Nonlinear development of the R-mode instability and the maximum rotation rate of neutron stars

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Abstract: We describe how the nonlinear development of the R-mode instability of neutron stars influences spin up to millisecond periods via accretion. When nearly resonant interactions of the $l = m = 2$ R-mode with pairs of "daughter modes" are included, the R-mode saturates at the lowest amplitude which leads to significant excitation of a pair of modes. The lower bound for this threshold amplitude is proportional to the damping rate of the particular daughter modes that are excited parametrically. We show that if dissipation occurs in a very thin boundary layer at the crust-core boundary, the R-mode saturation amplitude is too large for angular momentum gain from accretion to overcome loss to gravitational radiation. We find that lower dissipation is required to explain spin up to frequencies much higher than 300 Hz. We conjecture that if the transition from the fluid core to the crystalline crust occurs over a distance much longer than 1 cm, then a sharp viscous boundary layer fails to form. In this case, damping is due to shear viscosity dissipation integrated over the entire star. We estimate the lowest parametric instability threshold from first principles. The resulting saturation amplitude is low enough to permit spin up to higher frequencies. The requirement to allow continued spin up imposes an upper bound to the frequencies attained via accretion that plausibly may be about 750 Hz. Within this framework, the R-mode is unstable for all millisecond pulsars, whether accreting or not.

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NONLINEAR DEVELOPMENT OF THE R MODE INSTABILITY AND THE MAXIMUM ROTATION RATE OF NEUTRON STARS

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ABSTRACT

We describe how the nonlinear development of the R mode instability of neutron stars influences spin up to millisecond periods via accretion. Our arguments are based on nearly-resonant interactions of the R mode with pairs of “daughter modes.” The amplitude of the R mode saturates at the lowest value for which parametric instability leads to significant excitation of a particular pair of daughters. The lower bound on this limiting amplitude is proportional to the damping rate of the daughter modes that are excited parametrically. Based on this picture, we show that if modes damp because of dissipation in a very thin boundary layer at the crust-core boundary then spin up to frequencies larger than about 300 Hz does not occur. Within this conventional scenario the R mode saturates at an amplitude that is too large for angular momentum gain from accretion to overcome gravitational loss to gravitational radiation. We conclude that lower dissipation is required for spin up to frequencies much higher than 300 Hz. We conjecture that if the transition from the fluid core to the crystalline crust occurs over a distance much longer than ~ 1 cm then a sharp viscous boundary layer fails to form. In this case, damping is due to shear viscosity dissipation integrated over the entire star; the rate is slower than if a viscous boundary layer forms. We use statistical arguments and scaling relations to estimate the lowest parametric instability threshold from first principles. The resulting saturation amplitudes are low enough to permit spin up to higher frequencies. Further, we show that the requirement that the lowest parametric instability amplitude be small enough to allow continued spin up imposes an upper bound to the frequencies that may be attained via accretion that may plausibly be about 750 Hz. Within this framework, the R mode is unstable for all millisecond pulsars, whether accreting or not.

1. THE R MODE INSTABILITY VERSUS THE SPIN UP LINE

The fastest spinning radio pulsar has a rotational frequency $\nu = 716$ Hz ($\text{Hessels et al. 2006}$) and 39 have been detected with $\nu > 400$ Hz ($\text{Manchester et al. 2005}$). See ATNF Pulsar Catalogue at http://www.atnf.csiro.au/research/pulsar/psrcat/

Moreover, there are 14 pulsars in X-ray binaries with inferred $\nu > 400$ Hz, but none demonstrated convincingly to be faster than 620 Hz ($\text{Watts 2012}$; $\text{Patruno & Watts 2012}$). $\text{Chakrabarty has argued that the population of neutron star spins cuts off sharply around 730 Hz (Chakrabarty 2005, 2008, 2012).}$ In the standard picture, millisecond pulsars are thought to be spun up via accretion ($\text{Alpar et al. 1982}$) and the $P - \dot{P}$ diagram for radiopulsars is consistent with the idea that accreting neutron stars reach spin equilibrium ($\text{e.g. Bildsten et al. 1997}$) in that there appear to be no neutron stars outside the boundary set by the “spinup line” ($\text{e.g. Arzoumanian et al. 1999}$)

$$\nu_{eq} = \frac{\omega_s}{2\pi} \sqrt{\frac{GM}{R_{acc}^3}} \approx \frac{760 \text{Hz} \omega_s M_9^{3/7} M_{1.4}^{5/7}}{\eta_{acc}^{3/2} \mu_{26}^{6/7}} \leq 716 \text{Hz}$$

where the accretion radius is $R_{acc} \approx 20h_{acc}^{4/7} M_9^{1/7} M_{1.4}^{1/7} \text{km}$, $\dot{M} = 10^{-9} M_9 M_{\odot} \text{yr}^{-1}$ is the mass accretion rate, $M = 1.4 M_1 M_{\odot}$ is the stellar mass, $\mu = 10^{26} \mu_{26} \text{G cm}^3$ is the stellar magnetic moment, and $\omega_s \approx 1$ and $\eta_{acc} \approx 1$ are parameters that are determined by the magnetohydrodynamics of disk accretion. However, there is no particular reason for there to be a spin frequency cutoff as low as 730 Hz: although there are exceptions, for most representative equations of state of dense nuclear matter accretion can spin up a neutron star from $M = 1.4 M_1 M_{\odot}$ and $\nu = 0$ to $\nu \approx 1000 - 1500$ Hz before instability ensues ($\text{e.g. Cook et al. 1994}$).

The R mode instability can prevent neutron stars from spinning up to such high frequencies that either dynamical instability or viscosity-driven secular instability occurs. The R mode instability, which is driven slowly by gravitational radiation but stabilized by viscosity, is reviewed briefly below. However, in the presence of a crust-core boundary layer the R mode prevents spin-up too efficiently: instability sets in at $\nu \approx 300$ Hz (see Eq. (1)), which is too low to allow for observed frequencies of up to 716 Hz since nonlinear effects prevent substantial spin-up while the star is unstable. We call this "The Spin-Up Problem". Phenomenologically, the absence of millisecond pulsars outside the spin up line up to at least 660 Hz and the inference that some LMXBs are spinning
faster than 500 Hz suggest that spin up via slow equilibrium accretion is responsible for the highest spin frequencies observed, but the R mode instability appears to suppress spin up beyond about 300 Hz.

2. R MODE DYNAMICS AND THE SPIN UP PROBLEM

The inertial modes of a rotating star may be thought of as zero frequency “gauge modes” of a nonrotating star (δρ = −∇·(rξ) = 0) that acquire frequencies |ω| < 2Ω in a rotating star to O(1) (Papaloizou & Pringle 1978; Friedman & Schutz 1978; Lee & Strohmayer 1996; Schenk et al. 2002): the l = |m| R modes are a subset that are axial to O(1) for Newtonian stars, and have (rotating frame) frequencies |ω| = 2Ω/(l + 1). (Relativistic modifications have been discussed by Lockitch et al. (2003).) For the R modes, O(1) displacement fields can be expressed in terms of a single (magnetic) vector spherical harmonic; decompositions of the other inertial modes are more complicated even at O(1), generally involving a sum of vector spherical harmonics up to a maximum ℓ (e.g. Lockitch & Friedman 1999; Yoshida & Lee 2000a; 2000b; Lockitch et al. 2003; 2003a).

The R-modes of rotating neutron stars are destabilized by the emission of gravitational radiation because their rotating and inertial frame frequencies have opposite signs, implying that the rotating frame energy increases as the star radiates energy and angular momentum in the inertial frame (Chandrasekhar 1970; Friedman & Schutz 1978; Lindblom et al. 1998; Bildsten & Ushomirsky 1999; Andersson & Bildsten 1999). For the most unstable ℓ = m = 2 R mode the instability grows at a rate (numerical coefficients are for the Newtonian N = 1 polytrope)

$$\gamma_{GR} \approx \frac{M_{1.4} R^{1.4} \nu_{S}^{6}}{2900 s} \approx 1.6 \times 10^{-7} M_{1.4} R_{10}^{1.4} \nu_{S}^{5} \omega_{R}$$

where M = 1.4M⊙ and R = 10R⊙ km are the stellar mass and radius, and ν = 500ν/ωS Hz: ωR = 2Ω/3 = 4πν/3 (Andersson 1998; Friedman & Morsink 1998; Lindblom et al. 1998; Bildsten & Andersson 1999).

The first basic principle is a consequence of the relatively small value of |ω| below 716 Hz. Heating proceeds until balanced by cooling, whereupon recent calculations of the shear viscosity in the core of a neutron star with improved treatment of dynamical screening change η by a factor of a few and also alter its temperature scaling compared to “traditional” expressions (Cutler et al. 1990; Andersson et al. 2003; Shishkin & Yakovlev 2008). While these refinements alter Eq. (4) slightly, the weak viscosity dependence, K₄/11, still implies that νS is well below 716 Hz.

The small value of νS would not be problematic if spinup were to continue largely unabated within the unstable regime. However, detailed nonlinear three mode evolutions using representative input physics do not support this: the stellar frequency changes little Bondarescu et al. 2007.

Two basic principles emerged from our work on multimode (Schenk et al. 2002; Brink et al. 2004, 2004b) and three mode (Bondarescu et al. 2005, 2006) nonlinear models for saturation of the R mode instability:

1. The R mode amplitude does not grow beyond the first or second lowest parametric instability threshold amplitude |CR|_{PTT} for interactions with a pair of daughter modes. |CR|_{PTT} depends on the detuning δω = ωR − ω₂ − ω₃ between the R mode (ωR) and daughter (ω₂,₃) frequencies, the damping rates of the daughters (γ₂,₃) and the three mode coupling κ,

$$|CR|_{PTT}^{2} = \frac{9}{4(\kappa \Omega)^{2}} \left[\frac{\gamma R^{2}}{4} \left(\frac{\delta \omega}{\gamma_{2}^{2} + \gamma_{3}^{2}}\right)^{2}\right]$$

Parity and triangle selection rules for the interactions require that the principal mode numbers of the daughters satisfy the constraint n₂ = n₃ ± 1, and for large n₁ we expect the viscous damping rates of the daughter modes to have similar values γᵢ ≈ γD (i = 2, 3); we have also defined 4ω₂ω₃κ² = ωR²κ² = 4Ω²κ²/9.

2. Dissipation of the multitude of daughter modes heats the star a rate

$$H_{R} = 2MR^{2} \gamma_{GR} |CR|^{2}$$.

Heating proceeds until balanced by cooling, whereupon evolution tends to settle onto curves in the ν − T plane where thermal balance is maintained.

The first basic principle is a consequence of the relatively sparse couplings of the R mode to the sea of daughters...
The star crosses into the unstable regime at a spin saturation of the R mode instability prevents spin.

2.5

2.5

2

3

4

Gravitational radiation spins the star down at a regime. ally the same as if there were no spin-up in the unstable heating and cooling balance, so the end point is virtually ceases, the star spins down along the curve where heating, and the star heats up to \( T > T_S \). Because the cooling, which is dominated by Cooper pair formation at \( T_S \approx 1 \) (e.g. Flowers et al. 1976; Yakovlev et al. 1999; Kolomeitsev & Voskresensky 2008; Page et al. 2009), accelerates rapidly, fast heating of the star halts eventually. Subsequently, the star evolves relatively slowly.

The physical reasons that the evolution is constrained so tightly can be understood from considering three different characteristic R mode amplitudes:

1. From Eq. (5), the lowest parametric instability threshold is \( |C_R|_{\text{ PIT}} \geq 3 \gamma D / 2 \Omega \). For damping in a shearing boundary layer this inequality implies

\[
|C_R|_{\text{ PIT}} \geq \frac{3 \gamma D}{2 \kappa_D \Omega} \approx \frac{3 S_D^2 (dE_D / dr)_b}{4 \kappa_D E_D} \approx 1.3 \times 10^{-6} S_D^2 a_4^{1/2} (RdE_D / dr)_b \kappa_D T_S \nu_0^{1/2} R_{10} E_D,
\]

Here \( S_D < 1 \) is the fractional velocity jump across the crust-core boundary for daughter mode \( D \), and \( \ell = (\eta / \Omega)^{1/2} \) is the boundary layer thickness. At principal mode numbers \( n_D \geq \omega_R / c_t \approx 30 \nu_0 R_{10} / c_t S \), where \( c_t \) is the transverse shear mode speed in the crust, we expect \( S_D \approx 1 \); for lower \( n_D, S_D \ll 1 \). The fractional velocity jump for the \( n = 3 \) R-mode is \( S_R \approx 0.1 \). The lowest \( |C_R|_{\text{ PIT}} \) arises from modes with \( \delta \omega \lesssim \gamma_D \), which are likeliest at large \( n \). Explicit evaluation for modes of an incompressible star as well as WKB calculations for a compressible star imply that \((R/E_D)(dE_D/dr)_b\) is independent of \( n_D \) for \( n_D \gg 1 \) (see Appendix A.4). For incompressible stars, calculations by Brink (2005) show that \( |\kappa_D| \lesssim 1 \) is insensitive to \( n_D \), although larger values are likelier at large \( n_D \); moreover, \( \kappa_D \) is independent of \( \Omega \) (see Schenk et al. 2002; Arras et al. 2003). Thus, the lower bound in Eq. (4) is roughly independent of \( n_D \). In fact, because the lowest expected \( \delta \omega \) decreases with \( n \) while \( \gamma_D \) is roughly independent of \( n_D \), ultimately \( |C_R|_{\text{ PIT}} \approx 3 \gamma D / 2 \kappa_D \Omega \) is the lowest parametric instability threshold for damping in a shearing boundary layer.

2. Gravitational radiation spins the star down at a rate \( -J_{\text{GR}} = 6MR^2 \Omega_\gamma C_R |C_R|^2 \). If \( -J_{\text{GR}} < J_{\text{acc}} \), the star spins up until \( J_{\text{GR}} = -J_{\text{acc}} \). Otherwise, if \( -J_{\text{GR}} > J_{\text{acc}} \), the star will spin down and re-enter the region in which the R-mode is stable. In spin equilibrium, \( J = I \Omega_{eq} \propto IM^{5/7} \mu^{-6/7} \),

Fig. 1.— Schematic \( \nu - T \) evolutions are shown when the dissipation is dominated by (a) boundary layer viscosity (b) shear viscosity. It can be seen that in the latter case the star spins to much higher frequencies. The final spin frequency where the accretion torque is balanced by gravitational emission is given by Eq. (19). The CFS instability curve occurs when the gravitational driving equals the viscous damping of the R-mode.
where \( \Omega_{eq} = 2\pi\nu_{eq} \) and \( \nu_{eq} \) is given by Eq. (1). As accretion proceeds, the magnetic moment decreases and \( \nu_{eq} \) increases (e.g., Shibazaki et al. 1989; Zhang & Kojima 2000). A good approximation is \( \mu \propto (\Delta M)^{-\beta} \), where \( \Delta M \) is the total mass accreted, and so \( J \propto IM^{5/7}(\Delta M)^{6/7} \). Shibazaki et al. (1989) originally suggested \( \beta = 1 \), but Zhang & Kojima (2000) advocate \( \beta = 7/4 \) until \( \mu \) "bottoms out" at \( \nu_{eq} \approx 1 \) (see also Wang et al. 2011). In general, the accretion torque is defined to be \( \dot{J}_{acc} = M\dot{d}J/dM \), which can be written as \( \dot{J}_{acc} = J\sigma J M / \sigma J = 6\beta M / 7\Delta M + 5/7 + d\ln I / d\ln M \); spin up is faster before \( \mu \) bottoms out and \( \beta \to 0 \) and slows as mass accretes and \( \nu \) increases. As a simple model, we adopt \( \dot{J}_{acc} / J = \gamma_{acc}(v/\nu)^{2s} \); for numerical estimates, we take \( \gamma_{acc}(v/\nu)^{2s} = 10^{-5} y^{-1} \gamma_{acc,5} v_{500}^{2s} \), where \( s \approx 1/3 \) and \( s \approx 0.7 \) respectively before and after \( \mu \) bottoms out. The parameter \( \gamma_{acc,8} \) is different for each accreting neutron star. With this simplified model, \( \dot{J}_{GR} = -\dot{J}_{acc} \) at an R mode amplitude

\[
|C_R|J \approx \frac{2.1 \times 10^{-7} (I_{0,3}\gamma_{acc,8})^{1/2}}{\nu_{500}^{3/4} M_{1.4}^{1/4} R_{10}^{2}}
\]

where the moment of inertia of the star is \( I = 0.3 I_{0,3} M R^2 \). For numerical estimates, we shall use \( s = 1/3 \), since most of the spin up occurs in this regime. Comparing Eq. (7) with Eq. (4), we see that \( |C_R|J \lesssim 0.1|C_R|_{ITT} \), which means that \( J_{GR} > -J_{acc} \), and spin up is prevented.

3. The amplitude at which heating by the R mode balances heating via accretion is \( H_{acc} = \epsilon_{acc} M c^2 \) with \( \epsilon_{acc} = 10^{-5} \epsilon_{acc,3} \) (Brown 2000), is

\[
|C_R|H \approx \frac{5.5 \times 10^{-7} (I_{0,3}\gamma_{acc,8})^{1/2}}{\nu_{500}^{3/4} M_{1.4}^{1/4} R_{10}^{2}}
\]

For \( |C_R|_{ITT} > |C_R|H \), heating by the R mode dominates. Comparing Eq. (3) with Eq. (4) implies that heating by the R mode is more important than accretional heating for damping in a shearing boundary layer.

4. A fourth important amplitude comes from equating gravitational radiation spindown with \( \dot{J}_B = \eta_{mag} h^2 \Omega^3 / 3c^4 \), the rate of pulsar spindown,

\[
|C_R|B \approx \frac{1.5 \times 10^{-8} \mu_{26}^{1/2} \nu_{260}^{-1/2}}{\nu_{500}^{3/4} M_{1.4}^{1/4} R_{10}^{2}}
\]

This is relevant to the evolution after accretion ceases. If \( |C_R|_{ITT} > |C_R|B \) then spindown via gravitational radiation is faster than pulsar spindown. For damping in a shearing boundary layer, \( |C_R|_{ITT} > |C_R|J \) and so accretion spin-up is limited to about 300 Hz, which is inconsistent with observations of pulsars spinning up to 716 Hz. In Fig. 1 the left panel illustrates a typical evolution sequence in this case. Because \( |C_R|_{ITT} > |C_R|J \), the evolutionary track in the \( \nu - T \) plane is a rather tight cycle that is confined to a small range of frequencies \( \lesssim \nu_S \), the frequency at which the mode first becomes unstable, given in Eq. (4).

For spin up substantially beyond 300 Hz to be possible, \( |C_R|_{ITT} \) must remain below \( |C_R|_J \) up to frequencies well above \( \nu_S \). Eq. (7) shows that \( |C_R|_{ITT} \gtrsim 3\gamma_{D}/2\kappa \rho_8 \), but that for damping within a viscous shearing boundary layer \( \gamma_D \) is too large to allow significant spin up. However, Eq. (7) also suggests that lower \( \gamma_D \) would permit prolonged spin up. In 3, we examine what happens if a thin viscous shearing boundary layer does not form near the core-crust boundary, so that \( \gamma_D \) is due to shear viscosity damping distributed over the entire star. In Fig. 1, the right panel illustrates the sort of evolution that would become possible in this case. As can be seen from the figure, prolonged spin up is possible, but even in this case there is a maximum attainable spin frequency. We conjecture – but do not prove – that if the transition from core (super)fluid to crustal solid is gradual enough, a thin viscous shearing boundary layer does not form.

After accretion ceases, \( |C_R|_{ITT} \) must be small enough that gravitational radiation spin down timescales are \( \gtrsim 10^7 \) years in order for fast spin to be maintained on spin down timescales characteristic of the fastest millisecond pulsars. Otherwise, the spin up neutron star would simply spin down too rapidly via gravitational radiation, roughly retracing its steps down to the stable region, leaving a millisecond pulsar with spin frequency \( \nu_S \approx 300 \) Hz. Under these circumstances, heating due to the R mode will be less important than in Bondarescu et al. (2007) during spin up, but may still dominate over accretional heating. If these conditions can be met, the star heats modestly after becoming unstable, but continues to spin up by a significant factor. After accretion stops, the star cools and spins down within the unstable regime, but \( |C_R|_{ITT} \) is too small to accelerate spin down substantially, and the spin-up neutron star can become a long-lived millisecond pulsar. Such a scenario is unfavorable for gravitational radiation detection, but essential for understanding how pulsars spin up to frequencies \( \gtrsim 500 \) Hz.

3. CONDITIONS FOR A SUCCESSFUL OUTCOME OF SPIN-UP

In 2 we demonstrated that the R mode instability frustrates prolonged spin-up if dissipation is due to a viscous shearing boundary layer at the boundary between the stellar core and crust. The left panel of Fig. 1 illustrates the problem graphically. If, for some reason, such a thin viscous boundary layer does not arise, then damping would be due to shear viscosity damping distributed over the entire star. This would result in lower damping rates, and lower \( |C_R|_{ITT} \) during spin up, but may still dominate over accretional heating. If these conditions can be met, the star heats modestly after becoming unstable, but continues to spin up by a significant factor. After accretion stops, the star cools and spins down within the unstable regime, but \( |C_R|_{ITT} \) is too small to accelerate spin down substantially, and the spin-up neutron star can become a long-lived millisecond pulsar. Such a scenario is unfavorable for gravitational radiation detection, but essential for understanding how pulsars spin up to frequencies \( \gtrsim 500 \) Hz.
η/ℓ² ∝ ℓ⁻¹.

Suppose that instead of an abrupt transition, the shear modulus of the star grows from zero near \( r_b \) to its value at the inner edge of the crust over a radial zone of thickness \( ∆r \gg ℓ \). Then we expect the velocity jump to occur over this relatively extended region. We do not present a rigorous calculation of how this happens, since there are many uncertainties, principally in how the shear modulus grows within the transition region. The crude toy model developed in Appendix B illustrates the salient features. Fig. 2 shows how the displacement field evolves smoothly across the layer in this toy model. The dissipation associated with this smooth transition is \( E \sim 4πη_0^2 Δr (Δν)^2 / √{r} \sim ω (Δr)^2 ∝ (Δr)^{-1} \). (For the specific case of the toy model computed in Appendix B the constant of proportionality is about two.) For \( Δr \gg ℓ \), the dissipation rate associated with a smooth transition layer is much smaller than the dissipation rate that would arise in a thin boundary layer. For moderate velocity jumps, such as the relatively small jump associated with the R mode, the extended transition layer contributes relatively little compared with the energy dissipation associated with shear viscosity across the entire star.

In the remainder of this section, we assume that dissipation is due to the distributed effect of shear viscosity. We consider two cases: a “permeable” limit where the daughter modes penetrate into the crust, and an “impermeable” limit where they do not. We estimate the lowest value of \(|C_R|_{\text{ITT}}\) for each of these cases, taking full account of the two factors in Eq. (5). In order to get an estimate, we need scaling relations for \( κ_D \), \( δω \), and \( γ_D \) with \( n_D \). As was mentioned above, explicit calculations by Brink et al. (2004) (see also Brink 2003) indicate that \( κ_D \) does not rise systematically with \( n_D \), the principal mode number of the daughters, although larger values become likelier as \( n_D \) increases. We use statistical arguments for the expected smallest values of \( δω \) as a function of \( n_D \). We use WKB calculations presented in (3,5) plus explicit numerical evaluations (Brink et al. 2004; Brink 2003) for \( γ_D \). The upshot is that \( δω \) tends to decrease with \( n_D \), whereas \( γ_D \) tends to increase, so there is a minimum value of \(|C_R|_{\text{ITT}}\) at large values of \( n_D \). We shall demonstrate that the minimum occurs at \( n_D \gg 100 \), considerably beyond the ranges computed explicitly even for the modes of an incompressible star.

3.1. Permeable Crust

Let us consider the non-rigid case first. The WKB calculation detailed in Appendix A,B implies that

\[
γ_D \simeq \frac{p_D^2 (\eta_0 - \eta_{\text{crust}}) (r_b/R)^2}{R^2 √{1 - (r_b/R)^2}} + \frac{2p_D^2 \eta_{\text{crust}}}{3R^2} \tag{10}
\]

assuming different kinematic viscosities \( \eta_0 \) and \( \eta_{\text{crust}} \) in the core (\( r \leq r_b \)) and crust (\( r_b < r \leq 1 \)), respectively; here \( p_D = √{n_D (n_D + 1)} - |mD| (|mD| + 1) ≃ n_D \). The second term dominates for sufficiently large values of \( n_D \), but since \( \eta_{\text{crust}} \ll \eta_0 \) (Shternin & Yakovlev 2008) for practical purposes almost all of the dissipation occurs in the core. (For uniform \( \eta_0 = \eta_{\text{crust}} \) the second term dominates, and Eq. (10) agrees with results in Brink et al. 2004.) Moreover, comparing Eqs. (157) and (10), with \( \eta_0 \sim \eta_{\text{core}} \), we see that dissipation in the bulk of the star dominates over dissipation in the transition region as long as \( p_D \simeq n_D \gg S_D √{R/ΔR} = 10S_0 √{R/100ΔR} \). Tentatively, we assume that this inequality holds for the daughter modes involved in the lowest \(|C_R|_{\text{ITT}}\); we shall see that this is likely to be true. Thus we adopt \( γ_D = γ_0 n_D^2 \) for estimating the lowest \(|C_R|_{\text{ITT}}\) from the first term in Eq. (10) with \( p_D \simeq n_D \) (as WKB requires)

\[
\frac{γ_0}{Ω} = \frac{η_{\text{core}} (r_b/R)^2}{Ω R^2 √{1 - (r_b/R)^2}} = \frac{5.9 \times 10^{-12} K_4}{ρ_{500} T_8^2 R_{10}^2} \tag{11}
\]

where \( η_{\text{core}} = 10^5 K_4 T_8^{-2} c^2 m^2 s^{-1} = η_b \) and we set \( r_b = 0.9 R \).

The other factor in Eq. (5) is the detuning. The minimum \( δω/Ω \) up to principal quantum number \( n \) is expected to be approximately \( 2\sqrt{S}/N(\lesssim n) \), where \( N(\lesssim n) \approx n^4 \) is the number of couplings to the R mode consistent with selection rules for the transitions (Brink 2005). Since we are seeking an estimate of the lowest \(|C_R|_{\text{ITT}}\) we substitute this into Eq. (5) to get

\[
|C_R|^2 = \frac{9}{4κ_D} \left( \frac{γ_0 n_D^4}{Ω^2} + \frac{72}{R_D^2} \right) \tag{12}
\]

recalling that \( κ_D \) is relatively insensitive to \( n_D \) we minimize the quantity in brackets over \( n_D \) and find the lowest value of the threshold at

\[
n_D = 1.5 \left( \frac{Ω}{γ_0} \right)^{1/6} \simeq 110/√{500} T_8^{1/3} R_{10}^{1/3} K_4^{1/6} \tag{13}
\]

where we have used \( r_b = 0.9 R \) and therefore

\[
|C_R|_{\text{ITT}, \text{min}} = \frac{4.2}{κ_D} \left( \frac{γ_0}{Ω} \right)^{2/3} \simeq 1.4 \times 10^{-7} K_4^{2/3} \tag{14}
\]

Requiring that \(|C_R|_{\text{ITT}} < |C_R|_I\) implies that nonlinear dynamics limits spin up to frequencies

\[
ν \lesssim \left( \frac{590 Hz K_4^{3/8} T_8^{5/2} (J_0, 3, \eta_{\text{core}, 8})^{1/4}}{K_4^{1/4} J_0, 3, 10^{3/8} R_{10}^{1/4}} \right) \tag{15}
\]

where we have used \( s = 1/3 \) to obtain the numerical value. The existence of a maximum spin frequency limit for spin up via accretion is a generic feature of the dynamics: \(|C_R|_{\text{ITT}, \text{min}}\) is determined by a competition between the decrease of the smallest expected detuning \( δω \) and the increase of the dissipation \( γ_D\) with increasing \( n_D \).

Including other physical features that we have neglected here will not do away with this key feature of the dynamics. Two physical features we shall study subsequently are buoyancy and relativistic corrections. Buoyancy shifts mode frequencies, but note that of the R mode (Said 1982, Yoshida & Lec 2000a), and also activates the \( n \neq |m| + 1 \) r modes in the star (Said 1982, Yoshida & Lec 2000). Relativistic corrections also shift mode frequencies (e.g. Lockitch et al. 2000, 2003), and may also generate non-axial contributions to the R mode eigenfunction that permit additional couplings that would be forbidden non-relativistically (see e.g. Lockitch et al. 2003, 2003). Studies that combine buoyancy and relativistic corrections are tricky (Koilam 1998, Koilam & Hosonuma 1999, Boutloukos & Nollert 2007, Passamonti et al. 2008, e.g.)
but detailed calculations seem to support the existence of a mode structure very similar to the Newtonian case (Lockitch et al. 2004, 2001; Lockitch et al. 2003; Pons et al. 2005; Villain et al. 2005). In any event, including both buoyancy and relativistic corrections will not alter the key feature of the network of interacting modes, namely that there exists a dense set of frequencies bounded above and below, which permits an increasing number of near resonances as \( n_p \) increases. Moreover, additional couplings may become possible that would be forbidden otherwise, which could lower the value of the maximum spin rate. The saturation amplitude of the R mode is not an adjustable parameter, but is the value of the maximum spin rate. The saturation amplitude implies nonlinear dynamics is a basic conclusion of this paper. We investigate such effects in detail.

The existence of a maximum frequency dictated by the nonlinear dynamics is a basic conclusion of this paper. The actual value of the maximum frequency depends on the external parameter, \( \gamma_{\text{acc,}8} \), even though the dependence is weak. Each individual neutron star has its own value of \( \gamma_{\text{acc,}8} \), so the maximum spin rate that is attainable is not the same for all neutron stars. We emphasize that our estimate of \(|C_R|_{\text{PTT, min}}\) is the key to determining the value of the maximum spin rate. The saturation amplitude of the R mode is not an adjustable parameter, but is determined by the nonlinear hydrodynamics of the network of interacting modes.

It is reassuring that the value of the maximum frequency in Eq. (19) is close to 700 Hz, but detailed calculations seem to support the existence of a more precise upper bound

\[
\nu \lesssim 950 \text{ Hz} \frac{5/2}{f_\nu^{3/4}} \left( \frac{M_{\text{acc}}}{M_1} R_{10} \right)^{3/11} \equiv \nu_{\text{max}}.
\]

The full solution of Eq. (17) would give a slightly lower value.

Because \( \gamma_D \) is low, spin up begins at a significantly lower frequency than Eq. (14), typically \( \nu_\varphi \approx 100 - 150 \) Hz, so prolonged spin up via accretion is required. Throughout much of this evolution, the R mode plays almost no role because of the strong frequency dependences of \( H_R/H_{\text{acc}} \) and \(|C_R|_{\text{PTT, min}}/|C_R|_{\text{J}}\). Spin up ends either because accretion ceases or because spin equilibrium \(|C_R|_{\text{PTT}} = |C_R|_{\text{J}}\) is achieved. In the former case, spin up proceeds almost as it would if there were no R mode instability. At its maximum spin frequency, a neutron star is in spin balance, with equal and opposite gravitational radiation and accretion torques, and remains in that state until accretion ends. Depending on the detailed evolution, spin equilibrium can occupy a substantial fraction of the time during which a neutron star accretes. In thermal balance, Eq. (17) shows that the neutron star’s internal temperature is an increasing function of frequency, but also depends on \( \dot{M}_9 \), which is different for each accreting neutron star, and \( \kappa_D \). Although we expect similar values of \( \kappa_D \) for different neutron stars, they need not be identical, because \( \dot{M}_D \) is not the same for all neutron stars affected by the R mode instability. Thus, there may be some variability in internal and effective temperatures for neutron stars in the unstable domain. Intermittent accretion is unlikely to affect these conclusions: cooling timescales are \( \sim 100 - 1000 \) years so if the heating rate fluctuates at much shorter timescales the time averaged heating rate is all that matters. Similarly, the detailed time dependent dynamical evolution of the R mode proceeds on timescales that are too short, \( \sim 1/\delta \nu \sim 1/|C_R|_{\text{PTT}} \Omega \), to be important for the secular evolution of spin and internal temperature; whether there are any observable effects of the dynamics is beyond the scope of this paper.

Once accretion ends, the fast rotating neutron star cools and spins down. Because the cooling timescale is short compared with the spin down timescale (\( \gtrsim 10^{8} \) years) at the end of spin up, the neutron star first cools at fixed spin frequency. Cooling ends when \( H_R \) is balanced by cooling. Once this point is reached, the neutron star spins down along the curve given by Eq. (15), and

\[
|C_R|_{\text{PTT, min}} \approx 6.6 \times 10^{-8} \frac{K_{4}^{1/2}}{\nu_{500}^{2/3} M_{1.4}^{3/4} R_{10}^{7/4}} \nu_{\varphi}^{1/8}.
\]

Slow down evolution along this curve is driven by spin down: if there were no change in \( \nu \) the star would remain at a single point in the \( \nu - T \) plane. The total spin down rate is the sum of contributions from gravitational radiation and electromagnetic radiation \( \dot{J} = \dot{J}_{\text{GR}} + \dot{J}_{\text{B}} \). The spin-down rate at the lowest PIT given by Eq. (20) is

\[
- \frac{\dot{J}}{\Omega} = \frac{\nu/\nu_{\varphi}}{|C_{\text{acc,}8}|} \left( \frac{M_{\text{acc}}}{M_1} R_{10} \right)^{3/11} \kappa_D^{1/2} \left( \frac{\nu^{3/4}}{f_\nu} \right) \left( \frac{M_{\text{acc}}}{M_1} R_{10} \right)^{3/11} \frac{\eta_{\varphi}}{10^{8} \nu_{\varphi}^{2/3} \nu_{\text{max}}^{11/2} \frac{3/11}{3/11}} + \frac{\eta_{\varphi}^{11/2}}{3/11} \left( \frac{M_{\text{acc}}}{M_1} R_{10} \right)^{3/22}.
\]

Using Eq. (21) in Eq. (15) implies a more precise upper bound

\[
\nu \lesssim 950 \text{ Hz} \frac{5/2}{f_\nu^{3/4}} \left( \frac{3/11}{3/11} \frac{22}{22} \left( \frac{M_{\text{acc}}}{M_1} R_{10} \right)^{3/22} \equiv \nu_{\text{max}}.
\]
but there is an important difference: Eq. (27) requires $\kappa_\gamma 9^{4/7} K_4^{1/7} \gtrsim 6.6$ for $\nu_{\text{max}} \approx 750 \text{ Hz}$, holding all other parameters fixed. Eq. (28) implies a spin down rate

$$\frac{j}{I} \Omega \approx \left(\frac{\nu}{\nu_{\text{max}}}\right)^4 \gamma_{\text{acc,8}} \frac{\eta_{\text{mag}}^2 \nu_{\text{看电影}}^2}{10^8 y^{2/3} \max\, 8.0 + 14.5 \times 10^3 y^{2/3} \Omega_{10}^4 R_{10}^8}.$$  

(29)

Just as we found for the nonrigid case, we need to cut down the gravitational radiation contribution in order to be consistent with pulsar data: requiring a spin down timescale due to gravitational radiation $\gtrsim 10^7 \text{ years}$ near $\nu_{\text{max}}$ implies $\gamma_{\text{acc,8}} \lesssim 0.05$, and for $\nu_{\text{max}} \approx 750 \text{ Hz}$ we would then require $\kappa_\gamma 9^{4/7} K_4^{1/7} \gtrsim 45$, which is a more stringent constraint than we found in (3.1).

4. CONCLUSIONS

Our examination of the nonlinear dynamics of rotational modes of a neutron star suggests that in the conventional picture, where modes damp in a thin viscous boundary layer, spin up beyond about 300 Hz is not possible: not only is the frequency at which the R mode first destabilizes about 300 Hz (see Eq. (14) but the R mode amplitude saturates at a level large enough that gravitational radiation spindown prevents significant spin up subsequently (see Eq. (7)). Thus, we consider what happens if a thin shearing boundary layer cannot form. We conjecture that this may happen if the transition between core and crust occurs in a region thicker than $\sim 1 - 2 \text{ cm}$, and justify that assumption partially with the toy model in Appendix B. If a thin boundary layer does not form, damping of all modes is dominated by the distributed effects of shear viscosity throughout the star, which leads naturally to a lower R mode saturation amplitude.

Using scaling relations found by a combination of exact calculations for incompressible stars (Brink et al. 2004, Brink 2005), statistical arguments (Brink 2005) and approximate WKB calculations (Appendix A) we estimate the lowest parametric instability threshold $|C_R|_{\text{PIT, min}}$ analytically for coupling of the R mode to pairs of daughter modes. We find that the daughter modes for which this occurs are at principal mode quantum $n_D \approx 100$, typically; this is beyond the range for which explicit calculations exist, even for incompressible stars. We stress that the lowest parametric instability threshold sets the amplitude at which the R mode amplitude saturates during evolution of a network of rotational modes of a neutron star (Brink et al. 2005). Thus, our estimate of $|C_R|_{\text{PIT, min}}$ represents a first principles calculation of the saturation amplitude. We stress that this is not an adjustable parameter, but rather arises from the nonlinear hydrodynamics. Although it may seem counterintuitive, when there are many nearly resonant modes, as is the case for a rotating neutron star, nonlinear effects become important at low amplitude, and lead to saturation.

With the lower $|C_R|_{\text{PIT, min}}$ that arises when shear viscosity dominates the damping, prolonged spin up to frequencies above 500 Hz is possible. A basic conclusion is that the nonlinear development of the R mode instability naturally gives rise to an upper spin frequency limit. This bound arises from the requirement that $|C_R|_{\text{PIT, min}}$ be smaller than $|C_R|_j$, the amplitude where gravitational radiation spin down balances accretion spin up. Eq. (19) and Eq. (27) provide rough estimates for the maximum
spin frequency $\nu_{\text{max}}$ that can be attained under the assumption that the crust is permeable and impermeable to small scale modes, respectively. It is plausible that $\nu_{\text{max}} \simeq 750$ Hz, but consistency with observations of millisecond pulsars requires relatively strong (but not outrageously strong) coupling $\kappa_D$; smaller values are allowed for the permeable case, which may argue in its favor. This suggests that nonlinear interactions among the rotational modes of a neutron star may naturally imply a maximum spin frequency below what one might expect from dynamical instabilities of the star. This conclusion is compatible with studies that suggest that LMXBs are not spun up beyond about 730 Hz (Chakrabarty 2003, 2008, 2012) as well as the fact that the fastest spinning neutron star yet discovered spins at 716 Hz.

A second conclusion of our study is that after accretion ceases, fast spinning millisecond pulsars cool until they reach a balance between neutrino cooling and heating that results from the energy sent to smaller scale modes from the unstable R mode. The result is slow evolution along a curve in the $\nu - T$ plane, Eq. (18). Spindown timescales are sufficiently long that once spun up a millisecond pulsar ought to remain close to the upper part of this curve. This means that millisecond pulsars remain stuck in the domain where the R mode is unstable, and are therefore radiating gravitational radiation. However, the emission rate is very low, and strain amplitudes at Earth are correspondingly low, $\sim 10^{-26}/D_{\text{kpc}}t_{\text{sd},9}$, for a source at $D = D_{\text{kpc}}$ kpc with a spin down time $10^9t_{\text{sd},9}$ years. Although gravitational radiation may dominate the spin down, because the accretion spin up rate generally sets torque amplitudes we expect millisecond pulsars to be near the conventional spin up line but possibly slightly above it.

**APPENDIX**

**ASSORTED WKB RESULTS**

**Preliminaries: Coordinates**

The Bryan coordinates $x_{1,2}$ for a mode with $\omega = 2\Omega|\mu| \equiv 2\Omega \cos \theta|\mu| \leq 2\Omega$ are

$$\varpi = \sqrt{x^2 + y^2} = \sqrt{\frac{(1-x_1^2)(1-x_2^2)}{1-\mu^2}} = \frac{\sin \theta_1 \sin \theta_2}{\sqrt{1-\mu^2}}$$

$$z = \frac{x_1 x_2}{|\mu|} \equiv \frac{\cos \theta_1 \cos \theta_2}{|\mu|}$$

$$x_1 \in [\mu, 1], \quad x_2 \in [-|\mu|, |\mu|]$$

$$\theta_1 = \cos^{-1}(x_1) \in [0, \theta|\mu|], \quad \theta_2 = \cos^{-1}(x_2) \in [\theta|\mu|, \pi - \theta|\mu|]. \tag{A1}$$

We use units in which the radius of the star is $R = 1$. The following are useful definitions: with $\theta_\pm = \theta \pm \theta|\mu|$

$$\cos(\theta_2 - \theta_1) = z|\mu| + \varpi \sqrt{1-\mu^2} = r \cos \theta_-$$

$$\cos(\theta_2 + \theta_1) = z|\mu| - \varpi \sqrt{1-\mu^2} = r \cos \theta_+ \tag{A2}$$

For finding mode displacements, we will want derivatives of $\theta_1$ or $\theta_2$ with respect to coordinates. In cylindrical coordinates Eq. (A1) implies

$$\sqrt{1-\mu^2} d\varpi = \cos \theta_1 \sin \theta_2 d\theta_1 + \sin \theta_1 \cos \theta_2 d\theta_2$$

$$|\mu| dz = - \sin \theta_1 \cos \theta_2 d\theta_1 - \cos \theta_1 \sin \theta_2 d\theta_2$$

$$d\theta_1 = \frac{\cos \theta_1 \sin \theta_2 d\varpi \sqrt{1-\mu^2} + \sin \theta_1 \cos \theta_2 dz|\mu|}{\cos^2 \theta_1 \sin^2 \theta_2 - \sin^2 \theta_1 \cos^2 \theta_2}$$

$$d\theta_2 = \frac{\cos \theta_2 \sin \theta_1 d\varpi \sqrt{1-\mu^2} + \sin \theta_2 \cos \theta_1 dz|\mu|}{\cos^2 \theta_2 \sin^2 \theta_1 - \sin^2 \theta_2 \cos^2 \theta_1} \tag{A3}$$

Using Eqs. (A3) we find the area element

$$dA = \varpi d\varpi dz = \frac{\sin \theta_1 \sin \theta_2 (\cos \theta_1^2 - \cos \theta_2^2) d\theta_1 d\theta_2}{|\mu|(1-\mu^2)} = \frac{(x_1^2 - x_2^2)dx_1 dx_2}{|\mu|(1-\mu^2)} \tag{A4}$$

(The integral $\int dA$ over the ranges of $\theta_{1,2}$ or $x_{1,2}$ is 2/3.) The stellar surface $r = 1$ is patched together in the following way:

$$\theta_1 = \theta|\mu| \quad \text{and} \quad \theta_2 = \theta \in [\theta|\mu|, \pi - \theta|\mu|]$$

$$\theta_2 = \theta|\mu| \quad \text{and} \quad \theta_1 = \theta \in [0, \theta|\mu|]$$

$$\theta_2 = \pi - \theta|\mu| \quad \text{and} \quad \theta_1 = \pi - \theta \in [0, \theta|\mu|] \tag{A5}$$

There are special points where $x_1^2 - x_2^2 = 0 = dA$; at these points, $\cos \theta_1 = \pm \cos \theta_2 = |\mu|$.
WKB Approximation to Displacements

From [Arras et al. (2003) §3.2](#) we take the WKB Eulerian enthalpy perturbation to be

\[
\Psi \approx P_{nm}(x_1)P_{nm}(x_2) \exp[i(m\phi + \omega t)] \approx \frac{\cos(p\theta_1 + \alpha_1) \cos(p\theta_2 + \alpha_2) \exp[i(m\phi + \omega t)\sqrt{\rho \sin \theta_1 \sin \theta_2}]}{\sqrt{\rho \sin \theta_1 \sin \theta_2}} \tag{A6}
\]

where \(\rho = \rho(r)\) is the density profile and \(p = \sqrt{n(n+1) - m(m+1)} \approx n\). The first approximation assumes that the density scale height is large compared with characteristic scales on which \(\Psi\) varies. The second approximation is for the associated Legendre functions, and holds at sufficiently large values of \(p\). The values of the phases \(\alpha_i\) depend on the parity of the mode: based on asymptotic properties of the \(P_{nm}(z)\), [Arras et al. (2003)](#) adopted \(\alpha_1 = \alpha_2 = -p\pi/2\) or \(= -(p+1)\pi/2\) even or odd parity, respectively, but

[where](#) the parity of the mode: based on asymptotic properties of the \(P_{nm}(z)\), [Arras et al. (2003)](#) adopted \(\alpha_1 = \alpha_2 = -p\pi/2\) or \(= -(p+1)\pi/2\) even or odd parity, respectively, but

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where

\[\hat{\xi} \cdot \nabla \Psi = \frac{i \hat{z} \times \nabla \Psi}{\mu}
\]

and \(\mu = \sqrt{\rho \sin \theta_1 \sin \theta_2 (1 - \frac{1}{\mu^2})}\) from the equation

\[\nabla \Psi = \frac{d\Psi}{dx_i} = -\sin \theta_i \nabla \theta_i = -\sqrt{1 - x_i^2} \nabla \theta_i \tag{A8}\]

up to an overall normalization factor. With the approximation that the density scale height is large, we do not include derivatives of \(\rho\) in computing the displacements; thus we write

\[
\nabla \xi = \nabla \Psi - \frac{\hat{z} \hat{z} \cdot \nabla \Psi}{\mu^2} + \frac{i \hat{z} \times \nabla \Psi}{\mu} \tag{A7}
\]

For evaluating the derivatives, we use

\[
\nabla P_{nm}(x_i) = \frac{dP_{nm}}{dx_i} \nabla x_i
\]

where \(\nabla \theta_i\) are computed in Eqs. [A3](). If we further invoke the large \(p\) approximation to the associated Legendre polynomials, then we ignore the variation of the \(\sin \theta_i\) factors in computing derivatives; in this approximation

\[
\sqrt{\rho \sin \theta_1 \sin \theta_2 (1 - \frac{1}{\mu^2})} \xi \approx \exp(+i\omega t) \left\{ \nabla \left[ P_{nm}(x_1)P_{nm}(x_2) \exp(im\phi) \right] \right. - \frac{i \hat{z} \hat{z} \cdot \nabla \left[ P_{nm}(x_1)P_{nm}(x_2) \exp(im\phi) \right]}{\mu^2} + \left. \frac{i \hat{z} \times \nabla \left[ P_{nm}(x_1)P_{nm}(x_2) \exp(im\phi) \right]}{\mu} \right\} \tag{A9}
\]

In Eq. (A10) gradients are computed via

\[
\nabla_a [\cos(p\theta_i + \alpha_i)] = -p \nabla_a \sin(p\theta_i + \alpha_i) \tag{A11}
\]

where \(\nabla \theta_i\) are computed from Eqs. [A3](). The components of the displacement are

\[
\left(1 - \frac{1}{\mu^2}\right) \xi = \frac{\partial \Psi}{\partial \omega} + \frac{im \Psi}{\mu \omega} \approx \frac{\partial \Psi}{\partial \omega}
\]

\[
\left(1 - \frac{1}{\mu^2}\right) \xi = \frac{\partial \Psi}{\partial \omega} + \frac{im \Psi}{\mu \omega} \approx \frac{\partial \Psi}{\partial \omega}
\]

\[
\left(1 - \frac{1}{\mu^2}\right) \xi = \frac{im \Psi}{\omega} + \frac{i \partial \Psi}{\mu \omega} \approx \frac{im \Psi}{\omega} \tag{A12}
\]

However, we use the convention that the mode is proportional to \(\exp(+i\omega t)\). [Arras et al. (2003)](#) employed modes \(\propto \exp(-i\omega t)\). The sign of the last term here is opposite to Eq. (29) in [Arras et al. (2003)](#) because of the different sign convention for frequency used here.
where the approximations are valid within the WKB limit. The necessary derivatives are

\[
\frac{\partial \Psi}{\partial \phi} = -\frac{p\sqrt{1-\mu^2}e^{i(m\phi+\omega t)}}{2\sqrt{\rho \sin \theta_1 \sin \theta_2}} \left( \frac{\sin \eta_+ - \sin \eta_-}{\sin \theta_+} \right),
\]

\[
\frac{\partial \Psi}{\partial z} = \frac{p|\mu|e^{i(m\phi+\omega t)}}{2\sqrt{\rho \sin \theta_1 \sin \theta_2}} \left( \frac{\sin \eta_+ + \sin \eta_-}{\sin \theta_+} \right),
\]

(A13)

where \(\eta_\pm = p(\theta_2 \pm \theta_1) + \alpha_2 \pm \alpha_1\) and \(\tilde{\theta}_\pm = \theta_2 \pm \theta_1\).

Define

\[
N = \int d^3 x \rho |\xi|^2;
\]

(A14)

using Eqs. (A12) and (A13) as well as Eq. (A3) we get

\[
N = \frac{\pi |\mu| p^2}{(1 - \mu^2)^2} \int d\theta_1 d\theta_2 \left( \frac{\sin^2 \eta_+ \sin \tilde{\theta}_-}{\sin \theta_+} + \frac{\sin^2 \eta_- \sin \tilde{\theta}_+}{\sin \theta_-} - 2\mu^2 \sin \eta_+ \sin \eta_- \right).\]

(A15)

We replace the rapidly oscillating terms \(\sin^2 \eta_\pm \to \frac{1}{2}\) and \(\sin \eta_- \to 0\). Judiciously substitute \(\sin \tilde{\theta}_\pm = \sin(\tilde{\theta}_+ \pm 2\theta_1) = \sin \theta_\mp \cos 2\theta_1 \pm \cos \theta_\mp \sin 2\theta_1\), with which the integral becomes

\[
N = \frac{\pi p^2 |\mu|}{2(1 - \mu^2)^2} \int_{\theta_1}^{\theta_1} d\theta_1 \int_{\theta_1}^{\pi - \theta_1} d\theta_2 \left[ 2 \cos 2\theta_1 + \sin 2\theta_1 \left( \frac{\cos \tilde{\theta}_- - \cos \tilde{\theta}_+}{\sin \theta_- + \sin \theta_+} \right) \right].
\]

(A16)

The remaining integrals may all be done analytically; the result is

\[
N = \frac{\pi^2 p^2 |\mu|^2}{(1 - \mu^2)^{3/2}}.
\]

(A17)

See Arras et al. (2002), Eq. (47); the exact result for Bryan modes is in Brink et al. (2004), §II.D.

**Damping in a Viscous Boundary Layer**

For evaluating damping via boundary layer viscosity, we will need to compute the surface integral of \(\rho |\xi|^2\). We use Eq. (A2) to write \(\tilde{\theta}_\pm = \theta_2 \pm \theta_1 = \cos^{-1}(r \cos \theta_\pm)\) and therefore

\[
\rho |\xi|^2 \approx \frac{p^2 |\mu|^2}{2(1 - \mu^2) \sin \theta_1 \sin \theta_2} \left\{ \frac{\sin^2[p \cos^{-1}(r \cos \theta_+)] - \pi}{1 - r^2 + r^2 \sin^2 \theta_+} + \frac{\sin^2[p \cos^{-1}(r \cos \theta_-)]}{1 - r^2 + r^2 \sin^2 \theta_-} \right\}.
\]

\[
- \frac{\sigma r}{\pi} \int_0^{\pi} d\theta \sin \left[ p \cos^{-1}(r \cos \theta_+) - \pi \right] \sin \left[ p \cos^{-1}(r \cos \theta_-) \right]
\]

(A18)

where \(\sigma = +1\) for even parity and \(\sigma = -1\) for odd parity, and in the first two terms we used \(\sin^2[p \cos^{-1}(r \cos \theta_\pm)] = 1 - r^2 \cos^2 \theta_\pm = 1 - r^2 + r^2 \sin^2 \theta_\pm\). Using Eq. (A11) to eliminate \(\sin \theta_1 \sin \theta_2\) we get

\[
\frac{r^2}{2N} \int d\Omega |\xi|^2 \approx \frac{r}{2 \pi} \int_0^\pi d\theta \left\{ \frac{\sin^2[p \cos^{-1}(r \cos \theta_+) - \pi]}{1 - r^2 + r^2 \sin^2 \theta_+} + \frac{\sin^2[p \cos^{-1}(r \cos \theta_-)]}{1 - r^2 + r^2 \sin^2 \theta_-} \right\}
\]

(A19)

where \(\sigma = +1\) for even parity and \(\sigma = -1\) for odd parity. In the notation of Eq. (17), \(R(dE_D/dr)_dE_D\) is twice Eq. (A19). In Eq. (A19), the integrands in \{\ldots\} have large values near \(\theta_+ = \pi\) and \(\theta_- = 0\), respectively. Writing \(\theta_+ = \pi + \delta\) and \(\theta_- = \delta\), we get \(1 - r^2 + r^2 \sin^2 \theta_\pm = 1 - r^2 + r^2 \delta^2\), and \(\cos^{-1}(r \cos \theta_\pm) = \pi - \sqrt{1 - r^2 + r^2 \delta^2}\) and \(\cos^{-1}(r \cos \theta_\pm) = \sqrt{1 - r^2 + r^2 \delta^2}\), respectively, near those points; in both cases, then

\[
\frac{\sin^2[p \cos^{-1}(r \cos \theta_+) - \pi]}{1 - r^2 + r^2 \sin^2 \theta_+} \approx \frac{\sin^2[p \sqrt{1 - r^2 + r^2 \delta^2}]}{1 - r^2 + r^2 \delta^2} = \frac{\sin^2[p \sqrt{1 - r^2 + (1 + u^2)^{1/2}}]}{(1 - r^2)(1 + u^2)^{1/2}},
\]

(A20)

where \(u^2 = r^2 \delta^2/(1 - r^2)\). If \(1 - r^2 \ll 1\) we can approximate the integrals by

\[
\frac{1}{r \sqrt{1 - r^2}} \int_{-\infty}^{+\infty} du \frac{\sin^2[p \sqrt{1 - r^2 + (1 + u^2)^{1/2}}]}{1 + u^2} \approx \frac{\pi K(p \sqrt{1 - r^2})}{r \sqrt{1 - r^2}};
\]

(A21)
Since the contribution to the integral from the cross term is smaller, we find that
\[
\frac{r^2}{2N} \int d\Omega p|\xi|^2 \approx \frac{K(p\sqrt{1-r^2})}{\sqrt{1-r^2}}
\]
\[(A23)\]
for \(1 - r^2 \ll 1\). The damping rate from a viscous shearing boundary layer is therefore proportional to \(R(dE_D/dr_b)/2E_D \approx K(p\sqrt{1-r^2})/\sqrt{1-r^2}^2\); at large \(p\), the damping rate is proportional to \(1/2\sqrt{1-r^2}\), which is independent of \(p\).

Shear Viscosity Damping within \(r = r_b\)

For computing shear viscosity damping, we need the square of the shear tensor:
\[
\sigma_{ab} = \frac{\partial \xi_a}{\partial x_b} + \frac{\partial \xi_b}{\partial x_a} - \frac{2\delta_{ab} \nabla \cdot \xi}{3} \approx \frac{\partial \xi_a}{\partial x_b} + \frac{\partial \xi_b}{\partial x_a} = S_{ab} + S_{ba}
\]
\[(A24)\]
where the approximation \(\nabla \cdot \xi = 0\) holds in WKB. Using Eq. (A12), we can compute the shear tensor components; then
\[
\sigma^2 \equiv \sum_{ab} \sigma_{ab} \sigma_{ab} = \left[\frac{8\mu^4 + 2\mu^2}{(1 - \mu^2)^2}\right] \left(\frac{\partial^2 \Psi}{\partial \varphi^2}\right)^2 + \frac{2(1 - 3\mu^2 + 4\mu^4)}{(1 - \mu^2)^2} \left(\frac{\partial^2 \Psi}{\partial \varphi \partial \theta}\right)^2.
\]
\[(A25)\]
In the WKB limit
\[
\frac{\partial^2 \Psi}{\partial u \partial v} = -\frac{p^2}{\sqrt{\rho \sin \theta_1 \sin \theta_2}} \left[\cos(p\theta_1 + \alpha_1) \cos(p\theta_2 + \alpha_2) \left(\frac{\partial \theta_1}{\partial u} \frac{\partial \theta_1}{\partial v} + \frac{\partial \theta_2}{\partial u} \frac{\partial \theta_2}{\partial v}\right) - \sin(p\theta_1 + \alpha_1) \sin(p\theta_2 + \alpha_2) \left(\frac{\partial \theta_1}{\partial u} \frac{\partial \theta_2}{\partial v} + \frac{\partial \theta_1}{\partial v} \frac{\partial \theta_2}{\partial u}\right)\right];
\]
\[(A26)\]
using Eqs. (A3) and \(\sin^2(\theta_1 \pm \theta_2) = 1 - r^2 \cos^2 \theta_{\pm}\) we get
\[
\sigma^2 = \frac{p^4 \mu^2}{2pr \sin \theta (1 - \mu^2)^{3/2}} \left\{ \frac{\cos^2[p(\theta_1 + \theta_2) + \alpha_1 + \alpha_2]}{(1 - r^2 \cos^2 \theta_+)^2} + \frac{\cos^2[p(\theta_1 - \theta_2) + \alpha_1 - \alpha_2]}{(1 - r^2 \cos^2 \theta_-)^2} + \frac{4\mu^2(3 - 4\mu^2) \cos^2[p(\theta_1 + \theta_2) + \alpha_1 + \alpha_2] \cos^2[p(\theta_1 - \theta_2) + \alpha_1 - \alpha_2]}{(1 - r^2 \cos^2 \theta_+)(1 - r^2 \cos^2 \theta_-)} \right\};
\]
\[(A27)\]
the terms on the second line of Eq. (A27) oscillate rapidly and will be dropped in our detailed calculation of the damping rate.

We will assume that the main contribution to the damping rate is from the core of the neutron star, \(r \lesssim r_b < 1\); this means that we will never encounter the exactly two points on the surface where Eq. (A27) is singular. Fig. 6 in Shternin & Yakovlev (2008) suggests that the shear viscosity grows perhaps linearly in the core of a neutron star, so we let \(\eta = \eta_{\text{core}}\) in the core, where \(\eta_{\text{core}}\) is independent of density; this roughly cancels the \(1/\rho\) factor in Eq. (A27) from the WKB form of the modes. For large values of \(p\), we approximate the first two terms in Eq. (A27) by replacing \(\cos^2[p(\theta_1 \pm \theta_2) + \alpha_1 \pm \alpha_2] \to 1/2\), and drop the cross term entirely. Then
\[
\int d^3r \rho \sigma^2 \approx \frac{\pi p^4 \mu^2}{2(1 - \mu^2)^{3/2}} \int_0^\pi d\theta \int_0^{r_b} dr \int_0^{2\pi} d\theta \frac{1}{(1 - r^2 \cos^2 \theta_+)^2} + \frac{1}{(1 - r^2 \cos^2 \theta_-)^2};
\]
\[(A28)\]
\[
= \frac{\pi p^4 r_b^2 \mu^2}{4(1 - \mu^2)^{3/2}} \sum_{s=\pm} \int_{\theta_s}^{2\pi} d\theta \frac{1 - r_b \cos \theta_s}{1 - r^2 \cos \theta_s}
\]
\[
\int d^3r \rho \sigma^2 = \frac{\pi^2 p^4 r_b^2 \mu^2}{(1 - \mu^2)^{3/2} \sqrt{1 - r_b}}
\]
\[(A29)\]
Dividing by \(N\) and multiplying by \(\eta_{\text{core}}\) gives the damping rate
\[
\gamma_{\text{core}} = \frac{\eta_{\text{core}} p^2 (r_b/R)^2}{R^2 \sqrt{1 - (r_b/R)^2}}
\]
\[(A30)\]
where we have restored dimensional units. Note that Eq. A30 would diverge as \( r_b \to R \). That case requires a more careful treatment. 

Brink et al. (2004) included the entire star in the calculation of shear damping: Eq. (31) in that paper is an accurate analytic fit, which we reproduce here: for kinematic viscosity \( \eta \),

\[
\gamma R^2 \eta \equiv \frac{2n + 1}{3} \left[ (n + 3)(n - 2) - \frac{m(m - 2\mu)}{1 - \mu^2} \right]
\]

(A31)

which is approximately \( \gamma/\eta = 2n^3/3 \) for \( n \gg |m| \), which is typical of couplings of large \( n \) modes to the R mode. We have also done a WKB calculation that gives \( \gamma R^2/\eta \approx 2p^3/3 \). That calculation is rather complicated because the result is dominated by contributions from near the special points on the surface where \( \cos \theta_1 = \pm \cos \theta_2 = |\mu| \). The procedure is to return to the displacement field \( \xi \), introduce approximations valid near the special points, and then compute the shear tensor by direct differentiation. This last step deviates from the strict WKB approximation in that if \( k_i = \nabla \theta_i \) it includes terms arising from \( \nabla k_i \) that would be discarded ordinarily. The expression that results can then be integrated analytically, and the result is what we quoted above.

To get the expression for shear viscosity damping in the main text, we divide the star into a core out to \( r_b \) and crust outside \( r_b \), with separate viscosities \( \eta_{\text{core}} \) and \( \eta_{\text{crust}} \), respectively.

TOY MODEL FOR DISPLACEMENT EVOLUTION AS SHEAR MODULUS RISES

We consider the transition region of thickness \( \Delta r \) within which the shear modulus \( \mu(x) \) rises from zero to its value in the crystalline crust. We ignore density variation, and consider planar displacement fields only with \( \nabla \cdot \xi = 0 \). We orient the radial direction along \( x \) and define \( c_i(x) = \mu(x)/\rho \).

We assume that displacements are proportional to functions of \( x \) times \( \exp[i(k_y y + k_z z) - i\omega t] \); the divergenceless condition implies that \( \xi_y \) and \( \xi_z \) are both \( \mathcal{O}(\partial \xi_x/\partial x) \) and hence much larger than \( \xi_x \). Then if we systematically ignore \( k_y,z \) compared with \( \partial/\partial x \) in this region both \( \xi_y \) and \( \xi_z \) obey the approximate linear differential equation

\[
-\omega^2 \xi = \frac{\partial}{\partial x} \left( c_i^2 \frac{\partial \xi}{\partial x} \right).
\]

(B1)

This equation describes the evolution of the jumps in these displacement components. Let \( c_i^2 = c_{i,S}^2 f(u) \), where within the layer \( x = x_{\text{inner}} + u \Delta r \) and in the solid \( c_i^2 = c_{i,S}^2 \). The function \( f(u) \) may be determined from microphysics. Written in terms of \( u \) Eq. (B1) is

\[
0 = \frac{\partial}{\partial u} \left[ f(u) \frac{\partial \xi}{\partial u} \right] + \frac{\omega^2(\Delta r)^2 \xi}{c_{i,S}^2} \equiv \frac{\partial}{\partial u} \left[ f(u) \frac{\partial \xi}{\partial u} \right] + q^2 \xi.
\]

(B2)

Provided that both \( f(u) \) and \( \xi(u) \) are monotonic, we can regard \( \xi \) as a function of \( f \).

Realistically, we would solve Eq. (B2) for a specified \( f(u) \). To get a rough idea of what a solution might look like, we pursue an illustrative toy calculation: let \( \xi = f^p \), where \( p \) is some powerlaw index, to get

\[
0 = d^2 f^{1+p} \frac{du}{d^2} + q^2 (1+p) f^p;
\]

(B3)

rescale so that \( g = Af^{1+p} \) to get

\[
0 = \frac{d^2 g}{du^2} + g^{1+p}
\]

(B4)

where we have chosen the scaling constant so that \( q^2(1+p)A^{1+p} = 1 \). We solve Eq. (B4) with \( q = 0 \) at \( u = 0 \) but \( (dg/du)_0 \neq 0; \) we impose the condition that \( (dg/du)_1 = 0 \) at \( u = 1 \), which follows since \( (df/du)_1 = 0 \) for \( p > 0 \). Consequently \( (dg/du)_0 \) is an eigenvalue. Since we require that \( f(1) = 1 \), it follows that \( g(1) = A \), so we have \( q^2(1+p)g(1)^{1+p} = 1 \), which determines \( q^2 \). Thus, it should be clear that this choice of \( \xi(f) \) is hardly general, and would only hold for a very specific \( f(u) \) and \( q^2 \).

Note that Eq. (B4) can be integrated once to yield

\[
\frac{dg}{du} \left( \frac{dg}{du} \right)_0 \sqrt{1 - \left( \frac{g}{g_0} \right)^{1+p}}
\]

(B5)

where \( g_0^{1+p} = (1+2p)(dg/du)_0^2/2(1+p) \). Eq. (B5) can be solved via quadrature. An acceptable solution has \( g(1) = g_0 \).

The viscous dissipation rate within the layer is

\[
\dot{E} = 4\pi \rho_b r_b^2 \omega^2 \int dx \xi \frac{\partial}{\partial x} \left( \eta \frac{\partial \xi}{\partial x} \right) = \frac{4\pi \rho_b r_b^2 \omega^2}{\Delta r} \left[ \left. \xi \eta(u) \frac{\partial \xi}{\partial u} \right|_0^1 - \int_0^1 d\eta(u) \left( \frac{\partial \xi}{\partial u} \right)^2 \right].
\]

(B6)
Let the kinematic viscosity be $\eta(u) = \eta_{\text{core}} \hat{\eta}(u)$, where $\hat{\eta}(0) = 1$ and $\hat{\eta}(1) = \eta_{\text{crust}}/\eta_{\text{core}} \ll 1$; we assume $\hat{\eta}(u) \leq 1$ to get an upper bound on $E$. Since $\omega \xi = \Delta v [g/g(1)]^{1/\tau}$ in our toy model

$$E = -\frac{4\pi p^2 \eta_{\text{core}} \rho v_b^2 (\Delta v)^2}{\Delta r (1 + p)^2 [g(1)]^{1/\tau}} \left\{ \frac{(1 + p) g^{1/\tau}}{p} \frac{dg}{du} \right\}_0 + \int_0^1 \frac{d\hat{\eta}(u)}{[g(u)]^{1/\tau}} \left( \frac{dg(u)}{du} \right)^2 \right\}.$$  

(B7)

Since $g(u) \sim u$ at $u \ll 1$ the integral diverges for $p \leq 1$.

We have solved Eq. (B1) for $p = 2$; the eigenvalue is $(dg/du)_0 \approx 0.135164405635$, and $g(1) \approx 0.0811944$; consequently, $3\omega^2(\Delta r)^2 g[1]^{1/3}/c_{\ell S} = 1$, or $c_{\ell S} = 1.14 \Delta r = 3.5 \times 10^7 \text{ cm s}^{-1} \nu_{500}(\omega/\Omega)(\Delta r/100 \text{ m})$, which is a plausible value but cannot be right for all modes, each of which has its own value of $\omega$. For a given $f(u)$ the function $\xi(f)$ must differ among modes and generally the problem does not scale as it does when $\xi(f) = f^p$. Nevertheless, this toy model illustrates the salient features of how a transition might occur. The solution is shown in Fig. 2. The dissipation rate in this model is $E \leq 4\pi \eta_{\text{core}} \rho v_b^2 (\Delta v)^2 / \Delta r \times 1.99$.

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