework, the variance swap rate can be solved from the Laplace transform in equation (2):

\[ E_Q^t [V_{t,T}] = L_V(u) \left( \left[ \frac{\partial b(\tau)}{\partial u} \right] X_t + \left[ \frac{\partial c(\tau)}{\partial u} \right] \right) \bigg|_{u=0} = B(\tau)^\top X_t + C(\tau), \]

which is affine in the current level of the state vector \( X_t \). Note that \( L_{V,t}(u) \bigg|_{u=0} = 1 \) and the coefficients \( B(\tau) \) and \( C(\tau) \) are defined as the partial derivatives of \( b(\tau) \) and \( c(\tau) \) with respect to \( u \).

**Proposition 2** Under the affine stochastic variance framework as specified in (3), the time-\( t \) variance swap rates are affine in the state vector \( X_t \),

(6) \[ VS_t(T) = \frac{1}{\tau} \left[ B(\tau)^\top X_t + C(\tau) \right], \]

with

(7) \[ B(\tau) = \left( I - e^{-\kappa^\top \tau} \right) \left( \kappa^\top \right)^{-1} b_v, \quad C(\tau) = \left( c_v + b_v^\top \theta \right) \tau - B(\tau)^\top \theta. \]

From Proposition 2 we obtain the following corollary:

**Corollary 1** Under the affine stochastic variance framework, the term structure of the return variance swap rate only depends on the specification of the drift of the state vector, but does not depend upon the type and specification of the martingale component of these factors. Holding constant the long run mean and the reverting speed, the term structure remains the same whether the martingale component is a pure diffusion, a pure jump martingale, or a mixture of both.

The corollary follows readily by inspecting the solutions of the coefficients \( \{B(\tau), C(\tau)\} \) in equation (7) that determine the variance swap rate. The parameters controlling the covariance matrix of the diffusion component \( (\alpha, \beta) \) and the parameters for the jump component \( (\alpha_\lambda, \beta_\lambda, \mathcal{L}_q(\cdot)) \) do not enter the coefficients.

Corollary 1 implies that from the term structure of the return variance swap, one can identify the risk-neutral drift of the state vector that controls the dynamics of the return variance. Nevertheless, the innovation (martingale) specifications of the instantaneous variance rate play little role in determining the term structure

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1 The proof of the proposition as well as the proofs of all subsequent results are delegated to an appendix that can be obtained upon request.
of the variance swap, although they do affect the time-series behavior of the variance swap rates and the pricing of variance swap options.

We use the affine specification to illustrate the corollary, but the variance swap rate only depends on the risk-neutral drift of the instantaneous variance rate \( \mu(v) \) under any instantaneous variance rate dynamics:

\[
\text{VS}_t(T) = \frac{1}{\tau} \mathbb{E}_t^Q \left[ \int_t^T v_s ds \right] = \frac{1}{\tau} \mathbb{E}_t^Q \left[ \int_t^T \mu(v_s) ds \right],
\]

as the expectation of the martingale component equals zero. This conclusion is a result of the linear relation between the variance swap rate and future instantaneous variance rates. It is also this linear relation that makes variance swap contracts the most direct instruments for spanning variance risks.

B. Model design

Proposition \( \mathbb{I} \) identifies a set of conditions that generates the affine stochastic variance class. Based on these conditions, we design both a one-factor and a two-factor model for the variance risk dynamics and compare their empirical performance in matching the time-series and term structure behaviors of variance swap rates.

B.1. A one-factor variance rate model

In the one-factor setting, we let the variance rate follow the square-root dynamics as in Heston (1993). Under the risk-neutral measure \( \mathbb{Q} \), the instantaneous variance rate dynamics are:

\[
dv_t = \kappa_v (\theta_v - v_t) dt + \sigma_v \sqrt{v_t} dB^v_t.
\]

Comparing equation (9) to the general conditions in (3), we have \( b_v = 1, c_v = 0, \alpha = 0, \beta = \sigma_v^2, \lambda = 0 \). Plugging these parameterizations into equation (7) and rearranging, we have the variance swap rate as:

\[
\text{VS}_t(T) = \phi_v(\tau) v_t + (1 - \phi_v(\tau)) \theta_v,
\]

with \( \phi_v(\tau) = (1 - e^{-\kappa_v \tau})/(\kappa_v \tau) \). With a stationary risk-neutral variance rate dynamics (\( \kappa_v > 0 \)), the coefficient \( \phi_v(\tau) \) is between zero and one, and the variance swap rate is a weighted average of the instantaneous variance rate \( v_t \) and its risk-neutral long-run mean \( \theta_v \). The weight depends on the time-to-maturity (\( \tau \)) of the variance swap contract and the risk-neutral mean-reversion speed of the variance rate (\( \kappa_v \)). The linear structure further
dictates that under this one-factor setting, variance swap rates of all maturities show the same statistical persistence as does the instantaneous variance rate \( v_t \).

Holding a fixed risk-neutral mean-reversion speed \( \kappa > 0 \), the coefficient \( \phi_v(\tau) \) starts at one at \( \tau = 0 \) and declines to zero with increasing maturity. Hence, the variance swap rate converges to the instantaneous variance rate as the maturity goes to zero and converges to the risk-neutral long-run mean as the maturity goes to infinity. Holding the maturity fixed, as \( \kappa \) declines, the risk-neutral dynamics of the instantaneous variance rate becomes more persistent, \( \phi_v(\tau) \) increases, and the current variance rate \( v_t \) has a larger impact on the variance swap rate of longer maturities. On the other hand, as the risk-neutral mean-reversion speed increases, shocks on the instantaneous variance die out quickly as the variance swap maturity increases.

Taking expectations on both sides of equation (10) under the statistical measure \( P \), we obtain the mean term structure of variance swap rates as:

\[
E^P[VS_t(T)] = \phi_v(\tau)\theta^P_v + (1 - \phi_v(\tau))\theta_v,
\]

which is a weighted average of the statistical mean \( \theta^P_v \equiv E^P[v_t] \) and the risk-neutral mean \( \theta_v \) of the instantaneous variance rate. Since \( \phi_v(\tau) \) declines monotonically with increasing maturity, the risk-neutral mean has increasing weights at longer maturities. Therefore, to generate an upward or downward sloping mean term structure for the variance swap rates, we need the statistical mean and the risk-neutral mean of the instantaneous variance rate to be different. The difference between the two mean values dictates the sign and magnitude of the variance risk premium.

B.2. A two-factor variance rate model

We consider a two-factor variance risk dynamics controlled by the following stochastic differential equations under the risk-neutral measure \( Q \),

\[
\begin{align*}
dv_t &= \kappa_v (m_t - v_t) \, dt + \sigma_v \sqrt{v_t} dB^v_t, \\
dm_t &= \kappa_m (\theta_m - m_t) \, dt + \sigma_m \sqrt{m_t} dB^m_t,
\end{align*}
\]

where the instantaneous variance rate \( v_t \) reverts to a stochastic mean level \( m_t \). The mean level follows another square-root process. Analogous to Balduzzi, Das, and Foresi (1998) for interest rate modeling, we label \( m_t \) as the stochastic central tendency of the instantaneous variance rate, with \( \theta_m \) being the unconditional long-
run mean for both $v_t$ and $m_t$ under the risk-neutral measure. Under the two-factor variance risk specification, the variance swap rates are given by:

$$ \text{VS}_T(T) = \phi_v(\tau)v_t + \phi_m(\tau)m_t + (1 - \phi_v(\tau) - \phi_m(\tau))\theta_m, $$

with

$$ \phi_v(\tau) = \frac{1 - e^{-\kappa_v \tau}}{\kappa_v \tau}, \quad \phi_m(\tau) = \frac{1 + \frac{\kappa_m}{\kappa_v}e^{-\kappa_v \tau} + \frac{\kappa_v}{\kappa_v - \kappa_m}e^{-\kappa_m \tau}}{\kappa_m \tau}. $$

The variance swap rate in equation (13) is a weighted average of the instantaneous variance rate $v_t$, its stochastic central tendency $m_t$, and the risk-neutral long-run mean $\theta_m$. The weight on the instantaneous variance rate is the same as in the one-factor case. The weight converges to one as the maturity goes to zero and converges to zero as the maturity goes to infinity. The weight on $m_t$ also converges to zero as the maturity goes to infinity. Hence, the variance swap rate starts at the instantaneous variance rate at zero maturity and converges to the risk-neutral long-run mean $\theta_m$ as maturity goes to infinity. The stochastic central tendency factor plays a role at intermediate maturities, with the weighting coefficient $\phi_m(\tau)$ showing a hump-shaped term structure.

Under this two-factor structure, swap rates at different maturities can show different degrees of persistence. In particular, the central tendency factor is usually more persistent than the instantaneous variance rate. In this case, the short-term swap rate becomes less persistent than the long-term swap rate. For the variance process $v_t$ to be stationary under $Q$, we furthermore require that $\kappa_v, \kappa_m > 0$.

C. Market prices of variance risks

For both models, we assume that the market price on each source of risk is proportional to the square root of the risk level:

$$ \gamma(B_t^v) = \gamma_v \sqrt{v_t}, \quad \gamma(B_t^m) = \gamma_m \sqrt{m_t}. $$

Under the one-factor model, the statistical dynamics of the variance rate become,

$$ dv_t = \kappa_v \left( \theta_v - v_t \right) dt + \sigma_v \sqrt{v_t} dB_t^v. $$
with $\kappa^p_v = (\kappa_v - \gamma \sigma_v)$ and $\theta^p_v = \kappa_v \theta_v / \kappa^p_v$. Thus, a negative market price of variance risk makes the statistical variance rate process more mean reverting than its risk-neutral counterpart ($\kappa^p_v > \kappa_v$) and makes the statistical mean variance rate lower than its risk-neutral counterpart ($\theta^p_v < \theta_v$).

Under the two-factor model, the statistical dynamics of the variance rate become,

$$
\begin{align*}
  dv_t &= \kappa^p_v \left( \frac{\kappa_v m_t - v_t}{\kappa^p_v} \right) dt + \sigma_v \sqrt{v_t} dB^p_v, \\
  dm_t &= \kappa^p_m \left( \theta^p_m - m_t \right) dt + \sigma_m \sqrt{m_t} dB^m_v,
\end{align*}
$$

with $\kappa^p_m = \kappa_m - \gamma^m \sigma_m$ and $\theta^p_m = \kappa_m \theta_m / \kappa^p_m$. In this case, the long-run statistical mean of the variance rate becomes $\theta^p_v = \kappa^p_v \theta^p_m / \kappa^p_v = \kappa_v \kappa_m \theta_m / (\kappa^p_v \kappa^p_m)$. Similar to the one-factor case, negative market price of variance risk ($\gamma^v$) increases the mean-reversion speed ($\kappa^p_v$) and reduces the long-run mean ($\theta^p_v$) under the statistical measure. Negative market price of the central tendency risk ($\gamma^m$) has similar effects on the central tendency dynamics, and further reduces the long-run statistical mean of the instantaneous variance rate.

D. Comparison with the literature

Analogous to the interest rate term structure literature that starts with the the instantaneous interest rate (e.g., Duffie and Kan (1996), and Duffie, Pan, and Singleton (2000)), our modeling approach starts with the instantaneous variance rate as a function of a finite number of dynamic factors. Furthermore, as in affine interest rate term structure models, we require that the Laplace transform of the aggregate variance be exponential affine in the state vector, which is a stronger requirement than purely requiring the variance swap rate to be affine in the state vector. The affine variance swap rate requirement only constrains the risk-neutral drift specification of the affine factors, whereas the Laplace transform exponential affine requirement also constrains the specification of the martingale component of the factor dynamics. The latter constraint is important for generating tractable solutions to variance swap investment decisions.

To price interest-rate options, the literature often takes the yield curve as given and focus exclusively on the specification of the volatility structure. Analogously, several researchers have proposed to take the initial forward variance rate curve as given and directly model the volatility structure of the forward variance rate. $^2$Prominent examples include the forward rate models of Ho and Lee (1986), Hull and White (1993), and Heath, Jarrow, and Morton (1992), market rate models of Brace, Gatarek, and Musiela (1997), Jamshidian (1997), Milhersen, Sandmann, and Sondersmann (1997), and Musiela and Rutkowski (1997), and string models of Goldstein (2000), Santa-Clara and Sornette (2001), and Longstaff, Santa-Clara, and Schwartz (2001).
Take Bergomi (2005, 2008) as an example, who directly specifies the volatility structure of the instantaneous forward variance rate, \( \xi^T_t \equiv \mathbb{E}_t^Q [v_T] = \partial \left[ \text{VS}_t(T)(T - t) \right] / \partial T \).

In his one-factor example, Bergomi (2005) assumes that \( \xi^T_t \) is log-normally distributed with its volatility a function of the time to maturity,

\[
(17) \quad d\xi^T_t = e^{-\kappa_v(T-t)} \sigma_v \xi^T_t dB^v_t.
\]

Under our one-factor specification in (9), the forward variance rate is related to the instantaneous variance rate by \( \xi^T_t = \theta_v + e^{-\kappa_v(T-t)} (v_t - \theta_v) \), and it has the following risk-neutral dynamics,

\[
(18) \quad d\xi^T_t = e^{-\kappa_v(T-t)} \sigma_v \sqrt{\xi^T_t} dB^v_t = e^{-\kappa_v(T-t)} \sigma_v \sqrt{\xi^T_t - \theta_v} + \theta_v dB^v_t.
\]

Comparing (18) with (17), we observe that both models have the volatility of the forward variance rate decay exponentially with time to maturity. The difference between the two models lies in their respective variance rate level dependence. The volatility function is proportional to the forward variance rate level in Bergomi’s model but is proportional to the square root of the instantaneous variance rate, or the square root of an affine function of the forward variance rate \( \sqrt{e^{\kappa_v(T-t)} (\xi^T_t - \theta_v)} + \theta_v \) in our one-factor affine model.

In addition to the subtle differences in the forward variance rate volatility functions, our modeling framework maintains a finite-dimensional factor structure that generates fair valuation for the variance swap rate term structure. By contrast, Bergomi’s forward rate approach takes the forward variance rate curve as inputs and can be used to generate fair valuation for variance derivatives.

IV. Estimating Variance Swap Term Structure Models

We estimate the variance risk dynamics using over-the-counter quotes on a term structure of variance swap rates on the S&P 500 index. From a major broker dealer, we obtain daily closing quotes on variance swap rates with fixed time to maturities at two, three, six, 12, and 24 months starting January 10, 1996, and ending March 30, 2007, spanning over 11 years. To avoid the effect of weekday patterns on the dynamics estimation, we sample the data weekly on every Wednesday. When Wednesday is a holiday, we use the quotes from the previous business day. The data contain 586 weekly observations for each series.

\[3\] Analogously to the one-factor model, we could link Bergomi’s two-factor specification to our two-factor specification.
To sharpen the variance risk premium identification, we also obtain the time series data on the S&P 500 index and compute the annualized realized variance over different horizons based on daily log returns,

\[ \text{RV}_{t,T} = \frac{365}{D} \sum_{d=1}^{D} \left( \ln \left[ \frac{S_{t+d}}{S_{t+d-1}} \right] \right)^2, \]

where \( S_t \) denotes the index level at time \( t \) and \( D \) denotes the number of days between time \( t \) and \( T \). Following industry standard in variance swap payoff calculations, we compute the realized variance using un-demeaned log daily returns. At each date \( t \), we compute realized variance over fixed horizons of 7, 30, 60, 90, 120, and 150 days, and use these realized variance series together with the variance swap rates to identify the variance risk dynamics and variance risk premium. Ideally, we want to include realized variance over horizons that match the whole spectrum of the variance swap maturity. Yet, incorporating realized variance of longer horizons would severely shorten our sample length. The choice of horizon from seven to 150 days is a tradeoff between the two considerations.

A. Summary statistics of variance swap rates

Figure 1 plots the time series of the variance swap rates in the left graph at three selected maturities of two (solid line), six (dashed line), and 24 (dotted line) months. The right graph plots representative variance swap term structures at different dates. Per industry convention, the variance swap rates are represented in volatility percentage points. The time series plots show that the variance swap rates started at relatively low levels, but experienced a spike during the 1997 Asian crisis, and another even larger spike during the hedge fund crisis in late 1998. The series witnessed another two spikes between 2001 and 2003, but otherwise have been declining to very low levels. Over the course of the sample period, the variance swap rate level has varied greatly from 10.39% to 50.48%.

[FIGURE 1 about here.]

The right graph of Figure 1 shows that the term structure of variance swap rates can take a wide variety of shapes at different times, including upward sloping, downward sloping, and hump-shaped term structures. A successful model of variance risk dynamics must capture not only the large variation in the volatility levels, but also the different shapes of the term structure.
Table 1 reports the summary statistics of the variance swap rates across different maturities. The mean variance swap rates increase with maturity from 20.804% at two-month maturity to 22.866% at two-year maturity, thus generating an upward sloping mean term structure. The standard deviation estimates decline as maturity increases. The variance swap rates show positive skewness but mild kurtosis. The weekly autocorrelation estimates are high and increasing with swap maturities. According to the theoretical analysis in the previous section, the upward sloping mean term structure is evidence for non-zero market price of variance risk, and the downward sloping term structure on the standard deviation is evidence for mean reversion in the variance rate dynamics. On the other hand, the upward sloping term structure on the autocorrelation estimates points to the existence of multiple variance risk factors.

B. Estimation methodology

To estimate the variance risk dynamics, we cast the model into a state-space form and extract the variance risk factors ($v_t$ and $m_t$) from the observed variance swap rates using the classic Kalman (1960) filter. We estimate the model parameters by maximizing the aggregate log likelihood on the forecasting errors of the swap rate series and the realized variance.

We build the state propagation equation based on the statistical dynamics of the variance rates. Under the two factor model, we set $X_t = [v_t, m_t]^\top$ and construct the state propagation equation based on the Euler approximation of the statistical dynamics in equation (17),

\begin{equation}
X_{t+1} = A + \Phi X_t + \Sigma(X_t) \sqrt{\Delta t} \varepsilon_t,
\end{equation}

with $\varepsilon$ denoting a two-dimensional iid standard normal innovation vector,

\begin{align*}
A &= (I - \Phi) \theta^p, \quad \Phi = e^{-\kappa^p \Delta t}, \quad \theta^p = \begin{bmatrix} \theta_{vp}^p \\ \theta_{mp}^p \end{bmatrix}, \\
\kappa^p &= \begin{bmatrix} \kappa_{vp} \\ 0 \end{bmatrix}, \quad \Sigma(X_t) = \begin{bmatrix} \sigma_v \sqrt{v_t} & 0 \\ 0 & \sigma_m \sqrt{m_t} \end{bmatrix},
\end{align*}

and $\Delta t = 7/365$ being the weekly time interval of the discretization. The one-dimensional state propagation equation for the one-factor model is defined analogously.
We construct the measurement equation based on the observed variance swap rates:

\[ y_t = VS_t(T, X_t) + e_t, \quad T - t = 2, 3, 6, 12, 24 \text{ months.} \]  

(21)

where \( y_t \) denotes the observed variance swap series, \( VS_t(T, X_t) \) denotes their corresponding model values as a function of the variance risk factors \( X_t \), and \( e_t \) denotes the measurement error. We assume that the measurement error is independent of the state vector and that the measurement error on each of the five series is mutually independent but with distinct variance.

Since the state propagation equation is Gaussian linear and the measurement equation is linear in the state vector, the Kalman filter provides the efficient forecasts and updates on the state vector and the observed variance swap rates. We build the likelihood on the variance swap rates based on the forecasting errors from the Kalman filter. Specifically, let \( (\overline{y}_t, \overline{Q}_t) \) denote the time- \((t-1)\) Kalman filter forecasts on the conditional mean and the conditional variance of the variance swap rates at time \( t \), the time- \((t-1)\) conditional log likelihood on the variance swap rates at time \( t \) is,

\[ l_{t-1}(y_t, \Theta) = -\frac{1}{2} \left[ \log |\overline{Q}_t| + \left( (y_t - \overline{y}_t)^\top (\overline{Q}_t)^{-1} (y_t - \overline{y}_t) \right) \right], \]

(22)

where \( \Theta \) denotes the set of model parameters.

Furthermore, given the variance risk factors \( (X_t) \) extracted from the Kalman filter, we can also predict the annualized realized variance based on the statistical dynamics of the risk factors,

\[ \mathbb{E}^P_{t} [RV_{t,T}] = \frac{1}{\tau} \left[ (B(\tau)^P)^\top X_t + C(\tau)^P \right], \]

(23)

where the coefficients are analogous to those defined in Proposition 2.

(24) \[ B(\tau)^P = \left( I - e^{-(\kappa^P)^\top \tau} \right) \left( \kappa^P \right)^\top b_v, \quad C(\tau)^P = \left( c_v + b_v^\top \theta^P \right) \tau - (B(\tau)^P)^\top \theta^P. \]

Under the two-factor model, we have \( b_v = [1, 0]^\top \) and \( c_v = 0 \). Under the one-factor model, we have \( b_v = 1 \), \( c_v = 0 \), \( \theta^P = \theta_v^P \), and \( \kappa^P = \kappa_v^P \).

Given the variance forecasts, we build the likelihood function on the realized variances assuming that the forecasting errors on the realized variance, \( e_{RV, t+D}^R = RV_{t,t+D} - \mathbb{E}^P_{t} [RV_{t,t+D}] \) with \( D = 7, 30, 60, 90, 120, 150 \) days, are normally distributed with constant covariance matrix \( Q_{RV} \). Thus, the time- \( t \) conditional log likeli-
hood on the realized variance becomes:

\[
I_t(e_{t+D_*, \Theta}) = -\frac{1}{2} \left[ \log |Q_{RV}| + \left( e_{t+D_*, RV} \right)^\top \left( Q_{RV} \right)^{-1} \left( e_{t+D_*, RV} \right) \right].
\]

For estimation, we assume that the forecasting errors on variance swaps and realized variances are independent but with distinct variance. Thus, the aggregate log likelihood becomes the summation of the log likelihood values on the two sets of data series. We numerically maximize the aggregate log likelihood value to estimate the model parameters.

C. Model performance

Table 2 reports the summary statistics of the pricing errors, defined as the difference between the variance swap rate quotes and the model-implied values, both in volatility percentage points. We report the mean error (Mean), root mean squared error (RMSE), weekly autocorrelation (Auto), and the maximum absolute pricing error (Max). The last column \( R^2 \) in each panel reports the explained variation, defined as one minus the ratio of the pricing error variance to the variance of the original swap rate series, in percentage points. The last row of Table 2 reports the maximized log likelihood values.

The one-factor model fits the six-month variance swap to near perfection, but the pricing errors increase at other maturities. The performance of the two-factor model is more uniform across different maturities. The explained variations range from 98.77% to 100.00%. The root mean squared pricing errors range from practically zero to 0.8 volatility percentage points, no larger than the average bid-ask spreads for the over-the-counter variance swap rate quotes.\(^4\) The two-factor model also performs significantly better than the one-factor model in terms of the log likelihood values. A formal likelihood ratio test rejects the one-factor model over any reasonable confidence level.

D. Variance risk dynamics and the term structure of variance swap rates

Exploiting the information in the time series and term structure of variance swap rates and the realized variance, we can accurately identify the variance risk dynamics under both the statistical and the risk-neutral
measures. Table reports the parameter estimates and the absolute magnitudes of the \(t\)-statistics in parentheses. Focusing on the two-factor model specification, we observe that the risk-neutral mean-reversion speed for the instantaneous variance rate \((κ_v = 4.373)\) is much higher than that for the central tendency risk factor \((κ_m = 0.1022)\). To gain more intuition, we define the half life \(H\) of a first-order autoregressive process as the number of weeks for the autocorrelation of the process to decay to half of its weekly autocorrelation level, \(H = \ln(\phi/2)/\ln(\phi)\), with \(\phi = \exp(−κΔt), Δt = 7/365\) denoting the weekly autocorrelation of a series. Under the risk-neutral measure, the mean-reversion speed estimates imply a half life of less than ten weeks for the instantaneous variance rate \(v_t\), but almost seven years for the central tendency factor \(m_t\).

The different risk-neutral persistence dictates that the two risk factors have different impacts across the term structure of the variance swap rates. We can convert the risk-neutral persistence estimates into factor loadings coefficients \(φ_v(τ)\) and \(φ_m(τ)\) as in equation (14):

\[
φ_v(τ) = \frac{1 - e^{-κ_vτ}}{κ_v}, \quad φ_m(τ) = \frac{1 + \frac{κ_v - κ_m}{κ_v - κ_m}e^{-κ_vτ} - \frac{κ_v - κ_m}{κ_m}e^{-κ_mτ}}{κ_m},
\]

with which the variance swap rates of different maturities are linked to the two risk factors:

\[
VS_T(T) = φ_v(τ)v_t + φ_m(τ)m_t + (1 - φ_v(τ) - φ_m(τ))θ_m.
\]

The loading coefficients measure the contemporaneous responses of the variance swap term structure to unit shocks in the two variance risk factors \(v_t\) and \(m_t\). Figure 2 plots the term structure of the two responses in the left graph, with the solid line denoting the response to \(v_t\) and the dashed line denoting the response to \(m_t\). The dotted line captures the remaining weight on the risk-neutral mean \((θ_m)\) of the variance rate and the central tendency. The impact of the transient variance rate factor \((v_t)\) is mainly at short maturities. Its impact declines as maturity increases. On the other hand, the contribution of the persistent central tendency factor \((m_t)\) starts at zero, but increases progressively as the variance swap maturity increases. The remaining weight on the risk-neutral mean also starts at zero and increases monotonically with increasing maturity. This increasing weight on a constant is responsible in generating a downward sloping term structure on the standard deviation of the variance swap rate (Table I).

[FIGURE 2 about here.]
The right graph of Figure 2 plots the mean term structure of variance swap rates in volatility percentage points. The five circles represent the sample averages of the five data series and the solid lines are computed from the estimated two-factor variance risk model. The upward sloping mean term structure is consistent with the negative market price estimates ($\gamma$) on the variance risks in Table 3. The market price of the instantaneous variance rate is strongly negative; the market price of the central tendency factor is also negative, but with smaller absolute magnitude. The negative market prices on the two risk factors make the statistical mean-reversion speeds ($\kappa_P$) larger and the statistical long-run means ($\theta_P$) lower than their risk-neutral counterparts. The three long-run means show the following ranking: $\theta_P^v < \theta_P^m < \theta_m$. The statistical mean of the variance rate is lower than the statistical mean of the central tendency factor. Both are lower than their common risk-neutral mean $\theta_m$.

Taking unconditional expectations on both sides of equation (13) under the statistical measure $\mathbb{P}$, we obtain the mean term structure of variance swap rates as,

$$E^\mathbb{P}[VS_t(T)] = \phi_v(\tau)\theta_P^v + \phi_m(\tau)\theta_P^m + (1 - \phi_v(\tau) - \phi_m(\tau))\theta_m,$$

which is a weighted average of the statistical mean of the instantaneous variance rate $\theta_P^v$, the statistical mean of the central tendency factor $\theta_P^m$, and the common risk-neutral mean of the variance rate and the central tendency $\theta_m$. The factor loading in the left graph of Figure 2 suggests that $\theta_P^v$ has the highest weighting at short maturities, whereas both $\theta_P^m$ and $\theta_m$ have increasing weights as the swap maturity increases. The factor loadings and the ranking of the three long-run mean values generate the upward sloping mean term structure shown in the right graph of Figure 2.

The two-factor variance risk structure also generates different statistical persistence for swap rates at different maturities. Table 3 shows that the statistical mean-reversion speed for the instantaneous variance rate is much larger than that for the central tendency factor. The statistical half life of the instantaneous variance rate is just about four weeks, whereas the statistical half life of the central tendency factor is over three years. The increasing weight on the central tendency factor for longer maturity swaps suggests that the variance swap rate becomes increasingly persistent as the swap maturity increases. This observation is consistent with the weekly autocorrelation estimates reported in Table 1.
E. Model stability and out-of-sample performance

To gauge the stability of the model and its out-of-sample performance, we divide the data into two subsample periods. The first subsample is from January 10, 1996 to June 27, 2001, 286 weekly observations for each series. The second subsample is the remaining sample period from July 4, 2001 to March 28, 2007, 300 weekly observations for each series. Each sample contains about five and a half years of data.

We repeat the model estimation on the two subsamples. Table 4 reports the subsample model parameter estimates. The estimates are largely in line with the whole-sample estimates. Comparing the parameter estimates of the two-factor variance risk model during the two subsamples, we observe that the risk-neutral mean-reversion speed for the instantaneous variance rate is smaller during the first subsample than during the second sample. The opposite is true for the central tendency factor as the estimate for its mean-reversion speed is no longer significantly different from zero during the second subsample. The different estimates suggest that the roles of the two risk factors become more separated during the second sample. The impacts of the instantaneous variance rate are mainly at short maturities and die out quickly as the maturity increases. The impacts of the central tendency factor become even more persistent across the variance swap term structure. On the other hand, the market price estimates are relatively stable over the two sample periods.

[Table 4 about here.]

To gauge the out-of-sample pricing performance of the two models, we use the parameters estimated from the first subsample to price the variance swap rates during the second subsample. Table 5 reports the summary statistics on the out-of-sample pricing errors. Compared with the in-sample pricing error statistics in Table 2, the out-of-sample errors are not much different. The average explained variation for the one-factor model is 97.36% compared to the in-sample performance of 95.81%. The average explained variation for the two-factor model is 99.49% compared to the in-sample estimate of 99.54%. These statistics show that the model generates stable performance over time.

[Table 5 about here.]
V. Optimal Variance Swap Investment Decisions

The availability of variance swap contracts makes it convenient for investors to either hedge away variance risk or achieve additional exposures in it and receive variance risk premium. How does this availability alter an investor’s asset allocation decision? To answer this question, we study the optimal asset allocation problem for an investor who has access to the money market account, the stock index, and a series of return variance swaps on the index across different maturities.

We assume that investment in the money market generates a constant riskfree interest rate \( r \), and that the stock index evolves under the statistical measure \( \mathbb{P} \) according to the stochastic differential equation,

\[
dS_t / S_t = (r + \gamma S_t v_t) dt + \sqrt{v_t} dB_{S_t},
\]

where \( B_{S_t} \) denotes a Brownian motion that measures the stock index return risk and \( \gamma S_t v_t \) denotes the instantaneous risk premium on the index return, which we assume is proportional to the instantaneous variance rate level. We analyze the allocation problem under both the one-factor and two-factor variance risk specifications that we have estimated in the previous section.

A. The one-factor stochastic variance model

The statistical dynamics of the one-factor variance rate is specified in equation (16). Under the one-factor variance risk dynamics, the model values of the variance swap rates of all maturities are perfectly correlated and are all affine functions of the instantaneous variance rate. Therefore, the investor can choose any one variance swap contract to span the variance risk. The statistical evolution of the value of this variance swap contract, \( VS_t(T) \), is given by,

\[
dVS_t(T) = \phi_v(T - t) \gamma v_t dt + \phi_v(T - t) \sigma_v \sqrt{v_t} dB_{S_t}^v.
\]

To capture the well-known leverage effect, we decompose the Brownian motion in the variance dynamics into two components,

\[
dB_{v_t}^v = \rho dB_{S_t}^S + \sqrt{1 - \rho^2} dB_{z_t}^v,
\]
where \( \rho \) measures the instantaneous correlation between the return risk and the variance risk and \( B_t \) denotes the independent component of the variance risk. If we let the market price of the independent variance risk also be proportional to the square root of the instantaneous variance rate level,

\[
\gamma(B_t^z) = \gamma \sqrt{\gamma_z v_t},
\]

we can decompose the instantaneous total variance risk premium into two components, one from the return risk and the other from the independent variance risk,

\[
\gamma v_t = \rho \gamma v_t + \sqrt{1 - \rho^2} \gamma v_t.
\]

We assume that an investor allocates her initial wealth \( W_t \) at time \( t \) among the money market account, the stock index \( S_t \), and a variance swap contract with expiry date \( T \). Let \( W_t^S \) and \( W_t^B \) denote the amount of money invested in the stock index and the money market, respectively, and let \( N \) denote the dollar notional amount invested in the variance swap contract. The investor’s budget constraint becomes,

\[
W_t = W_t^S + W_t^B + N \left( V S_t(T) - K \right),
\]

where \( K \) is the delivery price of the variance swap contract. A negative notional amount implies a short position on the variance swap contract. We assume that at the time of decision making \( (t) \), we initiate new variance swap contracts with the delivery price set to the prevailing variance swap rate \( K = V S_t(T) \). In this case, the variance swap contract has zero initial value, and we have \( W_t^B = W_t - W_t^S \). In reality, margin requirements and leverage constraints can limit the actual notional amount of variance swap contracts that an investor can sign on. We do not directly consider this and other financial market constraints in our optimization.

If we use share prices and wealth fractions instead of dollar amounts, we can write the wealth dynamics as,

\[
\frac{dW_t}{W_t} = w_t \frac{dS_t}{S_t} + (1 - w_t) \gamma dW_t + n_t dV S_t,
\]

where \( w_t \) denotes the fraction of wealth invested in the stock index and \( n_t \) denotes the fraction of wealth invested as notional in the variance swap investment. Plugging in the dynamics for the stock index and the
variance swap contract, we have

\[ \frac{dW_t}{W_t} = rd\tau + (w_t\gamma^S + n_t\phi_v(T - t)\sigma_v\gamma^v) \nu_t dt + w_t\sqrt{\nu_t} dB^S_t + n_t\phi_v(T - t)\sigma_v\sqrt{\nu_t} dB^v_t. \] (35)

The investor chooses the allocation weights to maximize her utility of wealth \( W_T \) at the terminal time \( T \). Assuming constant relative risk aversion (CRRA) utility with relative risk aversion coefficient \( \eta > 0 \), we can write the indirect utility function as,

\[ J(t, W, v) = \sup_{(w_t, n_t)} \mathbb{E} \left( \frac{W_T^{1-\eta}}{1-\eta} \bigg| W_t = W, v_t = v \right), \quad \eta \neq 1, \] (36)

subject to the budget constraint in equation (35). The corresponding Hamilton-Jacobi-Bellman (HJB) equation is,

\[ 0 = \sup_{(w_t, n_t)} \left\{ J_t + J_t \kappa^F \left( \theta^F - v \right) + J_W W r + J_W W (w_t\gamma^S + n_t\phi_v(T - t)\sigma_v\gamma^v) \nu \right. \\
+ \frac{1}{2} J_{WW} W^2 \left[ w_t^2 + n_t^2\phi_v(T - t)^2\sigma_v^2 + 2w_t n_t\phi_v(T - t)\sigma_v \rho \right] v \\
\left. + J_{Wv} W [\sigma_v \rho w_t + \sigma_v^2 n_t\phi_v(T - t)] v + \frac{1}{2} \sigma_v^2 v J_{vv} \right\}, \] (37)

where \( J_t, J_v, J_W, J_{Wv}, J_{vv}, \) and \( J_{WW} \) denote the first, second, and cross derivatives with respect to \( t, v, \) and \( W \). From the first order conditions with respect to \( (w_t, n_t) \), we obtain the optimal asset allocation weights in the following generic forms.

**Proposition 3** Given the optimization problem in (36) under the budget constraint in (35), the stock dynamics in (29), and the variance dynamics in (30), the optimal allocations to the stock index \( (w_t) \) and the variance swap contract \( (n_t) \) are:

\[ w_t = -\frac{J_W}{W_t J_{WW}} \left( \gamma^S - \frac{\rho}{\sqrt{1 - \rho^2}} \gamma^v \right). \] (38)

\[ n_t = -\frac{1}{\phi_v(T - t)} \left( \frac{J_W}{W_t J_{WW}} \frac{\gamma^S}{\sqrt{1 - \rho^2} \sigma_v} + \frac{J_{Wv}}{W_t J_{WW}} \right). \] (39)

For comparison, we also derive analogously the generic optimal allocation decisions when the investor can only invest in one of the two risky assets:
Proposition 4 If the investor can only invest in the money market account and the stock index, the optimal fraction of wealth invested in the stock index is

\[ w_t = -J_W \frac{\gamma^s}{W_t J_{WW}} - \rho \sigma_v J_{Wv} \frac{\gamma^v}{W_t J_{WW}}. \]  

(40)

If the investor can only invest in the money market account and the variance swap contract, the optimal fraction of wealth invested as notional in the variance swap contract is

\[ n_t = -\frac{1}{\phi_v(T-t)} \left( J_W \frac{\gamma^v}{W_t J_{WW} \sigma_v} + J_{Wv} \right). \]  

(41)

As shown in Merton (1971), the optimal allocation to risky assets includes two components: a myopic component that is proportional to the mean excess return and an intertemporal hedging demand that is proportional to the covariance between the risky asset returns and the state variables that govern the stochastic investment opportunity, both scaled by the covariance matrix of the asset return. In our application, the stochastic variance risk represents the stochastic investment opportunity, which induces an intertemporal hedging demand when we invest in either the stock index alone as in equation (40) or the variance swap contract alone as in equation (41).

Interestingly, when we invest in both the stock index and the variance swap contract to span both the return risk and the variance risk, the optimal investment in the stock index in equation (38) no longer includes an intertemporal hedging demand. Stochastic investment opportunities ask for intertemporal hedging demand; yet an appropriately designed derivative contract can be used to span the risk inherent in the stochastic investment opportunities and to eliminate the need for intertemporal hedging with the primary security.

To better understand what type of derivative contracts can help eliminate the need for intertemporal hedging with the primary security, we consider a generic example of two risky assets, with the first security as the primary security (a stock index) and the second security as a specially designed derivative contract to hedge against the stochastic investment opportunity, which is governed by a one-factor state variable \( X_t \).

If we use \( \Sigma_{SS}, \mu_S, \) and \( \Sigma_{SX} \) to denote the return covariance matrix of the two risky assets, the mean excess return vector, and the covariance between the two risky assets and the state variable, respectively,

\[ \Sigma_{SS} = \begin{bmatrix} \sigma_1^2 & \sigma_1 \sigma_2 \rho_{12} \\ \sigma_1 \sigma_2 \rho_{12} & \sigma_2^2 \end{bmatrix}, \quad \mu_S = \begin{bmatrix} \gamma_1 \sigma_1 \\ \gamma_2 \sigma_2 \end{bmatrix}, \quad \Sigma_{SX} = \begin{bmatrix} \sigma_1 \sigma_x \rho_{1x} \\ \sigma_2 \sigma_x \rho_{2x} \end{bmatrix}, \]
we can write the optimal allocation weights as,

\[ w = -\frac{J_W}{W_tJ_{WW}}\Sigma_{SS}^{-1}HS - \frac{J_{WX}}{W_tJ_{WW}}\Sigma_{SS}^{-1}\Sigma_{SX}. \]

Plugging in the elements of the matrices, we have:

\begin{align*}
    w_1 &= -\frac{J_W}{W_tJ_{WW} (1 - \rho_{12}^2)} \gamma_1 - \rho_{12} \gamma_2 - \frac{J_{WX}}{W_tJ_{WW} (1 - \rho_{12}^2)} \rho_1 \sigma_x - \rho_{12} \rho_2 \sigma_x, \\
    w_2 &= -\frac{J_W}{W_tJ_{WW} (1 - \rho_{12}^2)} \gamma_2 - \rho_{12} \gamma_1 - \frac{J_{WX}}{W_tJ_{WW} (1 - \rho_{12}^2)} \rho_2 \sigma_x - \rho_{12} \rho_1 \sigma_x.
\end{align*}

To exclude the investment in the first risky asset from having an intertemporal hedging demand, we need

\[ \rho_{1x} - \rho_{12} \rho_{2x} = 0. \]

In our application, the stock index return has an instantaneous correlation of \( \rho \) with the stochastic variance risk. The variance swap contract is an affine function of the variance rate and the contract value does not depend on any other random variables or risk factors. Thus, the variance swap contract has a perfect instantaneous correlation with the stochastic variance risk factor. By having \( \rho_{2x} = 1 \), we have \( \rho_{12} = \rho_{1x} = \rho \) and hence \( \rho_{1x} - \rho_{12} \rho_{2x} = 0 \). The condition in (44) is satisfied.

Therefore, when we design a derivative instrument that shows perfect instantaneous correlation with the state variable, it eliminates the intertemporal hedging need from the primary security investment. Under the one-factor stochastic variance risk model, variance swap contracts play this role perfectly as they are only a function of the variance rate itself. By contrast, although an option contract can also be used to span the variance risk, it cannot be used to exclude the intertemporal hedging demand from the underlying primary security, because the variation of the option value not only depends on the variance risk, but also depends on the stock index level and hence the return risk.

Under the one-factor variance risk dynamics, we can solve the indirect utility function and the optimal allocation weights in analytical forms.

**Proposition 5** In the one-factor stochastic variance model, the indirect utility function has the analytical
representation:

\[(45)\]

\[
J(t, W, v) = \frac{W_{t}^{1-\eta}}{1-\eta} \exp(h_{t}(u)v_{t} + k(u)),
\]

with \(u \equiv T - t\) denoting the investment horizon. The optimal allocation weights in the stock index and the variance swap contract with maturity \(T\) are:

\[(46)\]

\[
w_{t} = \frac{1}{\eta} \left( \gamma^{S} - \frac{\rho}{\sqrt{1-\rho^{2}}} \gamma^{v} \right),
\]

\[(47)\]

\[
n_{t} = \frac{1}{\eta \phi_{v}(T-t)} \left( \frac{\gamma^{S}}{\sqrt{1-\rho^{2}}} + h_{t}(u) \right),
\]

where the intertemporal hedging demand for the variance swap contract is determined by the coefficient \(h_{t}(u)\) solving the following ordinary differential equation,

\[(48)\]

\[
h'_{v}(u) = \alpha h_{v}(u)^{2} + \beta h_{v}(u) + \zeta,
\]

with

\[(49)\]

\[
\alpha = \frac{1}{2\eta} \sigma_{v}^{2}, \quad \beta = \frac{1-\eta}{\eta} \sigma_{v} \gamma' - \kappa_{v}^{2}, \quad \zeta = \frac{1-\eta}{2\eta} \left( (\gamma^{S})^{2} + (\gamma^{v})^{2} \right),
\]

starting at \(h_{v}(0) = 0\).

The derivation and the expression for the coefficient \(k(u)\) in the indirect utility function are given in the appendix that can be obtained by the authors on request.

For a myopic investor, the intertemporal hedging demand is zero. The investments in the stock index and the variance swap contract depend crucially on the risk premia on the two sources of risks: the return risk and the variance risk. In the absence of a risk premium on the independent variance risk \(\gamma^{v} = 0\), the investor invests positively in the stock index (given that the index generates positive risk premium, \(\gamma^{S} > 0\)) but does not invest in the variance swap contract. Given the stylized fact of a highly negative correlation \(\rho\), a positive risk premium on the independent variance risk induces in equation (47) a positive investment in the variance swap contract and also increases the investment in the stock index. On the other hand, when the risk premium on the independent risk component is negative, the investor short sells the variance swap contract and also reduces her investment in the stock index accordingly. When the market price of the independent variance
risk is highly negative and its magnitude dominates the market price of the return risk, the negative variance risk premium effect can dominate the positive return risk premium and the optimal investment in the stock index in equation (46) can also become negative.

When the investor has a non-myopic investment horizon, her investment in the variance swap contract varies with the investment horizon, but her investment in the stock index does not. The hedging coefficient $h_v(u)$ starts at zero when the investment horizon $u$ is zero. With $\eta > 1$ and hence $\zeta < 0$ in equation (49), the hedging demand turns negative as the investment horizon increases.

The next proposition summarizes the portfolio allocation problem for an investor who can only invest in the stock or a variance swap contract alone.

**Proposition 6** In the one-factor stochastic variance model and for an investor with access only to the money market account and the stock index, the fraction of wealth invested in the stock index is

$$w_t = \frac{1}{\eta} \left( \gamma^S + \rho \sigma_v h_v(u) \right),$$

where the intertemporal hedging demand coefficient $h_v(u)$ solves,

$$h_v'(u) = \alpha h_v(u)^2 + \beta h_v(u) + \zeta, \quad h_v(0) = 0,$$

with

$$\alpha = \frac{\rho^2 (1 - \eta) + \eta \sigma_v^2}{2\eta}, \quad \beta = \frac{1 - \eta}{\eta} \sigma_v \rho \gamma^S - \kappa^p, \quad \zeta = \frac{1 - \eta}{2\eta} (\gamma^S)^2.$$

If the investor can only invest in the money market account and a variance swap contract, the optimal fraction of wealth used as the notional for the variance swap investment is

$$n_t = \frac{1}{\eta \phi_v(T-t)} \left( \frac{\gamma^S}{\sigma_v} + h_v(u) \right),$$

where $h_v(u)$ solves an ordinary differential equation that has the same form as equation (51), but with different coefficients:

$$\alpha = \frac{1}{2\eta} \sigma_v^2, \quad \beta = \frac{1 - \eta}{\eta} \sigma_v \gamma^S - \kappa^p, \quad \zeta = \frac{1 - \eta}{2\eta} (\gamma^S)^2.$$
B. The two-factor stochastic variance model

Under the two-factor stochastic variance model, both the instantaneous variance rate and its central tendency are stochastic. To span these two sources of variance risk, we need to invest in two variance swap contracts with different maturities \( T_1 \) and \( T_2 \). We can solve an analogous utility optimization problem,

\[
J(t, W, v, m) = \sup_{(w_t, n_{1t}, n_{2t})} \mathbb{E}\left( \frac{W_I^1 - \eta}{1 - \eta} \mid W_t = W, v_t = v, m_t = m \right), \quad \eta \neq 1,
\]

with the budget constraint,

\[
dW_t \frac{dW_t}{W_t} = w_t \frac{dS_t}{S_t} + n_{1t} dVS_t(T_1) + n_{2t} dVS_t(T_2) + (1 - w_t) rdt,
\]

where \( w_t \) denotes the fraction of wealth invested in the stock index and \( (n_{1t}, n_{2t}) \) denote the fractions of wealth used as the notional amounts for the two variance swap contracts.

**Proposition 7** Under the two-factor variance risk model, the optimal fractions of wealth invested in the stock index and the two variance swap contracts are:

\[
w_t = \frac{1}{\eta} \left( \frac{\gamma^v}{\sqrt{1 - \rho^2}} \right),
\]

\[
n_{1t} = \frac{1}{\eta D} \left[ \left( \frac{\gamma^v}{\sigma_v \sqrt{1 - \rho^2}} + h_v(u) \right) \phi_m(T_2 - t) - \left( \frac{\gamma^m}{\sigma_m} + h_m(u) \right) \phi_v(T_2 - t) \right],
\]

\[
n_{2t} = \frac{1}{\eta D} \left[ - \left( \frac{\gamma^v}{\sigma_v \sqrt{1 - \rho^2}} + h_v(u) \right) \phi_m(T_1 - t) + \left( \frac{\gamma^m}{\sigma_m} + h_m(u) \right) \phi_v(T_1 - t) \right],
\]

with \( D = \phi_v(T_1 - t) \phi_m(T_2 - t) - \phi_v(T_2 - t) \phi_m(T_1 - t) \). The intertemporal hedging demand coefficients \( h_v(u) \) and \( h_m(u) \) solve a set of ordinary differential equations.

Even in the presence of two variance risk factors, investing in the variance swap contracts can still eliminate the intertemporal hedging demand in the stock index, further illustrating that variance swap contracts are the simplest and most direct contracts for spanning variance risks.

At short investment horizons, the optimal allocations in the two variance swap contracts depend on the market prices of the independent variance rate risk \( (\gamma^v) \) and the central tendency risk \( (\gamma^m) \), as well as the exposures of the two contracts to the two sources of risk, \( \phi_v(T_i - t) \) and \( \phi_m(T_i - t) \), with \( i = 1, 2 \). If we
assume $T_1 < T_2$ such that the first contract is a short-term variance swap contract and the second is a long-term contract, we have $\phi_v(T_1 - t) > \phi_v(T_2 - t)$ and $\phi_m(T_1 - t) < \phi_m(T_2 - t)$. The common denominator $D$ in (57) is a positive quantity. Thus, investment in the long-term contract depends more on the market price of the central tendency factor, and investment in the short-term contract depends more on the market price of the independent variance rate risk.

When the second variance swap contract has a sufficiently long maturity, its loading on the central tendency factor dominates its loading on the instantaneous variance rate, $\phi_m(T_2 - t) \gg \phi_v(T_2 - t)$. In this case, investment in the first variance swap contract is mainly determined by the market price of the independent variance rate risk $\gamma^c$. Analogously, when the first variance swap contract has a sufficiently short maturity, its loading on the instantaneous variance rate dominates its loading on the central tendency factor, $\phi_v(T_1 - t) \gg \phi_m(T_1 - t)$. In this case, investment in the second variance swap contract is mainly determined by the market price of the central tendency risk $\gamma^m$. Negative market prices on the two sources of risks lead to short positions in both variance swap contracts.

When the maturity separation between the two contracts is moderate, optimal allocations to the two variance swap contracts also depend on the relative magnitude of the two market prices $\gamma^c$ and $\gamma^m$. For example, our estimate for $\gamma^c$ is much larger than our estimate for $\gamma^m$ in absolute magnitude. In this case, the $\gamma^c$ term can dominate the investment decision, and the optimal allocations can include short positions in the short-term variance swap contract, but long positions in the long-term variance swap contract.

The intertemporal hedging demands for the two variance swap contracts are determined by the two coefficients $h_v(u)$ and $h_m(u)$ for the two variance risk exposures, which start at zero for a myopic investor ($u = 0$). Since the constant terms in the two ordinary differential equations are both negative, the hedging demand coefficients grow increasingly negative as the investment horizon increases.

For completeness, the following proposition presents the optimal allocation results when the investor can only invest in the stock index alone or two variance swap contracts alone.

**Proposition 8** Under the two-factor variance risk model and for an investor with access to the stock index alone, the optimal fraction of wealth invested in the stock is the same as in the one-factor variance risk model.
in (50). If the investor has only access to the variance swap market, the optimal allocation policy is:

\begin{align}
    n_{1t} &= \frac{1}{\eta D} \left[ \left( \frac{\gamma^m}{\sigma_v} + h_v(u) \right) \phi_m(T_2 - t) - \left( \frac{\gamma^m}{\sigma_m} + h_m(u) \right) \phi_v(T_2 - t) \right], \\
    n_{2t} &= \frac{1}{\eta D} \left[ -\left( \frac{\gamma^v}{\sigma_v} + h_v(u) \right) \phi_m(T_1 - t) + \left( \frac{\gamma^m}{\sigma_m} + h_m(u) \right) \phi_v(T_1 - t) \right],
\end{align}

with \( D = \phi_v(T_1 - t) \phi_m(T_2 - t) - \phi_v(T_2 - t) \phi_m(T_1 - t) \). The intertemporal hedging demand coefficients \( h_v(u) \) and \( h_m(u) \) solve a set of ordinary differential equations.

### VI. Empirical Analysis of Variance Swap Investments

In this section, we combine the theoretical results on optimal allocations in Section V with the parameter estimates on the variance risk dynamics in Section IV to analyze the optimal allocations to the stock index and the variance swap contracts. We also perform historical analysis on the investment performance of different investment strategies.

#### A. Optimal allocation weights

We focus our empirical analysis on the optimal allocation under the two-factor variance risk dynamics. Section IV has estimated the variance risk dynamics during different subsample periods. To compute the optimal allocation weights, we also need to estimate the index return dynamics. Through an Euler approximation of the dynamics, we have:

\begin{align}
    ER_{t+1} &= \left( \gamma^S - \frac{1}{2} \right) v_t \Delta t + \sqrt{v_t} \Delta t \epsilon^S_{t+1}, \\
    v_{t+1} &= \left( \kappa_v m_t - \kappa_v^p v_t \right) \Delta t + \sigma_v \sqrt{v_t} \Delta t \epsilon_v^v, \\
    m_{t+1} &= \kappa_m^p \left( \Delta t \epsilon^m_{t+1} \right) \Delta t + \sigma_m \sqrt{m_t} \Delta t \epsilon^m_{t+1},
\end{align}

where \( ER_{t+1} \) denotes the weekly log excess return on the S&P 500 index, \( \Delta t = 7/365 \) denotes the weekly sampling interval, and \((\epsilon^S_{t+1}, \epsilon_v^v, \epsilon^m_{t+1})\) denotes three standard normal variables with \( \text{cov}(\epsilon^S_{t+1}, \epsilon_v^v) = \rho \) and with \( \epsilon^m_{t+1} \) independent of the other two normal variables. We adjust the index level for dividend payments and define the excess return against the corresponding one-week US dollar LIBOR rate. Holding fixed the previous estimates on the parameters that govern the variance dynamics, we can estimate \( \gamma^S \) and \( \rho \) with a
simple maximum likelihood method by regarding the extracted time series on \((v_t, m_t)\) as observables. The conditional likelihood for each set of observation is given by,

\[
l_t(y_{t+1}, \Theta) = -\frac{1}{2} \left[ \ln |\Sigma_t| + (y_{t+1} - \mu_t)^\top (\Sigma_t)^{-1} (y_{t+1} - \mu_t) \right],
\]

with

\[
y_{t+1} = \begin{bmatrix} E R_{t+1} \\ v_{t+1} \\ m_{t+1} \end{bmatrix}, \quad \Sigma_t = \begin{bmatrix} v_t & v_t \sigma_t \rho & 0 \\ v_t \sigma_t \rho & \sigma_t^2 v_t & 0 \\ 0 & 0 & \sigma_m^2 m_t \end{bmatrix} \Delta t, \quad \mu_t = \begin{bmatrix} (\gamma^d - \frac{1}{2}) v_t \\ (\kappa_m m_t - \kappa_m^p v_t) \\ \kappa_m^p (\theta_m - m_t) \end{bmatrix} \Delta t.
\]

Table \ref{table6} reports the estimates on \(\gamma^d\) and \(\rho\) during different sample periods. For the full sample period, the market price of return risk \((\gamma^d)\) is estimated at 1.6886 and the correlation \((\rho)\) is estimated at \(-0.7463\). From the estimates, we also compute the market price of the independent variance risk \(\gamma^z = (\gamma^r - \rho \gamma^d) / \sqrt{1 - \rho^2}\) and report the value in the last row. For the full sample period, the market price of the independent variance risk is strongly negative at \(-22.7105\). When we re-estimate the dynamics for two subsamples, we obtain a higher market price of return risk estimate but a slightly lower correlation estimate for the first subsample than for the second subsample. Accordingly, the market price of the independent variance risk becomes larger during the second sample period.

Table \ref{table6} about here.

The market price of variance risk has been found highly negative in several studies, e.g., Bakshi and Kapadia (2003a), Bondarenko (2004), and ?. A new stream of literature tries to rationalize the magnitude of the risk premium based on biased beliefs and Peso-problems (Bondarenko (2003)), path-dependent preferences (Bates (2001)), ambiguity aversion (Liu, Pan, and Wang (2005)), net-buying and demand pressure (Bollen and Whaley (2004), Gärleanu, Pedersen, and Poteshman (2007)), short-selling constraints (Isaenko (2007)), and margin requirements (Santa-Clara and Saretto (2007)). In this paper, we do not attempt to rationalize the magnitude of the variance risk premium. Instead, we study how an investor alters her asset allocation decision to benefit from the risk premium by including variance swap contracts into her investment portfolios.

Given the parameter estimates, we compute the allocation weights to the stock index and the variance swap contracts. The key determinants of the allocation weights are the market prices of the three sources of risks: the return risk \((\gamma^d)\), the independent variance risk \((\gamma^r)\), and the central tendency risk \((\gamma^m)\). Intuitively, the market
price of the return risk determines the position in the stock index, the market price of the independent variance risk determines the position in the short-term variance swap contract, and the market price of the central tendency determines the allocation to the long-term variance swap contract. Through cross-correlations in the risky assets, the market price of variance risk also enters the allocation in the stock index, and the market prices on the two sources of variance risks both enter the investments in the two variance swap contracts, the relative degree of which is determined by the maturity differences.

To study how the allocation weights vary with the market price estimates, we fix an investment horizon of two months and we take the full-sample parameter estimates in Tables 3 and 6 as our benchmarks. We vary the market prices of the three sources of risks ($\gamma_S$, $\gamma_z$, and $\gamma_m$) around their respective estimates and investigate how the allocation weights change with the varying market prices. In our calculation, we choose a high relative risk aversion ($\eta = 200$) to counter the high negative market price of variance risk so that the investments are not overly leveraged.

Figure 3 plots the optimal portfolio weights on the stock index ($w$) as a function of the market prices of return risk ($\gamma^S$) and independent variance risk ($\gamma^z$). The left graph plots the optimal allocation in the absence of variance swap contracts in the decision. In this case, the fraction of wealth invested in the stock index is mainly determined by the market price of return risk. The positive market price of return risk ($\gamma^S$) generates long positions in the stock index. The fraction of wealth invested in the stock index increases as the market price of return risk $\gamma^S$ increases. The market price of variance risk affects the intertemporal hedging demand through its impact on $\kappa^p_v = \kappa_v - (\rho \gamma^S + \sqrt{1 - \rho^2} \gamma^z) \sigma_v$. The negative market price of variance risk makes the hedging demand more negative and hence reduces the investment in the stock index. Nevertheless, as Chacko and Viceira (2005) have observed, the intertemporal hedging demand on the stock index induced by stochastic volatility is small. Therefore, when the stock index is the only risky asset in the investment decision, its allocation is mainly determined by the market price of return risk.

The dependence structure changes dramatically when the variance swap contracts become available to the investor. The right graph of Figure 3 plots the optimal fraction of wealth invested in the stock index when two variance swap contracts (at two-month and two-year maturities) are also part of the investment portfolio. In this case, both the market price of return risk ($\gamma^S$) and the market price of variance risk ($\gamma^z$) enter the investment decision in the stock index. With zero market price on the variance risk, it remains optimal to take
long positions in the stock index and to increase the long position with increasing market price of the return risk. However, our estimate of the market price of variance risk is strongly negative. The strongly negative market price of variance risk makes it optimal to take short positions in the stock index. The short position in the stock index increases as the market price of variance risk $\gamma$ becomes more negative.

Figure 4 shows the dependence of the optimal variance swap investments on the market prices of the two sources of variance risks. Signing variance swap contracts has zero initiation costs. We measure the investment by the notional value as fractions of the total wealth ($n$). Under the two-factor variance risk model, any two variance swap contracts of different maturities can span the two sources of variance risks. We choose the two variance swap contracts at two-month and two-year maturities in the left graph, and at six-month and one-year maturities in the right graph. Since the market price estimate of the instantaneous variance risk ($\gamma$) is much more negative at around $-20$ than the market price estimate of the central tendency factor ($\gamma''$) at less than $-1$, it is generally optimal to take short positions in the short-term variance swap contract and long positions in the long-term variance swap contract. Hence, in Figure 4, the surface on top in each graph represents the optimal allocation weights to the long-term variance swap contract, and the surface below represents the optimal allocation weights to the short-term variance swap contract. The distance between the two surfaces increases as the difference between the market prices of the two sources of risks increases.

Figure 4 shows the dependence of the optimal variance swap investments on the market prices of the two sources of variance risks. Signing variance swap contracts has zero initiation costs. We measure the investment by the notional value as fractions of the total wealth ($n$). Under the two-factor variance risk model, any two variance swap contracts of different maturities can span the two sources of variance risks. We choose the two variance swap contracts at two-month and two-year maturities in the left graph, and at six-month and one-year maturities in the right graph. Since the market price estimate of the instantaneous variance risk ($\gamma$) is much more negative at around $-20$ than the market price estimate of the central tendency factor ($\gamma''$) at less than $-1$, it is generally optimal to take short positions in the short-term variance swap contract and long positions in the long-term variance swap contract. Hence, in Figure 4, the surface on top in each graph represents the optimal allocation weights to the long-term variance swap contract, and the surface below represents the optimal allocation weights to the short-term variance swap contract. The distance between the two surfaces increases as the difference between the market prices of the two sources of risks increases.

When the market price of the instantaneous variance risk becomes less negative than the market price of the central tendency factor, the two surfaces start to cross with each other and it becomes optimal to take short positions in the long-term contract and long positions in the short-term contract. With the market prices fixed, comparing the two graphs shows that the maturity differences between the two variance swap contracts also affect the distance between the two surfaces. The distance is larger when the maturity difference between the two contracts is smaller. Therefore, in practice, it is often appropriate to choose the two maturities to be wide apart to reduce the absolute positions (and hence leverage) in the investments.

Compared with the traditional stock-only portfolio, the optimal allocation decision in the presence of variance swap contracts reacts strongly to the highly negative variance risk premium. The investor reaps the large variance risk premium by taking short positions in the short-term variance swap contract. To hedge part of the variance risk, the investor goes long on the long-term variance swap contract and short on the stock index. The long position in the long-term variance swap contract helps to dampen the effect of persistent increases in the volatility level. Furthermore, given the large negative correlation between index returns and
return variance, a short position in the stock index serves as an additional hedge against adverse movements in the short-term variance factor. This hedge is particularly effective against volatility spikes during sharp market downturns.

B. Historical investment performance

To gauge the historical performance of different investment strategies, we perform the following investment exercise. First, we compute the optimal portfolio weights for three investment strategies using parameter estimates from the first half of the sample from January 10, 1996 to June 27, 2001. The three strategies are:

- \( S_1 \): Invest in the stock index and two variance swap contracts.
- \( S_2 \): Invest in two variance swap contracts only.
- \( S_3 \): Invest in the stock index only.

In all three strategies, we have access to the money market account to balance out the investments. We assume that cash saved in the money market account makes a riskfree return given by the corresponding US dollar LIBOR rate. For the first two strategies \( S_1 \) and \( S_2 \), we choose the two variance swap contracts that have the largest maturity difference at two-month and two-year maturities. As we have shown in the previous subsection, choosing swap contracts with wider maturity spreads lead to smaller absolute positions in both contracts.

Given the fixed model parameters, the optimal allocation weights as fractions of the total wealth are fixed. For all three strategies, we choose an investment horizon of two months. For the first two strategies that involve variance swap investments, we use a high relative risk aversion of \( \eta = 200 \) to counteract with the highly negative variance risk premium. Under these conditions, the portfolio for the first strategy \( S_1 \) includes a short position in the stock of \( w = -0.1061 \), a short position in the two-month variance swap of \( n_1 = -0.6717 \), and a long position in the two-year variance swap of \( n_2 = 0.1397 \). The portfolio for strategy \( S_2 \) includes a short position in the two-month variance swap contract of \( n_1 = -0.3513 \) and a long position in the two-year variance swap contract of \( n_2 = 0.0644 \). For the stock index only strategy \( S_3 \), since the market price of the return risk is relatively low, we use a lower relative risk aversion of \( \eta = 3 \) to generate an allocation in the stock index at \( w = 0.7265 \). The relative risk aversion coefficient controls the size and hence financial leverage of the risky investment, but it has little effect on normalized performance measures such as the Sharpe ratio.
Chacko and Viceira (2005) argue that the presence of stochastic volatility has only a small effect on the asset allocation decision. They are correct when the investor can only invest in the stock index. In our third strategy $S_3$ with the stock index as the only available risky asset, the fraction of wealth invested in the index is 0.7265, of which 0.7129 is from the myopic demand and only 0.0136 is from the intertemporal hedging demand induced by the presence of stochastic volatility. However, the picture changes drastically when the investor has access to variance swap contracts. In this case, the investor can use the variance swap contracts to gain access to the large variance risk premium. Without stochastic volatility and variance risk premium, there is no need to invest in the variance swap contract and the allocation to the stock index is positive given the positive return risk premium. With stochastic volatility and highly negative variance risk premium, it becomes optimal to take short positions in a short-term variance swap contract and also short positions in the stock index. With access to variance swap contracts, the presence of stochastic volatility and variance risk premium alter the investment decision drastically.

For each strategy, we perform an investment exercise starting on January 10, 1996 with an investment horizon of two months. After two months, we liquidate the portfolio at the available market prices, calculate the corresponding portfolio returns, and then start a new investment. This exercise generates a non-overlapping time series of two-month returns for each strategy. To make full use of the data series, we also repeat this investment exercise starting at the eight different Wednesdays during the first two months of the sample. We compare the statistics of the eight non-overlapping return series generated from different starting dates.

The return calculation on the stock index and the money market account is straightforward. The first two strategies $S_1$ and $S_2$ also include investments in two variance swap contracts. The first variance swap contract has a two-month maturity and hence expires at the end of each two-month investment horizon. The profit and loss can be calculated based on the difference between the annualized realized variance over the two month horizon and the variance swap rate at the start of the exercise. Multiplying the variance difference by the dollar notional amount invested in the contract generates the dollar profit and loss,

$$PL_1 = N_1 (RV_{t,T_1} - VS_t(T_1)),$$

where $RV_{t,T_1}$ denotes the annualized realized variance during the two-month period $[t, T_1]$, with $t$ being the investment time and $T_1$ being the end of the investment horizon. Given the fixed allocation fraction ($n_1$) in terms of the total wealth, the dollar notional invested at time $t$ is $N_1 = n_1 W_t$, where $W_t$ denotes the wealth level at time $t$. 

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The second variance swap contract has a two-year maturity. The profit and loss at the end of the two-month investment horizon can be computed as,

\[
\text{PL}_2 = N_2 \left( \omega \text{RV}_{T_1} + (1 - \omega) \text{VS}_{T_1}(T_2) - \text{VS}_{T_1}(T_2) \right),
\]

where \( \omega = (T_1 - t)/(T_2 - t) \) denotes the fraction of time passed over the original maturity of the swap contract and \( \text{VS}_{T_1}(T_2) \) is the time-\( T_1 \) variance swap rate that expires at \( T_2 \). In this case, the profit and loss comes from two sources: one is the realization of return variance over the past two months, the other is the new variance swap rate at the same expiry date \( \text{VS}_{T_1}(T_2) \). Since the variance swap rates are quoted at fixed time to maturities, we perform piece-wise linear total variance interpolation on the time-\( T_1 \) variance swap rate term structure to obtain the value for \( \text{VS}_{T_1}(T_2) \).

For each strategy, we repeat the exercise eight times with different starting dates to generate eight non-overlapping time series of two-month returns. We compute the summary statistics on the excess returns for each series and report the average statistics over the eight time series for each strategy in Table 7. Both the mean excess return and the standard deviations are reported in annualized percentages. For the first strategy \( S_1 \) that includes both the stock index and variance swaps, the mean excess return over the whole sample period averages at 5.153%, the standard deviation averages at 3.705%, and the Newey-West serial dependence adjusted standard deviation averages larger at 3.886%. The annualized Sharpe ratio, which is computed as the mean excess return over the Newey-West standard deviation, averages at 1.461. The excess return shows large excess kurtosis and positive skewness.

For the variance swap only strategy \( S_2 \), the mean excess return over the whole sample period averages at 2.922%, the standard deviation averages at 2.263%, and the annualized Sharpe ratio averages at 1.376, lower than the average Sharpe ratio from \( S_1 \). For the stock index only strategy \( S_3 \), the mean excess return averages lower at 2.842%, but the standard deviation averages much higher at 10.13%. As a result, the Sharpe ratio of 0.325 is much lower than that from the other two strategies.

The Sharpe ratio difference between the stock only strategy \( S_3 \) and the other two strategies shows the benefit of exploiting the large variance risk premia. The Sharpe ratio difference between \( S_1 \) and \( S_2 \) further shows that it is beneficial to use the stock index to hedge against the variance risk.
Since the portfolio weights are computed based on the parameter estimates from the first half of the data sample, returns during the first half of the sample represent in-sample returns and returns during the second half are out of sample. In panels B and C, we separately compute the summary statistics for the in-sample and the out-of-sample excess returns. For all three strategies, both the mean excess returns and the standard deviations are lower during the more recent out-of-sample period. The Sharpe ratio estimates are also lower. Nevertheless, the performance rankings among the three strategies remain the same during both the in-sample and the out-of-sample periods.

Figure 5 plots the cumulative wealth paths for the three strategies. Each graph contains one strategy. The eight solid lines in each graph represent the cumulative wealth paths of the eight exercises with different starting dates. For each exercise, we normalize the starting wealth at one dollar and compound the returns accordingly. For comparison, we also plot in the dashed line the ups and downs of the S&P 500 index. For ease of comparison, we scale the starting point of the index level to one. In each graph, we also use a dash-dotted vertical line to separate the in-sample from the out-of-sample period.

For each strategy, the different starting dates generate similar behaviors. Strategies S₁ and S₂ show similar cumulative wealth paths behaviors in that the wealth keeps increasing regardless of the ups and downs of the stock index. By contrast, the stock index only investment (S₃) sees the cumulative wealth going up and down with the fluctuation of the index level.

VII. Concluding Remarks

Using more than a decade worth of variance swap quotes on the S&P 500 index across five fixed time to maturities, we design and estimate affine models of variance risk dynamics to capture the historical behavior of the term structure of variance swap rates. We find that two stochastic variance risk factors are needed to explain the term structure variation of the variance swap rates, with one factor controlling the instantaneous variance rate while the other controlling the central tendency of the variance rate movements. The variance rate factor is much more transient than the central tendency factor under both the risk-neutral and the statistical measures, thus generating different loading patterns across the term structure from the two risk factors and different autocorrelation patterns for variance swap rates of different maturities. We also find that the market
prices of both variance risk factors are negative, but the absolute magnitude of the market price is much higher for the instantaneous variance rate risk than for the central tendency factor.

Embedding variance swaps into an optimal investment strategy, we find that, with variance swap contracts available to span the variance risk, an investor drastically changes her asset allocation decision. The intertemporal hedging demand is completely removed from the stock investment. Moreover, given the large and negative estimate for the market price of variance risk, it becomes optimal for the investor to take short position in both a short-term variance swap contract and the stock index while taking a long position in a long-term variance swap contract. A historical investment exercise shows that incorporating both the stock index and the variance swap contracts in the investment portfolio markedly improves the investment performance, both in sample and out of sample.

Compared to traditional mean variance analysis on primary securities, the modern financial industry has recognized the important impacts that stock price jumps and stochastic variance can have on an investor’s welfare. Accordingly, derivative securities such as options and variance swaps have been developed to span risks along these two dimensions. To simplify the problem and to gain a clear picture of their separate effects, the academic literature either assumes constant volatility and focuses on how to choose options at different strikes to span the random jump risk in the stock price (Carr and Madan (2001)), or assumes purely continuous dynamics (or jumps of fixed size) and focuses on how to choose options to span the stochastic variance risk (Liu and Pan (2003)). We contribute to the latter literature by showing that using variance swap contracts across different maturities represents a more direct way of identifying variance risk dynamics and spanning the variance risks. Integrating these two dimensions can be a challenging but interesting direction for future research.

Yet another line for future research is an integrated study of index options, variance swap term structures, and options on index return volatilities such as realized variances and VIX futures. While these contracts are all linked to the stock index dynamics, they have different degrees of informativeness about the different dimensions of the index dynamics. The index options implied volatility smiles across different strikes, especially at short maturities, are the most informative about the jump structure of the return innovation. The variance swap quotes across a wide range of maturities reveal little about the return innovation structure, but are informative about the multi-dimensional transition dynamics of the index return variance. Finally, options on volatilities across different strikes are the most informative about the martingale component of the variance risk dynamics, including whether the variance risk variation is driven by a diffusion and/or a jump process.
and how the variation depends on the variance risk levels. An integrated analysis of these different contracts can provide a complete picture on the stock index dynamics.
References


### TABLE 1
Summary Statistics of Variance Swap Rates

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Mean</th>
<th>Std</th>
<th>Skew</th>
<th>Kurt</th>
<th>Auto</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>20.804</td>
<td>6.779</td>
<td>0.787</td>
<td>0.794</td>
<td>0.945</td>
</tr>
<tr>
<td>3</td>
<td>20.892</td>
<td>6.495</td>
<td>0.716</td>
<td>0.672</td>
<td>0.961</td>
</tr>
<tr>
<td>6</td>
<td>21.489</td>
<td>6.301</td>
<td>0.745</td>
<td>0.863</td>
<td>0.972</td>
</tr>
<tr>
<td>12</td>
<td>22.258</td>
<td>6.061</td>
<td>0.607</td>
<td>0.164</td>
<td>0.979</td>
</tr>
<tr>
<td>24</td>
<td>22.866</td>
<td>5.909</td>
<td>0.556</td>
<td>−0.218</td>
<td>0.983</td>
</tr>
</tbody>
</table>

Entries report the mean, standard deviation (Std), skewness (Skew), excess kurtosis (Kurt) and weekly autocorrelation (Auto) of the variance swap rates quotes (in volatility percentage points) on the S&P 500 index at different maturities. Data are weekly (every Wednesday) from January 10, 1996, to March 30, 2007 (586 observations for each series).

### TABLE 2
Summary Statistics of the Pricing Errors on the Variance Swap Rates

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Mean</th>
<th>RMSE</th>
<th>Auto</th>
<th>Max</th>
<th>$R^2$</th>
<th>Mean</th>
<th>RMSE</th>
<th>Auto</th>
<th>Max</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>−0.15</td>
<td>1.77</td>
<td>0.82</td>
<td>7.86</td>
<td>93.19</td>
<td>0.27</td>
<td>0.80</td>
<td>0.62</td>
<td>4.20</td>
<td>98.77</td>
</tr>
<tr>
<td>3</td>
<td>−0.20</td>
<td>1.13</td>
<td>0.89</td>
<td>4.49</td>
<td>97.05</td>
<td>−0.00</td>
<td>0.00</td>
<td>0.35</td>
<td>0.01</td>
<td>100.00</td>
</tr>
<tr>
<td>6</td>
<td>0.00</td>
<td>0.00</td>
<td>0.44</td>
<td>0.00</td>
<td>100.00</td>
<td>−0.09</td>
<td>0.40</td>
<td>0.75</td>
<td>2.59</td>
<td>99.61</td>
</tr>
<tr>
<td>12</td>
<td>0.04</td>
<td>1.08</td>
<td>0.90</td>
<td>3.88</td>
<td>96.82</td>
<td>0.00</td>
<td>0.00</td>
<td>0.18</td>
<td>0.00</td>
<td>100.00</td>
</tr>
<tr>
<td>24</td>
<td>−0.60</td>
<td>1.77</td>
<td>0.95</td>
<td>5.05</td>
<td>92.01</td>
<td>−0.05</td>
<td>0.49</td>
<td>0.68</td>
<td>3.39</td>
<td>99.32</td>
</tr>
<tr>
<td>Average</td>
<td>−0.18</td>
<td>1.15</td>
<td>0.80</td>
<td>4.25</td>
<td>95.81</td>
<td>0.03</td>
<td>0.34</td>
<td>0.52</td>
<td>2.04</td>
<td>99.54</td>
</tr>
</tbody>
</table>

Likelihood −5793.4 −3318.4

Entries report the summary statistics of the model pricing errors on the variance swap rates, including the sample average (Mean), root mean squared error (RMSE), weekly autocorrelation (Auto), maximum absolute error (Max), and explained percentage variation ($R^2$), defined as one minus the variance of the pricing error to the variance of the original swap rate quotes. The pricing errors are defined as the difference between the variance swap rate quotes and the corresponding model-implied values, both in volatility percentage points. The last row reports the maximized log likelihood values for the two models.
### TABLE 3
Parameter Estimates of Affine Stochastic Variance Models

<table>
<thead>
<tr>
<th>$X_t$</th>
<th>$\kappa$</th>
<th>$\theta$</th>
<th>$\sigma$</th>
<th>$\gamma$</th>
<th>$\kappa^p$</th>
<th>$\theta^p$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>A: One-factor variance risk model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$v_t$</td>
<td>0.1547</td>
<td>0.1220</td>
<td>0.2550</td>
<td>-17.0141</td>
<td>4.4929</td>
<td>0.0042</td>
</tr>
<tr>
<td></td>
<td>(13.81)</td>
<td>(32.55)</td>
<td>(47.47)</td>
<td>(43.06)</td>
<td>(71.42)</td>
<td>(21.52)</td>
</tr>
<tr>
<td><strong>B: Two-factor variance risk model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$v_t$</td>
<td>4.3730</td>
<td>-</td>
<td>0.4221</td>
<td>-16.3746</td>
<td>11.2851</td>
<td>0.0158</td>
</tr>
<tr>
<td></td>
<td>(38.72)</td>
<td></td>
<td>(44.10)</td>
<td>(37.70)</td>
<td>(51.96)</td>
<td>(3.71)</td>
</tr>
<tr>
<td>$m_t$</td>
<td>0.1022</td>
<td>0.0838</td>
<td>0.1581</td>
<td>-0.6844</td>
<td>0.2104</td>
<td>0.0407</td>
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<tr>
<td></td>
<td>(9.30)</td>
<td>(24.31)</td>
<td>(47.64)</td>
<td>(1.84)</td>
<td>(3.40)</td>
<td>(3.68)</td>
</tr>
</tbody>
</table>

Entries report the maximum likelihood parameter estimates and the absolute magnitudes of t-values (in parentheses) of the one-factor (Panel A) and the two-factor (Panel B) affine stochastic variance models. The estimation employs weekly data on variance swap rates at maturities of two, three, six, 12, and 24 months and ex post realized variances at maturities of seven, 30, 60, 90, and 150 days. The sample is from January 10, 1996, to March 28, 2007, 586 observations for each series.
TABLE 4
Subsample Parameter Estimates of Affine Stochastic Variance Models

<table>
<thead>
<tr>
<th></th>
<th>$X_t$</th>
<th>$\kappa$</th>
<th>$\theta$</th>
<th>$\sigma$</th>
<th>$\gamma$</th>
<th>$\kappa^p$</th>
<th>$\theta^p$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A:</strong> One-factor variance risk model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Subsample period: 1996-2001</td>
<td>$v_t$</td>
<td>0.0885</td>
<td>0.2442</td>
<td>0.3056</td>
<td>-17.0344</td>
<td>5.3097</td>
<td>0.0041</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3.51)</td>
<td>(4.47)</td>
<td>(29.93)</td>
<td>(27.07)</td>
<td>(44.08)</td>
<td>(13.23)</td>
</tr>
<tr>
<td>Subsample period: 2001-2007</td>
<td>$v_t$</td>
<td>0.7281</td>
<td>0.0495</td>
<td>0.2316</td>
<td>-12.1940</td>
<td>3.5517</td>
<td>0.0101</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(21.82)</td>
<td>(87.41)</td>
<td>(32.66)</td>
<td>(25.54)</td>
<td>(33.96)</td>
<td>(28.19)</td>
</tr>
<tr>
<td><strong>B:</strong> Two-factor variance risk model</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Subsample period: 1996-2001</td>
<td>$v_t$</td>
<td>3.3945</td>
<td>-</td>
<td>0.4635</td>
<td>-16.2472</td>
<td>10.9250</td>
<td>0.0153</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(16.53)</td>
<td>(27.27)</td>
<td>(25.12)</td>
<td>(29.84)</td>
<td>(3.88)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$m_t$</td>
<td>0.1857</td>
<td>0.0715</td>
<td>0.2086</td>
<td>-0.4029</td>
<td>0.2697</td>
<td>0.0492</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(7.61)</td>
<td>(15.14)</td>
<td>(26.98)</td>
<td>(1.11)</td>
<td>(3.17)</td>
<td>(3.84)</td>
</tr>
<tr>
<td>Subsample period: 2001-2007</td>
<td>$v_t$</td>
<td>5.3789</td>
<td>-</td>
<td>0.3476</td>
<td>-15.6418</td>
<td>10.8160</td>
<td>0.0239</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(33.83)</td>
<td>(27.86)</td>
<td>(19.44)</td>
<td>(33.91)</td>
<td>(0.46)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$m_t$</td>
<td>0.0010</td>
<td>4.2187</td>
<td>0.1058</td>
<td>-0.8444</td>
<td>0.0903</td>
<td>0.0480</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.06)</td>
<td>(0.06)</td>
<td>(24.42)</td>
<td>(0.46)</td>
<td>(0.46)</td>
<td>(0.46)</td>
</tr>
</tbody>
</table>

Entries report the maximum likelihood parameter estimates and the absolute magnitudes of t-values (in parentheses) of the one-factor (Panel A) and the two-factor (Panel B) affine stochastic variance models over two subsample periods. The first subsample is from January 10, 1996 to June 27, 2001, 286 weekly observation for each series. The second subsample is from July 4, 2001 to March 28, 2007, 300 weekly observations for each series. The estimation employs weekly data on variance swap rates at maturities of two, three, six, 12, and 24 months and ex post realized variances at maturities of seven, 30, 60, 90, and 150 days.
We estimate the model during the first subsample from January 10, 1996 to June 27, 2001, and use the model parameters to price variance swap rates out of sample from July 4, 2001 to March 28, 2007. The pricing errors are defined as the difference between the variance swap rate quotes and the corresponding model-implied values, both in volatility percentage points. Entries report the summary statistics of the model pricing errors on the variance swap rates, including the sample average (Mean), root mean squared error (RMSE), weekly autocorrelation (Auto), maximum absolute error (Max), and explained percentage variation ($R^2$), defined as one minus the variance of the pricing error to the variance of the original swap rate quotes.

### TABLE 5
Out-of-sample Pricing Performance on the Variance Swap Rates

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Mean</th>
<th>RMSE</th>
<th>Auto</th>
<th>Max</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.52</td>
<td>1.59</td>
<td>0.84</td>
<td>8.09</td>
<td>95.29</td>
</tr>
<tr>
<td>3</td>
<td>0.27</td>
<td>0.86</td>
<td>0.89</td>
<td>3.96</td>
<td>98.33</td>
</tr>
<tr>
<td>6</td>
<td>0.00</td>
<td>0.00</td>
<td>0.36</td>
<td>0.00</td>
<td>100.00</td>
</tr>
<tr>
<td>12</td>
<td>-0.62</td>
<td>0.96</td>
<td>0.91</td>
<td>3.95</td>
<td>97.57</td>
</tr>
<tr>
<td>24</td>
<td>-2.13</td>
<td>2.32</td>
<td>0.90</td>
<td>5.87</td>
<td>95.60</td>
</tr>
<tr>
<td>Average</td>
<td>-0.39</td>
<td>1.15</td>
<td>0.78</td>
<td>4.37</td>
<td>97.36</td>
</tr>
</tbody>
</table>

Entries report the maximum likelihood parameter estimates and the absolute magnitudes of t-values (in parentheses) of the market price of return risk ($\gamma^S$) and the instantaneous correlation ($\rho$) between the return risk and the instantaneous variance rate risk. The estimation is performed on weekly returns, with the extracted variance risk factors ($v_t, m_t$) treated as observables. We perform the estimation at both two subsamples and the full sample. The first subsample is from January 10, 1996 to June 27, 2001, 286 weekly observation for each series. The second subsample is from July 4, 2001 to March 28, 2007, 300 weekly observations for each series. The last row reports the market price of independent risk $\gamma^C = (\gamma^e - \rho \gamma^S) / \sqrt{1 - \rho^2}$, where the corresponding estimates for $\gamma^e$ are from Tables 3 and 4.

### TABLE 6
Market Price of Return Risk and Correlation

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma^S$</td>
<td>2.1386</td>
<td>0.8125</td>
<td>1.6886</td>
</tr>
<tr>
<td></td>
<td>(0.99)</td>
<td>(1.33)</td>
<td>(1.68)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.7339</td>
<td>-0.7924</td>
<td>-0.7463</td>
</tr>
<tr>
<td></td>
<td>(11.29)</td>
<td>(0.27)</td>
<td>(14.11)</td>
</tr>
<tr>
<td>$\gamma^C$</td>
<td>-21.6082</td>
<td>-24.5863</td>
<td>-22.7105</td>
</tr>
</tbody>
</table>

Entries report the maximum likelihood parameter estimates and the absolute magnitudes of t-values (in parentheses) of the market price of return risk ($\gamma^S$) and the instantaneous correlation ($\rho$) between the return risk and the instantaneous variance rate risk. The estimation is performed on weekly returns, with the extracted variance risk factors ($v_t, m_t$) treated as observables. We perform the estimation at both two subsamples and the full sample. The first subsample is from January 10, 1996 to June 27, 2001, 286 weekly observation for each series. The second subsample is from July 4, 2001 to March 28, 2007, 300 weekly observations for each series. The last row reports the market price of independent risk $\gamma^C = (\gamma^e - \rho \gamma^S) / \sqrt{1 - \rho^2}$, where the corresponding estimates for $\gamma^e$ are from Tables 3 and 4.
TABLE 7
Summary Statistics of Investment Strategies

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Mean</th>
<th>Std</th>
<th>Skew</th>
<th>Kurt</th>
<th>Newey</th>
<th>Sharpe</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A. Whole sample period: January 1996 - March 2007</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_1$</td>
<td>5.153</td>
<td>3.705</td>
<td>1.210</td>
<td>7.552</td>
<td>3.886</td>
<td>1.461</td>
</tr>
<tr>
<td>$S_2$</td>
<td>2.922</td>
<td>2.263</td>
<td>1.083</td>
<td>7.978</td>
<td>2.338</td>
<td>1.376</td>
</tr>
<tr>
<td>$S_3$</td>
<td>2.842</td>
<td>10.130</td>
<td>-0.175</td>
<td>0.678</td>
<td>9.601</td>
<td>0.325</td>
</tr>
<tr>
<td>B. In-sample period: January 1996 - June 2001</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_1$</td>
<td>6.369</td>
<td>4.331</td>
<td>1.365</td>
<td>5.796</td>
<td>4.141</td>
<td>1.716</td>
</tr>
<tr>
<td>$S_2$</td>
<td>3.694</td>
<td>2.611</td>
<td>1.405</td>
<td>6.418</td>
<td>2.492</td>
<td>1.664</td>
</tr>
<tr>
<td>$S_3$</td>
<td>4.575</td>
<td>10.958</td>
<td>-0.035</td>
<td>0.383</td>
<td>9.881</td>
<td>0.509</td>
</tr>
<tr>
<td>$S_1$</td>
<td>3.996</td>
<td>2.878</td>
<td>0.015</td>
<td>4.512</td>
<td>3.429</td>
<td>1.287</td>
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<tr>
<td>$S_2$</td>
<td>2.188</td>
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<td>-0.214</td>
<td>5.721</td>
<td>2.015</td>
<td>1.203</td>
</tr>
<tr>
<td>$S_3$</td>
<td>1.193</td>
<td>9.361</td>
<td>-0.516</td>
<td>1.005</td>
<td>9.262</td>
<td>0.142</td>
</tr>
</tbody>
</table>

Entries report the sample averages of the mean, standard deviation (Std), skewness (Skew), excess kurtosis (Kurt), and Newey-West serial dependence adjusted standard deviation (Newey) of the excess returns from three different investment strategies. The mean and standard deviations are in annualized percentages. The last column reports the annualized Sharpe ratio defined as the mean excess return divided by the Newey-West standard deviation. The first strategy $S_1$ invests in both the stock index and two variance swap contracts at two-month and two-year maturities, respectively. The second strategy $S_2$ invests in the two variance swaps only. The third strategy $S_3$ invests in the stock index only. All strategies have an investment horizon of two months. We start each strategy at eight different Wednesdays during the first two months of our sample to generate eight time series of non-overlapping two-month excess returns. The statistics in the table represent the average statistics of the eight series for each strategy. We report the full-sample statistics in Panel A, the in-sample statistics from January 1996 to June 2001 in Panel B, and the out-of-sample statistics for the remaining sample period in Panel C.
FIGURE 1
Time Series and Term Structure of the Return Variance Swap Rates

The left graph plots the time series of the variance swap rates in volatility percentage points at three selected time to maturities: two months (solid line), six months (dashed line), and 24 months (dotted line). The right graph plots representative variance swap rate term structures at different dates.
The left graph plots the contemporaneous response of the variance swap term structure to unit shocks on the instantaneous variance rate $v_t$ (solid line) and the central tendency factor $m_t$ (dashed line). The dotted line represents the remaining loading on the common unconditional risk-neutral mean ($\theta_m$) of the variance rate and the central tendency. The right graph plots the mean term structure of the variance swap rate in volatility percentage points. The circles are sample averages of the data, and the solid line represents values computed from the estimated two-factor variance risk model.
The two graphs plot the optimal fractions of wealth invested in the stock index ($w$) as the market prices of return risk ($\gamma^S$) and the market price of variance risk ($\gamma^z$) vary. The left graph plots the investment without the presence of variance swap contracts. The right graph plots the investment with the presence of two variance swap contracts at two-month and two-year maturities.
The two graphs plot the optimal investment in variance swap contracts (notional in fractions of total wealth, $n$) as a function of the market price of the instantaneous variance risk ($\gamma$) and the market price of the central tendency risk ($\gamma^m$). The left graph plots the investment in two-month and two-year variance swap contracts. The right graph plots the investment in six-month and one-year variance swap contracts.
FIGURE 5
Cumulative Wealth Paths of Different Investment Strategies

Solid lines denote the cumulative wealth paths for the stock index-variance swap strategy $S_1$ (top graph), the variance swap only strategy $S_2$ (middle graph), and stock index only strategy $S_3$ (bottom graph). The eight solid lines in each graph represent the results from eight different starting dates. The dashed line in each graph represents the scaled S&P 500 index level. The dash-dotted vertical line indicates the beginning of the out-of-sample period.