

Ambiguity Aversion in Ellsberg Frameworks*

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Abstract

We study optimal portfolio choice and equilibrium asset prices induced by α -maxmin expected utility (α -MEU) models. In the standard Ellsberg framework we prove that α -MEU preferences are equivalent to either maxmin, maxmax or subjective expected utility (SEU). We show how ambiguity aversion impacts equilibrium asset prices, and revisit the laboratory experimental findings in Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010). Only when there are three or more ambiguous states the α -MEU, maxmin, maxmax and SEU models induce different portfolio choices. We suggest criteria to discriminate among these models in laboratory experiments. Finally, we find that ambiguity seeking α -MEU agents may prevent the existence of market equilibrium. Our results indicate that ambiguity matters for portfolio choice and does not wash out in equilibrium.

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1 Introduction

Over the past decades the impact of Knightian uncertainty (Knight 1921), or ambiguity, on financial decision making has received significant attention in the academic community. Models with ambiguity averse agents can capture a variety of empirical phenomena such as non-participation, portfolio inertia and excess volatility of asset returns.¹ These models are also supported by experimental laboratory evidence that agents' preferences are heterogeneous and well approximated by ambiguity averse preferences; e.g., Bossaerts et al. (2010), and Ahn, Choi, Gale, and Kariv (2014).

The maxmin expected utility model (Gilboa and Schmeidler 1989) has been the pioneering and workhorse model to study the impact of ambiguity aversion on financial markets. One limitation of the model is that it does not allow for a distinction between ambiguity and ambiguity aversion.² The α -maxmin expected utility (α -MEU) model, defined in (2.1), generalizes the maxmin model and has a number of appealing features.³ Fixing the set of priors to be the one that describes the uncertainty of the setting, allows to interpret the one-dimensional parameter α as a measure of the agent's ambiguity aversion. Moreover, being a convex combination of the maxmax (0-MEU) and the maxmin (1-MEU) models, it represents a large spectrum of preferences ranging from ambiguity loving attitude of the 0-MEU to ambiguity aversion attitude of the 1-MEU. Thus the α -MEU has been used in many theoretical and experimental studies on agents' ambiguity attitudes.⁴ Despite its popularity, an in-depth analysis of portfolio choice and equilibrium asset prices of the α -MEU model

¹Studies investigating how nonparticipation may arise in equilibrium with ambiguity averse agents include Dow and Werlang (1992), Cao, Wang, and Zhang (2005), Easley and O'Hara (2009), and Illeditsch (2011). Studies relating ambiguity aversion to other market phenomena include Chen and Epstein (2002), Uppal and Wang (2003), Trojani and Vanini (2004), Epstein and Schneider (2008), Cao, Han, Hirshleifer, and Zhang (2011), and Boyle, Garlappi, Uppal, and Wang (2012). For a survey on this topic see Epstein and Schneider (2010).

²In the maxmin model any possible attitude towards ambiguity corresponds to a shrinkage of the set of priors relative to the maximal set of priors.

³The α -maxmin expected utility is a generalization of the Hurwicz's model introduced by Hurwicz (1951a,b); see also Arrow and Hurwicz (1972) and Jaffray (1988).

⁴Theoretical properties of the α -MEU model have been studied by Ghirardato, Klibanoff, and Marinacci (1998) and Marinacci (2002). For characterizations of subclasses of the α -MEU preferences see Ghirardato, Maccheroni, and Marinacci (2004), Olszewski (2007), Eichberger, Grant, Kelsey, and Koshevoy (2011), and Klibanoff, Mukerji, and Seo (2014). Chen, Katusčák, and Ozdenoren (2007) focus on sealed bid auctions and use the α -MEU to derive the equilibrium bidding strategy for α -MEU bidders. For recent experimental studies see, Ahn et al. (2014), Cubitt, van de Kuilen, and Mukerji (2014) and reference therein.

has not been carried out.⁵ While the maxmin model is always concave if the agent is risk averse, this is not the case for the α -MEU model, making its analysis more involved. Such an analysis is central to understand the attitudes towards ambiguity that the α -MEU model represents.

This paper theoretically studies the implications for optimal portfolio choice and equilibrium asset prices of the α -MEU model. We study the α -MEU model in a complete Arrow–Debreu market model where the future states of the economy correspond to draws from Ellsberg-type urns. The complete market setting provides an ideal framework to study portfolio choice because the agents can attain the desired amount of portfolio risk and ambiguity exposures, given their budget constraints. The separation between risky and ambiguous states and the interchangeability of the latter in the Ellsberg frameworks allow us to study the attitudes towards ambiguity of α -MEU agents.

We first consider the standard Ellsberg (1961) framework where the state space consists of three future states of the economy, one risky and two ambiguous. We find that, in this setting, α -MEU preferences coincide with either maxmin, maxmax, or subjective expected utility (SEU) preferences.⁶ This finding has implications for experimental studies. It indicates that standard Ellsberg frameworks (with two ambiguous states) are not enough to disentangle between maxmin and α -MEU preferences. Also, it rationalizes empirical evidence from recent laboratory experiments. These studies are carried out in the standard Ellsberg framework and use the α -MEU model to conclude that the experimental evidence point to a substantial heterogeneity in aversion to ambiguity; e.g., Bossaerts et al. (2010), and Ahn et al. (2014). Our result shows that these experimental studies could had come to the same conclusion by using maxmin preferences and varying the size of the set of priors, rather than α , to measure varying degrees of aversion to ambiguity.

We derive equilibrium asset prices for a market populated by ambiguity averse and SEU

⁵Bossaerts et al. (2010), and Ahn et al. (2014) derive the α -MEU model portfolio choice in the standard Ellsberg framework. However, as we show in Proposition 3.1, in that framework α -MEU preferences coincide with either maxmin, maxmax or SEU preferences.

⁶This result holds true for any number of risky states as long as that there are only two ambiguous states.

investors.⁷ We theoretically show through which channels ambiguity aversion impacts equilibrium asset prices, and revisit the laboratory experimental findings in Bossaerts et al. (2010) obtained in the standard Ellsberg. We show that our theoretically derived rankings of the state-price/state-probability ratios fully explain the empirical rankings observed by Bossaerts et al. (2010). This remarkable matching between theory and data indicates that ambiguity aversion does not wash out in equilibrium.

Next we consider an extended Ellsberg framework where the state space contains three or more ambiguous states. We show that, in this setting, the α -MEU preferences do not coincide with maxmin, maxmax or SEU preferences and yield portfolio choices that are not observationally equivalent. We consider the set of priors \mathcal{C}_{\max} that describes the uncertainty in the extended Ellsberg framework, and denote the corresponding class of models by α - \mathcal{C}_{\max} -MEU. Fixing \mathcal{C}_{\max} allows to interpret the parameter α as a measure of the agent's aversion toward ambiguity. Moreover, it makes the comparison between α - \mathcal{C}_{\max} -MEU and maxmin preferences meaningful by providing a utility specification common to the two classes of models.

We show that the α - \mathcal{C}_{\max} -MEU agents with $\alpha \in (0, 1)$ optimally choose only two types of portfolios: either an unambiguous portfolio (with no exposure to ambiguity, allocating equal wealth to all ambiguous states), or an ambiguous portfolio with one specific exposure to ambiguity (that allocates more wealth \bar{w} to one of the cheapest ambiguous state and less equal wealth \underline{w} to the other ambiguous states). The ambiguous portfolio can be seen as an unambiguous portfolio plus a bet of $\bar{w} - \underline{w}$ on one of the cheapest ambiguous states. The larger the difference $\bar{w} - \underline{w}$, the more the portfolio is exposed to ambiguity. If there is only one ambiguous state with minimum price, then the optimal portfolio is unique. If there are $n \geq 1$ ambiguous states with minimum price, then there are n optimal portfolios, when the optimal portfolio is the ambiguous one. The α - \mathcal{C}_{\max} -MEU agent finds equally optimal to bet $\bar{w} - \underline{w}$ on any of the n cheapest ambiguous states, because the ambiguous states with equal prices are indistinguishable and the optimal portfolios provide the same exposure to

⁷Bossaerts et al. (2010) and Ahn et al. (2014) run portfolio choice experiments in the standard Ellsberg framework and provide evidence of considerable heterogeneity in agents' preferences. They find that one half of the agents are well approximated by SEU preferences, while the remaining half has a significant degree of ambiguity aversion.

ambiguity.

The choice between unambiguous and ambiguous optimal portfolios only depends on α and the ratio of the minimum price to the total sum of prices of ambiguous states. The larger is α relative to the ratio above, the less the optimal portfolio is exposed to ambiguity, i.e., the smaller is $\bar{w} - \underline{w}$. The set of state prices for which an $\alpha\text{-}\mathcal{C}_{\max}$ -MEU agent chooses an unambiguous portfolio increases with the ambiguity aversion parameter α . The limiting case is the $1\text{-}\mathcal{C}_{\max}$ -MEU agent who always chooses an unambiguous portfolio.

To understand the various attitudes toward ambiguity expressed by the $\alpha\text{-}\mathcal{C}_{\max}$ -MEU model, we start by considering the optimal portfolio choice when the prices of the ambiguous states are all equal. In this setting the ambiguous states are equivalent from an informational point of view. We show that when α is small the $\alpha\text{-}\mathcal{C}_{\max}$ -MEU agent exhibits an ambiguity seeking behavior. Despite the ambiguous states are equivalent, the agent prefers portfolios exposed to ambiguity and makes the bet of $\bar{w} - \underline{w}$ indifferently on one of the ambiguous states. The number of optimal portfolios is equal to the number of ambiguous states. We show that when α is large $\alpha\text{-}\mathcal{C}_{\max}$ -MEU agents do not exhibit ambiguity seeking behavior, and prefer unambiguous portfolios. Maxmin agents also prefer unambiguous portfolios irrespective of their sets of priors.

Our theoretical findings can inform laboratory experiments to disentangle between ambiguity seeking $\alpha\text{-}\mathcal{C}_{\max}$ -MEU agents and agents who are not ambiguity seeking, and among the latter, between $\alpha\text{-}\mathcal{C}_{\max}$ -MEU and maxmin agents. We propose a multiple-stage experiment. In the first stage, setting the prices of ambiguous states all equal allows to identify the ambiguity seeking agents from their portfolio choices. Subsequent stages only involve the non-ambiguity seeking agents, and exploit the fact that optimal portfolios of $\alpha\text{-}\mathcal{C}_{\max}$ -MEU agents are not unique, while the optimal portfolio of maxmin agents is unique.

Finally, we study the equilibrium when SEU agents and ambiguity sensitive agents are both present in the market. We show that the existence of the market equilibrium depends on whether ambiguity seeking agents are present or not in the market. We find that the ambiguity seeking attitude of the $\alpha\text{-}\mathcal{C}_{\max}$ -MEU agents may prevent the existence of market equilibrium that otherwise would exist when non-ambiguity seeking $\alpha\text{-}\mathcal{C}_{\max}$ -MEU, maxmin and SEU agents are present. Intuitively, ambiguity seeking agents may take positions that

cannot be offset by SEU agents.

The structure of the paper is as follows. Section 2 introduces the setup. Section 3 shows that in the standard Ellsberg framework α -MEU preferences coincide with maxmin, maxmax and SEU preferences, derives equilibrium state prices, and revisits the experimental findings in Bossaerts et al. (2010). Section 4 studies the optimal portfolio choice of the α - \mathcal{C}_{\max} -MEU model in the extended Ellsberg framework, and the different attitudes towards ambiguity. Section 5 concludes. The Appendix collects proofs and technical results.

2 Setup

The utility of an α -MEU agent from some state dependent wealth $w = (w_\sigma)_{\sigma \in S}$ is

$$(2.1) \quad U(w) = \alpha \min_{\pi \in \mathcal{C}} \sum_{\sigma \in S} u(w_\sigma) \pi_\sigma + (1 - \alpha) \max_{\pi \in \mathcal{C}} \sum_{\sigma \in S} u(w_\sigma) \pi_\sigma$$

where $u : \mathbb{R} \rightarrow \mathbb{R}$ is a utility function, the set of priors \mathcal{C} is a closed and convex set on the state space S , and α can take any value between $[0, 1]$. For $\alpha = 1$, (2.1) reduces to the maxmin-expected utility (1-MEU) model, for $\alpha = 0$ to the maxmax-expected utility (0-MEU) model. All utility functions $u : \mathbb{R} \rightarrow \mathbb{R}$ are differentiable, strictly concave and strictly increasing. To keep the analysis tractable we assume that u is defined on the whole real line. The majority of results in this paper (e.g., the α -MEU portfolio characterization in Proposition 4.1) holds true also when u has a bounded domain, as long as the set of feasible portfolios remain convex and the utility differentiable.⁸

The market model considered in this paper is an Arrow–Debreu complete market for contingent claims with two dates, $t = 0$ and $t = 1$. S is the finite state space containing all possible states of the economy at time $t = 1$, and $|S|$ is the number of states. At time $t = 0$ the agents face both uncertainty (risk) and ambiguity since they neither know which state in S will realize at time $t = 1$ (uncertainty), nor what is the probability of the occurrence of some of the states in S (ambiguity). For any state there is an Arrow security traded in the market which pays at time $t = 1$ one unit of currency in that state and nothing in the other

⁸These properties can be insured, for instance, by requiring that the feasible portfolios are in the interior of the utility domain.

states. Pricing rules $p = (p_\sigma)_{\sigma \in S} \in \mathbb{R}_+^{|S|}$ are normalized so that the price of the risk-free and unambiguous portfolio $w = (1, \dots, 1)$ is 1, that is $\sum_{s=1}^{|S|} p_i = 1$.

Given N agents in the market, each agent n is characterized by an initial endowment $e^n \in \mathbb{R}^{|S|}$, where the i th coordinate of e^n corresponds to the number of Arrow securities that pay in the state i , and by a criterion U^n representing her preferences, $n = 1, \dots, N$. The total endowment in the market is $W := \sum_1^N e^n = (W_1, \dots, W_{|S|})$, where $:=$ denotes definition. Let \cdot denote the scalar product $x \cdot y = \sum_{i=1}^{|S|} x_i y_i$, $x, y \in \mathbb{R}^{|S|}$. Given the pricing rule p on the Arrow securities, a portfolio $w^n = (w_\sigma^n)_{\sigma \in S} \in \mathbb{R}^{|S|}$ is said to be optimal for agent n if w^n satisfies the budget constraint $p \cdot w^n \leq p \cdot e^n$ and maximizes the utility U^n over all portfolios $w \in \mathbb{R}^{|S|}$ subject to the budget constraint $p \cdot w \leq p \cdot e^n$, i.e.

$$U^n(w^n) = \max\{U^n(w) \mid w \in \mathbb{R}^{|S|}, p \cdot w \leq p \cdot e^n\}.$$

An equilibrium $(p; w^1, \dots, w^N)$ consists of a pricing rule p and individual portfolio choices w^n such that

- for each $n = 1, \dots, N$ the portfolio w^n is optimal for agent n given the pricing rule p , and
- the market clears: $\sum_1^N w^n = \sum_1^N e^n$.

3 Standard Ellsberg framework

Throughout this section we consider a standard Ellsberg framework, that is a three-dimensional state space $S = \{R, G, B\}$ where the states correspond to draws from the Ellsberg (1961) urn. The probability of the state R (red) is known and equal to $\pi_R \in (0, 1)$, while the probabilities of the two ambiguous states G (green) and B (blue) are unknown. Any closed convex set of priors \mathcal{D} , consistent with the above information on the Ellsberg framework, can be written as

$$(3.1) \quad \mathcal{D} = \{(\pi_R, q, 1 - q - \pi_R) : q \in [a, b]\}$$

where π_R , q , and $1 - q - \pi_R$ are the probability weights on the states R , G , and B , respectively, corresponding to a given prior in \mathcal{D} , and $0 \leq a \leq b \leq 1 - \pi_R$. Thus, any α -MEU utility U in (2.1) on the portfolio $w = (w_R, w_G, w_B) \in \mathbb{R}^3$ reads as

$$(3.2) \quad U(w) = \alpha \min_{q \in [a, b]} [\pi_R u(w_R) + q u(w_G) + (1 - q - \pi_R) u(w_B)] + \\ (1 - \alpha) \max_{q \in [a, b]} [\pi_R u(w_R) + q u(w_G) + (1 - q - \pi_R) u(w_B)]$$

for some $\alpha \in [0, 1]$.

3.1 Equivalence result

In the following Proposition 3.1, we show that α -MEU preferences are equivalent to either 1-MEU, 0-MEU or SEU preferences. The proof is provided in Appendix A.

Proposition 3.1. *Consider the utility U in (3.2) and let $c := \alpha a + (1 - \alpha)b$ and $d := (1 - \alpha)a + \alpha b$.*

(i) *If $\alpha > 1/2$, then U is a maxmin expected utility (1-MEU), i.e.*

$$(3.3) \quad U(w) = \min_{q \in [c, d]} [\pi_R u(w_R) + q u(w_G) + (1 - q - \pi_R) u(w_B)]$$

with set of priors $\mathcal{C} = \{(\pi_R, q, 1 - q - \pi_R) : q \in [c, d]\} \subset \mathcal{D}$.

(ii) *If $\alpha = 1/2$, then U is a subjective expected utility (SEU) with subjective prior*

$$(3.4) \quad (\pi_R, (a + b)/2, 1 - \pi_R - (a + b)/2).$$

(iii) *If $\alpha < 1/2$, then U is a maxmax expected utility (0-MEU), i.e.*

$$(3.5) \quad U(w) = \max_{q \in [d, c]} [\pi_R u(w_R) + q u(w_G) + (1 - q - \pi_R) u(w_B)]$$

with set of priors $\mathcal{C} = \{(\pi_R, q, 1 - q - \pi_R) : q \in [d, c]\} \subset \mathcal{D}$.

Proposition 3.1 shows that any α -MEU utility with $\alpha > 1/2$ ($\alpha < 1/2$) and a generic set of priors \mathcal{D} is equivalent to a unique maxmin utility (respectively, maxmax utility) over a set of priors \mathcal{C} , which is smaller than \mathcal{D} , and univocally characterized by α and \mathcal{D} .⁹ Consequently,

- the α -MEU preferences in the standard Ellsberg framework are indistinguishable from maxmin, maxmax or SEU preferences
- doing comparative statics with respect to α in the α -MEU model with a fixed set of prior is equivalent to doing comparative statics with respect to the size of the set of priors in the maxmin model.

These findings provide new insights into the implications of ambiguity for portfolio choice and have implications for experimental studies. For instance, they show that the standard Ellsberg framework is not the right setting to study the α -MEU model if one wants to use this model as a generalization of the maxmin, maxmax or SEU preferences.¹⁰ Moreover, they clarify some recent experimental studies carried out in the standard Ellsberg framework in which the α -MEU model is used to conclude that the experiment outcomes suggest a substantial heterogeneity in aversion to ambiguity; e.g. Bossaerts et al. (2010), and Ahn et al. (2014). Proposition 3.1 shows that these studies could have come to the same conclusions by using maxmin preferences and varying the size of the set of priors instead of α to measure varying degrees of aversion to ambiguity; see also Section 3.4.

The converse of Proposition 3.1 was already known from Siniscalchi (2006), namely that a maxmin model with a given set of priors \mathcal{C} can be rewritten as less parsimonious α -MEU models, with set of priors \mathcal{D} larger than \mathcal{C} , for many different (α, \mathcal{D}) .

The α -MEU preferences coincide with maxmin, maxmax, or SEU preferences also in a state space setting with more than one risky states, or with no risky states, as long as there are only two ambiguous states. When there are no risky states, Proposition 3.1 holds true by setting $\pi_R = 0$. When there are $m \geq 1$ risky states, R_1, \dots, R_m , with known probabilities $\pi_{R_i} \in (0, 1)$ which satisfy $\sum_{i=1}^m \pi_{R_i} < 1$, Proposition 3.1 holds true by replacing the prior in

⁹The set of priors \mathcal{C} equals \mathcal{D} when $\alpha = 1$ ($\alpha = 0$), and when α decreases (increases) to $1/2$ shrinks up to only containing the prior (3.4).

¹⁰In Section 4 we show that when there are more than two ambiguous states the equivalence result does not hold anymore and the α -MEU model induces different portfolio choice and expresses different attitude toward ambiguity than the maxmin model.

(3.4) by the prior $(\pi_{R_1}, \dots, \pi_{R_m}, (a+b)/2, 1 - \sum_{i=1}^m \pi_{R_i} - (a+b)/2)$, and $\pi_R u(w_R)$ in (3.3) and (3.5) by $\sum_{i=1}^m \pi_{R_i} u(w_{R_i})$.

3.2 Market equilibrium with ambiguity averse and SEU agents

Motivated by recent experimental evidences that investor's preferences are well approximated by SEU and ambiguity averse preferences, we study a simple market model populated by maxmin agents (or equivalently α -MEU agent with $\alpha > 1/2$, see Proposition 3.1 (i)) and SEU agents. We derive equilibrium asset prices and show theoretically how ambiguity aversion impacts equilibrium asset prices.

Let denote by $w = (w_R, w_G, w_B)$ the optimal portfolio of a maxmin agent and by $y = (y_R, y_G, y_B)$ the optimal portfolio of a SEU agent. Depending on the distribution of the total endowment $W = (W_R, W_G, W_B)$ in the market, only particular rankings of state-price/state-probability ratios can occur in equilibrium. The interesting case to study is when maxmin agents take an unambiguous portfolio, i.e. $w_G = w_B$, and the total endowment is $W_G \neq W_B$. Without loss of generality in the following we assume that $W_G > W_B$.

Appendix B provides the proof of Proposition 3.2 and a concise treatment of maxmin and SEU portfolio choice.

Proposition 3.2. *Suppose the market is in equilibrium and populated by maxmin agents (equivalently α -MEU preferences with $\alpha > 1/2$) who take unambiguous portfolios and SEU agents with prior $\pi = (\pi_R, \pi_G, \pi_B)$, with $\pi_R, \pi_G, \pi_B > 0$. Denote by $W = (W_R, W_G, W_B) \in \mathbb{R}^3$ the total endowment of the market.*

- 1) *If $W_R > W_G > W_B$, then two rankings of the state-price/state-probability ratios are possible:*

$$(3.6) \quad \frac{p_B}{\pi_B} > \frac{p_R}{\pi_R} > \frac{p_G}{\pi_G}$$

and the optimal portfolios y of any SEU agent and w of any maxmin agent satisfy

$$y_G > y_R > y_B \quad \text{and} \quad w_R > w_G = w_B.$$

The other possible ranking is:

$$(3.7) \quad \frac{p_B}{\pi_B} > \frac{p_G}{\pi_G} > \frac{p_R}{\pi_R} \quad \left(\text{or} \quad \frac{p_B}{\pi_B} > \frac{p_G}{\pi_G} = \frac{p_R}{\pi_R} \right)$$

and the optimal portfolios y of any SEU agent and w of any maximin agent satisfy

$$y_R > y_G > y_B \quad (\text{or} \quad y_R = y_G > y_B) \quad \text{and} \quad w_R > w_G = w_B.$$

2) If $W_G > W_R > W_B$, then the only possible ranking of the state-price/state-probability ratios is:

$$(3.8) \quad \frac{p_B}{\pi_B} > \frac{p_R}{\pi_R} > \frac{p_G}{\pi_G}$$

and the optimal portfolios y of any SEU agent and w of any maximin agent satisfy

$$y_G > y_R > y_B \quad \text{and} \quad w_R > w_G = w_B \quad \text{or} \quad w_R < w_G = w_B.$$

3) If $W_G > W_B > W_R$, then two rankings of the state-price/state-probability ratios are possible:

$$(3.9) \quad \frac{p_B}{\pi_B} > \frac{p_R}{\pi_R} > \frac{p_G}{\pi_G}$$

and the optimal portfolios y of any SEU agent and w of any maximin agent satisfy

$$y_G > y_R > y_B \quad \text{and} \quad w_R < w_G = w_B.$$

The other possible ranking is:

$$(3.10) \quad \frac{p_R}{\pi_R} > \frac{p_B}{\pi_B} > \frac{p_G}{\pi_G} \quad \left(\text{or} \quad \frac{p_R}{\pi_R} = \frac{p_B}{\pi_B} > \frac{p_G}{\pi_G} \right)$$

and the optimal portfolios y of any SEU agent and w of any maximin agent satisfy

$$y_G > y_B > y_R \quad (\text{or } y_G > y_B = y_R) \quad \text{and} \quad w_R < w_G = w_B.$$

Proposition 3.2 shows that when the total endowment $W = (W_R, W_G, W_B)$ satisfies $W_R > W_G > W_B$ or $W_G > W_B > W_R$, two rankings of the state-price/state-probability ratios of the SEU agents are possible in equilibrium. In the examples of Section 3.3 we show that all these rankings can indeed occur.

The actual occurrence of ranking (3.6) when $W_R > W_G > W_B$, and of ranking (3.9) when $W_G > W_B > W_R$, show that ambiguity aversion strongly impacts equilibrium prices. In these cases, the SEU agents in the market and the SEU representative agent (who rationalizes the market equilibrium) rank state-price/state-probability ratios in equilibrium differently.¹¹ Since the maximin agents take an unambiguous portfolio, i.e. $w_G = w_B$, and $W_G > W_B$, the market clearing implies that SEU agents have to hold portfolios which in aggregate pay strictly more on state G than on state B ¹². To induce the SEU agents to clear the supply difference $W_G - W_B$, the price of the Arrow security G in larger supply has to be comparatively lower and the price of the Arrow security in lower supply B has to comparatively higher than, for instance, in a market populated only by SEU agents sharing the same prior. When the supply differences $W_G - W_B$ will be above a certain level, the SEU agents will have to hold in equilibrium a state dependent portfolio that does not rank as the state dependent total endowment W , and consequently their state-price/state-probability ratios will not be ranked opposite to W . For instance, when $W_G > W_B > W_R$ and $W_G - W_B$ is “too large” compared to $W_B - W_R$, the optimal portfolio of the SEU agents will be $y_G > y_R > y_B$ and the ranking (3.9). Examples in Section 3.3 illustrate the mechanism through which ambiguity averse agent impacts prices in the CARA and Quadratic cases.

An ambiguity averse representative agent who rationalizes the market equilibrium may

¹¹Rankings (3.6) and (3.9) of the state-price/state-probability ratio of SEU agents in the market are not opposite to the ranking of the corresponding total endowments. By contrast, the state-price/state-probability of the SEU representative agent who rationalizes the market equilibrium is ranked opposite to total endowment: the representative agent has to hold the total endowment of the economy as optimal portfolio, thus (B.3) has to hold.

¹²This occurs if and only if $\frac{p_B}{\pi_B} > \frac{p_G}{\pi_G}$ (See (B.3) in Appendix B) and excludes all state-price/state-probability rankings in which $\frac{p_B}{\pi_B} \leq \frac{p_G}{\pi_G}$.

also be possible. However, if the aggregate total endowment does not happen to be an unambiguous portfolio, as it is in the case analyzed in Proposition 3.2, the representative agent will need to be less ambiguity averse (that is, to have a smaller set of priors) than the maxmin agents acting in the market.

3.3 Illustrating equilibrium results

We now illustrate Proposition 3.2 in the case of exponential and quadratic utilities. We recall that the market is populated by SEU agents with prior $\pi = (\pi_R, \pi_G, \pi_B)$, with $\pi_R, \pi_G, \pi_B > 0$, and by maxmin agents in (3.3) who are sufficiently ambiguity averse to hold an unambiguous portfolio, i.e. $w_G = w_B$.¹³ The total endowment $W = (W_R, W_G, W_B)$ is such that $W_G > W_B$.

3.3.1 CARA utility

There are L SEU agents and M maxmin agents, all having exponential utilities $u(z) = 1 - \frac{e^{-\delta z}}{\delta}$. Let $\delta = a$ and $\delta = b$ be the risk aversion parameter of the SEU agents and maxmin agents, respectively. Then, the equilibrium state prices are:

$$\begin{aligned} p_R &= \frac{\pi_R}{\pi_R + e^{\frac{ab(W_R - W_G)}{aM + bL}} \pi_G^{\frac{bL}{aM + bL}} q^{\frac{aM}{aM + bL}} + e^{\frac{ab(W_R - W_B)}{aM + bL}} \pi_B^{\frac{bL}{aM + bL}} (1 - \pi_R - q)^{\frac{aM}{aM + bL}}} \\ p_G &= \frac{\pi_G^{\frac{bL}{aM + bL}} q^{\frac{aM}{aM + bL}}}{\pi_G^{\frac{bL}{aM + bL}} q^{\frac{aM}{aM + bL}} + \pi_R e^{\frac{ab(W_G - W_R)}{aM + bL}} + e^{\frac{ab(W_G - W_B)}{aM + bL}} \pi_B^{\frac{bL}{aM + bL}} (1 - \pi_R - q)^{\frac{aM}{aM + bL}}} \\ p_B &= \frac{\pi_B^{\frac{bL}{aM + bL}} (1 - \pi_R - q)^{\frac{aM}{aM + bL}}}{\pi_B^{\frac{bL}{aM + bL}} (1 - \pi_R - q)^{\frac{aM}{aM + bL}} + \pi_R e^{\frac{ab(W_B - W_R)}{aM + bL}} + e^{\frac{ab(W_B - W_G)}{aM + bL}} \pi_G^{\frac{bL}{aM + bL}} q^{\frac{aM}{aM + bL}}} \end{aligned}$$

where

$$(3.11) \quad q := \pi_G \frac{\pi_G + \pi_B}{\pi_G + \pi_B e^{\frac{a}{L}(W_G - W_B)}}.$$

The above formulae show that the equilibrium prices are the same prices that would be obtained in a market populated only by SEU agents, L of which with prior (π_R, π_G, π_B)

¹³From (B.5) and the results that follow in this section (specifically equalities (3.12)) one can see that maxmin agents hold an unambiguous portfolio, i.e. $w_G = w_B$ if and only if q in (3.11) satisfies $c \leq q < \pi_G < d$.

and risk aversion a , and M of which with prior $(\pi_R, q, 1 - \pi_R - q)$ and risk aversion b . However, this equilibrium is significantly different than an equilibrium resulting from the interaction of SEU agents with heterogeneous beliefs (priors). The reason is that q in (3.11) is a function of $W_G - W_B$, of the number L of SEU agents in the market, and of their risk aversion a .¹⁴ In particular, the dependence of q on $W_G - W_B$ illustrates one channel through which ambiguity aversion impacts asset prices. In aggregate maxmin agents do not hold the imbalance $W_G - W_B$, which is left to the SEU agents to be absorbed.¹⁵

The impact of an increase of $\frac{a}{L}(W_G - W_B)$ on the securities prices that pay in the ambiguous states is clear: an increase of $\frac{a}{L}(W_G - W_B)$ will decrease q and increase $(1 - \pi_R - q)$, and consequently will decrease p_G , and increase p_B .¹⁶ The equilibrium price ratios

$$\begin{aligned}
 \frac{p_G}{p_B} &= \frac{\pi_G}{\pi_B} e^{-\frac{a}{L}(W_G - W_B)} \\
 \frac{p_G}{p_R} &= \frac{\pi_G}{\pi_R} \left(\frac{\pi_G + \pi_B}{\pi_G + \pi_B e^{\frac{a}{L}(W_G - W_B)}} \right)^{\frac{aM}{aM+bL}} e^{-\frac{ab}{aM+bL}(W_G - W_R)} \\
 \frac{p_B}{p_R} &= \frac{\pi_B}{\pi_R} \left(\frac{\pi_G + \pi_B}{\pi_G e^{-\frac{a}{L}(W_G - W_B)} + \pi_B} \right)^{\frac{aM}{aM+bL}} e^{-\frac{ab}{aM+bL}(W_B - W_R)}.
 \end{aligned}
 \tag{3.12}$$

show that all rankings of the state-price/state-probability ratios that are possible according to Proposition 3.2 can indeed occur. For instance, consider the case $W_G > W_B > W_R$. The first two equations in (3.12) show that always $\frac{p_B}{\pi_B} > \frac{p_G}{\pi_G}$ and $\frac{p_R}{\pi_R} > \frac{p_G}{\pi_G}$. The third equation in (3.12) shows that both $\frac{p_B}{\pi_B} > \frac{p_R}{\pi_R}$ and $\frac{p_B}{\pi_B} < \frac{p_R}{\pi_R}$ can occur and, consequently, the corresponding rankings (3.9) and (3.10).

¹⁴The fact that q does not depend neither on the number of M of maxmin agents nor on their risk aversion b is a peculiarity of the exponential utility.

¹⁵The prior $(\pi_R, q, 1 - \pi_R - q)$ is the only prior in the set \mathcal{C} of the maxmin agents which allows the L SEU agents to clear the imbalance $W_G - W_B$ and reach equilibrium.

¹⁶Depending on the particular rank of the total endowment $W = (W_R, W_B, W_G)$, the price of the Arrow security that pays in the state R will increase or decrease with $\frac{a}{L}(W_G - W_B)$.

3.3.2 Quadratic utility

The quadratic utility with parameter $c > 0$ reads

$$(3.13) \quad u_c(x) = \begin{cases} x - cx^2/2, & x \leq 1/c \\ 1/(2c), & x > 1/c \end{cases}$$

and feasible portfolios live on the strictly increasing part of the utility function. Suppose the SEU agents have utility u_a , the maxmin agents utility u_b and $a, b > 0$. The fixed point equations for the equilibrium prices are

$$\begin{aligned} p_R &= \frac{\pi_R(c - W_R)}{c - \pi \cdot W} \\ p_G &= \frac{\pi_G(c - W_G)}{c - \pi \cdot W} \left(\frac{D - (\frac{1}{b} - X^{\text{MEU}})(c - \pi \cdot W)(1 + \frac{W_G - W_B}{c - W_G} \frac{\pi_B}{1 - \pi_R})}{D - (\frac{1}{b} - X^{\text{MEU}})(c - \pi \cdot W)} \right) \\ p_B &= \frac{\pi_B(c - W_B)}{c - \pi \cdot W} \left(\frac{D - (\frac{1}{b} - X^{\text{MEU}})(c - \pi \cdot W)(1 - \frac{W_G - W_B}{c - W_B} \frac{\pi_G}{1 - \pi_R})}{D - (\frac{1}{b} - X^{\text{MEU}})(c - \pi \cdot W)} \right) \end{aligned}$$

where $c := \frac{1}{a} + \frac{1}{b}$, $D := \pi_R(c - W_R)^2 + (1 - \pi_R)(c - \frac{\pi_G W_G + \pi_B W_B}{1 - \pi_R})^2$ and X^{MEU} is the initial wealth of the maxmin agents. Note that p_G is lower (p_B is higher) than the price $\frac{\pi_G(c - W_G)}{c - \pi \cdot W}$ (respectively $\frac{\pi_B(c - W_B)}{c - \pi \cdot W}$) that would result in a market equilibrium with SEU agents sharing the same prior (π_R, π_G, π_B) .

Let $W_G > W_B > W_R$. Figure 1 shows the state-price/state-probability ratios of the equilibrium prices as a function of the difference $W_B - W_R$, computed for fixed $W_G = 272$ and $W_R = 81$ ¹⁷. The parameters a and b in (3.13) are set to 0.001 in the left graph, and to $a = 0.0015$ and $b = 0.001$ in the right graph.

[Include Figure 1 here]

In both cases there is a clear change of rankings of state-price/state-probability ratios: as $W_B - W_R$ increases, the ranking switches from (3.9) to (3.10).

¹⁷These values of the aggregate endowment W are the same values used by Bossaerts et al. (2010) in one of their experiments that we discuss in Section 3.4 and which empirical rankings are summarized in Figure 2 in our paper.

3.4 Revisiting laboratory experimental findings

Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010) run a series of laboratory experiments that reproduce a competitive financial market in the standard Ellsberg framework. The comparison of the experimental cross sectional distribution of the security holdings and empirical state-price/state-probability ratios, with and without ambiguity, provide clear evidence that ambiguity aversion matters for portfolio choices and equilibrium prices and does not wash out in aggregate. Bossaerts et al. (2010) summarize their experimental findings about equilibrium asset prices in Figures 6–8; Bossaerts et al. (2010, pages 1349 and 1350). These figures show the empirical distribution functions of the state-price/state-probability ratios obtained from experimental sessions with different total endowments. To support their experimental findings Bossaerts et al. (2010) use a theoretical market model involving SEU and ambiguity averse α -MEU agents with $\alpha > 1/2$. The theoretical portfolio choice supports the experimental data. However, Bossaerts et al. (2010) do not derive the equilibrium prices and some of their conjectures about the equilibrium state-price/state-probability ratio are not always supported by the data, as discussed by the authors; see Conclusion in Bossaerts et al. (2010). In the following we show that the rankings in equilibrium of the state-price/state-probability ratios we derive in Proposition 3.2 for a market populated by SEU and maxmin agents fully explain and thus theoretically justify all empirical rankings documented by Bossaerts et al. (2010). Note that the market model for which we derive the theoretical rankings is the same theoretical market model proposed by Bossaerts et al. (2010) to explain the experimental market.¹⁸

Figure 6 in Bossaerts et al. (2010), where the total endowment $W = (W_R, W_G, W_B)$ is such $W_G > W_R > W_B$, provides evidence of one ranking of the empirical state-price/state-probability ratio, which is exactly the ranking (3.8) predicted by our Proposition 3.2. Proposition 3.2 confirms the conjecture in Bossaerts et al. (2010, page 1339) that ranking (3.8) is more likely to occur when $W_G > W_R > W_B$. Indeed, this is the only theoretically possible ranking. However, Proposition 3.2 contradicts the conjecture that no matter how the total wealth W_R ranks with respect to the total wealth in the ambiguous states, W_G and W_B ,

¹⁸From Proposition 3.1 we know that in the standard Ellsberg framework α -MEU preferences with $\alpha > 1/2$ are equivalent to maxmin preferences.

any ranking of the state-price/state-probability ratio for the risky state p_R/π_R with respect to $p_B/\pi_B > p_G/\pi_G$ is theoretically possible; Bossaerts et al. (2010, page 1339). In fact, the theoretically possible rankings do depend on the ranking of W_R with respect to W_G and W_B .

Figure 2 below is a copy of Figure 8, right panel, in Bossaerts et al. (2010) where $W_G > W_B > W_R$.

[Include Figure 2 here]

The experimental findings summarized in Figure 2 provide evidence of two rankings: $p_R/\pi_R > p_B/\pi_B > p_G/\pi_G$, and $p_B/\pi_B > p_R/\pi_R > p_G/\pi_G$.¹⁹ Remarkably, Proposition 3.2 shows that these are exactly the two rankings possible, namely (3.9) and (3.10), when $W_G > W_B > W_R$.

Proposition 3.2 further shows that the ranking (3.10) prevails when $W_B - W_R$ is large enough to imply an optimal portfolio of the SEU agents with more Arrow securities that pay in the ambiguous state B than in the risky state R . This provides a potential explanation why in Figure 2 the prices do not settle in favor of one of the two rankings: the values W_R , W_G , and W_B in the experimental section in Figure 2 are close to the point at which the change from (3.9) to (3.10) takes place. Example 3.3.2 illustrates this point: When $W_B - W_R$ is approximately 81, the switch of the rankings occurs as in Figure 1. This confirms that to observe only one ranking of state-price/state-probability ratios in laboratory experiment, the difference in aggregate wealth $W_B - W_R$ should be chosen either relatively large or small.

Bossaerts et al. (2010) perform other experimental sessions in which $W_G > W_B > W_R$, summarized in Figure 7 in Bossaerts et al. (2010). Although the most common ranking of state-price/state-probability ratios is (3.9), the empirical distribution functions of p_R/π_R and p_B/π_B are very close. Proposition 3.2 predicts that to observe a clear separation of the rankings in (3.9) and (3.10), the aggregate wealth W_B should be chosen closer to W_R or W_G , respectively.

¹⁹Bossaerts et al. (2010, Page 1351) report that “the rankings appear anomalous,” because they only expect to see the second ranking (Bossaerts et al. 2010, Page 1339).

4 Extended Ellsberg framework

In this section we consider an extended Ellsberg framework, that is a state space S where the future states of the economy correspond to draws from an extended Ellsberg (1961) urn with m risky states with known probability, and $l \geq 3$ ambiguous states. $A \subset S$ denotes the set that contains the ambiguous states, thus $|A| = l$. The known probabilities $\pi_R \in (0, 1)$ of risky states $R \in S \setminus A$ satisfy $\sum_{R \in S \setminus A} \pi_R < 1$.

As set of priors \mathcal{C} of the α -MEU model in (2.1), we choose the set that contains all priors consistent with the uncertainty (i.e. the risk and the ambiguity) of the extended Ellsberg framework, namely all priors such that the probabilities on the risky states equal the known probabilities π_R , $R \in S \setminus A$. We call this set \mathcal{C}_{\max} and denote this class of models by α - \mathcal{C}_{\max} -MEU. Fixing $\mathcal{C} = \mathcal{C}_{\max}$ allows to interpret the parameter α as a measure of the agent's degree of ambiguity aversion and thereby allow for comparative statics.²⁰ Moreover, it makes the comparison between α -MEU and maxmin preferences meaningful by providing a utility specification common to the two classes of models (see Section 4.4).

4.1 The α - \mathcal{C}_{\max} -MEU model

The α -MEU utility in (2.1) with $\mathcal{C} = \mathcal{C}_{\max}$ can be rewritten as

$$(4.1) \quad U(w) = \sum_{R \in S \setminus A} \pi_R u(w_R) + (1 - \sum_{R \in S \setminus A} \pi_R) [\alpha u(w_{\min}^A) + (1 - \alpha) u(w_{\max}^A)]$$

where w_{\min}^A and w_{\max}^A is respectively the smallest and the largest wealth in the portfolio $w \in \mathbb{R}^{m+l}$ allocated among the l ambiguous states, that is

$$(4.2) \quad w_{\min}^A := \min_{\sigma \in A} w_{\sigma}, \quad \text{and} \quad w_{\max}^A := \max_{\sigma \in A} w_{\sigma}.$$

²⁰The interpretation of α could be lost by assuming a set of priors smaller than \mathcal{C}_{\max} . When \mathcal{C} is strictly smaller than \mathcal{C}_{\max} , it can reflect both additional information and less aversion towards ambiguity; see, e.g., Siniscalchi (2006), Gilboa and Marinacci (2013), and Machina and Siniscalchi (2014). For this reason, studies of agent's ambiguity aversion based on α -MEU model typically fix the set of priors equal to the one describing the uncertainty of the setting; see, e.g., Chen et al. (2007), and Ahn et al. (2014).

From (4.1) one can see that, when the number of ambiguous states is larger than two, the α - \mathcal{C}_{\max} -MEU utility with $\alpha \in (0, 1)$ cannot in general be rewritten neither as maxmin, maxmax nor as subjective expected utility.²¹ For instance, while SEU and maxmin utilities are always concave if u is concave, the α - \mathcal{C}_{\max} -MEU utility is concave if and only if $\alpha = 1$; see Appendix C.²² Thus the equivalence result in Proposition 3.1 holds only as long as there are only two ambiguous states. In Section 4.4 we show that, when the number of ambiguous states is larger than two, the α - \mathcal{C}_{\max} -MEU model and the maxmin model imply also different portfolio choices. Thus, in contrast to the standard Ellsberg framework, in the extended Ellsberg framework the α - \mathcal{C}_{\max} -MEU and maxmin models are not observationally equivalent.

4.2 The α - \mathcal{C}_{\max} -MEU portfolio choice

Before stating the proposition that characterizes the portfolio choice of α - \mathcal{C}_{\max} -MEU agents we introduce few notations that we use hereafter. Given a pricing rule $p \in \mathbb{R}^{m+l}$, p_{\min}^A denotes the lowest (minimum) price among the ambiguous state prices, that is

$$p_{\min}^A := \min_{\eta \in A} p_{\eta}$$

and I the set that contains all ambiguous states with lowest price, that is

$$I := \{\sigma \in A \mid p_{\sigma} = p_{\min}^A\}.$$

Finally, by $\tilde{\pi}$ we denote the prior which assigns to the risky states the corresponding known probabilities π_R , $R \in S \setminus A$, and to each ambiguous state equal probability $\tilde{\pi}_a := \frac{\sum_{R \in S \setminus A} \pi_R}{l}$.

²¹A way to see this is to observe that the α - \mathcal{C}_{\max} -MEU utility from a portfolio $w \in \mathbb{R}^{m+l}$ on the ambiguous states only depends on w_{\min}^A and w_{\max}^A . This is not the case for a maxmin (maxmax) utility model, as long as the state space contains more than two ambiguous states. The utility of the maxmin model from a portfolio $w \in \mathbb{R}^{m+l}$ will be a function of the portfolio's smallest wealth w_{\min}^A (respectively, the portfolio's largest wealth w_{\max}^A) and then, depending on the set of priors, of the second smallest wealth (respectively, the second largest wealth) and so on, until the sum of the probabilities of the states in which these wealths are allocated reaches $(1 - \sum_{R \in S \setminus A} \pi_R)$.

²²Also when the set of priors \mathcal{C} is a strict subset of \mathcal{C}_{\max} , the α -MEU utility cannot in general be rewritten neither as 1-MEU, 0-MEU nor SEU, and is not concave.

Proposition 4.1. *Suppose that the state price vector p satisfies $p_\sigma > 0$ for all $\sigma \in S$. Consider an α - \mathcal{C}_{\max} -MEU agent. Let $\alpha \in (0, 1]$ and suppose there is an optimal portfolio.*

- *If*

$$(4.3) \quad \alpha < 1 - \frac{p_{\min}^A}{1 - \sum_{R \in S \setminus A} p_R} = 1 - \frac{p_{\min}^A}{\sum_{\nu \in A} p_\nu}$$

there are $|I|$ optimal portfolios: all optimal portfolios coincide on the risky states whereas on the ambiguous states they only take two different values $\bar{w} \in \mathbb{R}$ and $\underline{w} \in \mathbb{R}$ with $\underline{w} < \bar{w}$, which are the same for all optimal portfolios. Every optimal portfolio is obtained by choosing a single ambiguous state $\nu \in I$ ($p_\nu = p_{\min}^A$) among the cheapest ones and then setting

$$(4.4) \quad \begin{cases} w_\nu = \bar{w} \\ w_\eta = \underline{w} \text{ for the remaining } (l-1) \text{ ambiguous states } \eta \in A \setminus \{\nu\}. \end{cases}$$

Hence, for all optimal portfolios $w_{\min}^A = \underline{w}$ and $w_{\max}^A = \bar{w}$.

- *If*

$$(4.5) \quad \alpha \geq 1 - \frac{p_{\min}^A}{\sum_{\nu \in A} p_\nu} \quad \text{or equivalently} \quad \alpha \geq 1 - \frac{p_\eta}{\sum_{\nu \in A} p_\nu}, \forall \eta \in A,$$

the optimal portfolio w is unique and unambiguous, i.e. $w_{\max}^A = w_{\min}^A$. In particular, when $\alpha = 1$, the optimal portfolio w is always unique and unambiguous.

If $\alpha = 0$, there is no optimal portfolio.

The proof of Proposition 4.1 is provided in Appendix D. The arguments used in the proof show that Proposition 4.1 holds true for any $\alpha \in (0, 1]$ also when the utility u in (2.1) has a bounded domain, as long as the set of feasible portfolios remains convex and the utility is differentiable.²³ The only difference that a utility with bounded domain would bring is the

²³These properties are used in the proofs of Lemmas D.1–D.5 that in turns prove Proposition 4.1, and can be ensured for instance by requiring that the feasible portfolios are in the interior of the utility domain.

existence of optimal portfolios of the 0-MEU agent.²⁴

The following example illustrates Proposition 4.1 when the state space S contains $m = 1$ risky state and $l = 3$ ambiguous states. Note that $l = 3$ yields $p_{\min}^A \leq \frac{1-p_R}{3}$ which is equivalent to $1 - \frac{p_{\min}^A}{1-p_R} \geq 2/3$. From the last inequality and Condition (4.3) follow that any agent with $\alpha \in (0, 2/3)$ prefers an ambiguous portfolio²⁵.

Example 4.2. Let $S = \{R\} \cup A$ where $A = \{G, B, Y\}$. Consider an α - \mathcal{C}_{\max} -MEU agent and let $w = (w_R, w_G, w_B, w_Y) \in \mathbb{R}^4$ be her optimal portfolio. Without loss of generality, let $0 < p_G \leq p_B \leq p_Y$.

- Let $\alpha \in (0, 2/3)$. Then the optimal portfolio is always exposed to ambiguity. In particular there are $w_R, \bar{w}, \underline{w} \in \mathbb{R}$ with $\bar{w} > \underline{w}$ such that:
 - (i) if $p_G < p_B$ ($p_{\min}^A = p_G$ and $I = \{G\}$), the optimal portfolio is unique and reads $w = (w_R, \bar{w}, \underline{w}, \underline{w})$
 - (ii) if $p_G = p_B < p_Y$ ($p_{\min}^A = p_G = p_B$ and $I = \{G, B\}$), then there are two optimal portfolios, namely $(w_R, \bar{w}, \underline{w}, \underline{w})$ and $(w_R, \underline{w}, \bar{w}, \underline{w})$
 - (iii) if $p_G = p_B = p_Y$ ($p_{\min}^A = p_G = p_B = p_Y = \frac{1-p_R}{3}$ and $I = A$), then there are three optimal portfolios: $(w_R, \bar{w}, \underline{w}, \underline{w})$, $(w_R, \underline{w}, \bar{w}, \underline{w})$, and $(w_R, \underline{w}, \underline{w}, \bar{w})$.
- Let $\alpha \in [\frac{2}{3}, 1]$. The optimal portfolio w is unambiguous (i.e. $w_G = w_B = w_Y$) if and only if $\alpha \geq 1 - \frac{p_{\min}^A}{1-p_R}$ or equivalently $\alpha \geq 1 - \frac{p_\eta}{1-p_R}, \forall \eta \in A = \{G, B, Y\}$. This is always the case if $p_G = p_B = p_Y$ or if $\alpha = 1$. Otherwise, i.e. if $\alpha < 1 - \frac{p_{\min}^A}{1-p_R}$, either (i) or (ii) holds.

Proposition 4.1 and Example 4.2 show that an α - \mathcal{C}_{\max} -MEU agent with $\alpha \in (0, 1)$, facing a state price vector $p \in \mathbb{R}^{m+l}$, optimally chooses only two type of portfolios: either an unambiguous portfolio (with no exposure to ambiguity, allocating equal wealth in each ambiguous

²⁴The non-existence of the optimal portfolio of 0-MEU agent is due to the fact that when the utility is defined on the whole real line, the agent can go arbitrarily long in one of the ambiguous states and still satisfy the budget constraint by going arbitrarily short in another ambiguous state. A utility with bounded domain would imply the existence of an optimal portfolio for the 0-MEU agent as the bounded domain will prevent the agent from going arbitrarily short; see Lemma D.5.

²⁵The dependence of the portfolio choice on the number of ambiguous states l is discussed in Section 4.2.4.

state), or an ambiguous portfolio with the specific exposure to ambiguity in (4.4), i.e. allocating more wealth \bar{w} to one of the cheapest ambiguous state and less equal wealth \underline{w} to each of the remaining $(l - 1)$ ambiguous states. This portfolio can be seen as an unambiguous portfolio with equal wealth \underline{w} in each ambiguous state, plus a bet of $(\bar{w} - \underline{w}) > 0$ on one of the cheapest state. The larger the difference $\bar{w} - \underline{w}$, the more the portfolio is exposed to ambiguity.

The choice between unambiguous and ambiguous portfolios only depends on α and the ratio of the lowest price among the ambiguous state prices p_{\min}^A , to the total sum of the ambiguous state prices, $\sum_{\eta \in A} p_{\eta}$. If (4.5) holds, the optimal portfolio is unambiguous and unique. Otherwise if (4.3) holds, the optimal portfolio is ambiguous. The ambiguous portfolio, when optimal, is unique if there is only one ambiguous state with price p_{\min}^A , i.e. $|I| = 1$. If the ambiguous states with price p_{\min}^A are more than one, i.e. $|I| > 1$, then there are $|I|$ optimal ambiguous portfolios. The α - \mathcal{C}_{\max} -MEU agent finds equally optimal to bet $\bar{w} - \underline{w}$ on any of the $|I|$ cheapest ambiguous states, because ambiguous states with equal prices are indistinguishable from an informational point of view. All optimal portfolios provide the same exposure to ambiguity. For an illustration, see Example 4.2.

In the following we discuss the dependence of the α - \mathcal{C}_{\max} -MEU agent's optimal portfolio on the ambiguity aversion, risk aversion and number of ambiguous states.

4.2.1 Impact of ambiguity aversion on portfolio choice

To understand how the ambiguity aversion parameter α determines the optimal exposure to ambiguity, we rewrite the α - \mathcal{C}_{\max} -MEU utility (4.1) from portfolio $w \in \mathbb{R}^{m+l}$ as

$$(4.6) \quad U(w) = \sum_{R \in S \setminus A} \pi_R u(w_R) + (1 - \sum_{R \in S \setminus A} \pi_R) [u(w_{\min}^A) + (1 - \alpha)(u(w_{\max}^A) - u(w_{\min}^A))].$$

This equation shows that the coefficient $(1 - \alpha)$ weights the utility $(u(w_{\max}^A) - u(w_{\min}^A))$ that the α - \mathcal{C}_{\max} -MEU agent derives from the maximal exposure to ambiguity of the portfolio w , that is from $w_{\max}^A - w_{\min}^A$. When α increases the utility from the exposure to ambiguity decreases: when $\alpha = 0$ this utility is highest, when $\alpha = 1$ the utility is zero. This implies that, the higher is α , the smaller is the exposure to ambiguity of the agent's optimal portfolio.

We now study how the parameter α and the utility u determine the optimal allocation of wealth to risky and ambiguous states. Note that the optimal allocation of wealth among the risky states only depend on the utility u ; see (4.1).

Let R denote the risky state, and set $m = 1$ for simplicity. The following holds true.

- *An increase of α decreases the demand for the ambiguous portfolio.*

Inequalities (4.5) show that an increase of α decreases the set of prices for which an $\alpha\mathcal{C}_{\max}$ -MEU agent prefers an ambiguous portfolio. The limit case is $\alpha = 1$ in which the optimal portfolio is always unambiguous.

- *An increase of α decreases the exposure to ambiguity of the ambiguous portfolio.*

This can be deduced from the first order conditions satisfied by the optimal ambiguous portfolio (see (D.2) in Lemma D.4 when $m = 1$)

$$(4.7) \quad \frac{u'(\underline{w})}{u'(\bar{w})} = \frac{\sum_{\nu \in A \setminus \{\sigma\}} p_{\nu} (1 - \alpha)}{p_{\sigma} \alpha} \quad \text{and} \quad \frac{u'(w_R)}{u'(\underline{w})} = \frac{\alpha(1 - \pi_R)p_R}{\sum_{\nu \in A \setminus \{\sigma\}} p_{\nu} \pi_R}$$

where σ denotes (one of) the cheapest state among the ambiguous states, i.e. $\sigma \in I$ and $p_{\sigma} = p_{\min}^A$, $w_{\sigma} = \bar{w}$ and $w_{\eta} = \underline{w}$ for all $\eta \in A \setminus \{\sigma\}$.

The closer α is to 0, the larger is $\bar{w} - \underline{w}$.²⁶ When α increases, the exposure to ambiguity of the agent's optimal portfolio $\bar{w} - \underline{w}$ decreases (as the ratio $\frac{1-\alpha}{\alpha}$ decreases). When $\alpha \uparrow 1 - \frac{p_{\min}^A}{\sum_{\nu \in A} p_{\nu}}$, the bet $\bar{w} - \underline{w} \downarrow 0$, that is the ambiguous portfolio becomes unambiguous; see Condition (4.5).²⁷

- *When the ambiguous portfolio is optimal, an increase of α always leads to an increase in the risk premium for the cheapest ambiguous state.*

An $\alpha\mathcal{C}_{\max}$ -MEU agent chooses an ambiguous portfolio if and only if among the ambiguous states there is at least one state σ (one of the cheapest ambiguous states) that

²⁶In the limit, when $\alpha \rightarrow 0$, $\bar{w} - \underline{w} \rightarrow \infty$, and thus there is no optimum; see discussion after Proposition 4.1.

²⁷When $\alpha \uparrow 1 - \frac{p_{\min}^A}{\sum_{\nu \in A} p_{\nu}}$, $1 - \alpha \downarrow \frac{p_{\min}^A}{\sum_{\nu \in A} p_{\nu}}$ and thus $\frac{u'(\underline{w})}{u'(\bar{w})} \downarrow 1$.

satisfies $p_\sigma < (1 - \alpha)(1 - p_R)$; see (4.3). Hence, the larger is α , the smaller p_σ must be in order to make the agent choose an ambiguous portfolio.

Equalities (4.7) also show how α impacts the allocation of wealth between the risky state R and the ambiguous states. An increase of α decreases the difference $\bar{w} - w_R$. When $\alpha \uparrow 1 - \frac{p_{\min}^A}{\sum_{\nu \in A} p_\nu}$ the optimal portfolio tends to the unambiguous portfolio, and the optimal allocation of wealth between the risky and the ambiguous states is the same as that of an SEU with the prior $\tilde{\pi}$.

Figure 3 illustrates the impact of the parameter α on the $\alpha\text{-}\mathcal{C}_{\max}$ -MEU agent's optimal portfolio when the agent's utility u in (4.1) is a CARA utility when $m = 1$ and $l = 4$.

[Include Figure 3 here]

4.2.2 Impact of risk aversion on portfolio choice

The utility function u in (4.1) that characterizes the risk aversion of the $\alpha\text{-}\mathcal{C}_{\max}$ -MEU agent also affects the extent to which the ambiguous portfolio in (4.4) is exposed to ambiguity. Equalities (4.7) show that, given a state price vector p and $\alpha \in (0, 1)$, the more the utility function u is concave (i.e. the faster u' decreases) the smaller is the portfolio exposure $\bar{w} - \underline{w}$ to ambiguity and the difference $w_R - \bar{w}$. The dependence of the ambiguity exposure on risk aversion is illustrated in Figure 4 assuming CARA utility. When risk aversion increases, the agent eventually invests in the risk free asset.

[Include Figure 4 here]

4.2.3 Portfolio inertia

The $\alpha\text{-}\mathcal{C}_{\max}$ -MEU model implies portfolio inertia both at the unambiguous and at the ambiguous portfolio, when optimal. The reason is that the optimal portfolio choice only depends on the values α , p_{\min}^A and $\sum_{\eta \in A} p_\eta = 1 - \sum_{R \in S \setminus A} p_R$. For example, consider the state space $S = \{R, 1, 2, \dots, l\}$ with one risky state ($m = 1$) denoted by R and $l \geq 3$ ambiguous states denoted respectively by $1, 2, \dots, l$. Let $\bar{p} = (\bar{p}_R, \bar{p}_1, \dots, \bar{p}_l)$ be the state price vector in the market.

- *Portfolio inertia at the ambiguous portfolio.*

Let the state 1 be the cheapest state among the ambiguous states, that is $p_{\min}^A = \bar{p}_1$. If (4.3) holds, that is if $\alpha < 1 - \frac{\bar{p}_1}{\sum_{\eta \in A} \bar{p}_\eta} = 1 - \frac{\bar{p}_1}{1 - \bar{p}_R}$, the α - \mathcal{C}_{\max} -MEU agent's optimal portfolio is $w^* = (w_R, \bar{w}, \underline{w}, \dots, \underline{w})$ for some $\bar{w} > \underline{w}$. Even if the prices of the ambiguous states $j = 2, \dots, l$ change, portfolio w^* remains optimal as long as \bar{p}_R and \bar{p}_1 do not change and \bar{p}_1 remains the cheapest price among the ambiguous states. Thus, the set of state price vectors for which there is portfolio inertia at w^* is

$$\{p = (\bar{p}_R, \bar{p}_1, p_2, \dots, p_l) \in \mathbb{R}^{m+l} \mid p_i \geq \bar{p}_1, \forall i = 2, \dots, l \text{ and } \sum_2^l p_i = 1 - \bar{p}_R - \bar{p}_1\}.$$

Rewriting (4.3) as $p_{\min}^A < (1 - \alpha)(1 - \bar{p}_R)$ shows that, the larger is α the smaller is this set.

- *Portfolio inertia at the unambiguous portfolio.*

If (4.5) holds, that is $\alpha \geq 1 - \frac{\bar{p}_{\min}^A}{\sum_{\eta \in A} \bar{p}_\eta}$, the α - \mathcal{C}_{\max} -MEU agent's optimal portfolio is the unambiguous $w^* = (w_R, \tilde{w}, \dots, \tilde{w})$ for some $\tilde{w} \in \mathbb{R}$. The first order conditions satisfied by this portfolio

$$\frac{\pi_R u'(w_R)}{p_R} = \frac{(1 - \pi_R) u'(\tilde{w})}{1 - p_R}$$

show that \bar{p}_R alone determines w_R and \tilde{w} . Then, even if the prices of all the ambiguous states $j = 1, \dots, l$ change portfolio w^* remains optimal as long as p_{\min}^A still satisfies (4.5) and \bar{p}_R does not change. Thus, the set of state price vectors for which there is portfolio inertia at w^* is

$$\{p = (\bar{p}_R, p_1, p_3, \dots, p_l) \in \mathbb{R}^{m+l} \mid p_i \geq (1 - p_R)(1 - \alpha), \forall i = 1, \dots, l \text{ and } \sum_1^l p_i = 1 - \bar{p}_R\}.$$

The larger is α , the larger is this set.

4.2.4 Portfolio choice and number of ambiguous states

We now show that an α - \mathcal{C}_{\max} -MEU agent with $\alpha \in (0, \frac{l-1}{l})$ always prefers the ambiguous portfolio (4.4) and never chooses an unambiguous portfolio. The reason is that in a complete

finite state space model when $\alpha \in (0, \frac{l-1}{l})$, Condition (4.3) is automatically satisfied. The normalization $\sum_{\eta \in A} p_\eta + \sum_{S \setminus A} p_R = 1$ yields

$$(4.8) \quad p_{\min}^A \leq \frac{\sum_{\eta \in A} p_\eta}{l} \quad \text{and} \quad p_{\min}^A = \frac{\sum_{\eta \in A} p_\eta}{l} \Leftrightarrow p_\nu = p_\eta \quad \forall \nu, \eta \in A.$$

As $p_{\min}^A \leq \frac{\sum_{\eta \in A} p_\eta}{l}$ is equivalent to $1 - \frac{p_{\min}^A}{\sum_{\eta \in A} p_\eta} \geq \frac{l-1}{l}$, any $\alpha \in (0, \frac{l-1}{l})$ satisfies Condition (4.3).

When ambiguous states have equal price, from the equality in (4.8) and Proposition 4.1 follow that the optimal portfolio of any α - \mathcal{C}_{\max} -MEU agent with $\alpha \in [\frac{l-1}{l}, 1]$ is unique and unambiguous. We formalize these concepts in the following corollaries.

Corollary 4.3. *In the setting of Proposition 4.1, any α - \mathcal{C}_{\max} -MEU agent with $\alpha \in (0, \frac{l-1}{l})$ always chooses a portfolio exposed to ambiguity of the type described in (4.4).*

Corollary 4.4. *In the setting of Proposition 4.1, suppose that the prices of the ambiguous states are all equal, i.e. $p_\nu = p_\eta$ for all $\nu, \eta \in A$.*

- (i) *If $\alpha \in (0, \frac{l-1}{l})$, the optimal portfolios are the ambiguous portfolios in (4.4). Since $|I| = |A| = l$, the number of optimal portfolios equals the number of ambiguous states.*
- (ii) *If $\alpha \in [\frac{l-1}{l}, 1]$, the optimal portfolio is unique and unambiguous, i.e. $w_{\max}^A = w_{\min}^A$.*

An increase of the number l of ambiguous states increases the interval of α -values $(0, \frac{l-1}{l})$ for which the corresponding α - \mathcal{C}_{\max} -MEU agent always chooses a portfolio exposed to ambiguity. The reason is that increasing l lowers the upper bound of p_{\min}^A in (4.8) and consequently increases the interval of α -values that satisfy (4.3). For example, setting $m = 0$ for simplicity, when $l = 4$, $p_{\min}^A \leq 0.25$ and the interval of α -values for which the α - \mathcal{C}_{\max} -MEU agent prefers exposure to ambiguity is $(0, 0.75)$. While when $l = 20$, $p_{\min}^A \leq 0.05$ and the interval of α -values is $(0, 0.95)$.

In Sections 4.3 and 4.4 we show how these results can be used in laboratory experiments to test the α -MEU model, measure the agent's degree of ambiguity aversion and distinguish between α - \mathcal{C}_{\max} -MEU agents and maxmin agents.

4.3 Different attitudes towards ambiguity of the α - \mathcal{C}_{\max} -MEU model

To understand the different attitudes toward ambiguity expressed by the α - \mathcal{C}_{\max} -MEU model we start with the optimal portfolio choice when the prices of the ambiguous states are all equal. In this setting all the ambiguous states are equivalent from an informational point of view, and thus indistinguishable.²⁸ From Corollary (4.4) we know that when $l \geq 3$ and ambiguous state prices are equal:

- any α - \mathcal{C}_{\max} -MEU agent with $\alpha \in (0, \frac{l-1}{l})$ optimally chooses an ambiguous portfolio allocating equal wealth \underline{w} on each ambiguous state plus a bet of size $\bar{w} - \underline{w} > 0$ made indifferently on one of the l ambiguous states. The number of optimal portfolios is equal to the number l of ambiguous states.
- any α - \mathcal{C}_{\max} -MEU agent with $\alpha \in [\frac{l-1}{l}, 1]$ optimally chooses an unambiguous portfolio with equal wealth in each ambiguous state. The optimal portfolio is unique.

The fact that α - \mathcal{C}_{\max} -MEU agents with $\alpha < \frac{l-1}{l}$ makes a bet on one among the l ambiguous states despite these states are all indistinguishable shows an ambiguity loving (or seeking) behavior of these agents. The ambiguity seeking behavior becomes more pronounced when α decreases. Indeed, as shown in Section 4.2.1, the smaller is α the larger is the exposure to ambiguity $\bar{w} - \underline{w}$ in the agent's optimal portfolio.

In contrast, α - \mathcal{C}_{\max} -MEU agents, with $\alpha \geq \frac{l-1}{l}$, do not show any ambiguity seeking behavior. When facing indistinguishable ambiguous states they optimally choose an unambiguous portfolio with equal wealth on each ambiguous state.

We observe that when $\alpha = \frac{l-1}{l}$, the α - \mathcal{C}_{\max} -MEU agent is not equivalent, not even observationally, to an ambiguity neutral SEU agent.²⁹ The $\frac{l-1}{l}$ - \mathcal{C}_{\max} -MEU agent and the SEU agent with prior $\tilde{\pi}$ choose the same unambiguous portfolio when ambiguous state prices are equal. However, the two agents choose different portfolios when the ambiguous states

²⁸A comparatively high price in one of the ambiguous state may make the agents believe that this state has a higher probability of occurrence than the other ambiguous states, even though in the Ellsberg framework an exact knowledge of the probabilities is not available and the ambiguous states are “equally ambiguous”.

²⁹In Standard Ellsberg framework, when $l = 2$, any α -MEU utility with $\alpha = \frac{l-1}{l} = \frac{1}{2}$ reduces to a SEU utility; see Proposition 3.1.

with cheapest price are more than one and less than $l - 1$ (i.e., $1 < |I| < l - 1$).³⁰ For example, when $m = 0$, $l = 3$, and two cheapest ambiguous states, G and B , there are two $\frac{2}{3}$ - \mathcal{C}_{\max} -MEU optimal portfolios: $(w_G, w_B, w_Y) = (\bar{w}, \underline{w}, \underline{w})$ and $(w_G, w_B, w_Y) = (\underline{w}, \bar{w}, \underline{w})$, for some $\bar{w} > \underline{w}$. While the SEU optimal portfolio is unique and equals $(y_G, y_B, y_Y) = (\bar{y}, \bar{y}, \underline{y})$ for some $\bar{y} > \underline{y}$.

4.4 Disentangling between α - \mathcal{C}_{\max} -MEU and maxmin agents

The utility of a maxmin agent from some state dependent wealth $w \in \mathbb{R}^{l+m}$ is

$$(4.9) \quad U(w) = \sum_{R \in S \setminus A} \pi_R u(w_R) + \min_{\pi \in \mathcal{C}} \sum_{\sigma \in A} \pi_\sigma u(w_\sigma)$$

where $\mathcal{C} \subseteq \mathcal{C}_{\max}$ is a convex and closed set of priors. When $\mathcal{C} = \mathcal{C}_{\max}$ this utility is equal to the α - \mathcal{C}_{\max} -MEU utility (4.1) when $\alpha = 1$, and provides the maximal degree of ambiguity aversion. Shrinking the set of priors \mathcal{C} in (4.9) decreases the exposure to ambiguity of the maxmin optimal portfolio, like decreasing the parameter α in (4.1) decreases the exposure to ambiguity of the α - \mathcal{C}_{\max} -MEU optimal portfolio.

The equivalence result in Proposition 3.1 shows that in the standard Ellsberg framework (two ambiguous states, i.e., $|A| = l = 2$) α -MEU utilities with $\alpha > \frac{1}{2}$ are maxmin utilities, and thus α -MEU preferences cannot be distinguished from maxmin preferences. In the following we show that in an extended Ellsberg framework (three or more ambiguous states, i.e., $|A| = l \geq 3$) this distinction can instead be achieved.³¹ To disentangle α -MEU from maxmin preferences we exploit the α - \mathcal{C}_{\max} -MEU portfolio choice derived in Section 4.2 and the following lemma.

Lemma 4.5. *Suppose that $p_\sigma = p_\eta$ for all $\sigma, \eta \in A$. Then any maxmin agent (4.9) with a set of priors \mathcal{C} such that $\tilde{\pi} \in \mathcal{C}$ takes an unambiguous portfolio.*

This lemma shows that when facing ambiguous states with equal prices, any maxmin

³⁰In fact, when $1 < |I| < l - 1$, the optimal portfolio of any α - \mathcal{C}_{\max} -MEU agent with $\alpha \in [0, 1]$ is different from the SEU optimal portfolio.

³¹In Section 4.2 we observe that the equivalence result does not hold when the number of ambiguity states is larger than two. However, the fact that α -MEU utilities cannot be rewritten as 1-MEU utilities does not imply that their portfolio choice may not be observationally equivalent.

agent with a set of priors \mathcal{C} that includes $\tilde{\pi}$ chooses a portfolio with no exposure to ambiguity. In Ellsberg frameworks where the ambiguous states are all “equally ambiguous” (i.e., indistinguishable from a probabilistic point of view) it is natural that the prior $\tilde{\pi}$ belongs to the maxmin agent’s set of priors.³²

In the following we point out the differences between the α -MEU optimal portfolio and the maxmin optimal portfolio that may be used in a multiple-stage laboratory experiment to disentangle ambiguity seeking from non-ambiguity seeking agents, and among the latter, $\alpha\mathcal{C}_{\max}$ -MEU from maxmin agents.

4.4.1 Ambiguity seeking and non-ambiguity seeking agents

The identification of ambiguity seeking $\alpha\mathcal{C}_{\max}$ -MEU agents, $\alpha \in (0, \frac{l-1}{l})$, and non-ambiguity seeking agents, $\alpha\mathcal{C}_{\max}$ -MEU agents with $\alpha \in [\frac{l-1}{l}, 1)$ and the maxmin agents, can be achieved by observing their different portfolio choices. For example, when $m = 1$, $l = 3$, and the ambiguous states have equal prices, ambiguity seeking $\alpha\mathcal{C}_{\max}$ -MEU agents ($\alpha < \frac{2}{3}$) should choose and be indifferent among the following three portfolios: $(w_R, \bar{w}, \underline{w}, \underline{w})$, $(w_R, \underline{w}, \bar{w}, \underline{w})$ and $(w_R, \underline{w}, \underline{w}, \bar{w})$ for some $\bar{w} > \underline{w}$. While $\alpha\mathcal{C}_{\max}$ -MEU agents with $\alpha \geq \frac{2}{3}$ and maxmin agents should choose one (unique) unambiguous portfolio.

Implications for equilibrium market prices depend on whether ambiguity seeking agents are present or not in the market. As shown in the following example, agents’ ambiguity seeking behavior may prevent the existence of market equilibrium that otherwise would exist when non-ambiguity seeking agents are in the market.

Example 4.6. *Consider a market with two agents and suppose that the state dependent total endowment $W \in \mathbb{R}^{m+l}$ is such that $W_\eta = W_\nu$, for all $\eta, \nu \in A$.*

- (i) *First suppose that in the market there is a SEU agent with prior $\tilde{\pi}$ and a non-ambiguity seeking agent, i.e. either an $\alpha\mathcal{C}_{\max}$ -MEU agent with $\alpha \in [\frac{l-1}{l}, 1)$, or a maxmin agent with $\tilde{\pi} \in \mathcal{C}$.³³ It is easy to see that in this market the equilibrium exists. Specifically,*

³²If a set of priors \mathcal{C} is symmetric (i.e., permutation invariant) in the ambiguous coordinates then $\tilde{\pi} \in \mathcal{C}$. Thus, if $\tilde{\pi} \notin \mathcal{C}$ some ambiguous states will be systematically overweighted and other underweighted. Note that \mathcal{C}_{\max} includes $\tilde{\pi}$ and is symmetric in the ambiguous states.

³³A necessary condition for the existence of the equilibrium is that beliefs are consistent across agents in the market. We recall that $\tilde{\pi}$ is in the set of prior \mathcal{C}_{\max} of the $\alpha\mathcal{C}_{\max}$ -MEU.

the state price vector in equilibrium is $p \in \mathbb{R}^{l+m}$ with $p_\eta = \frac{1 - \sum_{S \setminus A} p_R}{l}$, $\forall \eta \in A$; the SEU optimal portfolio is $y \in \mathbb{R}^{m+l}$ with $y_\eta = y_\nu$, $\forall \eta, \nu \in A$, and the ambiguity averse agent's optimal portfolio is $w \in \mathbb{R}^{m+l}$ with $w_\eta = w_\nu$, $\forall \eta, \nu \in A$, where y and w are such that $y_\eta + w_\eta = W_\eta$, $\forall \eta \in A$.³⁴

- (ii) Now suppose that in the market, together with the SEU agent, there is an ambiguity seeking α - \mathcal{C}_{\max} -MEU agent, i.e. $\alpha \in (0, \frac{l-1}{l})$. This agent only chooses ambiguous portfolio of the type (4.4) (see Corollary 4.3), that is $w \in \mathbb{R}^{l+1}$ such that $w_\sigma = \bar{w}$, $w_\eta = \underline{w}$, $\forall \eta \in A \setminus \{\sigma\}$, $\bar{w} > \underline{w}$, where σ is (one of the) the cheapest ambiguous state, i.e. $p_\sigma \leq p_\eta$, $\forall \eta \in A \setminus \{\sigma\}$. If the equilibrium exists, to clear the market, the SEU optimal portfolio $y \in \mathbb{R}^{l+1}$ has to satisfy $y_\sigma = W_\sigma - \bar{w} < W_\eta - \underline{w} = y_\eta$, $y_\eta = W_\eta - \underline{w} = W_\nu - \underline{w} = y_\nu$, $\forall \eta, \nu \in A \setminus \{\sigma\}$, that is $y_\sigma < y_\eta = y_\nu$, $\forall \eta, \nu \in A \setminus \{\sigma\}$. This portfolio will be optimal for the SEU agent only if (see (B.3)) the equilibrium state prices satisfy $p_\sigma > p_\eta = p_\nu$, $\forall \eta, \nu \in A \setminus \{\sigma\}$, but this condition is in contradiction with the α - \mathcal{C}_{\max} -MEU portfolio optimality condition $p_\sigma \leq p_\eta$, $\forall \eta, \nu \in A \setminus \{\sigma\}$. This implies that there is no equilibrium for this market.

To derive some economic intuition as to why the presence of ambiguity seeking α - \mathcal{C}_{\max} -MEU agents may prevent the existence of market equilibrium, note that in the α - \mathcal{C}_{\max} -MEU utility (4.1), $(1 - \sum_{R \in S \setminus A} \pi_R)(1 - \alpha)$ plays the role of the “fictitious” probability of the state σ on which the highest wealth w_{\max}^A is allocated. The more ambiguity seeking is the agent (i.e., the smaller is α), the higher is the probability of the state σ , and (ceteris paribus) the higher should be the price of state σ in equilibrium. However, the portfolio optimality condition of ambiguity seeking agents requires that σ is one of the cheapest ambiguous state in equilibrium. These potential contradicting conditions may prevent the existence of the equilibrium.

As an illustration of this point, we specify Example 4.6 (ii) to the CARA case, i.e., when the utility u of both the SEU and the ambiguity seeking α - \mathcal{C}_{\max} -MEU agents equal

³⁴When the ambiguous states have equal prices, the unambiguous portfolio is optimal both for the α - \mathcal{C}_{\max} -MEU (see Corollary 4.4) and the maxmin agent (see Lemma 4.5).

$u(z) = 1 - \frac{e^{-z}}{z}$. In this case, the existence of equilibrium requires the state prices to satisfy:

$$\begin{aligned}\frac{p_\sigma}{p_\eta} &= \left(\frac{(l-1)(1-\alpha)}{\alpha} \right)^{\frac{1}{2}}, \quad \forall \eta \in A \setminus \{\sigma\} \\ \frac{p_\sigma}{p_\eta} &\leq 1, \quad \forall \eta \in A \setminus \{\sigma\} \\ \frac{p_\eta}{p_\nu} &= 1, \quad \forall \eta, \nu \in A \setminus \{\sigma\}\end{aligned}$$

Any value of $\alpha \in (0, \frac{l-1}{l})$ implies a ratio of probability of the state σ to probability of the state $\eta \in A \setminus \{\sigma\}$ strictly larger than 1, i.e., $\frac{(1-\alpha)(l-1)}{\alpha} > 1$, and thus $p_\sigma > p_\eta$, which is in contradiction with the α - \mathcal{C}_{\max} -MEU portfolio optimality condition $\frac{p_\sigma}{p_\eta} \leq 1, \forall \eta \in A \setminus \{\sigma\}$.³⁵

For a non-ambiguity seeking α - \mathcal{C}_{\max} -MEU agent with $\alpha \in [\frac{l-1}{l}, 1)$, the probability $(1 - \sum_{R \in S \setminus A} \pi_R)(1 - \alpha)$ of the state on which w_{\max}^A is allocated is bounded from above by the probability $\tilde{\pi}_a$, and decreases when α increases. This is also true in the maxmin model. Indeed a necessary condition for having a maxmin agent to choose the ambiguous portfolio in (4.4) is that the prior π^* that realizes the minimum in (4.9) is bounded from above by the probability $\tilde{\pi}$.³⁶

Because non-ambiguity seeking agents have bounded probability of the state on which the highest wealth is allocated, and their optimal portfolios may also be unambiguous facilitate the equilibrium prices to settle, as in Example 4.6.

4.4.2 Non-ambiguity seeking and maxmin agents

Once the distinction between agents with and without ambiguity seeking attitudes is achieved, additional experiments involving only the agents with non-ambiguity seeking attitude can be carried out to disentangle maxmin agents from α - \mathcal{C}_{\max} -MEU agents with $\alpha \in [\frac{l-1}{l}, 1)$. This distinction can be achieved observing that the maxmin optimal portfolio is unique, while

³⁵The α - \mathcal{C}_{\max} -MEU utility assigns probability $(1 - \alpha)(1 - \sum_{R \in S \setminus A} \pi_R)$ to the state σ on which the highest wealth \bar{w} is allocated, and probability $\frac{\alpha(1 - \sum_{R \in S \setminus A} \pi_R)}{(l-1)}$ to each of the remaining states $\eta \in A \setminus \{\sigma\}$. The SEU-prior $\tilde{\pi}$ does not appear in the inequalities characterizing the ambiguous state prices because $\tilde{\pi}$ assigns equal probability $\tilde{\pi}_a$ to each ambiguous state and thus cancels out. The total endowment W also cancels out because $W_\eta = W_\nu$, for all $\eta, \nu \in A$.

³⁶This can be shown by observing that $w_\sigma > w_\eta, \forall \eta \in A \setminus \{\sigma\}$ implies $u(w_\eta) - u(w_\sigma) < 0, \forall \eta \in A \setminus \{\sigma\}$. Then the optimal prior π^* has to be a prior which maximizes the sum of the probability of the state $\eta \in A \setminus \{\sigma\}$. Therefore, since $\tilde{\pi} \in \mathcal{C}$, then π^* is such that $\sum_{\eta \in A \setminus \{\sigma\}} \pi_\eta^* \geq \frac{1 - \sum_{S \setminus A} \pi_R}{l} (l-1)$ or equivalently $\pi_\sigma^* \leq \frac{1 - \sum_{S \setminus A} \pi_R}{l}$.

the α - \mathcal{C}_{\max} -MEU ambiguous optimal portfolios are not unique when there are more than one ambiguous states with cheapest price and less than $l - 1$, i.e., $1 < |I| < l - 1$.³⁷ For example, when $m = 1$, $l = 3$, and the state price vector $p \in \mathbb{R}^{1+3}$ is such that $p_{\min}^A = p_G = p_B < p_Y$, any α - \mathcal{C}_{\max} -MEU agent with $\alpha \in [\frac{l-1}{l}, 1 - \frac{p_G}{1-p_R}] = [\frac{2}{3}, 1 - \frac{p_G}{1-p_R}]$ chooses and is indifferent between the two portfolios $(w_R, \bar{w}, \underline{w}, \underline{w})$ and $(w_R, \underline{w}, \bar{w}, \underline{w})$.³⁸ Thus, for instance by asking the agents in a sequence of experimental sections to choose their optimal portfolios without changing the prices, we expect to see the α - \mathcal{C}_{\max} -MEU agent switching its choice between the two optimal portfolios, while the maxmin agent chooses the same portfolio in any section.

The distinction between α - \mathcal{C}_{\max} -MEU agents with $\alpha \in [\frac{l-1}{l}, 1)$ and the maxmin agents cannot be achieved via the observation of one single portfolio choice. The reason is that depending on her set of priors a maxmin agent may also optimally choose one of the two portfolios which are optimal for the α - \mathcal{C}_{\max} -MEU agent with $\alpha \in [\frac{2}{3}, 1 - \frac{p_G}{1-p_R}]$, i.e., either the portfolio $(w_R, \bar{w}, \underline{w}, \underline{w})$ or $(w_R, \underline{w}, \bar{w}, \underline{w})$. Indeed, one can show that for any given portfolio there exists a set of priors \mathcal{C} for which the associated maxmin agent chooses that portfolio as optimal.

5 Conclusion

The α -MEU model has been used in many theoretical and experimntal studies to describe the behavior of agents under ambiguity. We show that in the standard Ellsberg framework (two ambiguous states) α -MEU preferences coincide with either maxmin, maxmax or subjective expected utility preferences, and derive equilibrium asset prices when the market is populated by ambiguity averse and subjective expected utility investors. Our theoretical results are strikingly in agreement with the laboratory experimental findings in Bossaerts et al. (2010), and show why ambiguity aversion does not wash out in equilibrium.

In an extended Ellsberg framework (three or more ambiguous states) we show that the α -MEU preferences do not coincide with maxmin, maxmax or subjective expected utility preferences and yield portfolio choices that are not observationally equivalent. We charac-

³⁷The optimal portfolio of a maxmin agent is typically unique. In particular this is always the case when the maxmin utility is strictly concave.

³⁸When $p_B = p_G$ approach zero the righthand side of the interval $[\frac{2}{3}, 1 - \frac{p_G}{1-p_R}]$ approaches 1.

terize the optimal portfolio choice of an $\alpha\text{-}\mathcal{C}_{\max}$ -MEU agent. This agent optimally chooses only between two types of portfolios: either an unambiguous portfolio, or an ambiguous portfolio with one specific exposure to ambiguity (that allocates more wealth to one of cheapest ambiguous states and less equal wealth to the other ambiguous states). The number of optimal ambiguous portfolios is equal to the number of ambiguous states with cheapest price. Our theoretical findings can inform laboratory experiments to disentangle between ambiguity seeking and non-ambiguity seeking agents, and among the latter, between $\alpha\text{-}\mathcal{C}_{\max}$ -MEU and maxmin agents. Finally, we find that when ambiguity seeking agents are present in the market they may prevent the existence of market equilibrium that otherwise would exist with non-ambiguity seeking agents.

A Proof of Proposition 3.1

Let U be as in (3.2). Then it follows that indeed

$$\begin{aligned} U(w) &= \begin{cases} \pi_R u(w_R) + \alpha[au(w_G) + (1 - \pi_R - a)u(w_B)] + \\ \quad (1 - \alpha)[bu(w_G) + (1 - \pi_R - b)u(w_B)] & \text{if } w_G \geq w_B \\ \pi_R u(w_R) + \alpha[bu(w_G) + (1 - \pi_R - b)u(w_B)] + \\ \quad (1 - \alpha)[au(w_G) + (1 - \pi_R - a)u(w_B)] & \text{if } w_G < w_B \end{cases} \\ &= \begin{cases} \pi_R u(w_R) + cu(w_G) + (1 - \pi_R - c)u(w_B) & \text{if } w_G \geq w_B \\ \pi_R u(w_R) + du(w_G) + (1 - \pi_R - d)u(w_B) & \text{if } w_G < w_B \end{cases} \end{aligned}$$

1st case: Suppose that $\alpha > 1/2$. Then

$$cu(w_G) + (1 - \pi_R - c)u(w_B) \leq du(w_G) + (1 - \pi_R - d)u(w_B)$$

whenever $w_G \geq w_B$ and

$$du(w_G) + (1 - \pi_R - d)u(w_B) \leq cu(w_G) + (1 - \pi_R - c)u(w_B)$$

if $w_G < w_B$. Thus the assertion of Proposition 3.1 (i) follows.

2nd case: If $\alpha = 1/2$, then $c = d$ and $U(w) = u(w_R) + cu(w_G) + (1 - \pi_R - c)u(w_B)$.

3rd case: If $\alpha < 1/2$, similar arguments as in the first case yield the assertion of Proposition 3.1 (iii).

B Proof of Proposition 3.2

In the following, we briefly summarize how the interaction among SEU and maxmin agents impacts the equilibrium asset prices. This will provide us the tools to prove Proposition 3.2. Assume that $(p; w^1, \dots, w^n)$ is an equilibrium with $p_\sigma > 0$ for all $\sigma \in \{R, G, B\}$. Then, the equilibrium price p satisfies

$$(B.1) \quad \lambda_n p \in \partial U^n(w^n)$$

for some $\lambda_n > 0$; see (F.3). Here $\partial U^n(w)$ denotes the supergradient of the criterion U^n of agent n at $w \in \mathbb{R}^3$. The supergradient of a SEU-agent with prior $\pi = (\pi_R, \pi_G, \pi_B)$ is simply the gradient

$$(B.2) \quad \partial U^n(w) = \{(\pi_R u'(w_R), \pi_G u'(w_G), \pi_B u'(w_B))\}.$$

From (B.2) and the strict concavity of the utility function follows the well known fact that the optimal portfolio $w = (w_R, w_G, w_B)$ of a SEU agent is always such that the optimal choices of state dependent wealth are ranked opposite to the state-price/state-probability ratios, i.e.

$$(B.3) \quad w_\sigma > w_\nu \Leftrightarrow \frac{p_\sigma}{\pi_\sigma} < \frac{p_\nu}{\pi_\nu}, \quad \sigma, \nu \in \{R, B, G\}.$$

The supergradient of an agent with maxmin (1-MEU) preferences represented as in (3.3) is

$$(B.4) \quad \partial U^m(w) = \begin{cases} \{(\pi_R u'(w_R), c u'(w_G), (1 - \pi_R - c) u'(w_B))\} & \text{if } w_G > w_B \\ \{(\pi_R u'(w_R), d u'(w_G), (1 - \pi_R - d) u'(w_B))\} & \text{if } w_G < w_B \\ \{(\pi_R u'(w_R), (\lambda c + (1 - \lambda)d) u'(w_G), \\ (1 - \pi_R - (\lambda c + (1 - \lambda)d)) u'(w_B)) \mid \lambda \in [0, 1]\} & \text{if } w_G = w_B. \end{cases}$$

Using (B.1) and the shape of the supergradients we easily obtain the optimal portfolio choices that were already derived in Bossaerts et al. (2010). In particular, from (B.4) and the strict concavity of u it follows that

$$(B.5) \quad \begin{cases} w_G > w_B & \text{if and only if } \frac{p_G}{p_B} < \frac{c}{1 - \pi_R - c} \\ w_G < w_B & \text{if and only if } \frac{p_G}{p_B} > \frac{d}{1 - \pi_R - d} \\ w_G = w_B & \text{if and only if } \frac{p_G}{p_B} \in \left[\frac{c}{1 - \pi_R - c}, \frac{d}{1 - \pi_R - d} \right] \end{cases}$$

where $x/0 := \infty$. The larger the set of priors \mathcal{C} in (3.4), the more likely a maxmin agent will take an unambiguous portfolio ($w_B = w_G$). In particular this will be always the case if $\mathcal{C} = \mathcal{C}_{\max} := \{(\pi_R, q, 1 - q - \pi_R) : q \in [0, 1 - \pi_R]\}$, because then the second respectively third coordinate of the supergradient in (B.4) will be 0 if either $w_G > w_B$ or $w_G < w_B$. Hence, $p_\sigma > 0$ for all $\sigma \in \{R, G, B\}$ and (B.1) imply that in equilibrium this agent will only take an unambiguous portfolios w . If $c > 0$ and/or $d < 1 - \pi_R$ in (3.4), then the multiple prior agent may also take an ambiguous portfolio in equilibrium. We observe that a maxmin agent holding an unambiguous optimal portfolio behaves as a SEU-agent who is not differentiating between the ambiguous states G and B , but merges them to an unambiguous state $\{G, B\}$ with probability $(1 - \pi_R)$. Indeed, from (B.4) and (B.1) it follows that

$$(B.6) \quad \frac{p_{\{G,B\}}}{p_R} = \frac{(1 - \pi_R) u'(w_{\{G,B\}})}{\pi_R u'(w_R)} \begin{cases} < \frac{(1 - \pi_R)}{\pi_R} & \text{iff } w_{\{G,B\}} > w_R \\ > \frac{(1 - \pi_R)}{\pi_R} & \text{iff } w_{\{G,B\}} < w_R \end{cases}$$

and thus

$$(B.7) \quad \frac{p_{\{G,B\}}}{(1 - \pi_R)} < \frac{p_R}{\pi_R} \Leftrightarrow w_{\{G,B\}} > w_R$$

$$(B.8) \quad \frac{p_{\{G,B\}}}{(1 - \pi_R)} > \frac{p_R}{\pi_R} \Leftrightarrow w_{\{G,B\}} < w_R \quad (\text{compare this to (B.3)}),$$

where $p_{\{G,B\}} := p_G + p_B$ and $w_{\{G,B\}} := w_G = w_B$.

Proof of Proposition 3.2

Case 1: Let $W_R > W_G > W_B$. Since the 1-MEU agents take an unambiguous portfolio, the optimal portfolio of some SEU agent must satisfy $y_G > y_B$ which according to (B.3) is equivalent to

$$(B.9) \quad \frac{p_B}{\pi_B} > \frac{p_G}{\pi_G}$$

which only leaves the ranking of p_R/π_R within (B.9) an open question. Suppose that the ranking of the ratios state-price/state-probability is as follows:

$$(B.10) \quad \frac{p_R}{\pi_R} \geq \frac{p_B}{\pi_B} > \frac{p_G}{\pi_G}.$$

Then (B.3) implies that $y_G > y_B \geq y_R$ for any SEU agent, and rearranging (B.10) yields

$$\frac{p_G + p_B}{1 - \pi_R} = \frac{p_G + p_B}{\pi_G + \pi_B} < \frac{p_R}{\pi_R}.$$

Consequently, according to (B.6), we must have for each 1-MEU agent that $w_R < w_G = w_B$. But this contradicts the clearing of the market and $W_R > W_G > W_B$. If the ranking is (3.6), then we have $y_G > y_R > y_B$ for each SEU agent according to (B.3). Denote by $y^\Sigma = (y_R^\Sigma, y_G^\Sigma, y_B^\Sigma)$ the sum over all optimal portfolios of the SEU agents and similarly by $w^\Sigma = (w_R^\Sigma, w_G^\Sigma, w_B^\Sigma)$ the sum over all optimal portfolios of the 1-MEU agents. The market clearing condition says $W_\sigma = y_\sigma^\Sigma + w_\sigma^\Sigma$ for every $\sigma \in \{R, G, B\}$. Since $y_G^\Sigma > y_R^\Sigma$ we conclude that

$$w_R^\Sigma = W_R - y_R^\Sigma > W_G - y_G^\Sigma = w_G^\Sigma.$$

Thus there must be at least one 1-MEU agent who's portfolio $w = (w_R, w_G, w_B)$ satisfies $w_R > w_G = w_B$ which implies that $(p_G + p_B)/p_R > (1 - \pi_R)/\pi_R$ due to (B.6). But then, again by (B.6), we must have $w_R > w_G = w_B$ for all 1-MEU agents. In case of (3.7) (B.3) and (B.6) imply the claimed ranking of payoffs in the portfolios y and w .

Case 2: Let $W_G > W_R > W_B$. As in case one we conclude that $y_G > y_B$. Assume that the ranking of the ratio state-price/state-probability is as follows:

$$(B.11) \quad \frac{p_R}{\pi_R} \geq \frac{p_B}{\pi_B} > \frac{p_G}{\pi_G}.$$

Then as in case 1 it follows that $y_G > y_B \geq y_R$ and $w_R < w_G = w_B$ which together with the clearing of the market contradicts $W_R > W_B$. Similarly it follows that the ranking

$$\frac{p_B}{\pi_B} > \frac{p_G}{\pi_G} \geq \frac{p_R}{\pi_R}$$

is not possible, since it would imply that $y_R \geq y_G > y_B$ and $w_R > w_G = w_B$ due to (B.6), again contradicting the assumed ranking of the aggregate wealth.

Case 3: Let $W_G > W_B > W_R$: Suppose that

$$\frac{p_B}{\pi_B} > \frac{p_G}{\pi_G} \geq \frac{p_R}{\pi_R}$$

then, $y_R \geq y_G > y_B$, and in view of (B.6) we obtain $w_R > w_G = w_B$ for every 1-MEU agent which again contradicts the market clearing and the assumed ranking $W_G > W_B > W_R$. Again (B.3), (B.6), and the clearing of the market imply the claimed ranking of payoffs in the portfolios y, w for the remaining possible rankings.

C Lack of concavity of the α -MEU utility

To see the lack of concavity of the α - \mathcal{C}_{\max} -MEU utility when $\alpha \neq 1$, consider portfolio w^1 such that $w_1^1 = 1, w_2^1 = 4$ and $w_j^1 = 2, \forall j = 2, \dots, l$, and portfolio w^2 such that $w_1^2 = 1, w_2^2 = 2, w_3^2 = 6$ and $w_j^2 = 2, \forall j = 4, \dots, l$. Let $w^\lambda = (w_1^\lambda, \dots, w_l^\lambda)$ be their convex combination, i.e. $w_j^\lambda = \lambda w_j^1 + (1 - \lambda)w_j^2, j = 1, \dots, l, \lambda \in [0, 1]$. Take for instance $\lambda = 1/2$. Then $w_1^\lambda = 1, w_2^\lambda = 3, w_3^\lambda = 4$ and $w_j^\lambda = 2, \forall j = 4, \dots, l$, and using (4.1)

$$\begin{aligned} \lambda U(w^1) + (1 - \lambda)U(w^2) &= \alpha u(1) + (1 - \alpha)\frac{1}{2}(u(4) + u(6)) > \\ \alpha u(1) + (1 - \alpha)u(4) &= \alpha u(w_{\min}^{\lambda A}) + (1 - \alpha)u(w_{\max}^{\lambda A}) = U(w^\lambda) = U((\lambda w^1 + (1 - \lambda)w^2)). \end{aligned}$$

D Proof of Proposition 4.1

Proposition 4.1 follows from Lemmas D.1–D.5 in the following.

Lemma D.1. *Suppose that the state price vector $p = (p_\sigma)_{\sigma \in S}$ satisfies $p_\sigma > 0$ for all $\sigma \in S$. Consider an α - \mathcal{C}_{\max} -MEU agent with $\alpha \in (0, 1)$. Let $w = (w_\sigma)_{\sigma \in S} \in \mathbb{R}^n$ be an optimal portfolio for the α - \mathcal{C}_{\max} -MEU agent. Then, either w takes the same value on all ambiguous states, or there exist two disjoint subsets \bar{A} and \underline{A} of the set of ambiguous states A such that $\bar{A} \cup \underline{A} = A$ and two values $\bar{w}, \underline{w} \in \mathbb{R}$ such that $w_\sigma = \bar{w} > \underline{w} = w_\eta$ for all $\sigma \in \bar{A}$ and all $\eta \in \underline{A}$.*

Proof. Note that the only portfolio values on the ambiguous states on which the utility U in (4.1) depends are w_{\max}^A and w_{\min}^A . We order the set of ambiguous states $A = \{\sigma_1, \dots, \sigma_l\}$ such that

$$(D.1) \quad w_{\sigma_1} \leq w_{\sigma_2} \leq \dots \leq w_{\sigma_l}.$$

Let s be the number of strict inequalities in (D.1). Consider states $\nu_1, \dots, \nu_{s+1} \in A$ such that $w_{\nu_1} < w_{\nu_2} < \dots < w_{\nu_{s+1}}$. Suppose there is a state $\eta \in A$ such that $w_\eta \neq w_{\max}^A$ and $w_\eta \neq w_{\min}^A$, namely suppose that $s \geq 2$. We now consider the function U in (4.1) as defined on \mathbb{R}^{m+s+1} , where we merge those ambiguous states in which w takes the same value. Let $\tilde{w} \in \mathbb{R}^{m+s+1}$ such that $\tilde{w}_R = w_R$ for all risky states $R \in S \setminus A$ and otherwise $\tilde{w}_{\sigma_i} = w_{\sigma_i}$ for $i = 1, \dots, s+1$. Then, \tilde{w} is a maximizer for the function U restricted to the open set $C := \{x \in \mathbb{R}^{m+s+1} \mid x_{\sigma_1} < x_{\sigma_2} < \dots < x_{\sigma_{s+1}}\}$, which we call U_C , given the budget constraint $\tilde{p} \cdot \tilde{w} \leq p \cdot e$. Here e is the initial portfolio and $\tilde{p} \in \mathbb{R}^{m+s+1}$ is obtained from p by summing up the prices of those states which are merged when forming \tilde{w} . As U_C is concave, according to (F.3), a multiple of \tilde{p} is in the supergradient of U_C at \tilde{w} . However, this supergradient is equal to zero in any x_{σ_i} -direction, $i \in \{2, \dots, s\}$, because only the largest value and the smallest value on the ambiguous states matter for U . This contradicts the assumption $p_{\sigma_i} > 0$ for $i \in \{2, \dots, s\}$. \square

Lemma D.2. *Assume Lemma D.1. If $p_\sigma < p_\eta$ for $\sigma, \eta \in A$, then the optimal portfolio w satisfies $w_\eta \leq w_\sigma$.*

Proof. Suppose that the optimal portfolio w is such that $w_\eta > w_\sigma$. Let \tilde{w} given by $\tilde{w}_\nu = w_\nu$ for all $\nu \in S \setminus \{\sigma, \eta\}$ and $\tilde{w}_\sigma = w_\eta$ and $\tilde{w}_\eta = w_\sigma$. Then $U(\tilde{w}) = U(w)$, but $p \cdot \tilde{w} < p \cdot w$ because $p \cdot (w - \tilde{w}) = (p_\eta - p_\sigma)(w_\eta - w_\sigma) > 0$. This contradicts the optimality of w , because increasing the wealth \tilde{w}_σ one could achieve a strictly higher utility while still respecting the budget constraint. \square

Lemma D.3. *Assume Lemma D.1. If the sets \bar{A} and \underline{A} associated to the optimal portfolio w are not empty, then $\bar{A} = \{\bar{\sigma}\}$ for a state $\bar{\sigma} \in I := \{\sigma \in A \mid p_\sigma = \min_{\eta \in A} p_\eta\}$. Moreover, any portfolio which equals w on the risky states and assigns the weight w_{\max}^A to a single state in I and w_{\min}^A to all the other ambiguous states is optimal. Hence, there are $|I|$ optimal portfolios.*

Proof. By contradiction suppose that there are two different states σ_1 and σ_2 in \bar{A} , i.e. that the optimal portfolio w is such that $w_{\sigma_1} = w_{\sigma_2} = w_{\max}^A$, and without loss of generality we assume that $p_{\sigma_1} \leq p_{\sigma_2}$. Consider \tilde{w} given by $\tilde{w}_\eta = w_\eta$ for all $\eta \in S \setminus \{\sigma_1, \sigma_2\}$ and $\tilde{w}_{\sigma_1} = 2w_{\max}^A - w_{\min}^A$ and $\tilde{w}_{\sigma_2} = w_{\min}^A$. Then $p \cdot \tilde{w} \leq p \cdot w$, so \tilde{w} satisfies the budget constraint, and $U(\tilde{w}) > U(w)$ since $\tilde{w}_{\max}^A = \tilde{w}_{\sigma_1} > w_{\max}^A$ and $\tilde{w}_{\min}^A = w_{\min}^A$. This is a contradiction to optimality of w . Lemma D.2 implies that $\bar{\sigma} \in I$. The last statement of the lemma follows by observing that all these portfolios share the same price and utility. \square

Lemma D.4. *Assume Lemma D.1 and let $\alpha < 1$. Then w is unambiguous, i.e. $w_\sigma = w_\nu$ for all $\sigma, \nu \in A$, if and only if (4.5) holds. In this case w is the only optimal portfolio. Condition (4.5) can only be satisfied if $\alpha \geq \frac{1}{T}$.*

Proof. Suppose $\bar{A} = \{\sigma\}$ and thus $\underline{A} = A \setminus \{\sigma\}$. Then, the first order conditions imply

$$(D.2) \quad \frac{p_R}{\pi_R u'(w_R)} = \frac{p_\sigma}{(1-\alpha)(1 - \sum_{R \in S \setminus A} \pi_R) u'(w_{\max}^A)} = \frac{\sum_{\nu \in A \setminus \{\sigma\}} p_\nu}{\alpha(1 - \sum_{R \in S \setminus A} \pi_R) u'(w_{\min}^A)}$$

where R denotes any risky state among the m ones. Thus,

$$(D.3) \quad \frac{p_\sigma}{\sum_{\nu \in A \setminus \{\sigma\}} p_\nu} = \frac{(1-\alpha)u'(w_{\max}^A)}{\alpha u'(w_{\min}^A)} < \frac{1-\alpha}{\alpha}$$

as $w_{\max}^A > w_{\min}^A$. Consequently, if there are no $\sigma \in A$ for which (D.3) is satisfied, i.e. if the condition (4.5) holds true, then w must be unambiguous. In order to prove necessity of (4.5), assume that (D.3) holds for some $\sigma \in A$. In the following we show that in this case the unambiguous portfolio cannot be optimal. To this end, suppose by contradiction that the unambiguous portfolio w is optimal and let $z := w_{\max}^A = w_{\min}^A$. Then $\epsilon = 0$ needs to maximize the function

$$F : \mathbb{R} \ni \epsilon \mapsto \alpha u(z - \epsilon) + (1 - \alpha)u(z + \delta(\epsilon))$$

over all $\epsilon \geq 0$, where $\delta(\epsilon) := \epsilon \frac{\sum_{\sigma \in A \setminus \{\sigma\}} p_\nu}{p_\sigma}$ is chosen such that the portfolio which invests $z - \epsilon$ in the states $\nu \in \underline{A}$, and $z + \delta(\epsilon)$ in the state σ satisfies the budget constraint (while the investment in the risky states is unaltered). F is a concave function and the first order

condition reads

$$\frac{u'(z + \delta(\epsilon))}{u'(z - \epsilon)} = \frac{\alpha}{(1 - \alpha)} \frac{p_\sigma}{\sum_{\nu \in A \setminus \{\sigma\}} p_\nu}.$$

By assumption, the right hand side of the above equation is strictly smaller than 1. Hence, F attains its optimum for $\epsilon > 0$, which contradicts the optimality at 0 over all $\epsilon \geq 0$.

Finally, note that summing up (4.5) over all $\sigma \in A$ yields:

$$\alpha \sum_{\sigma \in A} p_\sigma \geq (1 - \alpha)(l - 1) \sum_{\nu \in A} p_\nu \Leftrightarrow \alpha \geq \frac{l - 1}{l}.$$

□

Lemma D.5. *Assume Lemma D.1. If $\alpha = 1$, then w is unambiguous. If $\alpha = 0$, then there is no optimal portfolio.*

Proof. If $\alpha = 1$, then (4.1) is a maxmin agent and also $\tilde{\pi} \in \mathcal{C}_{\max}$. Hence, Lemma 4.5 proves the claim.

The optimization problem of a 0-MEU agent with the maximal set of priors \mathcal{C}_{\max} is

$$(D.4) \quad \begin{aligned} & \sum_{R \in S \setminus A} \pi_R u(w_R) + (1 - \sum_{R \in S \setminus A} \pi_R) u(w_{\max}^A) \rightarrow \max \\ & \text{subject to} \quad p \cdot w \leq p \cdot e \end{aligned}$$

where e denotes her initial endowment. Since the agent may go arbitrarily long in the ambiguous state σ with $w_\sigma = w_{\max}^A$ and satisfy the budget constraint by going arbitrarily short in an other ambiguous state, the optimal value in (D.4) cannot be attained. □

E Proof of Lemma 4.5

Let w be an optimal portfolio of the maxmin agent and assume that $w_\sigma \neq w_\eta$ for $\sigma, \eta \in A$. Consider the portfolio \hat{w} given by $\hat{w}_R = w_R$ for any risky state $R \in S \setminus A$ and $\hat{w}_\sigma = z$ for any ambiguous state $\sigma \in A$ where

$$z := \frac{\sum_{\sigma \in A} p_\sigma w_\sigma}{\sum_{\sigma \in A} p_\sigma} = \frac{1}{l} \sum_{\sigma \in A} w_\sigma.$$

The portfolio \hat{w} satisfies the budget constraint and

$$\begin{aligned} U(\hat{w}) &= \sum_{R \in S \setminus A} \pi_R u(w_R) + (1 - \sum_{R \in S \setminus A} \pi_R) u(z) \\ &> \sum_{R \in S \setminus A} \pi_R u(w_R) + \frac{1}{l} (1 - \sum_{R \in S \setminus A} \pi_R) \sum_{\sigma \in A} u(w_\sigma) \geq U(w) \end{aligned}$$

where the strict inequality follows from the strict concavity of u and the last inequality is due to $\tilde{\pi} \in \mathcal{C}$. This contradicts the optimality of w .

F Optimization in the partially concave case

Consider the optimization problem

$$(F.1) \quad \max_{x \in C} U(x) \quad \text{subject to} \quad px \leq pe$$

where $C \neq \emptyset$ is a convex subset of \mathbb{R}^n , $p, e \in \mathbb{R}^n$, and $U : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ is a concave function with $\text{dom } U = C$.

Lemma F.1. *If the optimal value in (F.1) is not $+\infty$ and if there exists at least one $\bar{x} \in \text{ri } C$ with $p\bar{x} \leq pe$, then there is a multiplier $\lambda \geq 0$ such that the supremum of $h_\lambda(x) = U(x) - \lambda p(x - e)$, $x \in \mathbb{R}^n$, is finite and equal to the optimal value in (F.1). Moreover, suppose that $\lambda > 0$ and that D is the set of points $x \in \mathbb{R}^n$ where h attains its maximum intersected with the set of points satisfying $px = pe$, then D is the set of all optimal solutions to (F.1).*

Proof. see Theorem 28.1 and Corollary 28.2.2 in Rockafellar (1997). \square

Now suppose that agent n with choice criterium $U^n : \mathbb{R}^{|S|} \rightarrow \mathbb{R}$ maximizes her utility over all portfolios $w \in \mathbb{R}^{|S|}$ satisfying the budget constraint $pw \leq pe^n$ for some $p \in \mathbb{R}^{|S|}$ with $p_i > 0$ for all $i = 1, \dots, |S|$. Furthermore, assume that an optimal portfolio \hat{w} exists and that $\hat{w} \in C$ for a convex set $C \subset \mathbb{R}^{|S|}$ such that the restriction U_C^n of U^n to C is concave. Then, we may view U_C^n as defined on all $\mathbb{R}^{|S|}$ by defining $U_C^n(x) := -\infty$ for $x \notin C$, and we are thus in the setting of Lemma F.1 where \hat{w} is a solution to problem (F.1) with $U = U_C^n$. Hence, if there exists $x \in \text{ri } C$ with $px \leq pe^n$, which is satisfied if for instance $\hat{w} \in \text{ri } C$, then there exists a multiplier $\lambda \geq 0$ such that

$$(F.2) \quad U_C^n(\hat{w}) = \sup_{x \in \mathbb{R}^n} h_\lambda(x)$$

with h_λ as in Lemma F.1. If $C = C + \mathbb{R}_+ \cdot (1, 0, \dots, 0)$ and given that the utility function u is strictly increasing we deduce that $\lambda > 0$, since otherwise

$$h_\lambda(\hat{w} + (1, 0, \dots, 0)) = U_C^n(\hat{w} + (1, 0, \dots, 0)) > U_C^n(\hat{w}).$$

Moreover, any solution \hat{x} to the right hand side of (F.2) with $p\hat{x} = pe^n$ is a solution to the portfolio optimization problem, and in particular \hat{w} is such a solution. Additionally, for any solution \hat{x} to the right hand side of (F.2) we have for all $y \in \mathbb{R}^{|S|}$ that

$$U_C^n(y) - \lambda p(y - e^n) \leq U_C^n(\hat{x}) - \lambda p(\hat{x} - e^n)$$

which shows that

$$(F.3) \quad \lambda p \in \partial U_C^n(\hat{x})$$

where $\partial U_C^n(w)$ denotes the supergradient of U_C^n at w , i.e.

$$\partial U_C^n(w) := \{\nu \in \mathbb{R}^{|S|} \mid \forall y \in \mathbb{R}^{|S|}, U_C^n(y) \leq U_C^n(w) + \nu \cdot (y - w)\}.$$

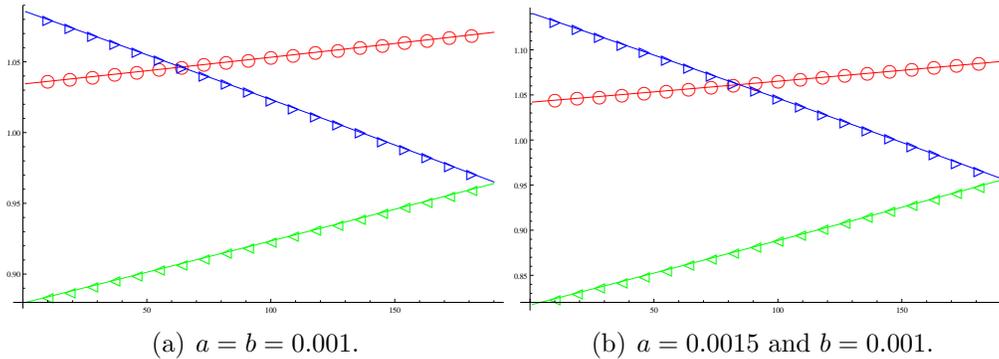


Figure 1: State-price/state-probability ratios (y -axis) of the equilibrium prices as a function of the difference $W_B - W_R$ (x -axis), computed for fixed $W_G = 272$ and $W_R = 81$, as in Figure 2. The line marked with circles represents p_R/π_R , the one marked with arrows pointing to the right represents p_B/π_B , and the one marked with arrows pointing to the left represents p_G/π_G . The SEU prior is $\pi_R = \pi_G = \pi_B = 1/3$. The parameters a and b in (3.13) are set to 0.001 in the left graph, and to $a = 0.0015$ and $b = 0.001$ in the right graph.

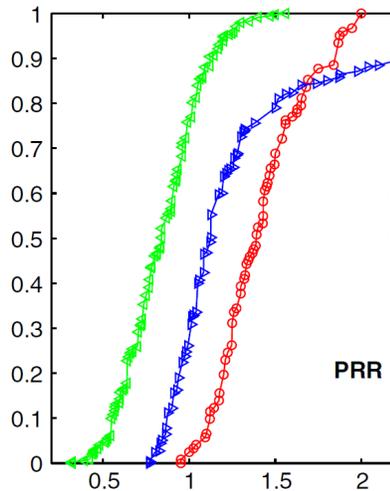


Figure 2: Empirical distribution functions of state-price/state-probability ratios from the experimental session of eight trading periods in Bossaerts et al. (2010) with $W_G = 272$, $W_B = 162$, and $W_R = 81$. The distribution function with circles is for p_R/π_R ; the one with arrows pointing to the right is for p_B/π_B ; the one with arrows pointing to the left is for p_G/π_G . This figure is a copy of Figure 8, right panel, in Bossaerts et al. (2010).

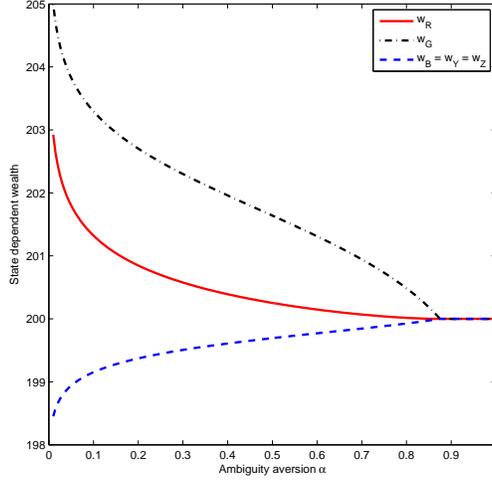


Figure 3: Optimal state dependent wealth α - \mathcal{C}_{\max} -MEU agent as a function of the degree of ambiguity aversion α , when the number of risk states $m = 1$, the number of ambiguous states $l = 4$, R is the risky state, and G, B, Y, Z are the ambiguous states. State prices are $p_R = 0.2$, $p_G = 0.1$, and p_B, p_Y, p_Z such that $p_B + p_Y + p_Z = 1 - p_R - p_G$ and $p_G = \min_{\nu \in \{G, B, Y, Z\}} p_\nu$. The agent's utility u in (4.1) is a CARA utility, $u(z) = 1 - e^{-\delta z}/\delta$, where $\delta = 1$.

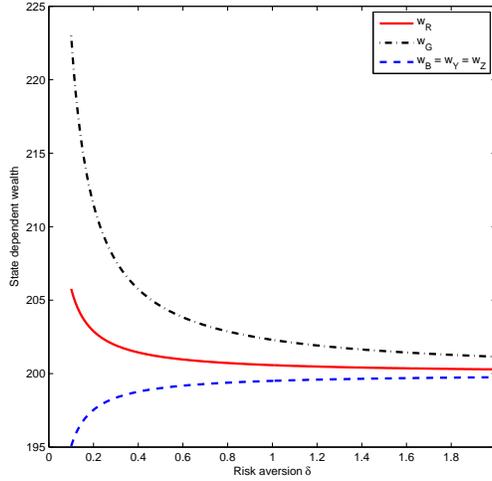


Figure 4: Optimal state dependent wealth of the α - \mathcal{C}_{\max} -MEU agent as a function of the degree of risk aversion δ , when the ambiguity aversion coefficient $\alpha = 0.3$, the number of risk states $m = 1$, the number of ambiguous states $l = 4$, R is the risky state, and G, B, Y, Z are the ambiguous states. State prices are $p_R = 0.2$, $p_G = 0.1$, and p_B, p_Y, p_Z such that $p_B + p_Y + p_Z = 1 - p_R - p_G$ and $p_G = \min_{\nu \in \{G, B, Y, Z\}} p_\nu$. The agent's utility u in (4.1) is a CARA utility, $u(z) = 1 - e^{-\delta z}/\delta$.

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