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Coauthorship Networks and Research Output

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Abstract

We study the impact of research collaborations in coauthorship networks on total research output. Through the links in the collaboration network researchers create spillovers not only to their direct coauthors but also to researchers indirectly linked to them. We characterize the interior equilibrium when multiple agents spend effort in multiple, possibly overlapping projects, and there are interaction effects in the cost of effort.

Key words: coauthor networks, economics of science
JEL: C72, D85, D43, L14, Z13

1. Introduction

We build a micro-founded model of scientific co-authorship that incorporates and generalizes previous ones in the literature [cf. e.g. Ballester et al., 2006; Cabrales et al., 2010; Jackson and Wolinsky, 1996]. We characterize the interior equilibrium when multiple agents spend effort in multiple, possibly overlapping projects, and there are interaction effects in the cost of effort. While we assume that the allocation of agents into different projects is exogenous (and determined by some underlying meeting process), the endogenous choice of efforts makes the network of positive efforts endogenous, and in this sense we consider an endogenous network formation model. The equilibrium solution to this model then allows us to study the impact of individual researchers on total research output.

There exists a growing literature, both empirical and theoretical, on coauthorship networks including Bosquet and Combes [2013]; Ductor [2011]; Ductor et al. [2013]; Fafchamps et al. [2010]; Goyal et al. [2006], Newman [2001a,b,c,d, 2004], König [2011], Ballester et al. [2006]; Cabrales et al. [2010]; Calvó-Armengol et al. [2009], Azoulay et al. [2010]; Waldinger [2010, 2012] and König et al. [2014]; Liu et al. [2011]. In particular, our paper is related to the recent ones by Baumann [2014] and Salonen [2014], where agents choose time to invest into bilateral relationships. Our model extends the setups considered in these papers to allow for

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investments into multiple projects involving more than two agents. Moreover, Bimpikis et al. [2014] analyze firms competing in quantities à la Cournot across different markets with a similar linear-quadratic payoff specification, and allow firms to choose endogenously the quantitites sold to each market. In contrast, the efforts invested by the agents in different projects in our model are strategic complements, and not substitutes as in their paper.

2. Production Function

Assume that there are $s = 1, \ldots, p$ research projects. Let the *production function* for project s be given by

$$Y_s(G, \mathbf{e}_s) = \sum_{i \in \mathcal{N}_s} \alpha_i e_{is} + \frac{\beta}{2} \sum_{i \in \mathcal{N}_s} \sum_{j \in \mathcal{N}_s \setminus \{i\}} e_{is} e_{js} = \sum_{i \in \mathcal{N}_s} e_{is} \left(\alpha_i + \frac{\beta}{2} \sum_{j \in \mathcal{N}_s \setminus \{i\}} e_{js} \right), \tag{1}$$

where e_{is} is the research effort of agent i in project s, $\mathcal{N}_s$ is the set of agents participating in project s, α_i is the ability/skill of researcher i and β is a spillover parameter from complementarities between the research efforts of coauthors. If efforts are measured in logs then Y_s corresponds to a *translog production function* [cf. Christensen et al., 1973, 1975]. The translog production function can be viewed as an exact production function, a second order Taylor approximation to a more general production function or a second order approximation to a CES production function [cf. Adams, 2006].[1]

3. Payoffs

In the following we study two alternative payoff specifications.[2] The assumption of a convex separable cost is similar to the model studied in Adams [2006]. The introduction of a quadratic cost with substitutes or complements, depending on the sign of the parameters $\phi_{ss'}$, is similar to Cohen-Cole et al. [2012], and it includes the case of a convex total cost as a special case when $\phi_{s,s'} = \gamma$, and the the case of a convex separable cost when $\phi_{s,s'} = \gamma \delta_{s,s'}$. A theoretical model with only two activities is studied in Belhaj and Deroïan [2014], and an empirical analysis is provided in Liu [2014].

[1] A related specification, however, without allowing agents to spend effort across different projects, can be found in Ballester et al. [2006].

[2] Table 1 gives an overview of possible extensions and alternative specifications.

3.1. Convex Separable Costs

The payoff of agent i is given by

$$
\begin{aligned}
\pi_i(G,\mathbf{e}) &= \sum_{s=1}^{p}\left(Y_s(G,\mathbf{e}_s)-\frac{\gamma}{2}e_{is}^2\right)\delta_{is} \\
&= \sum_{s=1}^{p}\left(\sum_{j\in\mathcal{N}_s}\alpha_j e_{js}+\frac{\beta}{2}\sum_{j\in\mathcal{N}_s}\sum_{k\in\mathcal{N}_s\backslash\{j\}}e_{js}e_{ks}-\frac{\gamma}{2}e_{is}^2\right)\delta_{is} \\
&= \sum_{s=1}^{p}\left(\sum_{j\in\mathcal{N}_s}e_{js}\left(\alpha_j+\frac{\beta}{2}\sum_{k\in\mathcal{N}_s\backslash\{j\}}e_{ks}\right)-\frac{\gamma}{2}e_{is}^2\right)\delta_{is}, \qquad (2)
\end{aligned}
$$

where $n_s = |\mathcal{N}_s|$ is the number of agents participating in project s, and it holds that $n_s = \sum_{i=1}^{n}\delta_{is}$ with $\delta_{is}\in\{0,1\}$ indicating whether i is participating in project s.

Proposition 1. *Let the payoff function for each agent $i = 1,\ldots,n$ be given by Equation (2). Then the unique interior Nash equilibrium effort levels are given by*

$$
e_{is} = \frac{\alpha_i}{\beta+\gamma}+\frac{\beta}{(\beta+\gamma)(\gamma-\beta(n_s-1))}\sum_{j\in\mathcal{N}_s}\alpha_j. \qquad (3)
$$

for each agent $i = 1,\ldots,n$ and each project $s = 1,\ldots,p$.

Inserting effort levels from Equation (21) into the production function from Equation (1) yields

$$
\begin{aligned}
Y_s(G) &= \sum_{i\in\mathcal{N}_s}e_{is}\left[\alpha_i+\frac{\beta}{2}\sum_{j\in\mathcal{N}_s\backslash\{i\}}e_{js}\right] \\
&= \frac{1}{2}\sum_{i\in\mathcal{N}_s}e_{is}\left(\alpha_i+\gamma e_{is}\right) \\
&= \frac{1}{2}\sum_{i\in\mathcal{N}_s}\left(\alpha_i e_{is}+\gamma e_{is}^2\right) \\
&= \frac{1}{2}\sum_{i\in\mathcal{N}_s}\left[\alpha_i\left(\frac{\alpha_i}{\beta+\gamma}+\frac{\beta}{\gamma-\beta(n_s-1)}\sum_{j\in\mathcal{N}_s}\frac{\alpha_j}{\beta+\gamma}\right)+\gamma\left(\frac{\alpha_i}{\beta+\gamma}+\frac{\beta}{\gamma-\beta(n_s-1)}\sum_{j\in\mathcal{N}_s}\frac{\alpha_j}{\beta+\gamma}\right)^2\right]. \qquad (4)
\end{aligned}
$$

The model in this section assumes that efforts invested by an agent across different projects are independent. This assumption is relaxed in the more general setup analyzed in the following section.

3.2. Quadratic Costs with Substitutes/Complements

In the following we introduce a cost given by the quadratic form, $\frac{1}{2}\sum_{s,s'=1}^{p}\phi_{s,s'}e_{is}e_{is'}\delta_{is}\delta_{is'} = \frac{1}{2}\tilde{\mathbf{e}}_i^\top\boldsymbol{\phi}\tilde{\mathbf{e}}_i$, where $\phi_{s,s'} = \phi_{s',s}$, $\tilde{\mathbf{e}}_i = (\tilde{e}_{i1},\ldots,\tilde{e}_{ip})^\top$ and $\tilde{e}_{is} = e_{is}\delta_{is}$. This cost is convex if and only if the $p \times p$ matrix $\boldsymbol{\phi}$ is positive definite. The case of a quadratic cost includes the case of a convex total cost as a special case when $\phi_{s,s'} = \gamma$, and the case of a convex separable cost discussed in Section 3.1 when $\phi_{s,s'} = \gamma\delta_{s,s'}$.

The payoff of agent i is given by

$$\begin{aligned}
\pi_i(G, \mathbf{e}) &= \sum_{s=1}^{p} Y_s(G, \mathbf{e}_s)\delta_{is} - \frac{1}{2}\sum_{s,s'=1}^{p} \phi_{s,s'} e_{is} e_{is'} \delta_{is}\delta_{is'} \\
&= \sum_{s=1}^{p} \left(\sum_{j \in \mathcal{N}_s} \alpha_j e_{js} + \frac{\beta}{2} \sum_{j \in \mathcal{N}_s} \sum_{k \in \mathcal{N}_s \setminus \{j\}} e_{js} e_{ks} \right) \delta_{is} - \frac{1}{2}\sum_{s,s'=1}^{p} \phi_{s,s'} e_{is} e_{is'} \delta_{is}\delta_{is'} \\
&= \sum_{s=1}^{p} \left(\sum_{j \in \mathcal{N}_s} e_{js} \left(\alpha_j + \frac{\beta}{2} \sum_{k \in \mathcal{N}_s \setminus \{j\}} e_{ks} \right) \right) \delta_{is} - \frac{1}{2}\sum_{s,s'=1}^{p} \phi_{s,s'} e_{is} e_{is'} \delta_{is}\delta_{is'}, \qquad (5)
\end{aligned}$$

where $n_s = |\mathcal{N}_s|$ is the number of agents participating in project s, and it holds that $n_s = \sum_{i=1}^{n} \delta_{is}$ with $\delta_{is} \in \{0, 1\}$ indicating whether i is participating in project s.

Proposition 2. *Let the payoff function for each agent $i = 1, \ldots, n$ be given by Equation (5) and assume that*

$$\phi_{ss'} = \begin{cases} \gamma, & \text{if } s' = s, \\ \rho, & \text{otherwise.} \end{cases}$$

Denote by

$$\begin{aligned}
\varphi_{is} &\equiv \frac{\rho\beta\delta_{is}}{(\beta(n_s - 1) + \rho - \gamma)(\beta + \gamma + \rho(d_i - 1))}, \\
\mu_s(\boldsymbol{\alpha}) &\equiv \sum_{i=1}^{n} \frac{\rho\alpha_i d_i \delta_{is}}{(\beta(n_s - 1) + \rho - \gamma)(\beta + \gamma + \rho(d_i - 1))} - \frac{1}{\beta(n_s - 1) + \rho - \gamma} \sum_{i=1}^{n} \alpha_i \delta_{is}, \\
\omega_{ss'} &\equiv \sum_{i=1}^{n} \varphi_{is}\delta_{is'}.
\end{aligned}$$

Further, let $\boldsymbol{\Omega} \equiv (\omega_{ss'})_{1 \le s,s' \le p}$, assume that the matrix $\mathbf{I}_p - \boldsymbol{\Omega}$ is invertible, and define by $\boldsymbol{\epsilon} \equiv (\mathbf{I}_p - \boldsymbol{\Omega})^{-1}\boldsymbol{\mu}(\boldsymbol{\alpha})$. Then, for β small enough, the unique interior Nash equilibrium effort levels are given by

$$e_{is} = \frac{1}{\beta + \gamma - \rho} \left[\beta\epsilon_s + \alpha_i - \frac{\rho}{\beta + \gamma + \rho(d_i - 1)} \left(\sum_{s'=1}^{p} \delta_{is'}\epsilon_{s'} + \alpha_i d_i \right) \right], \qquad (6)$$

if $\delta_{is} = 1$ for each agent $i = 1, \ldots, n$ and each project $s = 1, \ldots, p$. Further, the total effort spent in project s is given by ϵ_s.

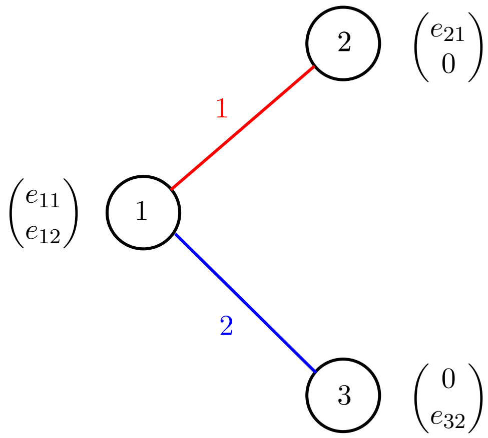

Figure 1: The network analyzed in Example 1. The effort levels of the individual agents for each project they are involved in are indicate next to the nodes.

Inserting Equation (6) into the production function from Equation (1) gives

$$\begin{aligned}
Y_s(G) &= \sum_{i \in \mathcal{N}_s} \alpha_i e_{is} + \beta \sum_{i \in \mathcal{N}_s} \sum_{j \in \mathcal{N}_s \setminus \{i\}} e_{is} e_{js} \\
&= \sum_{i \in \mathcal{N}_s} e_{is} \left(\alpha_i + \beta \sum_{j \in \mathcal{N}_s \setminus \{i\}} e_{js} \right) \\
&= \sum_{i \in \mathcal{N}_s} \frac{\delta_{is}}{\beta + \gamma - \rho} \left[\beta \epsilon_s + \alpha_i - \frac{\rho}{\beta + \gamma + \rho(d_i - 1)} \left(\sum_{s'=1}^{p} \delta_{is'} \epsilon_{s'} + \alpha_i d_i \right) \right] \\
&\times \left\{ \alpha_i + \beta \sum_{j \in \mathcal{N}_s \setminus \{i\}} \frac{\delta_{js}}{\beta + \gamma - \rho} \left[\beta \epsilon_s + \alpha_j - \frac{\rho}{\beta + \gamma + \rho(d_j - 1)} \left(\sum_{s'=1}^{p} \delta_{js'} \epsilon_{s'} + \alpha_j d_j \right) \right] \right\}.
\end{aligned} \tag{7}$$

We will illustrate the equilibrium characterization of Proposition 2 in several examples that follow.

Example 1. *Consider a network with* 2 *projects and* 3 *agents, where in the first project agents* 1 *and* 2 *are collaborating and in the second project agents* 1 *and* 3 *are collaborating. An illustration can be found in Figure 1. The payoffs of the agents are given by*

$$\begin{aligned}
\pi_1 &= e_{11}\left(\alpha_1 + \frac{\beta}{2} e_{21}\right) + e_{21}\left(\alpha_2 + \frac{\beta}{2} e_{11}\right) + e_{12}\left(\alpha_1 + \frac{\beta}{2} e_{32}\right) + e_{32}\left(\alpha_3 + \frac{\beta}{2} e_{12}\right) - \frac{\gamma}{2} e_{11}^2 - \frac{\gamma}{2} e_{12}^2 - \rho e_{11} e_{12} \\
\pi_2 &= e_{11}\left(\alpha_1 + \frac{\beta}{2} e_{21}\right) + e_{21}\left(\alpha_2 + \frac{\beta}{2} e_{11}\right) - \frac{\gamma}{2} e_{21}^2 \\
\pi_3 &= e_{12}\left(\alpha_1 + \frac{\beta}{2} e_{32}\right) + e_{32}\left(\alpha_3 + \frac{\beta}{2} e_{12}\right) - \frac{\gamma}{2} e_{32}^2.
\end{aligned}$$

The first order conditions are given by

$$\begin{aligned}
\frac{\partial \pi_1}{\partial e_{11}} &= \alpha_1 + e_{21}\beta - e_{11}\gamma - e_{12}\rho = 0 \\
\frac{\partial \pi_1}{\partial e_{12}} &= \alpha_1 + e_{32}\beta - e_{12}\gamma - e_{11}\rho = 0 \\
\frac{\partial \pi_2}{\partial e_{21}} &= \alpha_2 + e_{11}\beta - e_{21}\gamma = 0 \\
\frac{\partial \pi_3}{\partial e_{32}} &= \alpha_3 + e_{12}\beta - e_{32}\gamma = 0.
\end{aligned}$$

Solving this system of equations directly yields

$$\begin{aligned}
e_{11} &= -\frac{(\beta-\gamma)(\beta+\gamma)(\alpha_2\beta+\alpha_1\gamma)+\gamma(\alpha_3\beta+\alpha_1\gamma)\rho}{(\beta^2-\gamma^2)^2-\gamma^2\rho^2}, \\
e_{12} &= -\frac{(\beta-\gamma)(\beta+\gamma)(\alpha_3\beta+\alpha_1\gamma)+\gamma(\alpha_2\beta+\alpha_1\gamma)\rho}{(\beta^2-\gamma^2)^2-\gamma^2\rho^2}, \\
e_{21} &= -\frac{(\beta-\gamma)(\beta+\gamma)(\alpha_1\beta+\alpha_2\gamma)+\beta(\alpha_3\beta+\alpha_1\gamma)\rho+\alpha_2\gamma\rho^2}{(\beta^2-\gamma^2)^2-\gamma^2\rho^2}, \\
e_{32} &= -\frac{(\beta-\gamma)(\beta+\gamma)(\alpha_1\beta+\alpha_3\gamma)+\beta(\alpha_2\beta+\alpha_1\gamma)\rho+\alpha_3\gamma\rho^2}{(\beta^2-\gamma^2)^2-\gamma^2\rho^2}.
\end{aligned} \tag{8}$$

Next, we compute the above equilibrium effort levels using the equilibrium characterization in Equation (6). Note that $\mathbf{d} = (d_i)_{1\le i\le 3} = (2,1,1)^\top$, $\mathbf{n} = (n_s)_{1\le s\le 2} = (2,2)^\top$,

$$\boldsymbol{\delta} = (\delta_{is})_{1\le i\le 3, 1\le s\le 2} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\boldsymbol{\varphi} = (\varphi_{is})_{1\le i\le 3, 1\le s\le 2} = \begin{pmatrix} \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} & \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} \\ \frac{\beta\rho}{(\beta+\gamma)(\beta-\gamma+\rho)} & 0 \\ 0 & \frac{\beta\rho}{(\beta+\gamma)(\beta-\gamma+\rho)} \end{pmatrix}.$$

Further, we have that $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^\top$ *and*

$$\boldsymbol{\mu}(\boldsymbol{\alpha}) = \begin{pmatrix} \frac{\alpha_2\left(-1+\frac{\rho}{\beta+\gamma}\right)+\alpha_1\left(-1+\frac{2\rho}{\beta+\gamma+\rho}\right)}{\beta-\gamma+\rho} \\ \frac{\alpha_3\left(-1+\frac{\rho}{\beta+\gamma}\right)+\alpha_1\left(-1+\frac{2\rho}{\beta+\gamma+\rho}\right)}{\beta-\gamma+\rho} \end{pmatrix}.$$

Next, we have that

$$\boldsymbol{\Omega} = \begin{pmatrix} \frac{\beta\rho(2(\beta+\gamma)+\rho)}{(\beta+\gamma)(\beta-\gamma+\rho)(\beta+\gamma+\rho)} & \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} \\ \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} & \frac{\beta\rho(2(\beta+\gamma)+\rho)}{(\beta+\gamma)(\beta-\gamma+\rho)(\beta+\gamma+\rho)} \end{pmatrix},$$

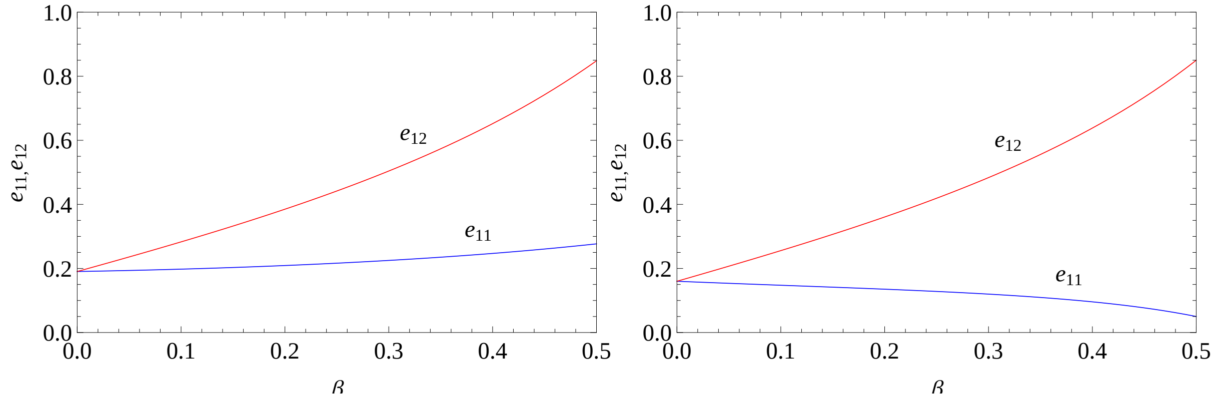

Figure 2: Equilibrium effort levels for agent 1 with $\alpha_1 = 0.2$, $\alpha_2 = 0.1$, $\alpha_3 = 0.9$, $\rho = 0.05$ (left panel), $\rho = 0.25$ (right panel) and $\gamma = 1$.

and hence

$$\boldsymbol{\epsilon} = (\mathbf{I}_2 - \boldsymbol{\Omega})^{-1}\boldsymbol{\mu}(\boldsymbol{\alpha}) = \begin{pmatrix} \frac{(\alpha_1+\alpha_2)(\beta-\gamma)(\beta+\gamma)^2+(\beta+\gamma)(\alpha_3\beta+\alpha_1\gamma)\rho+\alpha_2\gamma\rho^2}{\gamma^2\rho^2-(\beta^2-\gamma^2)^2} \\ \frac{(\alpha_1+\alpha_3)(\beta-\gamma)(\beta+\gamma)^2+(\beta+\gamma)(\alpha_2\beta+\alpha_1\gamma)\rho+\alpha_3\gamma\rho^2}{\gamma^2\rho^2-(\beta^2-\gamma^2)^2} \end{pmatrix}.$$

Inserting the above expressions into Equation (6) yields exactly the equilibrium effort levels of Equation (8). An illustration of the equilibrium effort levels for agent 1 *with* $\alpha_1 = 0.2$, $\alpha_2 = 0.1$, $\alpha_3 = 0.9$, $\gamma = 1$ *and two different values of* $\rho = 0.05$ *and* $\rho = 0.25$ *is shown in Figure 2. We observe that with increasing values of* β *the effort spent by agent* 1 *on project* 2 *is increasing more that the effort spent on project* 1*. The reason is that with increasing* β *the complementarity effects between efforts of collaborating agents become stronger, and this effect is more pronounced for the collaboration of agent* 1 *with the more productive agent* 3*, than with the less productive agent* 2*. Moreover, when the cost parameter* ρ *is high enough, then agent* 1 *may even spend less effort in equilibrium in the project with agent* 1 *than for higher values of* β *than for low values of* β*.*

Example 2. *Consider a network with* 2 *projects and* 4 *agents, where in the first project agents* 1*,* 2 *and* 3 *are collaborating while in the second project agents* 2 *and* 4 *are collaborating. An illustration can be found in Figure 3. The payoffs of the agents are given by*

$$\pi_1 = e_{11}\left(\alpha_1 + \frac{\beta}{2}(e_{21}+e_{31})\right) + e_{21}\left(\alpha_2 + \frac{\beta}{2}(e_{11}+e_{31})\right) + e_{31}\left(\alpha_3 + \frac{\beta}{2}(e_{11}+e_{21})\right) - \frac{\gamma}{2}e_{11}^2$$

$$\pi_2 = e_{21}\left(\alpha_2 + \frac{\beta}{2}(e_{11}+e_{31})\right) + e_{11}\left(\alpha_1 + \frac{\beta}{2}(e_{21}+e_{31})\right) + e_{31}\left(\alpha_3 + \frac{\beta}{2}(e_{11}+e_{21})\right) + e_{22}\left(\alpha_2 + \frac{\beta}{2}e_{42}\right)$$
$$+ e_{42}\left(\alpha_4 + \frac{\beta}{2}e_{22}\right) - \frac{\gamma}{2}e_{21}^2 - \frac{\gamma}{2}e_{22}^2 - \rho e_{21}e_{22}$$

$$\pi_3 = e_{31}\left(\alpha_3 + \frac{\beta}{2}(e_{11}+e_{21})\right) + e_{11}\left(\alpha_1 + \frac{\beta}{2}(e_{21}+e_{31})\right) + e_{21}\left(\alpha_2 + \frac{\beta}{2}(e_{11}+e_{31})\right) - \frac{\gamma}{2}e_{31}^2$$

$$\pi_4 = e_{42}\left(\alpha_4 + \frac{\beta}{2}e_{22}\right) + e_{22}\left(\alpha_2 + \frac{\beta}{2}e_{42}\right) - \frac{\gamma}{2}e_{42}^2.$$

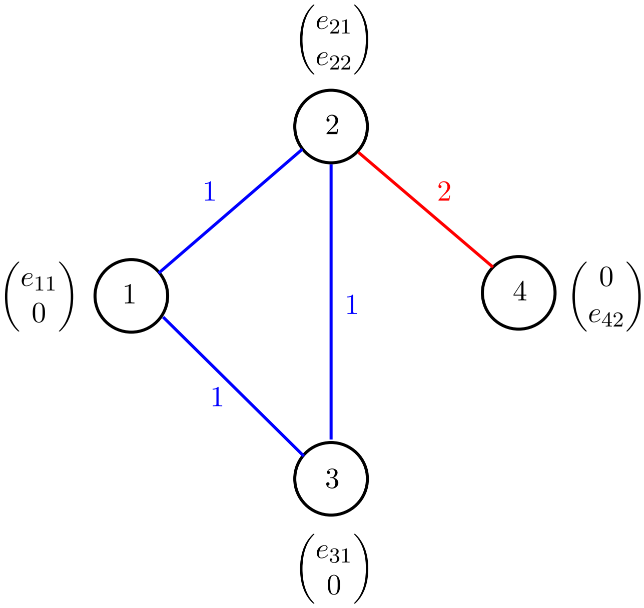

Figure 3: The network analyzed in Example 2. The effort levels of the individual agents for each project they are involved in are indicate next to the nodes.

The first order conditions are given by

$$\begin{aligned}
\frac{\partial \pi_1}{\partial e_{11}} &= \alpha_1 + (e_{21} + e_{31})\beta - e_{11}\gamma = 0 \\
\frac{\partial \pi_2}{\partial e_{21}} &= \alpha_2 + (e_{11} + e_{31})\beta - e_{21}\gamma - e_{22}\rho = 0 \\
\frac{\partial \pi_2}{\partial e_{22}} &= \alpha_2 + e_{42}\beta - e_{22}\gamma - e_{21}\rho = 0 \\
\frac{\partial \pi_3}{\partial e_{31}} &= \alpha_3 + (e_{11} + e_{21})\beta - e_{31}\gamma = 0 \\
\frac{\partial \pi_4}{\partial e_{42}} &= \alpha_4 + e_{22}\beta - e_{42}\gamma = 0.
\end{aligned}$$

Solving this system of equations directly yields

$$\begin{aligned}
e_{11} &= \frac{(-(\alpha_2+\alpha_3)\beta + \alpha_1(\beta-\gamma))(\beta-\gamma)(\beta+\gamma)^2 - \beta(\beta+\gamma)(\alpha_4\beta+\alpha_2\gamma)\rho - \gamma(\alpha_3\beta+\alpha_1\gamma)\rho^2}{(\beta-\gamma)(\beta+\gamma)\left((2\beta-\gamma)(\beta+\gamma)^2+\gamma\rho^2\right)}, \\
e_{21} &= \frac{-(\beta+\gamma)((\alpha_1-\alpha_2+\alpha_3)\beta+\alpha_2\gamma) + (\alpha_4\beta+\alpha_2\gamma)\rho}{(2\beta-\gamma)(\beta+\gamma)^2+\gamma\rho^2}, \\
e_{22} &= -\frac{(2\beta-\gamma)(\beta+\gamma)(\alpha_4\beta+\alpha_2\gamma) + \gamma((\alpha_1-\alpha_2+\alpha_3)\beta+\alpha_2\gamma)\rho}{(\beta-\gamma)\left((2\beta-\gamma)(\beta+\gamma)^2+\gamma\rho^2\right)}, \\
e_{31} &= -\frac{(\beta-\gamma)(\beta+\gamma)^2((\alpha_1+\alpha_2-\alpha_3)\beta+\alpha_3\gamma) + \beta(\beta+\gamma)(\alpha_4\beta+\alpha_2\gamma)\rho + \gamma(\alpha_1\beta+\alpha_3\gamma)\rho^2}{(\beta-\gamma)(\beta+\gamma)\left((2\beta-\gamma)(\beta+\gamma)^2+\gamma\rho^2\right)}, \\
e_{42} &= \frac{-(2\beta-\gamma)(\beta+\gamma)(\alpha_2\beta+\alpha_4\gamma) - \beta((\alpha_1-\alpha_2+\alpha_3)\beta+\alpha_2\gamma)\rho + \alpha_4(\beta-\gamma)\rho^2}{(\beta-\gamma)\left((2\beta-\gamma)(\beta+\gamma)^2+\gamma\rho^2\right)}. \qquad (9)
\end{aligned}$$

Next, we compute the above equilibrium effort levels using the equilibrium characterization in

Equation (6). Note that $\mathbf{d} = (d_i)_{1 \leq i \leq 3} = (1,2,1,1)^\top$, $\mathbf{n} = (n_s)_{1\leq s \leq 2} = (2,2)^\top$,

$$\boldsymbol{\delta} = (\delta_{is})_{1\leq i\leq 3, 1\leq s\leq 2} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\boldsymbol{\varphi} = (\varphi_{is})_{1\leq i\leq 3, 1\leq s\leq 2} = \begin{pmatrix} \frac{\beta\rho}{(\beta+\gamma)(2\beta-\gamma+\rho)} & 0 \\ \frac{\beta\rho}{(2\beta-\gamma+\rho)(\beta+\gamma+\rho)} & \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} \\ \frac{\beta\rho}{(\beta+\gamma)(2\beta-\gamma+\rho)} & 0 \\ 0 & \frac{\beta\rho}{(\beta+\gamma)(\beta-\gamma+\rho)} \end{pmatrix}.$$

Further, we have that $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^\top$ *and*

$$\boldsymbol{\mu}(\boldsymbol{\alpha}) = \begin{pmatrix} \frac{-(\alpha_1+\alpha_2+\alpha_3)(\beta+\gamma)^2+\alpha_2(\beta+\gamma)\rho+(\alpha_1+\alpha_3)\rho^2}{(\beta+\gamma)(2\beta-\gamma+\rho)(\beta+\gamma+\rho)} \\ \frac{\alpha_4\left(-1+\frac{\rho}{\beta+\gamma}\right)+\alpha_2\left(-1+\frac{2\rho}{\beta+\gamma+\rho}\right)}{\beta-\gamma+\rho} \end{pmatrix}.$$

Next, we have that

$$\boldsymbol{\Omega} = \begin{pmatrix} \frac{\beta\rho(3(\beta+\gamma)+2\rho)}{(\beta+\gamma)(2\beta-\gamma+\rho)(\beta+\gamma+\rho)} & \frac{\beta\rho}{(2\beta-\gamma+\rho)(\beta+\gamma+\rho)} \\ \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} & \frac{\beta\rho(2(\beta+\gamma)+\rho)}{(\beta+\gamma)(\beta-\gamma+\rho)(\beta+\gamma+\rho)} \end{pmatrix},$$

and hence

$$\boldsymbol{\epsilon} = (\mathbf{I}_2 - \boldsymbol{\Omega})^{-1}\boldsymbol{\mu}(\boldsymbol{\alpha}) = \begin{pmatrix} -\frac{(\alpha_1+\alpha_2+\alpha_3)(\beta-\gamma)(\beta+\gamma)^2+(\beta+\gamma)(\alpha_4\beta+\alpha_2\gamma)\rho+(\alpha_1+\alpha_3)\gamma\rho^2}{(\beta-\gamma)((2\beta-\gamma)(\beta+\gamma)^2+\gamma\rho^2)} \\ \frac{-(\alpha_2+\alpha_4)(2\beta-\gamma)(\beta+\gamma)^2-(\beta+\gamma)((\alpha_1-\alpha_2+\alpha_3)\beta+\alpha_2\gamma)\rho+\alpha_4(\beta-\gamma)\rho^2}{(\beta-\gamma)((2\beta-\gamma)(\beta+\gamma)^2+\gamma\rho^2)} \end{pmatrix}.$$

Inserting the above expressions into Equation (6) yields exactly the equilibrium effort levels of Equation (9). An example of the equilibrium effort levels for agents 3 *and* 4 *is shown in Figure 4. The figure illustrates that with differently skilled agents, the effort of an agent in one project might exceed the effort in another project when the complementarity parameter* β *increases.*

Example 3. *Consider a network with* 3 *projects and* 3 *agents, where in the first project agents* 1 *and* 2 *are collaborating, in the second project agents* 1 *and* 3 *are collaborating and in the third project agents* 2 *and* 3 *are collaborating. An illustration can be found in Figure 5. The payoffs of the agents are given by*

$$\pi_1 = e_{11}\left(\alpha_1 + \frac{\beta}{2}e_{21}\right) + e_{21}\left(\alpha_2 + \frac{\beta}{2}e_{11}\right) + e_{12}\left(\alpha_1 + \frac{\beta}{2}\, e_{32}\right) + e_{32}\left(\alpha_3 + \frac{\beta}{2}\, e_{12}\right) - \frac{\gamma}{2}e_{11}^2 - \frac{\gamma}{2}e_{12}^2 - \rho e_{11}e_{12}$$
$$\pi_2 = e_{11}\left(\alpha_1 + \frac{\beta}{2}\, e_{21}\right) + e_{21}\left(\alpha_2 + \frac{\beta}{2}e_{11}\right) + e_{23}\left(\alpha_2 + \frac{\beta}{2}\, e_{33}\right) + e_{33}\left(\alpha_3 + \frac{\beta}{2}e_{23}\right) - \frac{\gamma}{2}e_{21}^2 - \frac{\gamma}{2}e_{23}^2 - \rho e_{21}e_{23}$$
$$\pi_3 = e_{32}\left(\alpha_3 + \frac{\beta}{2}\, e_{12}\right) + e_{12}\left(\alpha_1 + \frac{\beta}{2}\, e_{32}\right) + e_{33}\left(\alpha_3 + \frac{\beta}{2}e_{23}\right) + e_{23}\left(\alpha_2 + \frac{\beta}{2}\, e_{33}\right) - \frac{\gamma}{2}e_{32}^2 - \frac{\gamma}{2}e_{33}^2 - \rho e_{32}e_{33}.$$

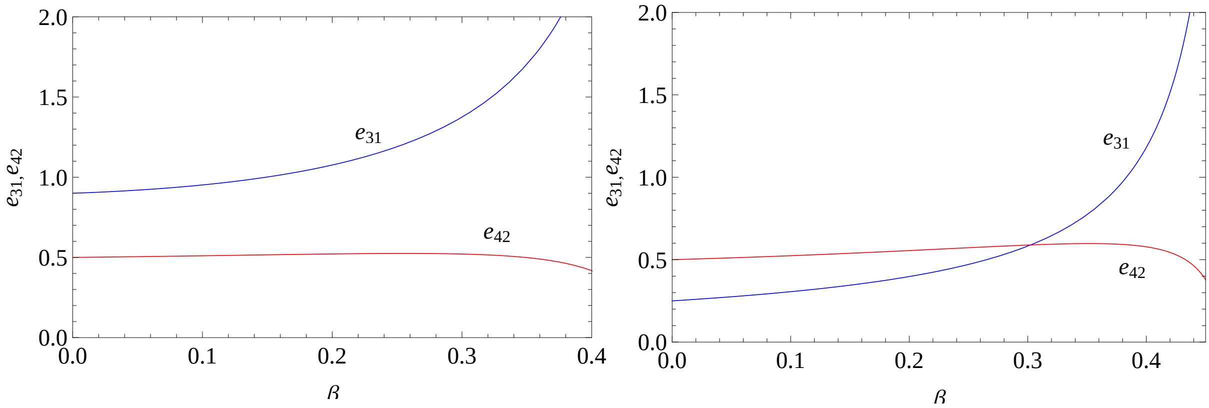

Figure 4: Equilibrium effort levels for agents 3 and 4 with $\rho = 0.05$, $\gamma = 1$, $\alpha_1 = 0.2$, $\alpha_2 = 0.1$, $\alpha_3 = 0.9$, $\alpha_4 = 0.5$ in the left panel, while $\alpha_1 = 0.25$, $\alpha_2 = 0.25$, $\alpha_3 = 0.25$, $\alpha_4 = 0.5$ in the right panel.

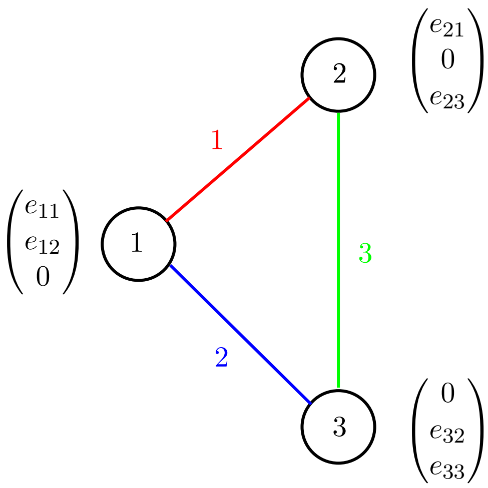

Figure 5: The network analyzed in Example 3. The effort levels of the individual agents for each project they are involved in are indicate next to the nodes.

The first order conditions are given by

$$\begin{aligned}
\frac{\partial \pi_1}{\partial e_{11}} &= \alpha_1 + e_{21}\beta - e_{11}\gamma - e_{12}\rho = 0 \\
\frac{\partial \pi_1}{\partial e_{12}} &= \alpha_1 + e_{32}\beta - e_{12}\gamma - e_{11}\rho = 0 \\
\frac{\partial \pi_2}{\partial e_{21}} &= \alpha_2 + e_{11}\beta - e_{21}\gamma - e_{23}\rho = 0 \\
\frac{\partial \pi_2}{\partial e_{23}} &= \alpha_2 + e_{33}\beta - e_{23}\gamma - e_{21}\rho = 0 \\
\frac{\partial \pi_3}{\partial e_{32}} &= \alpha_3 + e_{12}\beta - e_{32}\gamma - e_{33}\rho = 0 \\
\frac{\partial \pi_3}{\partial e_{33}} &= \alpha_3 + e_{23}\beta - e_{33}\gamma - e_{32}\rho = 0.
\end{aligned}$$

Solving this system of equations directly yields

$$\begin{aligned}
e_{11} &= -\frac{(\beta-\gamma)(\alpha_2\beta+\alpha_1\gamma)+\alpha_3\beta\rho+\alpha_1\rho^2}{(\beta-\gamma-\rho)\left(\beta^2-\gamma^2+\beta\rho+\rho^2\right)}, \\
e_{12} &= -\frac{(\beta-\gamma)(\alpha_3\beta+\alpha_1\gamma)+\alpha_2\beta\rho+\alpha_1\rho^2}{(\beta-\gamma-\rho)\left(\beta^2-\gamma^2+\beta\rho+\rho^2\right)}, \\
e_{21} &= -\frac{(\beta-\gamma)(\alpha_1\beta+\alpha_2\gamma)+\alpha_3\beta\rho+\alpha_2\rho^2}{(\beta-\gamma-\rho)\left(\beta^2-\gamma^2+\beta\rho+\rho^2\right)}, \\
e_{23} &= -\frac{(\beta-\gamma)(\alpha_3\beta+\alpha_2\gamma)+\alpha_1\beta\rho+\alpha_2\rho^2}{(\beta-\gamma-\rho)\left(\beta^2-\gamma^2+\beta\rho+\rho^2\right)}, \\
e_{32} &= -\frac{(\beta-\gamma)(\alpha_1\beta+\alpha_3\gamma)+\alpha_2\beta\rho+\alpha_3\rho^2}{(\beta-\gamma-\rho)\left(\beta^2-\gamma^2+\beta\rho+\rho^2\right)}, \\
e_{33} &= -\frac{(\beta-\gamma)(\alpha_2\beta+\alpha_3\gamma)+\alpha_1\beta\rho+\alpha_3\rho^2}{(\beta-\gamma-\rho)\left(\beta^2-\gamma^2+\beta\rho+\rho^2\right)}.
\end{aligned} \tag{10}$$

Next, we compute the above equilibrium effort levels using the equilibrium characterization in Equation (6). Note that $\mathbf{d} = (d_i)_{1\le i\le 3} = (2,2,2)^\top$, $\mathbf{n} = (n_s)_{1\le s\le 3} = (2,2,2)^\top$,

$$\boldsymbol{\delta} = (\delta_{is})_{1\le i\le 3, 1\le s\le 3} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

and

$$\boldsymbol{\varphi} = (\varphi_{is})_{1\le i\le 3, 1\le s\le 3} = \begin{pmatrix} \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} & \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} & 0 \\ \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} & 0 & \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} \\ 0 & \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} & \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} \end{pmatrix}.$$

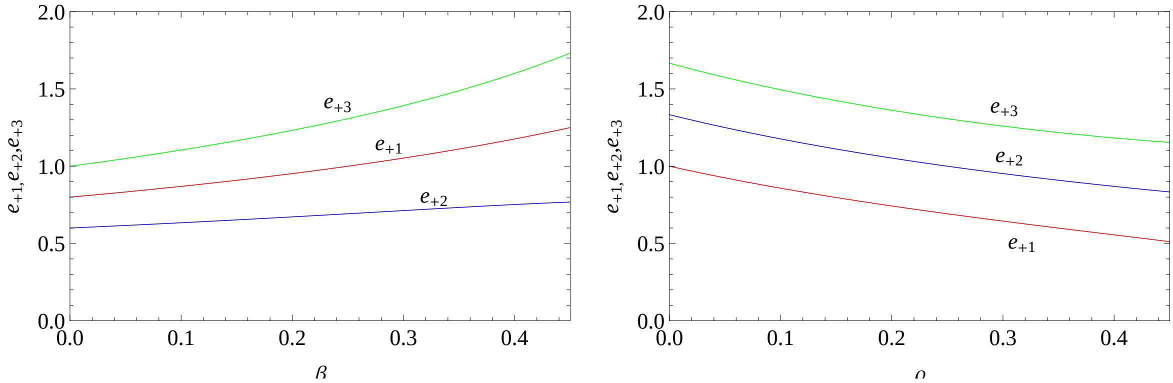

Figure 6: Total equilibrium effort levels, $e_{+s} = \sum_{i=1}^{p} e_{is}\delta_{is}$, for projects 1, 2 and 3 with $\gamma = 1$, $\alpha_1 = 0.25$, $\alpha_2 = 0.5$, $\alpha_3 = 0.75$ and varying values of β in the left panel, and varying values of ρ in the right panel.

Further, we have that $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^\top$ *and*

$$\boldsymbol{\mu}(\boldsymbol{\alpha}) = \begin{pmatrix} -\frac{(\alpha_1+\alpha_2)(\beta+\gamma-\rho)}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} \\ -\frac{(\alpha_1+\alpha_3)(\beta+\gamma-\rho)}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} \\ -\frac{(\alpha_2+\alpha_3)(\beta+\gamma-\rho)}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} \end{pmatrix}.$$

Next, we have that

$$\boldsymbol{\Omega} = \begin{pmatrix} \frac{2\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} & \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} & \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} \\ \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} & \frac{2\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} & \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} \\ \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} & \frac{\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} & \frac{2\beta\rho}{(\beta-\gamma+\rho)(\beta+\gamma+\rho)} \end{pmatrix},$$

and hence

$$\boldsymbol{\epsilon} = (\mathbf{I}_2 - \boldsymbol{\Omega})^{-1}\boldsymbol{\mu}(\boldsymbol{\alpha}) = \begin{pmatrix} -\frac{(\alpha_1+\alpha_2)(\beta-\gamma)(\beta+\gamma)+2\alpha_3\beta\rho+(\alpha_1+\alpha_2)\rho^2}{(\beta-\gamma-\rho)(\beta^2-\gamma^2+\beta\rho+\rho^2)} \\ -\frac{(\alpha_1+\alpha_3)(\beta-\gamma)(\beta+\gamma)+2\alpha_2\beta\rho+(\alpha_1+\alpha_3)\rho^2}{(\beta-\gamma-\rho)(\beta^2-\gamma^2+\beta\rho+\rho^2)} \\ -\frac{(\alpha_2+\alpha_3)(\beta-\gamma)(\beta+\gamma)+2\alpha_1\beta\rho+(\alpha_2+\alpha_3)\rho^2}{(\beta-\gamma-\rho)(\beta^2-\gamma^2+\beta\rho+\rho^2)} \end{pmatrix}.$$

Inserting the above expressions into Equation (6) yields exactly the equilibrium effort levels of Equation (10). The total equilibrium effort levels for projects 1, 2 *and* 3 *are shown in Figure 6 for varying values of* β *and* ρ.

A more compact characterization of the equilibrium effort levels of Proposition 2 can be obtained in the special case of complete network [cf. e.g. Salonen, 2014]. Assume that $\delta_{is} = 1$ for all $i = 1, \ldots, n$. From Equation (25) we then get

$$\sum_{s'=1}^{p} e_{is'}\phi_{s',s}\delta_{is}\delta_{is'} = \alpha_i\delta_{is} + \beta \sum_{j\in\mathcal{N}_s\backslash\{i\}} e_{js}\delta_{js}\delta_{is}, \tag{11}$$

for $i = 1, \ldots, n$ and $s = 1, \ldots, p$. Let $\mathbf{e}_s = [e_{1s}\delta_{1s}, \ldots, e_{ns}\delta_{ns}]^\top$ be a vector of effort levels that agents $1, \ldots, n$ put into project s. Let $\mathbf{E} = [\mathbf{e}_1, \ldots, \mathbf{e}_p]$ be the $n \times p$ matrix with columns given by $\mathbf{e}_s$ for $s = 1, \ldots, p$. Further, denote by $w_{ij,s} = \delta_{is}\delta_{js}$ for $i \neq j$ and $w_{ii,s} = 0$, and let

$\mathbf{W}_s = [w_{ij,s}]_{1\leq i,j\leq n}$ be an $n \times n$ matrix with elements $w_{ij,s} \in \{0,1\}$ indicating whether agents i and j are participating in project s. Moreover, we denote by $\tilde{\alpha}_{is} = \alpha_i \delta_{is}$, $\boldsymbol{\alpha}_s = (\tilde{\alpha}_{1s}, \ldots, \tilde{\alpha}_{ns})$ and let $\boldsymbol{A} = [\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_p]$ be the $n \times p$ matrix with columns given by α_s. Then we can write Equation (26) in matrix form as follows

$$\mathbf{E}\boldsymbol{\phi} = \boldsymbol{A} + \beta[\mathbf{W}_1\mathbf{e}_1, \ldots, \mathbf{W}_p\mathbf{e}_p]. \tag{12}$$

Note that the $n \times p$ matrix $[\mathbf{W}_1\mathbf{e}_1, \ldots, \mathbf{W}_p\mathbf{e}_p]$ is composed of $n \times 1$ column vectors $\mathbf{W}_s\mathbf{e}_s$ for $s = 1, \ldots, p$. We next apply a vectorization to both sides of Equation (12) to obtain[3]

$$\text{vec}(\mathbf{E}\boldsymbol{\phi}) = \text{vec}(\boldsymbol{A}) + \beta \begin{bmatrix} \mathbf{W}_1\mathbf{e}_1 \\ \mathbf{W}_2\mathbf{e}_2 \\ \vdots \\ \mathbf{W}_p\mathbf{e}_p \end{bmatrix}. \tag{13}$$

Using the fact that

$$\text{vec}(\mathbf{E}\boldsymbol{\phi}) = (\boldsymbol{\phi} \otimes \mathbf{I}_n)\text{vec}(\mathbf{E}),$$

and

$$\begin{bmatrix} \mathbf{W}_1\mathbf{e}_1 \\ \mathbf{W}_2\mathbf{e}_2 \\ \vdots \\ \mathbf{W}_p\mathbf{e}_p \end{bmatrix} = \begin{bmatrix} \mathbf{W}_1 & 0 \cdots & \\ 0 & \mathbf{W}_2 & \\ 0 & \cdots & \ddots \\ 0 & \cdots & \mathbf{W}_p \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_p \end{bmatrix} = \text{diag}(\mathbf{W}_s)_{s=1}^p \text{vec}(\mathbf{E}),$$

we can write Equation (13) as follows

$$(\boldsymbol{\phi} \otimes \mathbf{I}_n)\text{vec}(\mathbf{E}) = \text{vec}(\boldsymbol{A}) + \beta \text{diag}(\mathbf{W}_s)_{s=1}^p \text{vec}(\mathbf{E}). \tag{14}$$

Next, denoting by $\mathbf{e} = \text{vec}(\mathbf{E})$, $\mathbf{W} = \text{diag}(\mathbf{W}_s)_{s=1}^p$ and $\boldsymbol{\alpha} = \text{vec}(\boldsymbol{A})$, we can write Equation (14) as

$$(\boldsymbol{\phi} \otimes \mathbf{I}_n)\mathbf{e} = \boldsymbol{\alpha} + \beta\mathbf{W}\mathbf{e}. \tag{15}$$

When the matrix $(\boldsymbol{\phi} \otimes \mathbf{I}_n + \beta\mathbf{W})^{-1}$ is invertible, we then obtain the equilibrium effort levels given by

$$\mathbf{e} = (\boldsymbol{\phi} \otimes \mathbf{I}_n + \beta\mathbf{W})^{-1}\boldsymbol{\alpha}. \tag{16}$$

An alternative compact form of the equilibrium effort levels can be obtain using the *line*

[3]For example, for the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the vectorization is $\text{vec}(A) = \begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix}$ [cf. e.g. Dhrymes, 1984, Chap. 4].

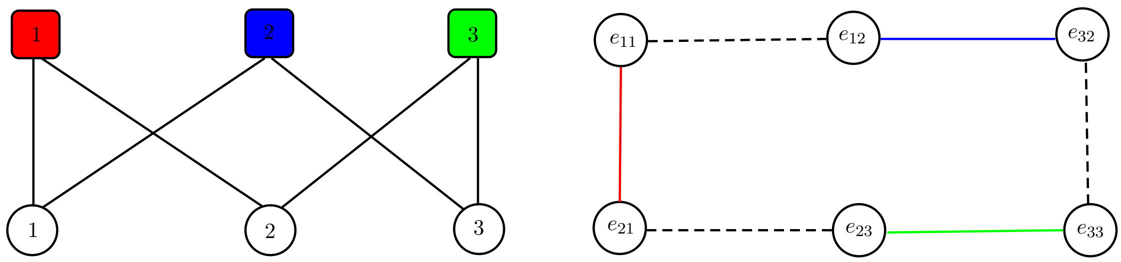

Figure 7: (Left panel) The bipartite collaboration network G corresponding to the network shown in Figure 5, where round circles represent authors, squares represent projects and lines indicate the efforts agents invest into the different projects. (Right panel) The line graph $L(G)$ associated with the collaboration network G in which each node represents the effort an author invests into different projects. Solid lines indicate nodes sharing a project while dashed lines indicate nodes with the same author.

graph[4] representation of the collaboration network G similar to Bimpikis et al. [2014]. An example is shown in Figure 7. The network corresponds to the one shown in Figure 5. First note that the FOC of the effort levels can be written as (see Equation (29) in the proof of Proposition 2 in Appendix A)

$$e_{is}\delta_{is} + \frac{\rho\delta_{is}}{\beta+\gamma-\rho}\sum_{s=1}^{p} e_{is}\delta_{is} - \frac{\beta\delta_{is}}{\beta+\gamma-\rho}\sum_{j\in\mathcal{N}_s} e_{js}\delta_{js} = \frac{1}{\beta+\gamma-\rho}\alpha_i\delta_{is}.$$

Then, introducing the matrix

$$\Gamma_{is,jk} = \begin{cases} \rho & \text{if } i=j, s\neq k, \\ -\beta & \text{if } i\neq j, s=k, \\ 0 & \text{otherwise,} \end{cases}$$

we can write for the vector $\mathbf{e}$ of stacked agent-project effort levels, e_{is}, the following

$$\left(\mathbf{I} + \frac{1}{\beta+\gamma-\rho}\mathbf{\Gamma}\right)\mathbf{e} = \frac{1}{\beta+\gamma-\rho}\boldsymbol{\alpha},$$

where $\boldsymbol{\alpha}$ is a stacked vector with elements $\alpha_i\delta_{is}$, so that, when the matrix $\mathbf{I} + \frac{1}{\beta+\gamma-\rho}\mathbf{\Gamma}$ is invertible, we can write the equilibrium effort levels as follows

$$\mathbf{e} = \frac{1}{\beta+\gamma-\rho}\left(\mathbf{I} + \frac{1}{\beta+\gamma-\rho}\mathbf{\Gamma}\right)^{-1}\boldsymbol{\alpha}.$$

Observe that the matrix $\mathbf{\Gamma}$ represents a weighted matrix of the line graph $L(G)$ of the bipartite collaboration network G, where each link between nodes sharing a project has weight $-\beta$, and each link between nodes sharing an author have weight ρ.

[4]Given a graph G, its line graph $L(G)$ is a graph such that each node of $L(G)$ represents an edge of G, and two nodes of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G [cf. e.g. West, 2001].

4. Conclusion

We have analyzed the equilibrium efforts of agents involved in multiple, possibly overlapping projects. We show that, given an allocation of researchers to different projects, the Nash equilibrium can be completely characterized.

We have focussed on two particular specifications, the case of convex separable costs (see Section 3.1) and the case of a quadratic costs with substitutes or complements (see Section 3.2). The assumption of a convex separable cost is similar to the model studied in Adams [2006]. The introduction of a quadratic cost with substitutes or complements, depending on the sign of the parameters $\phi_{ss'}$, is similar to Cohen-Cole et al. [2012], and it includes the case of a convex total cost as a special case when $\phi_{s,s'} = \gamma$, and the the case of a convex separable cost when $\phi_{s,s'} = \gamma\delta_{s,s'}$. For the special case of only two activities, a theoretical model is studied in Belhaj and Deroïan [2014], and an empirical analysis is provided in Liu [2014].

Our analysis can be extended along several directions. First, we can allow the returns of an agent from participating in a project to be split equally among the participants of the project similar to the models studied in Jackson and Wolinsky [1996]; Kandel and Lazear [1992]. Second, instead of a convex cost, we can introduce a time constraint as in Baumann [2014]; Salonen [2014]. These extensions and the relation to the current setup are summarized in Table 1.

	return independent of number of authors	return decreasing with number of authors
convex separable cost	$\pi_i = \sum_{s=1}^{p} \left(Y_s - \frac{\gamma}{2} e_{is}^2\right) \delta_{is}$	$\pi_i = \sum_{s=1}^{p} \left(\frac{1}{n_s} Y_s - \frac{\gamma}{2} e_{is}^2\right) \delta_{is}$
convex total cost	$\pi_i = \sum_{s=1}^{p} Y_s \delta_{is} - \frac{\gamma}{2} \left(\sum_{s=1}^{p} e_{is} \delta_{is}\right)^2$	$\pi_i = \sum_{s=1}^{p} \frac{1}{n_s} Y_s \delta_{is} - \frac{\gamma}{2} \left(\sum_{s=1}^{p} e_{is} \delta_{is}\right)^2$
quadratic cost with substitutes/complements	$\pi_i = \sum_{s=1}^{p} Y_s \delta_{is} - \frac{1}{2} \sum_{s,s'=1}^{p} \phi_{s,s'} e_{is} e_{is'} \delta_{is} \delta_{is'}$	$\pi_i = \sum_{s=1}^{p} \frac{1}{n_s} Y_s \delta_{is} - \frac{1}{2} \sum_{s,s'=1}^{p} \phi_{s,s'} e_{is} e_{is'} \delta_{is} \delta_{is'}$
time constraint	$\pi_i = \sum_{s=1}^{p} Y_s \delta_{is}$ s.t. $\sum_{s=1}^{p} e_{is} \delta_{is} = 1$	$\pi_i = \sum_{s=1}^{p} \frac{1}{n_s} Y_s \delta_{is}$ s.t. $\sum_{s=1}^{p} e_{is} \delta_{is} = 1$

Table 1: The alternative payoff specifications, π_i, for $i = 1, \ldots, n$ analyzed in Section 3. The output Y_s of project s is given in Equation (1). The case of convex separable costs is studied in Section 3.1. The case of a quadratic costs with substitutes or complements is studied in Section 3.2. The case of a quadratic cost includes the case of a convex total cost as a special case when $\phi_{s,s'} = \gamma$, and the case of a convex separable cost when $\phi_{s,s'} = \gamma \delta_{s,s'}$.

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Appendix

A. Proofs

Proof of Proposition 1. The first order condition (FOC) wrt e_{is} is given by[5]

$$\frac{\partial \pi_i(G,\mathbf{e})}{\partial e_{is}} = \sum_{s'=1}^{p} \left(\frac{\partial Y_{s'}(G,\mathbf{e}_{s'})}{\partial e_{is}} - \gamma e_{is'} \right) \delta_{is'} = \left(\alpha_i + \beta \sum_{j \in \mathcal{N}_s \setminus \{i\}} e_{js} \right) \delta_{is} - \gamma e_{is} \delta_{is} = 0, \quad (17)$$

where we have used the fact that

$$\frac{\partial Y_{s'}(G,\mathbf{e}_{s'})}{\partial e_{is}} = \begin{cases} \alpha_i + \beta \sum_{j \in \mathcal{N}_s \setminus \{i\}} e_{js}, & \text{if } s = s', \\ 0, & \text{otherwise.} \end{cases}$$

From Equation (30) we get

$$e_{is} = \frac{\alpha_i}{\gamma} + \frac{\beta}{\gamma} \sum_{j \in \mathcal{N}_s \setminus \{i\}} e_{js}, \quad (18)$$

for all projects s in which i is participating. Further, Equation (18) can be written as

$$\left(1 + \frac{\beta}{\gamma}\right) e_{is} = \frac{\alpha_i}{\gamma} + \frac{\beta}{\gamma} \sum_{j \in \mathcal{N}_s} e_{js}, \quad (19)$$

and

$$e_{is} = \frac{\alpha_i}{\beta + \gamma} + \frac{\beta}{\beta + \gamma} \sum_{j \in \mathcal{N}_s} e_{js}. \quad (20)$$

Summation over $i \in \mathcal{N}_s$ gives

$$\sum_{j \in \mathcal{N}_s} e_{js} = \frac{1}{\beta + \gamma} \sum_{j \in \mathcal{N}_s} \alpha_j + \frac{\beta n_s}{\beta + \gamma} \sum_{j \in \mathcal{N}_s} e_{js},$$

and we get

$$\left(1 - \frac{\beta n_s}{\beta + \gamma}\right) \sum_{j \in \mathcal{N}_s} e_{js} = \frac{1}{\beta + \gamma} \sum_{j \in \mathcal{N}_s} \alpha_j.$$

Hence

$$\sum_{j \in \mathcal{N}_s} e_{js} = \frac{1}{\gamma - \beta(n_s - 1)} \sum_{j \in \mathcal{N}_s} \alpha_j.$$

Inserting into Equation (20) yields

$$e_{is} = \frac{\alpha_i}{\beta + \gamma} + \frac{\beta}{(\beta + \gamma)(\gamma - \beta(n_s - 1))} \sum_{j \in \mathcal{N}_s} \alpha_j. \quad (21)$$

[5]Observe that the second order condition (SOC) is given by $\frac{\partial^2 \pi_i(G,\mathbf{e})}{\partial e_{is}^2} = -\gamma \delta_{is} \leq 0$.

This allows us to determine the individual effort e_{is} of agent i in project s. Denoting by $\tilde{\alpha}_i \equiv \frac{\alpha_i}{\beta+\gamma}$ we can write Equation (21) as

$$e_{is} = \tilde{\alpha}_i + \frac{\beta}{\gamma - \beta(n_s - 1)} \sum_{j \in \mathcal{N}_s} \tilde{\alpha}_j. \tag{22}$$

Further, denoting by $\tilde{\beta}_s \equiv \frac{\beta}{\gamma-\beta(n_s-1)}$ this can be simplified to

$$e_{is} = \tilde{\alpha}_i + \tilde{\beta}_s \sum_{j \in \mathcal{N}_s} \tilde{\alpha}_j. \tag{23}$$

From Equation (18) we have that

$$\beta \sum_{j \in \mathcal{N}_s \setminus \{i\}} e_{js} = \gamma e_{is} - \alpha_i,$$

Inserting into the production function from Equation (1) yields

$$\begin{aligned} Y_s(G) &= \sum_{i \in \mathcal{N}_s} e_{is} \left[\alpha_i + \frac{\beta}{2} \sum_{j \in \mathcal{N}_s \setminus \{i\}} e_{js} \right] \\ &= \frac{1}{2} \sum_{i \in \mathcal{N}_s} e_{is} \left(\alpha_i + \gamma e_{is} \right) \\ &= \frac{1}{2} \sum_{i \in \mathcal{N}_s} \left(\alpha_i e_{is} + \gamma e_{is}^2 \right) \\ &= \frac{1}{2} \sum_{i \in \mathcal{N}_s} \left(\alpha_i \left(\tilde{\alpha}_i + \tilde{\beta}_s \sum_{j \in \mathcal{N}_s} \tilde{\alpha}_j \right) + \left(\tilde{\alpha}_i + \tilde{\beta}_s \sum_{j \in \mathcal{N}_s} \tilde{\alpha}_j \right)^2 \right) \\ &= \frac{1}{2} \sum_{i \in \mathcal{N}_s} \left[\alpha_i \left(\frac{\alpha_i}{\beta+\gamma} + \frac{\beta}{\gamma - \beta(n_s-1)} \sum_{j \in \mathcal{N}_s} \frac{\alpha_j}{\beta+\gamma} \right) + \gamma \left(\frac{\alpha_i}{\beta+\gamma} + \frac{\beta}{\gamma - \beta(n_s-1)} \sum_{j \in \mathcal{N}_s} \frac{\alpha_j}{\beta+\gamma} \right)^2 \right]. \end{aligned} \tag{24}$$

□

Proof of Proposition 2. The first order condition (FOC) wrt. e_{is} is given by[6]

$$\frac{\partial \pi_i(G, \mathbf{e})}{\partial e_{is}} = \sum_{s'=1}^{p} \delta_{is'} \frac{\partial Y_{s'}(G, \mathbf{e}_{s'})}{\partial e_{is}} - \delta_{is} \sum_{s'=1}^{p} \phi_{s,s'} e_{is'} \delta_{is'} = \left(\alpha_i + \beta \sum_{j \in \mathcal{N}_s \setminus \{i\}} e_{js} \right) \delta_{is} - \delta_{is} \sum_{s'=1}^{p} \phi_{s,s'} e_{is'} \delta_{is'} = 0, \tag{25}$$

where we have used the fact that

$$\frac{\partial Y_{s'}(G, \mathbf{e}_{s'})}{\partial e_{is}} = \begin{cases} \alpha_i + \beta \sum_{j \in \mathcal{N}_s \setminus \{i\}} e_{js}, & \text{if } s = s' \\ 0, & \text{otherwise.} \end{cases}$$

[6] Observe that the second order condition (SOC) is given by $\frac{\partial^2 \pi_i(G,\mathbf{e})}{\partial e_{is}^2} = -\phi_{ss}\delta_{is} \leq 0$.

From Equation (25) we get

$$\sum_{s'=1}^{p} e_{is'}\phi_{s',s}\delta_{is}\delta_{is'} = \alpha_i\delta_{is} + \beta \sum_{j\in\mathcal{N}_s\setminus\{i\}} e_{js}\delta_{js}\delta_{is}, \tag{26}$$

for $i = 1, \ldots, n$ and $s = 1, \ldots, p$. In the following we denote by

$$\tilde{e}_{is} = \begin{cases} e_{is}, & \text{if } i \in \mathcal{N}_s, \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

That is, we define $\tilde{e}_{is} \equiv \delta_{is}e_{is}$. Then we can write

$$\delta_{is}\sum_{s'=1}^{p} \tilde{e}_{is'}\phi_{s',s} = \alpha_i\delta_{is} + \beta\delta_{is} \sum_{j\in\mathcal{N}_s\setminus\{i\}} \tilde{e}_{js}, \tag{28}$$

In the following we assume that

$$\phi_{ss'} = \begin{cases} \gamma, & \text{if } s' = s, \\ \rho, & \text{otherwise.} \end{cases}$$

Then we obtain from Equation (28) that

$$\gamma\tilde{e}_{is} + \rho\delta_{is}\sum_{s'\neq s}^{p} \tilde{e}_{is'} = \alpha_i\delta_{is} + \beta\delta_{is} \sum_{j\in\mathcal{N}_s\setminus\{i\}} \tilde{e}_{js}, \tag{29}$$

which can be written as follows

$$(\gamma - \rho)\tilde{e}_{is} + \rho\delta_{is}\sum_{s'=1}^{p} \tilde{e}_{is'} = \alpha_i\delta_{is} + \beta\delta_{is} \sum_{j\in\mathcal{N}_s\setminus\{i\}} \tilde{e}_{js}.$$

We can write this as

$$(\beta + \gamma - \rho)\tilde{e}_{is} + \rho\delta_{is}\sum_{s'=1}^{p} \tilde{e}_{is'} = \alpha_i\delta_{is} + \beta\delta_{is} \sum_{j\in\mathcal{N}_s} \tilde{e}_{js},$$

and hence

$$\delta_{is}\tilde{e}_{+s} = \frac{\beta + \gamma - \rho}{\beta}\tilde{e}_{is} - \frac{\alpha_i}{\beta}\delta_{is} + \frac{\rho}{\beta}\delta_{is}\tilde{e}_{i+}, \tag{30}$$

where we have denoted by

$$\tilde{e}_{+s} \equiv \sum_{j\in\mathcal{N}_s} \tilde{e}_{js},$$

$$\tilde{e}_{i+} \equiv \sum_{s=1}^{p} \tilde{e}_{is}.$$

Summing over all $i \in \mathcal{N}_s$ yields

$$n_s\tilde{e}_{+s} = \frac{\beta + \gamma - \rho}{\beta}\tilde{e}_{+s} - \frac{1}{\beta}\sum_{i\in\mathcal{N}_s} \alpha_i\delta_{is} + \frac{\rho}{\beta}\sum_{i\in\mathcal{N}_s} \tilde{e}_{i+}.$$

Solving for $\tilde{e}_{+s}$ gives

$$\tilde{e}_{+s} = \frac{\rho}{\beta(n_s-1)+\rho-\gamma}\sum_{i\in\mathcal{N}_s}\tilde{e}_{i+} - \frac{1}{\beta(n_s-1)+\rho-\gamma}\sum_{i\in\mathcal{N}_s}\alpha_i. \tag{31}$$

Next, summation over all projects s involving agent i in Equation (30) yields

$$\sum_{s=1}^{p}\delta_{is}\tilde{e}_{+s} = \frac{\beta+\gamma-\rho}{\beta}\sum_{s=1}^{p}\tilde{e}_{is} - \frac{\alpha_i}{\beta}\sum_{s=1}^{p}\delta_{is} + \frac{\rho}{\beta}\tilde{e}_{i+}\sum_{s=1}^{p}\delta_{is},$$

and denoting by

$$d_i \equiv \sum_{s=1}^{p}\delta_{is},$$

we get

$$\sum_{s=1}^{p}\delta_{is}\tilde{e}_{+s} = \frac{\beta+\gamma+\rho(d_i-1)}{\beta}\tilde{e}_{i+} - \frac{1}{\beta}\alpha_i d_i. \tag{32}$$

Solving for $\tilde{e}_{i+}$ gives

$$\tilde{e}_{i+} = \frac{\beta}{\beta+\gamma+\rho(d_i-1)}\sum_{s=1}^{p}\delta_{is}\tilde{e}_{+s} + \frac{1}{\beta+\gamma+\rho(d_i-1)}\alpha_i d_i.$$

Summation over all $i \in \mathcal{N}_s$ yields

$$\sum_{i\in\mathcal{N}_s}\tilde{e}_{i+} = \sum_{i\in\mathcal{N}_s}\frac{\beta}{\beta+\gamma+\rho(d_i-1)}\sum_{s'=1}^{p}\delta_{is'}\tilde{e}_{+s'} + \sum_{i\in\mathcal{N}_s}\frac{1}{\beta+\gamma+\rho(d_i-1)}\alpha_i d_i. \tag{33}$$

Inserting Equation (33) into Equation (31) gives

$$\begin{aligned}\tilde{e}_{+s} &= \sum_{i=1}^{n}\frac{\rho\beta\delta_{is}}{(\beta(n_s-1)+\rho-\gamma)(\beta+\gamma+\rho(d_i-1))}\sum_{s'=1}^{p}\delta_{is'}\tilde{e}_{+s'} \\ &+ \sum_{i=1}^{n}\frac{\rho\alpha_i d_i\delta_{is}}{(\beta(n_s-1)+\rho-\gamma)(\beta+\gamma+\rho(d_i-1))} \\ &- \frac{1}{\beta(n_s-1)+\rho-\gamma}\sum_{i=1}^{n}\alpha_i\delta_{is}.\end{aligned} \tag{34}$$

In the following we denote by

$$\begin{aligned}\varphi_{is} &\equiv \frac{\rho\beta\delta_{is}}{(\beta(n_s-1)+\rho-\gamma)(\beta+\gamma+\rho(d_i-1))}, \\ \mu_s(\boldsymbol{\alpha}) &\equiv \sum_{i=1}^{n}\frac{\rho\alpha_i d_i\delta_{is}}{(\beta(n_s-1)+\rho-\gamma)(\beta+\gamma+\rho(d_i-1))} - \frac{1}{\beta(n_s-1)+\rho-\gamma}\sum_{i=1}^{n}\alpha_i\delta_{is}.\end{aligned}$$

Then we can write Equation (34) as follows

$$\begin{aligned}\tilde{e}_{+s} &= \sum_{i=1}^{n} \varphi_{is} \sum_{s'=1}^{p} \delta_{is'} \tilde{e}_{+s'} + \mu_s(\boldsymbol{\alpha}) \\ &= \sum_{s'=1}^{p} \tilde{e}_{+s'} \sum_{i=1}^{n} \varphi_{is} \delta_{is'} \\ &= \sum_{s'=1}^{p} \omega_{ss'} \tilde{e}_{+s'} + \mu_s(\boldsymbol{\alpha}),\end{aligned}$$

where we have denoted by

$$\omega_{ss'} \equiv \sum_{i=1}^{n} \varphi_{is} \delta_{is'}.$$

Further, let $\boldsymbol{\epsilon} \equiv (\tilde{e}_{+1}, \dots, \tilde{e}_{+p})^\top$ and $\boldsymbol{\Omega} \equiv (\omega_{ss'})_{1 \le s, s' \le p}$, then we can write the above equation in vector-matrix form as

$$\boldsymbol{\epsilon} = \boldsymbol{\Omega}\boldsymbol{\epsilon} + \boldsymbol{\mu}(\boldsymbol{\alpha}).$$

That is

$$(\mathbf{I}_p - \boldsymbol{\Omega})\boldsymbol{\epsilon} = \boldsymbol{\mu}(\boldsymbol{\alpha}).$$

When the matrix $\mathbf{I}_p - \boldsymbol{\Omega}$ is invertible, then we can write

$$\boldsymbol{\epsilon} = (\mathbf{I}_p - \boldsymbol{\Omega})^{-1} \boldsymbol{\mu}(\boldsymbol{\alpha}). \tag{35}$$

Next, inserting Equation (35) into Equation (32) gives

$$\sum_{s=1}^{p} \delta_{is} \epsilon_s = \frac{\beta + \gamma + \rho(d_i - 1)}{\beta} \tilde{e}_{i+} - \frac{1}{\beta} \alpha_i d_i,$$

so that

$$\tilde{e}_{i+} = \frac{\beta}{\beta + \gamma + \rho(d_i - 1)} \sum_{s=1}^{p} \delta_{is} \epsilon_s + \frac{1}{\beta + \gamma + \rho(d_i - 1)} \alpha_i d_i. \tag{36}$$

Moreover, note that Equation (30) can be written as

$$\tilde{e}_{is} = \frac{\beta}{\beta + \gamma - \rho} \delta_{is} \tilde{e}_{+s} + \frac{1}{\beta + \gamma - \rho} \alpha_i \delta_{is} - \frac{\rho}{\beta + \gamma - \rho} \delta_{is} \tilde{e}_{i+}. \tag{37}$$

Inserting Equations (36) and (35) into Equation (37) gives

$$\begin{aligned}\tilde{e}_{is} = &\frac{\beta}{\beta + \gamma - \rho} \delta_{is} \epsilon_s + \frac{1}{\beta + \gamma - \rho} \alpha_i \delta_{is} \\ &- \frac{\rho\beta}{(\beta + \gamma - \rho)(\beta + \gamma + \rho(d_i - 1))} \delta_{is} \sum_{s'=1}^{p} \epsilon_{s'} \delta_{is'} \\ &- \frac{\rho}{(\beta + \gamma - \rho)(\beta + \gamma + \rho(d_i - 1))} \delta_{is} \alpha_i d_i.\end{aligned} \tag{38}$$

Equation (38) can be written as follows

$$\tilde{e}_{is} = \frac{\delta_{is}}{\beta + \gamma - \rho} \left[\beta \epsilon_s + \alpha_i - \frac{\rho}{\beta + \gamma + \rho(d_i - 1)} \left(\sum_{s'=1}^{p} \delta_{is'} \epsilon_{s'} + \alpha_i d_i \right) \right]. \tag{39}$$

□