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DOI: https://doi.org/10.1002/cpa.21542

Originally published at:
Benedikter, Niels; de Oliveira, Gustavo; Schlein, Benjamin (2015). Quantitative erivation of the Gross-Pitaevskii equation. Communications on Pure and Applied Mathematics, 68(8):1399-1482.
DOI: https://doi.org/10.1002/cpa.21542

Quantitative Derivation of the Gross-Pitaevskii Equation

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Abstract

Starting from first principle many-body quantum dynamics, we show that the dynamics of Bose-Einstein condensates can be approximated by the time-dependent nonlinear Gross-Pitaevskii equation, giving a bound on the rate of the convergence. Initial data are constructed on the bosonic Fock space applying an appropriate Bogoliubov transformation on a coherent state with expected number of particles N. The Bogoliubov transformation plays a crucial role; it produces the correct microscopic correlations among the particles. Our analysis shows that, on the level of the one particle reduced density, the form of the initial data is preserved by the many-body evolution, up to a small error which vanishes as $N^{-1/2}$ in the limit of large N.

1 Introduction and main results

A Bose-Einstein condensate is a state of matter of a gas of bosons where a macroscopic fraction of the particles occupy the same one-particle state. The existence of Bose-Einstein condensation at small temperature was first predicted in 1925 by Bose and Einstein, who considered gases of non-interacting bosons. Seventy years later, in 1995, the existence of Bose-Einstein condensates was then confirmed by experiments; see [2, 11]. Since then, Bose-Einstein condensates have attracted a lot of attention in theoretical and in experimental physics. In particular, they have been used to explore fundamental questions in quantum mechanics, such as the emergence of interference, decoherence, superfluidity and quantized vortices. In experiments, condensates are initially trapped by strong magnetic fields and cooled down at extremely low temperatures (in the nano-kelvin scale). Then, the traps are released and the subsequent evolution of the condensate is observed. The goal of this paper is to study the dynamics of initially trapped Bose-Einstein condensates. In particular, starting from many-body quantum dynamics, we show rigorously that the evolution of the condensate can be described, in certain regimes, by the time-dependent Gross-Pitaevskii equation.

Communications on Pure and Applied Mathematics, Vol. 000, 0001–0079 (2000)

The model. We consider a trapped gas of N bosons, described by the Hamilton operator

$$H_N^{\text{trap}} = \sum_{j=1}^N \left(-\Delta_{x_j} + V_{\text{ext}}(x_j)\right) + \sum_{i<j}^N N^2 V(N(x_i - x_j)) \tag{1.1}$$

acting on the Hilbert space $L^2_s(\mathbb{R}^{3N}, d\mathbf{x})$, the subspace of $L^2(\mathbb{R}^{3N}, d\mathbf{x})$ consisting of all functions symmetric with respect to permutations of the N particles. The external potential V_{ext} confines the particles inside the trap. The interaction potential V is assumed to be non-negative, spherically symmetric and decaying sufficiently fast at infinity. We denote by a_0 the scattering length of the potential V. To define the scattering length, we consider the solution of the zero energy scattering equation

$$\left(-\Delta + \frac{1}{2}V\right) f = 0 \tag{1.2}$$

with the boundary condition $f(x) \to 1$ as $|x| \to \infty$. The scattering length is then given by

$$8\pi a_0 = \int dx\, V(x) f(x)\,. \tag{1.3}$$

It is easy to check that, for large $|x|$,

$$f(x) = 1 - \frac{a_0}{|x|} + O(|x|^{-2})\,. \tag{1.4}$$

By simple scaling, we find then

$$\left(-\Delta_x + \frac{N^2}{2} V(Nx)\right) f(Nx) = 0$$

which implies that the scattering length of the rescaled potential $N^2 V(N\cdot)$ appearing in (1.1) is given by $a = a_0/N$.

Ground state properties. Let

$$E_N = \min_{\substack{\psi_N \in L^2_s(\mathbb{R}^{3N}) \\ \|\psi_N\|=1}} \langle \psi_N, H_N^{\text{trap}} \psi_N \rangle$$

denote the ground state energy of (1.1). It was proven in [31] that

$$\lim_{N\to\infty} \frac{E_N}{N} = \min_{\substack{\varphi \in L^2(\mathbb{R}^3) \\ \|\varphi\|_2=1}} \mathscr{E}_{\text{GP}}(\varphi)$$

where

$$\mathscr{E}_{\text{GP}}(\varphi) = \int dx \left(|\nabla \varphi(x)|^2 + V_{\text{ext}}(x)|\varphi(x)|^2 + 4\pi a_0 |\varphi(x)|^4\right) \tag{1.5}$$

is the Gross-Pitaevskii energy functional. Hence, in the leading order, the ground state energy per particle depends on the interaction potential V only through its scattering length a_0. In [30], it was also shown that the ground state of (1.1) exhibits complete Bose-Einstein condensation in the minimizer of the Gross-Pitaevskii energy functional (1.5), in the sense that

$$\gamma_N^{(1)} \to |\phi_{\mathrm{GP}}\rangle\langle\phi_{\mathrm{GP}}| \tag{1.6}$$

where $|\phi_{\mathrm{GP}}\rangle\langle\phi_{\mathrm{GP}}|$ is the orthogonal projection onto the (normalized) minimizer $\phi_{\mathrm{GP}} \in L^2(\mathbb{R}^3)$ of (1.5) and where $\gamma_N^{(1)}$ denotes the one-particle reduced density associated with the ground state $\psi_N \in L_s^2(\mathbb{R}^{3N})$ of (1.1), which is defined as the non-negative trace class operator with integral kernel

$$\gamma_N^{(1)}(x;x') := \int dx_2 \dots dx_N \, \psi_N(x,x_2,\dots,x_N)\overline{\psi}_N(x',x_2,\dots,x_N). \tag{1.7}$$

We assume here that $\|\psi_N\| = 1$, and therefore that $\operatorname{Tr}\gamma_N^{(1)} = 1$. The convergence in (1.6), which hold in the trace-class topology, implies that, in the ground state of (1.1), all particles, up to a fraction which vanishes in the limit of large N, are in the same one particle state, described by the orbital ϕ_{GP}. The results of [31, 30] show that the Gross-Pitaevskii theory correctly describes the ground state properties of the Hamiltonian (1.1).

Time evolution. When the traps are switched off, the system starts to evolve, the dynamics being governed by the N-particle Schrödinger equation

$$i\partial_t \psi_{N,t} = H_N \psi_{N,t} \tag{1.8}$$

with the translation invariant Hamiltonian

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{i<j}^{N} N^2 V(N(x_i - x_j)). \tag{1.9}$$

It turns out that the time evolution of an initial data exhibiting complete condensation can be described, in the limit of large N, by the Gross-Pitaevskii theory. In fact, the following result was established in [14, 15, 16, 17, 18], and, in a slightly different form, in [33]. Consider a family $\psi_N \in L_s^2(\mathbb{R}^{3N})$ with bounded energy per particle

$$\langle \psi_N, H_N \psi_N \rangle \leq CN$$

and exhibiting complete condensation in a one-particle state $\varphi \in H^1(\mathbb{R}^3)$, in the sense that the one-particle reduced density $\gamma_N^{(1)}$ associated with ψ_N (defined as in (1.7)) satisfies

$$\gamma_N^{(1)} \to |\varphi\rangle\langle\varphi| \qquad \text{as } N \to \infty.$$

Then, the solution $\psi_{N,t}$ of the Schrödinger equation (1.8) still exhibits complete Bose-Einstein condensation for any fixed $t \in \mathbb{R}$, in the sense that the reduced one-particle density $\gamma_{N,t}^{(1)}$ associated with $\psi_{N,t}$ satisfies

$$\gamma_{N,t}^{(1)} \to |\varphi_t\rangle\langle\varphi_t| \qquad \text{as } N \to \infty, \tag{1.10}$$

where φ_t is the solution of the time-dependent Gross-Pitaevskii equation

$$i\partial_t \varphi_t = -\Delta\varphi_t + 8\pi a_0 |\varphi_t|^2 \varphi_t. \tag{1.11}$$

This result establishes the stability of complete Bose-Einstein condensation with respect to the time-evolution for times of order one, independent of N, and the fact that the condensate wave function evolves according to (1.11).

Mean field regime. The method used in [14, 15, 16, 17, 18] to prove (1.10) relies on techniques first developed to understand the mean field limit of many-body quantum dynamics. This regime is achieved when considering the time-evolution

$$i\partial_t \psi_{N,t} = H_N^{\text{mf}} \psi_{N,t} \tag{1.12}$$

generated by the Hamiltonian

$$H_N^{\text{mf}} = \sum_{j=1}^N -\Delta_{x_j} + \frac{1}{N}\sum_{i<j}^N V(x_i - x_j) \tag{1.13}$$

in the limit of large N. Also in this case, under appropriate assumptions on the potential V, complete condensation is preserved by the time-evolution. Here, the evolution of the condensate wave function is governed by the Hartree equation

$$i\partial_t \varphi_t = -\Delta\varphi_t + \left(V * |\varphi_t|^2\right)\varphi_t\,. \tag{1.14}$$

The first rigorous derivation of (1.14) was obtained in [35] for bounded interaction potential, i.e. under the assumption $\|V\|_\infty < \infty$. The basic idea in [35] was to study the time-evolution of the family of reduced densities $\gamma_{N,t}^{(k)}$, $k = 1,\dots,N$. The operator $\gamma_{N,t}^{(k)}$ acts on $L^2(\mathbb{R}^{3k})$ and it is defined, similarly to (1.7) by the integral kernel

$$\gamma_N^{(k)}(\mathbf{x};\mathbf{x}') := \int dx_{k+1}\dots dx_N\, \psi_N(\mathbf{x},x_{k+1},\dots,x_N)\overline{\psi}_N(\mathbf{x}',x_{k+1},\dots,x_N),$$

where $\mathbf{x} = (x_1,\dots,x_k) \in \mathbb{R}^{3k}$; the normalization is chosen so that $\operatorname{Tr}\gamma_{N,t}^{(k)} = 1$. From the Schrödinger equation (1.12), it is possible to derive a hierarchy of N coupled equation, known as the BBGKY hierarchy (for Bogoliubov, Born, Green, Kirkwood and Yvon), given by

$$\begin{aligned} i\partial_t \gamma_{N,t}^{(k)} &= \sum_{j=1}^k \left[-\Delta_j, \gamma_{N,t}^{(k)}\right] + \frac{1}{N}\sum_{i<j}^k \left[V(x_i - x_j), \gamma_{N,t}^{(k)}\right] \\ &\quad + \frac{N-k}{N}\sum_{j=1}^k \operatorname{Tr}_{k+1}\left[V(x_j - x_{k+1}), \gamma_{N,t}^{(k+1)}\right], \end{aligned} \tag{1.15}$$

where $k = 1,\dots,N$, Tr_{k+1} denotes the partial trace over the $(k+1)$-th particle, and where we used the convention $\gamma_{N,t}^{(k)} = 0$ for $k > N$.

As $N \to \infty$, the BBGKY hierarchy converges, at least formally, towards an infinite hierarchy of coupled equations, which is solved by products of the solution of the Hartree equation (1.14). The problem of proving the convergence towards the Hartree dynamics essentially reduces to showing the uniqueness of the solution of the infinite hierarchy. This general scheme, first introduced in [35] for bounded interactions, was later extended to more singular potentials; see, for example, [19, 1, 12, 26, 6, 9, 10].

In [15, 16, 17, 18], the same strategy was then applied to analyze the dynamics generated by the Hamiltonian (1.9) and to obtain a rigorous derivation of the Gross-Pitaevskii equation (1.11). Writing (1.9) as

$$H_N = \sum_{j=1}^N -\Delta_{x_j} + \frac{1}{N}\sum_{i<j}^N N^3 V(N(x_i - x_j)) \tag{1.16}$$

one can interpret the Gross-Pitaevskii regime as a very singular mean field limit, where the interaction converges, as $N \to \infty$, towards a delta-function. From the point of view of physics, however, these two regimes are very different. While in the mean field limit particles experience a large number of weak collisions, the evolution generated by (1.1) is characterized by very rare and strong interactions (two particles only interact when they are very close, at distances of order N^{-1} from each others, which is much smaller than the typical interparticle distance, of order $N^{-1/3}$). As a consequence, it turns out that correlations among particles, which are negligible in the mean field limit, play a crucial role in the Gross-Pitaevskii regime. To explain this point, let us consider the evolution of the one-particle reduced density $\gamma^{(1)}_{N,t}$ which is governed by

$$\begin{aligned} i\partial_t \gamma^{(1)}_{N,t}(x;x') &= (-\Delta_x + \Delta_{x'})\, \gamma^{(1)}_{N,t}(x;x') \\ &\quad + \int dx_2\, \big((N-1)N^2 V(N(x-x_2)) - (N-1)N^2 V(N(x'-x_2))\big) \\ &\quad \times \gamma^{(2)}_{N,t}(x,x_2;x',x_2) \end{aligned} \tag{1.17}$$

which is the analogous of the first equation in (1.15). If we assume the initial state to exhibit complete condensation and we accept that condensation is preserved by the time evolution, we should expect $\gamma^{(1)}_{N,t}$ and $\gamma^{(2)}_{N,t}$ to be approximately factorized. In $\gamma^{(2)}_{N,t}$, however, we also want to take into account the correlations between the two particles. Describing the correlations through the solution of the zero-energy scattering equation (1.2), we are led to the ansatz

$$\begin{aligned} \gamma^{(1)}_{N,t}(x;x') &\simeq \varphi_t(x)\overline{\varphi}_t(x'), \\ \gamma^{(2)}_{N,t}(x_1,x_2;x_1',x_2') &\simeq f(N(x_1-x_2)) f(N(x_1'-x_2')) \\ &\quad \times \varphi_t(x_1)\varphi_t(x_2)\overline{\varphi}_t(x_1')\overline{\varphi}_t(x_2'). \end{aligned} \tag{1.18}$$

Plugging this ansatz into (1.17), we obtain a self-consistent equation for φ_t, namely

$$i\partial_t \varphi_t = -\Delta\varphi_t + \left(N^2(N-1)V(N\cdot)f(N\cdot) * |\varphi_t|^2\right)\varphi_t. \tag{1.19}$$

From (1.3), we have $N^2(N-1)V(Nx)f(Nx) \simeq N^3V(Nx)f(Nx) \simeq 8\pi a_0\delta(x)$ as $N \to \infty$; therefore, in this limit, φ_t must be a solution of the Gross-Pitaevskii equation (1.11). We observe here that the presence of the factor $f(N\cdot)$, describing the correlations among the particles, is crucial in this argument to understand the emergence of the scattering length in (1.11). While in the mean field limit it is possible to close the BBGKY hierarchy assuming the factorization $\gamma^{(2)}_{N,t} \simeq \gamma^{(1)}_{N,t} \otimes \gamma^{(1)}_{N,t}$, here is is important to take into account the singular correlation structure developed by the many-body evolution. In fact, understanding the correlations and adapting the techniques of [35, 19, 12] to deal with them (for example, when proving a priori regularity of the limiting densities) was one of the main challenges faced in [14, 15, 16, 17, 18]. As we will discuss shortly, correlations also play a major role in the approach presented in this paper.

The coherent states approach. A drawback of the approach used in [14, 15, 16, 17, 18] is the fact that it does not give any control on the rate of the convergence of the many-body quantum evolution towards the limiting dynamics (described by the Hartree or the Gross-Pitaevskii equation). The problem of controlling the rate of convergence is by no means of purely academic interest. In experiments, the number of particles N is large ($N \simeq 10^3$ in typical samples of Bose-Einstein condensates), but of course finite. For fixed N, the statement (1.10) is empty. Only an explicit bound on the difference $\gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|$ can tell us whether the limiting (Hartree or Gross-Pitaevskii) dynamics is a good approximation for the many-body evolution. In [34], a different approach to the study of the many-body quantum dynamics in the mean field regime was developed, starting from ideas introduced in a slightly different context in [24, 21]. To explain this approach, we consider the bosonic Fock space

$$\mathscr{F} = \bigoplus_{n\geq 0} L^2_s(\mathbb{R}^{3n}, dx_1 \dots dx_n).$$

The idea here is that on $\mathscr{F}$ we can describe states with variable number of particles. The normalized Fock-state $\psi = \{\psi^{(n)}\}_{n\geq 1}$ contains n particles with probability $\|\psi^{(n)}\|^2$. For $f \in L^2(\mathbb{R}^3)$ we introduce, as usual, creation and annihilation operators $a^*(f)$ and $a(f)$ which act on $\mathscr{F}$ by creating and annihilating a particle with wave function f, respectively (precise definitions and basic properties are given in Section 2). For $x \in \mathbb{R}^3$, we can also introduce operator valued distributions a^*_x and a_x, creating and annihilating a particle at x. In terms of these distributions, we define the mean field Hamiltonian

$$\mathscr{H}_N^{\text{mf}} = \int dx \nabla_x a^*_x \nabla_x a_x + \frac{1}{2N}\int dx dy V(x-y) a^*_x a^*_y a_y a_x.$$

The operator $\mathscr{H}_N$ commutes with the number of particles operator $\mathscr{N}$, whose action on $\psi = \{\psi^{(n)}\}_{n\geq 0} \in \mathscr{F}$ is given by $(\mathscr{N}\psi)^{(n)} = n\psi^{(n)}$. Moreover, when restricted to the sector with exactly N particles, $\mathscr{H}_N^{\text{mf}}$ coincides exactly with the mean field Hamiltonian (1.13) introduced above. The advantage of working on the Fock space is that we have more freedom in the choice of the initial state. We will use this freedom, by choosing coherent initial states. The coherent state with wave function $f \in L^2(\mathbb{R}^3)$ is defined as the vector $W(f)\Omega \in \mathscr{F}$, where $\Omega = \{1, 0, \dots\}$ is the Fock vacuum, and $W(f) = \exp(a^*(f) - a(f))$ is the Weyl operator with wave function f. A simple computation shows that

$$W(f)\Omega = e^{-\|f\|_2^2/2} \sum_{j\geq 0} \frac{(a^*(f))^j}{j!}\Omega = e^{-\|f\|_2^2/2}\left\{1, f, \frac{f^{\otimes 2}}{\sqrt{2!}}, \dots\right\}.$$

Coherent states do not have a fixed number of particles. However, the expected number of particles is given by $\|f\|_2^2$. To recover the mean field regime discussed above, we consider therefore the time evolution of an initial coherent state $W(\sqrt{N}\varphi)\Omega$, for $\varphi \in L^2(\mathbb{R}^3)$ with $\|\varphi\|_2 = 1$. It turns out that coherent states have especially nice algebraic properties (due to the fact that they are eigenvectors of all annihilation operators). Making use of these properties, one can show that, for large N, the mean field evolution $\Psi_{N,t} = e^{-i\mathscr{H}_N^{\text{mf}}t}W(\sqrt{N}\varphi)\Omega$ of an initial coherent state will again be approximately coherent, of the form $W(\sqrt{N}\varphi_t)\Omega$, where φ_t is the solution of the Hartree equation (1.14) with initial data $\varphi_{t=0} = \varphi$.

Moreover, it is possible to express the difference between the reduced densities associated with the evolved coherent state and the orthogonal projection onto φ_t in terms of a so called fluctuation dynamics, defined as the two-parameter group of unitary transformations

$$\mathscr{U}^{\text{mf}}(t;s) = W^*(\sqrt{N}\varphi_t)e^{-i\mathscr{H}_N^{\text{mf}}(t-s)}W(\sqrt{N}\varphi_s). \tag{1.20}$$

Essentially, the problem of bounding the rate of the convergence of the many-body evolution towards the Hartree dynamics in the mean field limit reduces to the problem of controlling the growth of the expectation of the number of particles operator $\mathscr{N}$ with respect to $\mathscr{U}^{\text{mf}}(t;s)$, uniformly in N. To obtain such estimates, we observe that the fluctuation dynamics satisfies a Schrödinger type equation

$$i\partial_t \mathscr{U}^{\text{mf}}(t;s) = \mathscr{L}_N^{\text{mf}}(t)\mathscr{U}^{\text{mf}}(t;s)$$

with $\mathscr{U}^{\text{mf}}(s;s) = 1$ and with the time dependent generator

$$\begin{aligned}
\mathscr{L}_N^{\text{mf}}(t) &= \big(i\partial_t W^*(\sqrt{N}\varphi_t)\big)W(\sqrt{N}\varphi_t) + W^*(\sqrt{N}\varphi_t)\mathscr{H}_N^{\text{mf}}W(\sqrt{N}\varphi_t) \\
&= \int dx \nabla_x a_x^* \nabla_x a_x + \int dx (V * |\varphi_t|^2)(x) a_x^* a_x \\
&\quad + \int dx dy V(x-y)\varphi_t(x)\overline{\varphi}_t(y)\, a_x^* a_y \\
&\quad + \int dx dy V(x-y)\left(\varphi_t(x)\varphi_t(y)\, a_x^* a_y^* + \overline{\varphi}_t(x)\overline{\varphi}_t(y)\, a_x a_y\right) \\
&\quad + \frac{1}{\sqrt{N}} \int dx dy V(x-y) a_x^* \left(\varphi_t(y) a_y^* + \overline{\varphi}_t(y) a_y\right) a_x \\
&\quad + \frac{1}{2N}\int dx dy V(x-y) a_x^* a_y^* a_y a_x.
\end{aligned} \tag{1.21}$$

The terms on the third and fourth line do not commute with the number of particles operator (because the number of creation operator does not match the number of annihilation operators). As a consequence, the fluctuation dynamics $\mathscr{U}^{\text{mf}}(t;s)$ does not leave the number of particles invariant. This is not surprising since $\mathscr{U}^{\text{mf}}(t;s)$ describes fluctuations around the Hartree evolution, which are expected to grow with time. Still, under the assumption that the interaction V contains at most Coulomb singularities, bounds of the form

$$\langle \psi, \mathscr{U}^{\text{mf}}(t;0)^* \mathscr{N} \mathscr{U}^{\text{mf}}(t;0)\psi\rangle \leq Ce^{K|t|}\langle \psi, (\mathscr{N}+1)^4\psi\rangle$$

for the growth of the number of particles operator (and, actually, also for all its power) were proven in [34]. As a corollary, estimates of the form

$$\text{Tr}\left|\Gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\right| \leq \frac{Ce^{K|t|}}{N} \tag{1.22}$$

on the rate of convergence towards the Hartree dynamics followed. Here $\Gamma^{(1)}_{N,t}$ denotes the one-particle reduced density associated with the evolved state $\Psi_{N,t} = e^{-i\mathscr{H}_N t}W(\sqrt{N}\varphi)\Omega$. The one-particle reduced density associated with a Fock space vector Ψ is defined by the integral kernel

$$\Gamma^{(1)}_{\Psi}(x,y) = \frac{1}{\langle \Psi, \mathscr{N}\Psi\rangle}\left\langle \Psi, a_y^* a_x \Psi\right\rangle. \tag{1.23}$$

It is simple to check that, for N-particle states, (1.23) coincides with the definition (1.7). The analysis of the time evolution of initial coherent states is also useful to study the dynamics of initial data with a fixed number of particles N. For N-particle initial data $\psi_N \in L^2(\mathbb{R}^{3N})$ obtained projecting down a Fock-state of the form $W(\sqrt{N}\varphi)\Psi$ onto the N-particle sector, bounds of the form (1.22) were established in [5], extending the ideas developed in [34], for arbitrary $\Psi \in \mathscr{F}$ with finite number of particles and energy (note that this class of N-particle states include factorized wave functions of the form $\varphi^{\otimes N}$, which are obtained choosing $\psi = \Omega$). Using techniques similar to those proposed in [33], bounds for the rate of convergence towards the nonlinear Hartree dynamics were also obtained in [27], allowing also for potential with strong singularities. A different point of view on the mean field limit was given in [20], where the convergence towards the Hartree dynamics was interpreted as a Egorov-type theorem.

Bogoliubov transformations and the Gross-Pitaevskii regime. It seems natural to ask whether the coherent state approach introduced in [34] can be used to obtain a rigorous derivation of the Gross-Pitaevskii equation (1.11), providing at the same time bounds on the rate of the convergence of the form (1.22). A major difficulty in this program is immediately clear. There are two contributions to the generator (1.21), one arising from the derivative of the Weyl operator $W^*(\sqrt{N}\varphi_t)$, the other from the derivative of $e^{-i(t-s)\mathscr{H}_N^{\text{mf}}}$. To compute the second contribution, given by $W^*(\sqrt{N}\varphi_t)\mathscr{H}_N^{\text{mf}}W(\sqrt{N}\varphi)$, we recall that Weyl operators act as shifts on creation and annihilation operators (see (2.4)). We obtain that $W^*(\sqrt{N}\varphi_t)\mathscr{H}_N^{\text{mf}}W(\sqrt{N}\varphi)$ contains a term, linear in creation and annihilation operators, having the form

$$\sqrt{N}\int dx\left(-\Delta\varphi_t(x)+(V*|\varphi_t|^2)(x)\varphi_t(x)\right)a_x^*+\text{h.c.} \tag{1.24}$$

This term is large (of the order $N^{1/2}$) and does not commute with the number of particles operator. In the mean field regime, however, (1.24) is exactly canceled by the contribution proportional to the derivative of $W^*(\sqrt{N}\varphi_t)$, which contains the term

$$\begin{aligned}-\sqrt{N}\int dx\,(i\partial_t\varphi_t(x))a_x^*-\text{h.c.}\\ =-\sqrt{N}\int dx\left(-\Delta\varphi_t(x)+(V*|\varphi_t|^2)(x)\varphi_t(x)\right)a_x^*-\text{h.c.}\end{aligned}$$

where we used the Hartree equation (1.14). As a result, the generator $\mathscr{L}_N^{\text{mf}}(t)$ in (1.21) contains only terms which, at least formally, are order one or smaller.

To adapt this approach to the Gross-Pitaevskii regime, we define the Fock space Hamiltonian

$$\mathscr{H}_N=\int dx\,\nabla_x a_x^*\nabla_x a_x+\frac{1}{2}\int dxdy N^2V(N(x-y))a_x^*a_y^*a_ya_x \tag{1.25}$$

and, following (1.20), we naively introduce the fluctuation dynamics

$$\mathscr{U}^{\text{GP}}(t;s)=W^*(\sqrt{N}\varphi_t^{(N)})e^{-i(t-s)\mathscr{H}_N}W(\sqrt{N}\varphi_t^{(N)}) \tag{1.26}$$

where, inspired by (1.19) (and for technical reasons that will be clear later), we choose $\varphi_t^{(N)}$ to solve the modified Gross-Pitaevskii equation

$$i\partial_t\varphi_t^{(N)}(x)=-\Delta\varphi_t^{(N)}(x)+\left(N^3V(N\cdot)f(N\cdot)*|\varphi_t^{(N)}|^2\right)(x)\varphi_t^{(N)}(x) \tag{1.27}$$

where f is the solution of the zero-energy scattering equation (1.2) (we use the notation $\varphi_t^{(N)}$ to distinguish the solution of (1.27) from the solution φ_t of the Gross-Pitaevskii equation (1.11)). Since the solution of (1.27) can be shown to converge towards the solution of (1.11), with an error of the order N^{-1}, control of the fluctuations around the modified Gross-Pitaevskii equation also implies control of the fluctuations around the Gross-Pitaevskii equation. The generator of (1.26) is given by

$$\begin{aligned}\mathscr{L}_N^{\text{GP}}(t)=\left(i\partial_t W^*(\sqrt{N}\varphi_t^{(N)})\right)W(\sqrt{N}\varphi_t^{(N)})\\ +W^*(\sqrt{N}\varphi_t^{(N)})\mathscr{H}_N W(\sqrt{N}\varphi_t^{(N)}).\end{aligned} \tag{1.28}$$

As in the mean field regime, the term $W^*(\sqrt{N}\varphi_t^{(N)})\mathscr{H}_N W(\sqrt{N}\varphi_t^{(N)})$ contains a large contribution which is linear in creation and annihilation operators

$$\sqrt{N}\int dx\left(-\Delta\varphi_t^{(N)}(x)+(N^3V(N\cdot)*|\varphi_t^{(N)}|^2)(x)\varphi_t^{(N)}(x)\right)a_x^*+\text{h.c.}$$

On the other hand, $\big(i\partial_t W^*(\sqrt{N}\varphi_t^{(N)})\big)W(\sqrt{N}\varphi_t^{(N)})$ contains the linear term

$$\begin{aligned} & -\sqrt{N}\int dx\big((i\partial_t\varphi_t^{(N)}(x))a_x^* + \text{h.c.}\big) \\ & = -\sqrt{N}\int dx\left(-\Delta\varphi_t^{(N)}(x) + N^3V(N\cdot)f(N\cdot)|\varphi_t^{(N)}(x)|^2\varphi_t^{(N)}(x)\right)a_x^* - \text{h.c.} \end{aligned}$$

In contrast to the mean field regime discussed above, here, because of the factor f, there is no complete cancellation between the two large linear terms. Hence, the generator $\mathscr{L}_N^{\text{GP}}(t)$ of (1.26) contains a large contribution which is linear in the creation and annihilation operators and has the form

$$\sqrt{N}\int dx\left(N^3V(N\cdot)(1-f(N\cdot))*|\varphi_t^{(N)}|^2\right)(x)\left(\varphi_t^{(N)}(x)a_x^* + \text{h.c.}\right). \tag{1.29}$$

Because of this term it is impossible to obtain a uniform (in N) bound on the growth of the number of particles w. r. t. the fluctuation dynamics (1.26). The reason for this failure is that we are trying to control fluctuations around the wrong evolution. When we approximate $e^{-it\mathscr{H}_N}W(\sqrt{N}\varphi)\psi$ by an evolved coherent state $W(\sqrt{N}\varphi_t^{(N)})\psi$, we completely neglect the correlation structure developed by the many-body evolution. As a result, fluctuations around the coherent approximation are too strong to be bounded uniformly in N.

Since correlations are, in first order, an effect of two-body interactions, we are going to approximate them using a unitary operator of the form

$$T(k) = \exp\left(\frac{1}{2}\int dxdy\left(k(x,y)a_x^*a_y^* - \overline{k}(x,y)a_xa_y\right)\right) \tag{1.30}$$

for an appropriate $k \in L^2(\mathbb{R}^3\times\mathbb{R}^3)$, which will be interpreted as the integral kernel of a Hilbert-Schmidt operator (again denoted by k) on $L^2(\mathbb{R}^3)$. The operator $T(k)$ acts on creation and annihilation operators as a Bogoliubov transformation (see Section 2.2 for precise definitions and basic properties):

$$\begin{aligned} T^*(k)a(f)T(k) &= a(\text{ch}(k)(f)) + a^*(\text{sh}(k)(\overline{f})), \\ T^*(k)a^*(f)T(k) &= a^*(\text{ch}(k)(f)) + a(\text{sh}(k)(\overline{f})) \end{aligned} \tag{1.31}$$

where $\text{ch}(k)$ and $\text{sh}(k)$ are the bounded operators on $L^2(\mathbb{R}^3)$ defined by the absolutely convergent series

$$\text{ch}(k) = \sum_{n\geq 0}\frac{1}{(2n)!}(k\overline{k})^n \qquad \text{and} \qquad \text{sh}(k) = \sum_{n\geq 0}\frac{1}{(2n+1)!}(k\overline{k})^n k$$

where products of k and $\overline{k}$ have to be understood in the sense of operators. Inspired by the analysis of [15, 16, 17, 18], where correlations were successfully described by the solution of the zero-energy scattering equation (1.4), we define the (time-dependent) kernel

$$k_t(x,y) = -Nw(N(x-y))\varphi_t^{(N)}(x)\varphi_t^{(N)}(y) \tag{1.32}$$

where $\varphi_t^{(N)}$ is the solution of the modified Gross-Pitaevskii equation (1.27), and where

$$w(x) = 1 - f(x)$$

with f being the solution of the zero-energy scattering equation (1.2). We will consider initial data of the form $W(\sqrt{N}\varphi)T(k_0)\psi$, for $\psi \in \mathscr{F}$ with bounded number of particles and bounded energy (for example, ψ can be taken as the Fock space vacuum Ω), and we will approximate its time evolution by $W(\sqrt{N}\varphi_t)T(k_t)\psi$, leading to the fluctuation dynamics

$$\mathscr{U}(t;s) = T^*(k_t)W^*(\sqrt{N}\varphi_t)e^{-i(t-s)\mathscr{H}_N}W(\sqrt{N}\varphi_s)T(k_s). \tag{1.33}$$

In this way, the approximating dynamics takes into account the correct correlation structure, and we have a better chance to bound the fluctuations. Indeed, we will show in Section 4 that it is possible to obtain a uniform (in N) control for the growth of

$$\langle \mathscr{U}(t;0)\psi, \mathscr{N}\mathscr{U}(t;0)\psi\rangle.$$

To this end, we will study the generator

$$\begin{aligned} \mathscr{L}_N(t) = T^*(k_t)\big[\big(i\partial_t W^*(\sqrt{N}\varphi_t^{(N)})\big)W(\sqrt{N}\varphi_t^{(N)}) \\ + W^*(\sqrt{N}\varphi_t^{(N)})\mathscr{H}_N W(\sqrt{N}\varphi_t^{(N)})\big]T(k_t) + (i\partial_t T^*(k_t))\,T(k_t) \end{aligned} \tag{1.34}$$

of the dynamics (1.33). We will prove in Section 6 that $(i\partial_t T^*(k_t))\,T(k_t)$ is harmless (see Proposition 6.10). As for the other terms on the r.h.s. of (1.34), they will still contain the large contribution (1.29). In this case, however, this dangerous term will be compensated by a contribution arising from the conjugation of the cubic part of $W^*(\sqrt{N}\varphi_t^{(N)})\mathscr{H}_N W(\sqrt{N}\varphi_t^{(N)})$ with the Bogoliubov transformation $T(k_t)$. In fact, according to (1.31), conjugation with $T(k_t)$ will produce terms with creation operators standing to the right of annihilation operators; restoring normal order will generate linear terms, canceling (1.29). Other important cancellations will occur between the quadratic and the quartic part of $W^*(\sqrt{N}\varphi_t^{(N)})\mathscr{H}_N W(\sqrt{N}\varphi_t^{(N)})$ (again, after conjugation with the Bogoliubov transformation $T(k_t)$; see Section 6 for the details). The control of the growth of the number of particles w.r.t. (1.33) will imply convergence of the one-particle reduced density matrix associated with the fully evolved Fock state $e^{-i\mathscr{H}_N t}W(\sqrt{N}\varphi_t)T(k_t)\psi$ towards the orthogonal projection onto the solution of the Gross-Pitaevskii equation (1.11) as $N \to \infty$, with a bound on the rate of the convergence.

Notice that operators of the form (1.30) have been used in physics to describe the evolution of Bose gases; see [36, 37]. More recently this idea has been introduced in the mathematical literature in [22, 23] to obtain norm-approximations to the many body Fock space dynamics in the mean field regime. The same result has been shown in [7] for system interacting through a three body potential. In [28], Bogoliubov transformation play an important role to get a norm-approximation for the dynamics of N-particle systems (adapting techniques developed in the time-independent setting in [29]). A slightly different point of view is presented in [3, 4], where fluctuations around mean field Hartree dynamics are described by time-dependent Bogoliubov transformations. Similarly as explained above, also in [22, 23, 7] one can think that the effect of the Bogoliubov transformation consists in canceling certain terms from the generator of the fluctuation dynamics. However, it is important to stress the difference with respect to our work. In the mean field regime studied in [22, 23, 7, 28, 3, 4], the effect of the correlations is of lower order; the Bogoliubov transformation is only needed to get second order corrections. In the Gross-Pitaevskii regime that we are considering, on the other hand, because of the singularity of the interaction the correlations are a leading order effect.

The main theorem. We are now ready to state our main result.

Theorem 1.1. *Let $\varphi \in H^4(\mathbb{R}^3)$, with $\|\varphi\|_2 = 1$. Let $\mathcal{H}_N$ be the Hamilton operator defined in (1.25), with a non-negative and spherically symmetric potential $V \in L^1 \cap L^3(\mathbb{R}^3, (1+|x|^6)dx)$. Let $\psi \in \mathcal{F}$ (possibly depending on N) with $\|\psi\| = 1$ be such that*

$$\langle \psi, \mathcal{N} \psi \rangle, \frac{1}{N} \langle \psi, \mathcal{N}^2 \psi \rangle, \langle \psi, \mathcal{H}_N \psi \rangle \leq D \tag{1.35}$$

for a constant $D > 0$ (independent of N). Let $\Gamma^{(1)}_{N,t}$ denote the one-particle reduced density associated with the evolved state $e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi) T(k_0) \psi$, as defined in (1.23). Then there exist constants $C, c_1, c_2 > 0$, depending only on V, $\|\varphi\|_{H^4}$ and on the constant D appearing in (1.35), such that

$$\operatorname{Tr} \left| \Gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t| \right| \leq \frac{C \exp(c_1 \exp(c_2|t|))}{N^{1/2}} \tag{1.36}$$

for all $t \in \mathbb{R}$ and $N \in \mathbb{N}$. Here φ_t denotes the solution of the time-dependent Gross-Pitaevskii equation

$$i\partial_t \varphi_t = -\Delta \varphi_t + 8\pi a_0 |\varphi_t|^2 \varphi_t \tag{1.37}$$

with the initial condition $\varphi_{t=0} = \varphi$.

Remarks.

(i) Let us point out that we insert the correct correlation structure in the initial data. Our result implies the approximate stability of states of the form $W(\sqrt{N}\varphi)T(k_0)\psi$ with respect to the many-body evolution (in the sense that the evolution of $W(\sqrt{N}\varphi)T(k_0)\psi$ has approximately the same form, just with an evolved φ_t and up to an error which, for any fixed time t, becomes small for large N). It does not imply, on the other hand, that the correlation structure is produced by the time-evolution. This is in contrast with the results of [15, 16, 17, 18], which can also be applied to completely factorized initial data. It remains unclear, however, if it is possible to obtain convergence with a $N^{-1/2}$ rate (or with any rate) for initial data with no correlations (the problem of the creation of correlations was studied in [13]).

(ii) Initial data of the form $W(\sqrt{N}\varphi)T(k_0)\psi$, with ψ satisfying (1.35), arise naturally as approximation for the ground state of the Hamiltonian

$$\mathcal{H}_N^{\text{trap}} = \int dx\, a_x^* \left(-\Delta_x + V_{\text{ext}}(x)\right) a_x + \frac{1}{2}\int dx dy N^2 V(N(x-y)) a_x^* a_y^* a_y a_x$$

describing a Bose gas trapped by a confining potential V_{ext}, when the chemical potential is tuned so that the expected number of particles is N (the number of particles in $W(\sqrt{N}\varphi)T(k_0)\psi$ concentrates around N, up to errors of order $\sqrt{N}$). Hence, $W(\sqrt{N}\varphi)T(k_0)\psi$ models the state prepared in experiments by cooling the trapped Bose gas to very low temperatures. In fact, combining the results of Propositions 6.1, 6.3, 6.5 and 6.6, and assuming $\psi \in \mathcal{F}$ to satisfy (1.35) (with $\mathcal{H}_N$ replaced by $\mathcal{H}_N^{\text{trap}}$), one can easily show that

$$\begin{aligned}
&\left\langle W(\sqrt{N}\varphi)T(k_0)\psi, \mathcal{H}_N^{\text{trap}} W(\sqrt{N}\varphi)T(k_0)\psi\right\rangle \\
&= N\int dx\, \left(|\nabla\varphi(x)|^2 + V_{\text{ext}}(x)|\varphi(x)|^2 + 4\pi a_0|\varphi(x)|^4\right) + O(\sqrt{N}) \\
&= N\mathcal{E}_{\text{GP}}(\varphi) + O(\sqrt{N})
\end{aligned}$$

with the Gross-Pitaevskii energy functional defined in (1.5). Choosing φ as the normalized minimizer of $\mathcal{E}_{\text{GP}}$, it follows from [31] that the state $W(\sqrt{N}\varphi)T(k_0)\psi$ has, in leading order, the energy of the ground state.

(iii) The time dependence on the r.h.s. of (1.36) deteriorates fast for large t. This however is just a consequence of the fact that, in general, high Sobolev norms of the solution of (1.37) can grow exponentially fast. Assuming a uniform bound for $\|\varphi_t\|_{H^4}$, the time dependence on the r.h.s. of (1.36) can be replaced by $C\exp(K|t|)$.

(iv) To simplify a bit the notation and some computations, we did not include an external potential in the Hamiltonian (1.25) generating the evolution on the Fock space. In contrast to [15, 16, 17, 18], the approach presented in this paper can be extended with no serious obstacle to Hamilton operators with external potential. The inclusion of external potentials is important to describe experiments where the evolution of the condensate is observed after tuning the magnetic traps, rather than switching them off. Observe that, using the BBGKY approach, derivations of nonlinear Schrödinger equations with external potentials have been obtained in [8, 9].

(v) The convergence (1.36) and the fact that the limit is a rank-one projection immediately implies convergence of the higher order reduced density $\Gamma_{N,t}^{(k)}$, associated with the evolved state

$$\Psi_{N,t} = e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi)T(k_0)\psi.$$

Similarly to (1.23), $\Gamma^{(k)}_{N,t}$ is defined as the non-negative trace class operators on $L^2(\mathbb{R}^{3k})$ with integral kernel

$$\begin{aligned}&\Gamma^{(k)}_{N,t}(x_1,\dots,x_k;y_1,\dots,y_k)\\&=\frac{\langle \Psi_{N,t}, a^*_{y_1}\dots a^*_{y_k}a_{x_k}\dots a_{x_1}\Psi_{N,t}\rangle}{\langle \Psi_{N,t},\mathcal{N}(\mathcal{N}-1)\dots(\mathcal{N}-k+1)\Psi_{N,t}\rangle}.\end{aligned}$$

Following the arguments outlined in Section 2 of [27], (1.36) implies that, for every $k\in\mathbb{N}$,

$$\operatorname{Tr}\left|\Gamma^{(k)}_{N,t}-|\varphi_t\rangle\langle\varphi_t|^{\otimes k}\right|\le C\frac{k^{1/2}}{N^{1/4}}\exp\left(\frac{c_1}{2}\exp(c_2|t|)\right).$$

To obtain bounds for the convergence of the k-particle reduced density with the same $N^{-1/2}$ rate as in (1.36), following the same approach used below to study $\Gamma^{(1)}_{N,t}$ would require to control the growth of higher powers of the number of particle operator with respect to the fluctuation dynamics (1.33). This may be doable, but the analysis becomes more involved.

(vi) Theorem 1.1 and the method used in its proof can also be applied to deduce the convergence towards the Gross-Pitaevskii dynamics for certain initial data with a fixed number of particles. In Appendix C, we consider initial N-particle states of the form $P_N W(\sqrt{N}\varphi)T(k_0)\psi$, for $\psi\in\mathscr{F}$ (with $\|\psi\|=1$) satisfying (1.35), assuming that we have $\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|\gg N^{-1/2}$ for large N (it is explained in Appendix C why this is a reasonable condition). Here P_N denotes the orthogonal projection onto the N-particle sector of $\mathscr{F}$. It remains to be understood which class of N-particle states can be written as $P_N W(\sqrt{N}\varphi)T(k_0)\psi$, for a $\psi\in\mathscr{F}$ satisfying (1.35).

2 Operators on the Fock space

The bosonic Fock space over $L^2(\mathbb{R}^3)$ is the Hilbert space

$$\mathscr{F}=\bigoplus_{n\ge 0}L^2(\mathbb{R}^3)^{\otimes_s n}=\mathbb{C}\oplus\bigoplus_{n\ge 1}L^2_s(\mathbb{R}^{3n}),$$

with the convention that $L^2(\mathbb{R}^3)^{\otimes_s 0}=\mathbb{C}$. Here $L^2_s(\mathbb{R}^{3n})$ is the subspace of $L^2(\mathbb{R}^{3n})$ consisting of all functions that are symmetric with respect to arbitrary permutations of the n variables. Vectors in $\mathscr{F}$ are sequences $\psi=\{\psi^{(n)}\}_{n\ge 0}$ of n-particle wave functions $\psi^{(n)}\in L^2_s(\mathbb{R}^{3n})$. The inner product on $\mathscr{F}$ is defined as

$$\begin{aligned}\langle\psi_1,\psi_2\rangle&=\sum_{n\ge 0}\langle\psi_1^{(n)},\psi_2^{(n)}\rangle_{L^2(\mathbb{R}^{3n})}\\&=\overline{\psi_1^{(0)}}\psi_2^{(0)}+\sum_{n\ge 1}\int dx_1\cdots dx_n\,\overline{\psi_1^{(n)}}(x_1,\dots,x_n)\psi_2^{(n)}(x_1,\dots,x_n).\end{aligned}$$

On $\mathscr{F}$, we can describe states where the number of particles is not fixed. The Fock space vector $\psi=\{\psi^{(0)},\psi^{(1)},\dots\}$ describes a coherent superposition of states with different number of particles; the n-particle component is described by $\psi^{(n)}$, for any $n \in \mathbb{N}$ (the probability that the state has exactly n particles is given by $\|\psi^{(n)}\|^2$). A state with exactly N particles is described on the Fock space $\mathscr{F}$ by a sequence $\{\psi^{(n)}\}_{n\geq 0}$ where $\psi^{(n)} = 0$ for all $n \neq N$ and $\psi^{(N)} = \psi_N$. The vector $\{1,0,0,\dots\} \in \mathscr{F}$ is called the vacuum and is denoted by Ω. States with a fixed number of particles are eigenvectors of the number of particles operator, defined by

$$(\mathscr{N}\psi)^{(n)} = n\psi^{(n)}.$$

To define an evolution on $\mathscr{F}$, we introduce the Hamilton operator

$$(\mathscr{H}_N\psi)^{(n)} = \mathscr{H}_N^{(n)}\psi^{(n)}$$

with the n-th sector operator

$$\mathscr{H}_N^{(n)} = \sum_{j=1}^{n} -\Delta_{x_j} + \sum_{i<j}^{n} N^2 V(N(x_i - x_j)).$$

Note that the subscript N in the notation $\mathscr{H}_N$ is not related with the number of particles (since this is not fixed on the Fock space), but only reflects the scaling in the interaction potential (of course, we will choose the initial Fock state to have expected number of particles close to N; otherwise, there would be no relation with the regime discussed in Section 1). Observe that, by definition, the Hamiltonian $\mathscr{H}_N$ commutes with $\mathscr{N}$. As a consequence, the evolution generated by $\mathscr{H}_N$ leaves each n-particle sector invariant. In particular,

$$e^{-i\mathscr{H}_N t}\{0,\dots,0,\psi_N,0,\dots\} = \{0,\dots,0,e^{-iH_N t}\psi_N,0,\dots\}$$

where H_N is the N-particle Hamiltonian defined in (1.9). In this sense, the N-body dynamics is embedded in the Fock space representation.

It is very useful to introduce creation and annihilation operators on $\mathscr{F}$. For $f \in L^2(\mathbb{R}^3)$, the creation operator $a^*(f)$ and the annihilation operator $a(f)$ on $\mathscr{F}$ are defined as

$$(a^*(f)\psi)^{(n)}(x_1,\dots,x_n) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} f(x_j)\psi^{(n-1)}(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n),$$

$$(a(f)\psi)^{(n)}(x_1,\dots,x_n) = \sqrt{n+1}\int dx\,\overline{f(x)}\psi^{(n+1)}(x,x_1,\dots,x_n).$$

The operators $a^*(f)$ and $a(f)$ are unbounded, densely defined and closed. Note that $a^*(f)$ is linear in f, while $a(f)$ is anti-linear. The creation operator $a^*(f)$ is the adjoint of the annihilation operator $a(f)$, and they satisfy the canonical commutation relations

$$[a(f),a^*(g)] = \langle f,g\rangle_{L^2} \qquad \text{and} \qquad [a(f),a(g)] = [a^*(f),a^*(g)] = 0 \tag{2.1}$$

for $f,g \in L^2(\mathbb{R}^3)$. We also introduce the self-adjoint operator

$$\phi(f) = a^*(f) + a(f).$$

Although creation and annihilation operators are not bounded, they are bounded with respect to the number of particles operator $\mathscr{N}$ (actually, to its square root). This is the content of the next standard lemma (see [34] for a proof of this well-known result).

Lemma 2.1. *Let $f \in L^2(\mathbb{R}^3)$. Then, for any $\psi \in \mathscr{F}$,*

$$\begin{aligned} \|a(f)\psi\| &\leq \|f\|_2 \|\mathscr{N}^{1/2}\psi\|, \\ \|a^*(f)\psi\| &\leq \|f\|_2 \|(\mathscr{N}+1)^{1/2}\psi\|, \\ \|\phi(f)\psi\| &\leq 2\|f\|_2 \|(\mathscr{N}+1)^{1/2}\psi\|. \end{aligned} \tag{2.2}$$

We will make use of operator-valued distributions a_x^* and a_x, with $x \in \mathbb{R}^3$, defined so that

$$a^*(f) = \int dx\, f(x) a_x^* \qquad \text{and} \qquad a(f) = \int dx\, \overline{f(x)} a_x$$

for $f \in L^2(\mathbb{R}^3)$. The canonical commutation relations assume the form

$$[a_x, a_y^*] = \delta(x-y) \qquad \text{and} \qquad [a_x, a_y] = [a_x^*, a_y^*] = 0,$$

where δ is the Dirac delta distribution.

In terms of the operator valued distributions a_x^* and a_x, the number of particles operator $\mathscr{N}$ and the Hamilton operator $\mathscr{H}_N$ can be written as

$$\mathscr{N} = \int dx\, a_x^* a_x$$

and

$$\mathscr{H}_N = \int dx\, \nabla_x a_x^* \nabla_x a_x + \frac{1}{2} \int dx dy N^2 V(N(x-y)) a_x^* a_y^* a_y a_x. \tag{2.3}$$

The first term in the Hamiltonian is the kinetic energy; since it will play an important role in our analysis, we introduce the notation

$$\mathscr{K} = \int dx\, \nabla_x a_x^* \nabla_x a_x.$$

Note that, like $\mathscr{H}_N$, $\mathscr{K}$ leaves every n-particle sector invariant, and for any $\psi \in \mathscr{F}$

$$(\mathscr{K}\psi)^{(n)} = \sum_{j=1}^{n} -\Delta_{x_j} \psi^{(n)}.$$

2.1 Weyl operators and coherent states

For $f \in L^2(\mathbb{R}^3)$, we define the Weyl operator

$$W(f) = e^{a^*(f) - a(f)}$$

acting on the Fock space $\mathscr{F}$. In the following lemma we collect some well-known properties of Weyl operators.

Lemma 2.2. *Let $f, g \in L^2(\mathbb{R}^3)$.*

(1) *Weyl operators satisfy the Weyl relations*

$$W(f)W(g) = W(g)W(f)e^{-2i\,\mathrm{Im}\,\langle f,g\rangle_{L^2}} = W(f+g)e^{-i\,\mathrm{Im}\,\langle f,g\rangle_{L^2}}.$$

(2) *The operator $W(f)$ is unitary on $\mathscr{F}$ and*

$$W(f)^* = W(f)^{-1} = W(-f).$$

(3) *We have*

$$\begin{aligned} W(f)^* a(g) W(f) &= a(g) + \langle g, f\rangle_{L^2}, \\ W(f)^* a^*(g) W(f) &= a^*(g) + \langle f, g\rangle_{L^2}. \end{aligned} \tag{2.4}$$

Using Weyl operators, we construct coherent states on $\mathscr{F}$. For $f \in L^2(\mathbb{R}^3)$, the coherent state with wave function f is defined as $W(f)\Omega$, where $\Omega = \{1, 0, \dots\}$ is the vacuum vector in $\mathscr{F}$, describing a state with no particles. Since $W(f)$ is unitary, coherent states are always normalized. From the canonical commutation relations (2.1), it follows that

$$\begin{aligned} W(f)\Omega &= e^{-\|f\|_2^2/2} e^{a^*(f)}\Omega = e^{-\|f\|_2^2/2} \sum_{n\geq 0} \frac{(a^*(f))^n}{n!}\Omega \\ &= e^{-\|f\|_2^2/2} \left\{ 1, f, \frac{f^{\otimes 2}}{\sqrt{2!}}, \dots, \frac{f^{\otimes n}}{\sqrt{n!}}, \dots \right\}. \end{aligned}$$

In particular, coherent states do not have a fixed number of particles. Instead, they are given by linear combinations of states with all possible number of particles. From (2.4), the expected number of particles in the state $W(f)\Omega$ is given by

$$\langle W(f)\Omega, \mathscr{N} W(f)\Omega\rangle = \int dx\, \langle \Omega, (a_x^* + \overline{f}(x))(a_x + f(x))\Omega\rangle = \|f\|_2^2.$$

More precisely, one can show that the number of particles in the coherent state $W(f)\Omega$ is a Poisson random variable with average and variance given by $\|f\|_2^2$.

Coherent states have particularly nice algebraic properties, which also simplify the study of their time-evolution. These properties are a consequence of the fact that coherent states are eigenvectors of all annihilation operators; in fact, from (2.4), we find

$$a(g)W(f)\Omega = W(f)(a(g) + \langle g, f\rangle)\Omega = \langle g, f\rangle W(f)\Omega$$

for all $f, g \in L^2(\mathbb{R}^3)$.

2.2 Bogoliubov transformations

For $f, g \in L^2(\mathbb{R}^3)$, we introduce the notation

$$A(f,g) = a(f) + a^*(\overline{g}). \tag{2.5}$$

We observe that

$$(A(f,g))^* = A(\overline{g}, \overline{f}) = A\left(\begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} (f,g) \right) \tag{2.6}$$

where $J : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is the antilinear operator defined by $Jf = \overline{f}$. From the canonical commutation relations (2.1), we find that the operators $A(f,g)$ satisfy the commutation relations

$$[A(f_1,g_1), A^*(f_2,g_2)] = \langle (f_1,g_1), S(f_2,g_2)\rangle_{L^2\oplus L^2} \tag{2.7}$$

with

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A bounded linear map $\Theta : L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ satisfying

$$\Theta \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \Theta \tag{2.8}$$

and

$$S = \Theta^* S \Theta \tag{2.9}$$

is called a Bogoliubov transformation. Bogoliubov transformations are linear maps of the pairs $(f,g) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ with the property that the operators $b^*(f)$ and $b(f)$, defined similarly to (2.5) by the equation

$$b(f) + b^*(\overline{g}) = A(\Theta(f,g))$$

for any $f,g \in L^2(\mathbb{R}^3)$, are still creation and annihilation operators satisfying canonical commutation relations and being adjoint to each other. It is simple to check that a general Bogoliubov transformation can be written as

$$\Theta = \begin{pmatrix} U & \overline{V} \\ V & \overline{U} \end{pmatrix} \tag{2.10}$$

for bounded linear maps $U,V : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ satisfying $U^*U - V^*V = 1$ and $U^*\overline{V} = V^*\overline{U}$. Here we use the notation $\overline{U} = JUJ$ for any bounded operator U on $L^2(\mathbb{R}^3)$ (the integral kernel of $\overline{U} = JUJ$ is given by $\overline{U(x,y)}$).

For a kernel $k \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ with $k(x,y) = k(y,x)$, we define now the operator

$$T(k) = \exp\left(\frac{1}{2}\int dxdy\, \left(k(x,y)a_x^*a_y^* - \overline{k}(x,y)a_xa_y\right)\right) \tag{2.11}$$

acting on the Fock space $\mathscr{F}$. As remarked in the paragraph preceding Theorem 1.1, similar constructions have been used in [36, 37, 22, 23, 7] to analyze the mean field limit of many body quantum dynamics (but with a different choice of the kernel k). In particular, some of the results of the next lemma, stated in a slightly different form, can already be found in [22, Sections 4 and 5] and in [23, Section 6].

Lemma 2.3. *Let $k \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ be symmetric, in the sense that $k(x,y) = k(y,x)$.*

(i) *The operator $T(k)$ is unitary on $\mathscr{F}$ and*

$$T(k)^* = T(k)^{-1} = T(-k).$$

(ii) *For every $f, g \in L^2(\mathbb{R}^3)$, we have*

$$T(k)^* A(f,g) T(k) = A(\Theta_k(f,g)) \tag{2.12}$$

where $\Theta_k : L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ is the Bogoliubov transformation defined by the matrix

$$\Theta_k = \begin{pmatrix} ch(k) & sh(k) \\ sh(\bar{k}) & ch(\bar{k}) \end{pmatrix}.$$

Here $ch(k), sh(k) : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ are the bounded operators defined by

$$ch(k) = \sum_{n \geq 0} \frac{1}{(2n)!} (k\bar{k})^n \quad \text{and} \quad sh(k) = \sum_{n \geq 0} \frac{1}{(2n+1)!} (k\bar{k})^n k$$

where products of k and $\bar{k}$ have to be understood in the sense of operators.

(iii) *We decompose*

$$ch(k) = 1 + p(k) \qquad \text{and} \qquad sh(k) = k + r(k), \tag{2.13}$$

where 1 denotes the identity operator on $L^2(\mathbb{R}^3)$. Then $p(k)$ and $r(k)$ (and therefore $sh(k)$) are Hilbert-Schmidt operators, with

$$\|p(k)\|_2 \leq e^{\|k\|_2}, \quad \|r(k)\|_2 \leq e^{\|k\|_2}, \quad \|sh(k)\|_2 \leq e^{\|k\|_2}. \tag{2.14}$$

(Here $\|p(k)\|_2$ denotes the $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ norm of the kernel $p(k)(x,y)$, which agrees with the Hilbert-Schmidt norm of the operator $p(k)$.)

(iv) *Suppose now that $k \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ is s.t. $\nabla_1 k \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$. Then, by symmetry, also $\nabla_2 k \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$. (We use here the notation $(\nabla_1 k)(x,y) = \nabla_x k(x,y)$ and $(\nabla_2 k)(x,y) = \nabla_y k(x,y)$; note that $\nabla_1 k$ and $\nabla_2 k$ are the integral kernels of the operator products ∇k and $-k\nabla$.) Moreover*

$$\|\nabla_1 p(k)\|_2, \|\nabla_1 r(k)\|_2 \leq e^{\|k\|_2} \|\nabla_1 (k\bar{k})\|_2,$$
$$\|\nabla_2 p(k)\|_2, \|\nabla_2 r(k)\|_2 \leq e^{\|k\|_2} \|\nabla_2 (\bar{k}k)\|_2.$$

(v) *If the kernel k depends on a parameter t (later, it will depend on time), and if derivatives w.r.t. t are denoted by a dot, we have*

$$\|\dot{p}(k)\|_2, \|\dot{r}(k)\|_2 \leq \|\dot{k}\|_2 \, e^{\|k\|_2}$$

and

$$\begin{aligned} &\|\nabla_1 \dot{p}(k)\|_2, \|\nabla_1 \dot{r}(k)\|_2 \\ &\qquad \leq C e^{\|k\|_2} \left(\|\dot{k}\|_2 \|\nabla_1 (k\bar{k})\|_2 + \|\nabla_1 (\dot{k}\bar{k})\|_2 + \|\nabla_1 (k\dot{\bar{k}})\|_2 \right), \\ &\|\nabla_2 \dot{p}(k)\|_2, \|\nabla_2 \dot{r}(k)\|_2 \\ &\qquad \leq C e^{\|k\|_2} \left(\|\dot{k}\|_2 \|\nabla_2 (k\bar{k})\|_2 + \|\nabla_2 (\dot{k}\bar{k})\|_2 + \|\nabla_2 (k\dot{\bar{k}})\|_2 \right). \end{aligned}$$

Proof. (i) is clear. To prove (ii), we observe that, setting

$$B = \frac{1}{2}\int dxdy\,\left(k(x,y)a_x^* a_y^* - \overline{k}(x,y)a_x a_y\right),$$

we have, for any $f,g \in L^2(\mathbb{R}^3)$,

$$\begin{aligned} e^{-B}A(f,g)e^{B} &= A(f,g) + \int_0^1 d\lambda_1 \frac{d}{d\lambda_1} e^{-\lambda_1 B}A(f,g)e^{\lambda_1 B} \\ &= A(f,g) - \int_0^1 d\lambda_1\, e^{-\lambda_1 B}[B, A(f,g)]e^{\lambda_1 B}. \end{aligned}$$

Iterating, we find the Baker-Campbell-Hausdorff formula

$$\begin{aligned} e^{-B}A(f,g)e^{B} &= A(f,g) + \sum_{j=1}^{n} \frac{(-1)^j}{j!}\,\mathrm{ad}_B^j(A(f,g)) \\ &+ (-1)^{n+1}\int_0^1 d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \dots \int_0^{\lambda_n} d\lambda_{n+1}\, e^{-\lambda_{n+1}B}\mathrm{ad}_B^{n+1}(A(f,g))\, e^{\lambda_{n+1}B} \end{aligned} \tag{2.15}$$

where $\mathrm{ad}_B^1(C) = [B,C]$ and $\mathrm{ad}_B^{n+1}(C) = [B, \mathrm{ad}_B^n(C)]$. A simple computation shows that

$$\mathrm{ad}_B^1(A(f,g)) = [B, A(f,g)] = -A\left(\begin{pmatrix} 0 & k \\ \overline{k} & 0 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}\right)$$

and therefore that

$$\mathrm{ad}_B^j(A(f,g)) = (-1)^j A\left(\begin{pmatrix} 0 & k \\ \overline{k} & 0 \end{pmatrix}^j \begin{pmatrix} f \\ g \end{pmatrix}\right).$$

We have

$$\begin{pmatrix} 0 & k \\ \overline{k} & 0 \end{pmatrix}^{2m} = \begin{pmatrix} (k\overline{k})^m & 0 \\ 0 & (\overline{k}k)^m \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & k \\ \overline{k} & 0 \end{pmatrix}^{2m+1} = \begin{pmatrix} 0 & (k\overline{k})^m k \\ (\overline{k}k)^m \overline{k} & 0 \end{pmatrix}$$

for every $m \in \mathbb{N}$. Inserting all this in (2.15), we obtain (2.12), if we can show that the error converges to zero. We claim, more precisely, that the error term on the r.h.s. of (2.15) vanishes, as $n \to \infty$, when applied on the domain $D(\mathcal{N}^{1/2})$. To prove this claim, we start by observing that

$$\|(\mathcal{N}+1)^{1/2} e^{-\lambda B}(\mathcal{N}+1)^{-1/2}\| \leq e^{|\lambda|\|k\|_2} \tag{2.16}$$

for every $\lambda \in \mathbb{R}$. In fact, (2.16) follows from Gronwall's Lemma, since, for all $\psi \in \mathscr{F}$,

$$\begin{aligned}\frac{d}{d\lambda}\|(\mathcal{N}+1)^{1/2}e^{-\lambda B}\psi\|^2 &= \langle e^{-\lambda B}\psi, [B,\mathcal{N}]\, e^{-\lambda B}\psi\rangle \\ &= -\int dxdy\, k(x,y)\langle e^{-\lambda B}\psi, a_x^* a_y^*\, e^{-\lambda B}\psi\rangle \\ &\quad - \int dxdy\, \bar{k}(x,y)\langle e^{-\lambda B}\psi, a_x a_y\, e^{-\lambda B}\psi\rangle \\ &\leq 2\int dx\, \|a_x e^{-\lambda B}\psi\|\, \|a^*(k(x,.))e^{-\lambda B}\psi\| \\ &\leq 2\|k\|_2\, \|(\mathcal{N}+1)^{1/2}e^{-\lambda B}\psi\|^2.\end{aligned}$$

Assuming, for example, that n is odd, (2.16) implies that

$$\begin{aligned}&\left\|\int_0^1 d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \dots \int_0^{\lambda_n} d\lambda_{n+1}\, e^{-\lambda_{n+1}B}\mathrm{ad}_B^{n+1}(A(f,g))\, e^{\lambda_{n+1}B}(\mathcal{N}+1)^{-1/2}\right\| \\ &\quad\leq \frac{e^{\|k\|_2}}{(n+1)!}\|A\left((k\bar{k})^{(n+1)/2}f, (\bar{k}k)^{(n+1)/2}g\right)(\mathcal{N}+1)^{-1/2}\| \\ &\quad\leq (\|f\|_2+\|g\|_2)\, e^{\|k\|_2}\, \frac{\|(k\bar{k})^{(n+1)/2}\|_2}{(n+1)!} \leq C\frac{\|k\|_2^n}{(n+1)!},\end{aligned}$$

which vanishes as $n \to \infty$. The case of even n can be treated similarly. This concludes the proof of the claim.

To prove (iii), we notice that

$$\|p(k)\|_2 = \left\|\sum_{n\geq 1}\frac{(k\bar{k})^n}{(2n)!}\right\|_2 \leq \sum_{n\geq 1}\frac{\|(k\bar{k})^n\|_2}{(2n)!} \leq \sum_{n\geq 1}\frac{\|k\|_2^{2n}}{(2n)!} \leq e^{\|k\|_2}$$

where we used that, by Cauchy-Schwarz inequality, for any two kernels $K_1, K_2 \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$,

$$\begin{aligned}\|K_1K_2\|_2^2 &= \int dxdy\left|\int dz\, K_1(x,z)K_2(z,y)\right|^2 \\ &= \int dxdydz_1dz_2 K_1(x,z_1)\overline{K}_1(x,z_2)K_2(z_1,y)\overline{K}_2(z_2,y) \\ &\leq \int dxdydz_1dz_2\, |K_1(x,z_1)|^2\, |K_2(z_2,y)|^2 \\ &= \|K_1\|_2^2\, \|K_2\|_2^2.\end{aligned} \tag{2.17}$$

The bounds for $r(k)$ and $\mathrm{sh}(k)$ can be proven similarly.

To show (iv) we write, using the fact that the series for $p(k)$, $r(k)$ and $\mathrm{sh}(k)$ are absolutely convergent,

$$\begin{aligned}\nabla_1 p(k) &= \nabla_1 (k\bar{k}) \left[\sum_{n=1}^{\infty} \frac{1}{(2n)!} (k\bar{k})^{n-1} \right], \\ \nabla_1 r(k) &= \nabla_1 (k\bar{k}) \left[\sum_{n=1}^{\infty} \frac{1}{(2n+1)!} (k\bar{k})^{n-1} k \right].\end{aligned}$$

Applying (2.17), we find the desired bounds. The bounds for the derivative ∇_2 can be obtained similarly.

Finally, to show (v), we remark that

$$\|\dot{p}(k)\|_2 \leq \sum_{n\geq 1} \frac{1}{(2n)!} 2n \|\dot{k}\|_2 \|k\|_2^{2n-1} \leq \|\dot{k}\|_2 \, e^{\|k\|_2}.$$

The bound for $\dot{r}(k)$ can be proven analogously. From the product rule, we also find that

$$\begin{aligned}\|\nabla_1 \dot{p}(k)\|_2 \leq & \sum_{n=2}^{\infty} \frac{n-1}{(2n)!} \|k\|_2^{2(n-2)} \|\partial_t (k\bar{k})\|_2 \, \|\nabla_1 (k\bar{k})\|_2 \\ & + \sum_{n=1}^{\infty} \frac{1}{(2n)!} \|k\|_2^{2(n-1)} \|\nabla_1 \partial_t (k\bar{k})\|_2 \\ \leq & \, e^{\|k\|_2} \left(\|\dot{k}\|_2 \, \|\nabla_1 (k\bar{k})\|_2 + \|\nabla_1 (k\dot{\bar{k}})\|_2 + \|\nabla_1 (\dot{k}\bar{k})\|_2 \right).\end{aligned}$$

The other bounds are shown similarly. □

3 Construction of the fluctuation dynamics

In this section, we will construct an approximation to the full many-body evolution of an initial data of the form $W(\sqrt{N}\varphi)T(k_0)\psi$, as considered in Theorem 1.1. Our approximation will consist of two parts. In the first step, we construct the coherent component of the approximation. Later we will take care of the correlation structure.

For a given $\varphi \in H^1(\mathbb{R}^3)$, we define $\varphi_t^{(N)}$ as the solution of the modified time-dependent Gross-Pitaevskii equation

$$i\partial_t \varphi_t^{(N)} = -\Delta \varphi_t^{(N)} + \left(N^3 f(N\cdot) V(N\cdot) * |\varphi_t^{(N)}|^2 \right) \varphi_t^{(N)} \tag{3.1}$$

where f denotes the solution of the zero-energy scattering equation (1.2). For technical reason, which will become clear later on, it is more convenient for us to work with the solution $\varphi_t^{(N)}$ of the modified Gross-Pitaevskii equation, rather than directly with the solution φ_t of the Gross-Pitaevskii equation (1.37). Since $N^3 f(Nx)V(Nx) \to 8\pi a_0 \delta(x)$, the solution $\varphi_t^{(N)}$ converges towards the solution φ_t of (1.37) with the same initial data, as $N \to \infty$. This is proven, together with other important properties of the solutions of (3.1) and of (1.37), in the next proposition.

Proposition 3.1. *Let $V \in L^1 \cap L^3(\mathbb{R}^3, (1+|x|^6)dx)$ be non-negative and spherically symmetric. Let f denote the solution of the zero-energy scattering equation (1.2), with boundary condition $f(x) \to 1$ as $|x| \to \infty$. Then, by Lemma 3.2 below, we have $0 \leq f \leq 1$ and therefore $Vf \geq 0$ with $Vf \in L^1 \cap L^3(\mathbb{R}^3, (1+|x|^6)dx)$. Let $\varphi \in H^1(\mathbb{R}^3)$, with $\|\varphi\|_2 = 1$.*

(i) *Well-posedness. There exist unique global in time solutions $\varphi_{(\cdot)}$ and $\varphi^{(N)}_{(\cdot)}$ in $C(\mathbb{R}; H^1(\mathbb{R}^3))$ of the Gross-Pitaevskii equation (1.37) and, respectively, of the modified Gross-Pitaevskii equation (3.1) with initial data φ. These solutions are such that $\|\varphi_t\|_2 = \|\varphi^{(N)}_t\|_2 = 1$ for all $t \in \mathbb{R}$. Moreover, there exists a constant $C > 0$ such that*

$$\|\varphi_t\|_{H^1}, \|\varphi^{(N)}_t\|_{H^1} \leq C$$

for all $t \in \mathbb{R}$.

(ii) *Propagation of higher regularity. If we make the additional assumption that $\varphi \in H^n(\mathbb{R}^3)$, for some integer $n \geq 2$, then φ_t and $\varphi^{(N)}_t$ are in $H^n(\mathbb{R}^3)$ for every $t \in \mathbb{R}$. Moreover there exists a constant $C > 0$ depending on $\|\varphi\|_{H^n}$ and on n, and a constant $K > 0$ depending only on $\|\varphi\|_{H^1}$ and n, such that*

$$\|\varphi_t\|_{H^n}, \|\varphi^{(N)}_t\|_{H^n} \leq Ce^{K|t|} \tag{3.2}$$

for all $t \in \mathbb{R}$.

(iii) *Regularity of time derivatives. Suppose $\varphi \in H^4(\mathbb{R}^3)$. Then there exist a constant $C > 0$ depending on $\|\varphi\|_{H^4}$, and a constant $K > 0$ depending only on $\|\varphi\|_{H^1}$, such that*

$$\|\dot{\varphi}^{(N)}_t\|_{H^2}, \|\ddot{\varphi}^{(N)}_t\|_2 \leq Ce^{K|t|}$$

for all $t \in \mathbb{R}$.

(iv) *Comparison of dynamics. Suppose now $\varphi \in H^2(\mathbb{R}^3)$. Then there exist constants $C, c_1, c_2 > 0$, depending on $\|\varphi\|_{H^2}$ (c_2 actually depends only on $\|\varphi\|_{H^1}$) such that*

$$\|\varphi^{(N)}_t - \varphi_t\|_2 \leq \frac{C\exp(c_1\exp(c_2|t|))}{N}$$

for all $t \in \mathbb{R}$.

The proof of Proposition 3.1 can be found in Appendix A.

Using the solution $\varphi^{(N)}_t$ of (3.1), we are going to approximate the coherent part of the evolution. As explained in the introduction, however, this approximation is not good enough. The many-body evolution develops a singular correlation structure, which is completely absent in the evolved coherent state. As a consequence, fluctuations around the coherent approximation are too strong to be controlled. To solve this problem, we have to produce a better approximation of the many-body evolution, in particular an approximation which takes into account the short-scale correlation structure. To reach this goal, we are going to multiply the Weyl operator $W(\sqrt{N}\varphi_t^{(N)})$, which generates the coherent approximation to the many-body dynamics, with a unitary Bogoliubov transformation $T(k)$, having the form (2.11) of an exponential of a quadratic expression in creation and annihilation operators. The kernel $k \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ has to be chosen so that $T(k)$ creates the correct correlations among the particles. Since correlations are, in good approximation, two-body effects, we can describe them through the solution f of the zero-energy scattering equation (1.2). We write

$$f(x) = 1 - w(x) \tag{3.3}$$

with $\lim_{|x|\to\infty} w(x) = 0$. The scattering length of V is defined as

$$8\pi a_0 = \int dx V(x) f(x).$$

Equivalently, a_0 is given by

$$a_0 = \lim_{|x|\to\infty} w(x)|x|.$$

Note that, if V has compact support inside $\{x \in \mathbb{R}^3 : |x| < R\}$, then $a_0 \le R$ and $w(x) = a_0/|x|$ for $|x| > R$. In general, under our assumptions on V, one can prove the following properties of the function w.

Lemma 3.2. *Let $V \in L^1 \cap L^3(\mathbb{R}^3, (1+|x|^6)dx)$ be spherically symmetric, with $V \ge 0$. Denote by f the solution of the zero-energy scattering equation (1.2) and let $w = 1 - f$. Then*

$$0 \le w(x) \le 1 \qquad \text{for all } x \in \mathbb{R}^3.$$

Moreover, there is a constant $C > 0$ such that

$$w(x) \le \frac{C}{|x|+1} \qquad \text{and} \qquad |\nabla w(x)| \le \frac{C}{|x|^2+1}. \tag{3.4}$$

Proof. Standard arguments show that $0 \le f(x) \le 1$ holds for every $x \in \mathbb{R}^3$ ($f(x) \le 1$ follows from $V \ge 0$, because of the monotonic dependence of f on the potential; see [32, Appendix C]). This implies that $0 \le w(x) \le 1$ for all $x \in \mathbb{R}^3$. From the zero energy scattering equation, we have $-\Delta w = Vf/2$. This implies that

$$w(x) = C \int dy \frac{1}{|x-y|} V(y) f(y) \quad \text{and} \quad \nabla w(x) = C \int dy \frac{x-y}{|x-y|^3} V(y) f(y)$$

for an appropriate constant $C \in \mathbb{R}$. Using $|x| \le |x-y| + |y|$, the fact that $f \le 1$, and the Hardy-Littlewood-Sobolev inequality, we find

$$\begin{aligned} |(1+|x|)w(x)| &\le C \int dy \left(\frac{1}{|x-y|} + 1 + \frac{|y|}{|x-y|} \right) V(y) f(y) \\ &\le C(\|V\|_{3/2} + \|V\|_{L^1} + \||y|V(y)\|_{3/2}) \end{aligned}$$

and, analogously,

$$\begin{aligned}\left|(1+|x|^2)w(x)\right| &\leq C\int dy\left(\frac{1}{|x-y|^2}+1+\frac{|y|^2}{|x-y|^2}\right)V(y)f(y)\\ &\leq C\left(\|V\|_3+\|V\|_1+\||y|^2V\|_3\right).\end{aligned}$$

The right hand side of the last two equations is bounded under the assumption $V\in L^1\cap L^3((1+|x|^6)dx)$. □

The zero-energy scattering equation for the rescaled potential $N^2V(Nx)$ is then solved by $f(Nx)$. We define $w(Nx)=1-f(Nx)$. Clearly

$$\lim_{|x|\to\infty} w(Nx)|x| = \frac{a_0}{N},$$

showing that the scattering length of $N^2V(Nx)$ is a_0/N. Equivalently, this follows from $\int dx N^2V(Nx)f(Nx)=8\pi a_0/N$.

It follows immediately from Lemma 3.2 that $0<w(Nx)<c$ for some $c<1$ and for all $x\in\mathbb{R}^3$, and that there exists C with

$$w(Nx)\leq C\frac{1}{N|x|+1} \qquad \text{and} \qquad |\nabla_x w(Nx)|\leq C\frac{N}{N^2|x|^2+1}. \tag{3.5}$$

We will use the solution $f(Nx)$ of the scaled zero-energy scattering equation to approximate the correlations among the particles, arising on the microscopic scale. It is however important to keep in mind that these correlations are also modulated on the macroscopic scale. The macroscopic variation is described, or at least, this is what we expect, by the solution of the modified Gross-Pitaevskii equation (3.1). We define therefore the kernel

$$k_t(x,y)=-Nw(N(x-y))\varphi_t^{(N)}(x)\varphi_t^{(N)}(y) \tag{3.6}$$

and the corresponding unitary operator

$$T(k_t)=\exp\left(\frac{1}{2}\int dxdy\left(k_t(x,y)a_x^*a_y^*-\overline{k}_t(x,y)a_xa_y\right)\right).$$

In the next lemma, we collect several bounds for the kernel k_t which will be useful in the following.

Lemma 3.3. *Let $\varphi_t^{(N)}\in H^1(\mathbb{R}^3)$ be the solution of (3.1) with initial data $\varphi\in H^1(\mathbb{R}^3)$. Let $w(Nx)=1-f(Nx)$, where f solves the zero-energy scattering equation (1.2). Let the kernel k_t be defined as in (3.6)*[1].

[1] In fact, the lemma holds for any kernel $k(x,y)=-Nw(N(x-y))\varphi(x)\varphi(y)$ with $\varphi\in H^1(\mathbb{R}^3)$.

(1) *There exists a constant C, depending only on the norm*[2] $\|\varphi_t^{(N)}\|_{H^1}$ *such that*

$$\|k_t\|_2 \leq C,$$
$$\|\nabla_1 k_t\|_2, \|\nabla_2 k_t\|_2 \leq C\sqrt{N},$$
$$\|\nabla_1 (k_t \bar{k}_t)\|_2, \|\nabla_2 (k_t \bar{k}_t)\|_2 \leq C.$$

Defining $p(k_t)$ *and* $r(k_t)$ *as in (2.13), so that* $ch(k_t) = 1 + p(k_t)$ *and* $sh(k_t) = k_t + r(k_t)$*, it follows from Lemma 2.3, part (iii) and (iv), that*

$$\|p(k_t)\|_2, \|r(k_t)\|_2, \|sh(k_t)\|_2 \leq C,$$
$$\|\nabla_1 p(k_t)\|_2, \|\nabla_2 p(k_t)\|_2 \leq C,$$
$$\|\nabla_1 r(k_t)\|_2, \|\nabla_2 r(k_t)\|_2 \leq C.$$

(2) *For almost all* $x, y \in \mathbb{R}^3$*, we have the pointwise bounds*

$$|k_t(x,y)| \leq \min\left(N|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)|, \frac{1}{|x-y|}|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)| \right),$$
$$|r(k_t)(x,y)| \leq C|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)|,$$
$$|p(k_t)(x,y)| \leq C|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)|.$$

(3) *Suppose further that* $\varphi \in H^2(\mathbb{R}^3)$*. Then*

$$\sup_{x\in\mathbb{R}^3} \|k_t(\cdot,x)\|_2, \sup_{x\in\mathbb{R}^3} \|p(k_t)(\cdot,x)\|_2 \leq C\|\varphi_t^{(N)}\|_{H^2},$$
$$\sup_{x\in\mathbb{R}^3} \|r(k_t)(\cdot,x)\|_2, \sup_{x\in\mathbb{R}^3} \|sh(k_t)(\cdot,x)\|_2 \leq C\|\varphi_t^{(N)}\|_{H^2}.$$

The proof can be found in Appendix B. We will also need bounds on the time derivative of the kernels k_t, $p(k_t)$, $r(k_t)$. These are collected in the following lemma, whose proof can also be found in Appendix B.

Lemma 3.4. *Let* $\varphi \in H^4(\mathbb{R}^3)$*, and* $\varphi_t^{(N)} \in H^4(\mathbb{R}^3)$ *be the solution of (3.1), with initial data* φ*. Let* $w(Nx) = 1 - f(Nx)$*, where* f *is the solution of the zero-energy scattering equation (1.2). Let the kernel* k_t *be defined as in (3.6), so that*

$$\dot{k}_t(x,y) = -Nw(N(x-y))\big(\dot{\varphi}_t^{(N)}(x)\varphi_t^{(N)}(y) + \varphi_t^{(N)}(x)\dot{\varphi}_t^{(N)}(y)\big). \tag{3.7}$$

Then there are constants $C, K > 0$*, where* C *depends on the* $\|\varphi\|_{H^4}$ *and* K *only on* $\|\varphi\|_{H^1}$ *such that the following bounds hold:*

(i) $$\|\dot{k}_t\|_2, \|\ddot{k}_t\|_2, \|\dot{p}(k_t)\|_2, \|\dot{r}(k_t)\|_2 \leq Ce^{K|t|}.$$

(ii) $$\|\nabla_1 \dot{p}(k)\|_2, \|\nabla_2 \dot{p}(k)\|_2, \|\nabla_1 \dot{r}(k)\|_2, \|\nabla_2 \dot{r}(k)\|_2 \leq Ce^{K|t|}.$$

[2] By Proposition 3.1 this norm is uniformly bounded in t.

(iii)
$$\sup_x \|\dot{k}_t(\cdot,x)\|_2, \sup_x \|\dot{p}(k_t)(\cdot,x)\|_2 \le Ce^{K|t|},$$
$$\sup_x \|\dot{r}(k_t)(\cdot,x)\|_2, \sup_x \|\dot{sh}(k_t)(\cdot,x)\|_2 \le Ce^{K|t|}.$$

As explained in the introduction, we are going to approximate the many-body evolution
$$e^{-i\mathscr{H}_N t} W(\sqrt{N}\varphi)T(k_0)\psi$$
of an initial state which is almost coherent but has the correct short-scale structure generated by $T(k_t)$, by the Fock state $W(\sqrt{N}\varphi_t^{(N)})T(k_t)\psi$. This leads us to the fluctuation dynamics, defined as the two-parameter group of unitary transformations
$$\mathscr{U}(t;s) = T^*(k_t)W^*(\sqrt{N}\varphi_t^{(N)})e^{-i\mathscr{H}_N(t-s)}W(\sqrt{N}\varphi_s^{(N)})T(k_s) \tag{3.8}$$
where $\mathscr{U}(s;s) = 1$ for all $s \in \mathbb{R}$.

The fluctuation dynamics satisfies the Schrödinger-type equation
$$i\partial_t \mathscr{U}(t;s) = \mathscr{L}_N(t)\mathscr{U}(t;s)$$
with the time-dependent generator
$$\begin{aligned}\mathscr{L}_N(t) &= T^*(k_t)\big(i\partial_t W^*(\sqrt{N}\varphi_t^{(N)})\big)W(\sqrt{N}\varphi_t^{(N)})T(k_t)\\ &\quad + T^*(k_t)W^*(\sqrt{N}\varphi_t^{(N)})\mathscr{H}_N W(\sqrt{N}\varphi_t^{(N)})T(k_t) + \big(i\partial_t T^*(k_t)\big)T(k_t).\end{aligned}$$

The next theorem, whose proof is deferred to Section 6, is the main technical ingredient of this paper. It contains important estimates for the generator $\mathscr{L}_N(t)$, which will be used in the next section to control the growth of the expectation of the number of particles operator with respect to the fluctuation dynamics $\mathscr{U}(t;s)$.

Theorem 3.5. *Define the time-dependent constant $C_N(t)$ which collects all terms in $\mathscr{L}_N(t)$ containing neither creation nor annihilation operators by*
$$\begin{aligned}C_N(t) &= N\int dx \Big(N^3V(N\cdot)\Big(\frac{1}{2} - f(N\cdot)\Big) * |\varphi_t^{(N)}|^2\Big)(x)|\varphi_t^{(N)}(x)|^2\\
&+ \int dxdy\, |\nabla_x sh(k_t)(y,x)|^2 + \int dxdy \big(N^3V(N\cdot) * |\varphi_t^{(N)}|^2\big)(x)\,|sh(k_t)(y,x)|^2\\
&+ \int dxdydz N^3V(N(x-y))\varphi_t^{(N)}(x)\overline{\varphi}_t^{(N)}(y) sh(\overline{k}_t)(z,x)\, sh(k_t)(z,y)\\
&+ \mathrm{Re}\int dxdydz N^3V(N(x-y))\varphi_t^{(N)}(x)\varphi_t^{(N)}(y) sh(\overline{k}_t)(z,x) ch(k_t)(z,y)\\
&+ \int dxdy N^2V(N(x-y))\\
&\quad \times \Bigg[\Big|\int dz\, sh(\overline{k}_t)(z,x)\, ch(z,y)\Big|^2 + \Big|\int dz\, sh(\overline{k}_t)(z,x)\, sh(z,y)\Big|^2\\
&\qquad + \int dz_1 dz_2\, sh(\overline{k}_t)(z_1,x)\, sh(k_t)(z_1,y) sh(\overline{k}_t)(z_2,x)\, sh(k_t)(z_2,y)\Bigg].\end{aligned} \tag{3.9}$$

The precise value of $C_N(t)$ is not important here, since it only corresponds to an overall phase (notice however, that it is of order N). Let

$$\widetilde{\mathscr{L}_N}(t) = \mathscr{L}_N(t) - C_N(t). \tag{3.10}$$

Then we have, for some $K > 0$ depending only on $\|\varphi\|_{H^1}$,

$$\widetilde{\mathscr{L}_N}(t) \geq \frac{1}{2}\mathscr{H}_N - C\frac{\mathscr{N}^2}{N} - Ce^{K|t|}(\mathscr{N}+1) \tag{3.11}$$

and

$$\widetilde{\mathscr{L}_N}(t) \leq \frac{3}{2}\mathscr{H}_N + C\frac{\mathscr{N}^2}{N} + Ce^{K|t|}(\mathscr{N}+1). \tag{3.12}$$

Moreover,

$$\pm\left[\mathscr{N}, \widetilde{\mathscr{L}_N}(t)\right] \leq \mathscr{H}_N + C\frac{\mathscr{N}^2}{N} + Ce^{K|t|}(\mathscr{N}+1) \tag{3.13}$$

and

$$\pm\dot{\widetilde{\mathscr{L}}}_N(t) \leq \mathscr{H}_N + Ce^{K|t|}\left(\frac{\mathscr{N}^2}{N} + \mathscr{N} + 1\right). \tag{3.14}$$

4 Growth of fluctuations

The goal of this section is to prove a bound, uniform in N, for the growth of the expectation of the number of particles operator with respect to the fluctuation dynamics. The properties of the generator $\mathscr{L}_N(t)$ of the fluctuation dynamics, as established in Theorem 3.5, play here a crucial role.

Theorem 4.1. *Suppose $\psi \in \mathscr{F}$ (possibly depending on N) with $\|\psi\| = 1$ is such that*

$$\left\langle \psi, \left(\frac{\mathscr{N}^2}{N} + \mathscr{N} + \mathscr{H}_N\right)\psi\right\rangle \leq D \tag{4.1}$$

for a constant $D > 0$ (independent of N). Let $\varphi \in H^4(\mathbb{R}^3)$, and let $\varphi_t^{(N)}$ be the solution of the modified Gross-Pitaevskii equation (3.1) with initial data φ. Let $\mathscr{U}(t;s)$ be the fluctuation dynamics defined in (3.8). Then there exist constants $C, c_1, c_2 > 0$ such that

$$\langle \psi, \mathscr{U}^*(t;0)\mathscr{N}\mathscr{U}(t;0)\psi\rangle \leq C\exp(c_1\exp(c_2|t|)).$$

The strategy of the proof of Theorem 4.1, which will be given at the end of this section, consists in applying Gronwall's inequality. The derivative of the expectation of $\mathscr{N}$ is given by the expectation of the commutator $i[\mathscr{N}, \mathscr{L}_N(t)]$, where $\mathscr{L}_N(t)$ is the generator (6.1) of the fluctuation dynamics. By (3.13), this commutator is bounded in terms of the energy, of $(\mathscr{N}+1)$, and of $\mathscr{N}^2/N$ (the difference between $\mathscr{L}_N(t)$ and the generator $\widetilde{\mathscr{L}_N}(t)$ appearing in (3.13) is a constant and hence it does not contribute to the commutator). The growth of the energy is controlled with the help of (3.14). What remains to be done in order to apply Gronwall's inequality is to bound the term $\mathcal{N}^2/N$. In the next proposition, we show that the expectation of $\mathcal{N}^2/N$ at time t can be controlled by its expectation at time $t=0$ (a harmless constant, by the assumption (4.1)) and by the expectation of $(\mathcal{N}+1)$ (which fits well in the scheme of Gronwall's inequality).

Proposition 4.2. *Let the fluctuation dynamics $\mathcal{U}(t;s)$ be defined as in (3.8). Then there exists a constant $C>0$ such that*

$$\mathcal{U}^*(t;0)\mathcal{N}^2\,\mathcal{U}(t;0) \le C\big(N\,\mathcal{U}^*(t;0)\mathcal{N}\,U(t;0) + N(\mathcal{N}+1) + (\mathcal{N}+1)^2\big).$$

The next lemma is useful in the proof of Proposition 4.2.

Lemma 4.3. *Let $k_t \in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ be as defined in (3.6). Then there exists a constant C, depending only on $\|k_t\|_2$, such that*

$$T^*(k_t)\,\mathcal{N}\,T(k_t) \le C(\mathcal{N}+1), \tag{4.2}$$

$$T^*(k_t)\,\mathcal{N}^2\,T(k_t) \le C(\mathcal{N}+1)^2 \tag{4.3}$$

for all $t\in\mathbb{R}$. Furthermore, the same inequalities hold with k_t replaced by $-k_t$ (notice that $T(-k_t)=T^*(k_t)$).

Proof. We use the decomposition $\text{ch}(k_t) = 1 + p(k_t)$ and the shorthand notation $c_x(z) = \text{ch}(k_t)(z,x)$, $p_x(z) = p(k_t)(z,x)$ and $s_x(z) = \text{sh}(k_t)(z,x)$. We have

$$\begin{aligned}
&\langle\psi, T^*(k_t)\mathcal{N}T(k_t)\psi\rangle \\
&\quad = \int dx\,\langle\psi,(a^*(c_x)+a(s_x))(a(c_x)+a^*(s_x))\psi\rangle \\
&\quad = \int dx\,\|(a_x + a(p_x) + a^*(s_x))\psi\|^2 \\
&\quad \le C\int dx\,\|a_x\psi\|^2 + \int dx\,\|a(p_x)\psi\|^2 + \int dx\,\|a^*(s_x)\psi\|^2 \\
&\quad \le C(1+\|p(k_t)\|_2^2 + \|\text{sh}(k_t)\|_2^2)\|(\mathcal{N}+1)^{1/2}\psi\|,
\end{aligned}$$

and (4.2) follows by Lemma 2.3 (since $\|p(k_t)\|_2, \|\text{sh}(k_t)\|_2 \le e^{\|k_t\|_2}$). To prove (4.3), we observe that

$$\begin{aligned}
&\langle\psi, T^*(k_t)\mathcal{N}^2T(k_t)\psi\rangle \\
&\quad = \int dxdy\,\langle\psi, T^*(k_t)a_x^*a_xa_y^*a_yT(k_t)\psi\rangle \\
&\quad = \int dx\,\langle\psi, T^*(k_t)a_x^*\mathcal{N}a_xT(k_t)\psi\rangle + \langle\psi,T^*(k_t)\mathcal{N}T(k_t)\psi\rangle \\
&\quad = \int dx\,\langle(a(c_x)+a^*(s_x))\psi, T^*(k)\mathcal{N}T(k)(a(c_x)+a^*(s_x))\psi\rangle \\
&\qquad + \langle\psi,T^*(k_t)\mathcal{N}T(k_t)\psi\rangle.
\end{aligned}$$

Then, applying (4.2), we obtain

$$\begin{aligned}
&\langle \psi, T^*(k_t)\mathcal{N}^2 T(k_t)\psi\rangle \\
&\quad \leq C\int dx\, \|(\mathcal{N}+1)^{1/2}(a(c_x)+a^*(s_x))\psi\|^2 + C\langle\psi,(\mathcal{N}+1)\psi\rangle \\
&\quad \leq C\int dx\, (\|a_x\mathcal{N}^{1/2}\psi\|^2 + \|a(p_x)\mathcal{N}^{1/2}\psi\|^2 + \|a^*(s_x)(\mathcal{N}+2)^{1/2}\psi\|^2) \\
&\qquad + C\langle\psi,(\mathcal{N}+1)\psi\rangle \\
&\quad \leq C(1+\|p(k_t)\|_2^2 + \|\mathrm{sh}(k_t)\|_2^2)\langle\psi,(\mathcal{N}+1)^2\psi\rangle .
\end{aligned}$$

The bounds from Lemma 2.3 imply (4.3). ◻

Proof of Proposition 4.2. From Lemma 4.3 we find

$$\begin{aligned}
&\langle\psi,\mathcal{U}^*(t;0)\mathcal{N}^2\mathcal{U}(t;0)\psi\rangle \\
&\qquad \leq C\langle\psi,\mathcal{U}^*(t;0)T^*(k_t)(\mathcal{N}+1)^2 T(k_t)\mathcal{U}(t;0)\psi\rangle \\
&\qquad \leq C\langle\psi,\mathcal{U}^*(t;0)T^*(k_t)\mathcal{N}^2 T(k_t)\mathcal{U}(t;0)\psi\rangle + C\langle\psi,\psi\rangle .
\end{aligned} \tag{4.4}$$

We now show how to bound the r.h.s. of the last equation. Using the definition of the fluctuation dynamics

$$\mathcal{U}(t;0) = T^*(k_t)W^*(\sqrt{N}\varphi_t^{(N)})e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)T(k_0),$$

we find

$$\begin{aligned}
&\langle\psi,\mathcal{U}^*(t;0)T^*(k_t)\mathcal{N}^2 T(k_t)\mathcal{U}(t;0)\psi\rangle \\
&= \langle\mathcal{N}\,T(k_t)\mathcal{U}(t;0)\psi, \\
&\qquad W^*(\sqrt{N}\varphi_t^{(N)})\,(\mathcal{N}-\sqrt{N}\phi(\varphi_t^{(N)})+N)e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)T(k_0)\psi\rangle \\
&= \langle\mathcal{N}\,T(k_t)\mathcal{U}(t;0)\psi, W^*(\sqrt{N}\varphi_t^{(N)})\,\mathcal{N}\,e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)T(k_0)\psi\rangle \\
&\quad -\sqrt{N}\,\langle\mathcal{N}\,T(k_t)\mathcal{U}(t;0)\psi, W^*(\sqrt{N}\varphi_t^{(N)})\,\phi(\varphi_t^{(N)})e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)T(k_0)\psi\rangle \\
&\quad + N\langle\psi,\mathcal{U}^*(t;0)T^*(k_t)\mathcal{N}\,T(k_t)\mathcal{U}(t;0)\psi\rangle
\end{aligned} \tag{4.5}$$

where we used the notation $\phi(f) = a(f)+a^*(f)$, and the property (2.4) to show that

$$W(\sqrt{N}\varphi_t^{(N)})\mathcal{N}W^*(\sqrt{N}\varphi_t^{(N)}) = \mathcal{N}-\sqrt{N}\phi(\varphi_t^{(N)})+N,$$

since $W(\sqrt{N}\varphi_t^{(N)}) = W^*(-\sqrt{N}\varphi_t^{(N)})$. In the first term on the r.h.s. of (4.5), we use now the fact that $\mathcal{N}$ commutes with $\mathcal{H}_N$. In the second term, on the other hand, we move the factor $\phi(\varphi_t^{(N)})$ back to the left of the Weyl operator $W(\sqrt{N}\varphi_t^{(N)})$, using that

$$W^*(\sqrt{N}\varphi_t^{(N)})\phi(\varphi_t^{(N)}) = \left(\phi(\varphi_t^{(N)})+2\sqrt{N}\right)W^*(\sqrt{N}\varphi_t^{(N)}) .$$

We conclude that

(4.6)
$$\begin{aligned}
&\langle \psi, \mathscr{U}^*(t;0)T^*(k_t)\mathcal{N}^2 T(k_t)\mathscr{U}(t;0)\psi\rangle \\
&= \langle \mathcal{N}\, T(k_t)\mathscr{U}(t;0)\psi, \\
&\qquad W^*(\sqrt{N}\varphi_t^{(N)})\, e^{-i\mathscr{H}_N t} W(\sqrt{N}\varphi)(\mathcal{N}+\sqrt{N}\phi(\varphi)+N)\, T(k_0)\psi\rangle \\
&\quad - \sqrt{N}\,\langle \mathcal{N}\, T(k_t)\mathscr{U}(t;0)\psi, \\
&\qquad \big(\phi(\varphi_t^{(N)})+2\sqrt{N}\big) W^*(\sqrt{N}\varphi_t^{(N)})\, e^{-i\mathscr{H}_N t} W(\sqrt{N}\varphi) T(k_0)\psi\rangle \\
&\quad + N\langle \psi, \mathscr{U}^*(t;0)T^*(k_t)\mathcal{N}\, T(k_t)\mathscr{U}(t;0)\psi\rangle \\
&= \langle \mathcal{N}\, T(k_t)\mathscr{U}(t;0)\psi, W^*(\sqrt{N}\varphi_t^{(N)})\, e^{-i\mathscr{H}_N t} W(\sqrt{N}\varphi)\mathcal{N}\, T(k_0)\psi\rangle \\
&\quad + \sqrt{N}\langle \mathcal{N}\, T(k_t)\mathscr{U}(t;0)\psi, W^*(\sqrt{N}\varphi_t^{(N)})\, e^{-i\mathscr{H}_N t} W(\sqrt{N}\varphi)\phi(\varphi)\, T(k_0)\psi\rangle \\
&\quad - \sqrt{N}\,\langle \mathcal{N}\, T(k_t)\mathscr{U}(t;0)\psi, \phi(\varphi_t^{(N)}) W^*(\sqrt{N}\varphi_t^{(N)})\, e^{-i\mathscr{H}_N t} W(\sqrt{N}\varphi) T(k_0)\psi\rangle .
\end{aligned}$$

By Cauchy-Schwarz, we obtain

$$\begin{aligned}
&\langle \psi, \mathscr{U}^*(t;0)T^*(k_t)\mathcal{N}^2 T(k_t)\mathscr{U}(t;0)\psi\rangle \\
&\le \|\mathcal{N}\, T(k_t)\mathscr{U}(t;0)\psi\| \\
&\quad \times \big(\|\mathcal{N}\, T(k_0)\psi\| + \sqrt{N}\|\phi(\varphi)\, T(k_0)\psi\| + \sqrt{N}\|\phi(\varphi_t^{(N)}) T(k_t)\mathscr{U}(t;0)\psi\|\big) \\
&\le \frac{1}{2}\|\mathcal{N}\, T(k_t)\mathscr{U}(t;0)\psi\|^2 \\
&\quad + C\big(\|\mathcal{N}\, T(k_0)\psi\|^2 + N\|\phi(\varphi)\, T(k_0)\psi\|^2 + N\|\phi(\varphi_t^{(N)}) T(k_t)\mathscr{U}(t;0)\psi\|^2\big).
\end{aligned}$$

Subtracting the first term appearing on the r.h.s., and using $\|\phi(f)\psi\| \le \|f\|_2\|(\mathcal{N}+1)^{1/2}\psi\|$, we find that

$$\begin{aligned}
\langle \psi, \mathscr{U}^*(t;0)T^*(k_t)\mathcal{N}^2 T(k_t)\mathscr{U}(t;0)\psi\rangle
&\le C\big(\|\mathcal{N}\, T(k_0)\psi\|^2 + N\|(\mathcal{N}+1)^{1/2} T(k_0)\psi\|^2 \\
&\qquad + N\|(\mathcal{N}+1)^{1/2} T(k_t)\mathscr{U}(t;0)\psi\|^2\big).
\end{aligned}$$

By Lemma 4.3 we conclude that

$$\begin{aligned}
&\langle \psi, \mathscr{U}^*(t;0)T^*(k_t)\mathcal{N}^2 T(k_t)\mathscr{U}(t;0)\psi\rangle \\
&\quad \le C\|(\mathcal{N}+1)\psi\|^2 + CN\|(\mathcal{N}+1)^{1/2}\psi\|^2 + CN\|(\mathcal{N}+1)^{1/2}\mathscr{U}(t;0)\psi\|^2.
\end{aligned}$$

With (4.4), this concludes the proof of the proposition. □

We are now ready to show Theorem 4.1.

Proof of Theorem 4.1. Let $C_N(t)$ be defined as in (3.9) and define

$$\widetilde{\mathscr{U}}(t;s) = e^{i\int_s^t C_N(\tau)d\tau}\mathscr{U}(t;s).$$

Then $\widetilde{\mathscr{U}}(t;s)$ satisfies the Schrödinger type equation

$$i\partial_t \widetilde{\mathscr{U}}(t;s) = \widetilde{\mathscr{L}_N}(t)\widetilde{\mathscr{U}}(t;s), \qquad \text{with} \quad \widetilde{\mathscr{U}}(s,s) = 1$$

for all $s \in \mathbb{R}$, and with generator $\widetilde{\mathscr{L}_N}(t) = \mathscr{L}_N(t) - C_N(t)$, as defined in (3.10). On the other hand, since the two evolutions only differ by a phase, we have

$$\langle \psi, \mathscr{U}^*(t;0)\mathscr{N}\mathscr{U}(t;0)\psi\rangle = \langle \psi, \widetilde{\mathscr{U}}^*(t;0)\mathscr{N}\widetilde{\mathscr{U}}(t;0)\psi\rangle.$$

We now use the properties of $\widetilde{\mathscr{L}_N}(t)$, as established in (3.11)-(3.14). Eq. (3.11) implies

$$\mathscr{H}_N \leq 2\widetilde{\mathscr{L}_N}(t) + C\frac{\mathscr{N}^2}{N} + Ce^{K|t|}(\mathscr{N}+1). \tag{4.7}$$

From Proposition 4.2, we conclude that there exists a constant C_1 (which depends on $\langle \psi, (\mathscr{N}^2/N + \mathscr{N} + 1)\psi\rangle$), such that

$$\begin{aligned} 0 &\leq \left\langle \psi, \widetilde{\mathscr{U}}^*(t;0)\mathscr{H}_N\widetilde{\mathscr{U}}(t;0)\psi\right\rangle \\ &\leq \left\langle \psi, \widetilde{\mathscr{U}}^*(t;0)\left(2\widetilde{\mathscr{L}_N}(t) + C_1 e^{K|t|}(\mathscr{N}+1)\right)\widetilde{\mathscr{U}}(t;0)\psi\right\rangle. \end{aligned} \tag{4.8}$$

From (3.13), combined with (4.7) and Proposition 4.2, there exists moreover a constant $C_2 > 0$ (depending on $\langle \psi, (\mathscr{N}^2/N + \mathscr{N} + 1)\psi\rangle$) such that

$$\begin{aligned} &\frac{d}{dt}\left\langle \psi, \widetilde{\mathscr{U}}^*(t;0)\mathscr{N}\widetilde{\mathscr{U}}(t;0)\psi\right\rangle \\ &\qquad \leq \left\langle \psi, \widetilde{\mathscr{U}}^*(t;0)\left(2\widetilde{\mathscr{L}_N}(t) + C_2 e^{K|t|}(\mathscr{N}+1)\right)\widetilde{\mathscr{U}}(t;0)\psi\right\rangle. \end{aligned}$$

We now estimate the growth of the expectation of the generator $\widetilde{\mathscr{L}_N}(t)$. Using (3.14) together with (4.7) and Proposition 4.2, we conclude that there exists a constant $C_3 > 0$ (again, depending on $\langle \psi, (\mathscr{N}^2/N + \mathscr{N} + 1)\psi\rangle$), with

$$\begin{aligned} &\frac{d}{dt}\left\langle \psi, \widetilde{\mathscr{U}}^*(t;0)\widetilde{\mathscr{L}_N}(t)\widetilde{\mathscr{U}}(t;0)\psi\right\rangle \\ &\qquad = \left\langle \psi, \widetilde{\mathscr{U}}^*(t;0)\dot{\widetilde{\mathscr{L}}}_N(t)\,\widetilde{\mathscr{U}}(t;0)\psi\right\rangle \\ &\qquad \leq \left\langle \psi, \widetilde{\mathscr{U}}^*(t;0)\left(2\widetilde{\mathscr{L}_N}(t) + C_3 e^{K|t|}(\mathscr{N}+1)\right)\widetilde{\mathscr{U}}(t;0)\psi\right\rangle. \end{aligned}$$

We now fix $D := \max(C_1 + 1, C_2, C_3, K)$. Then, we have

$$\begin{aligned} &\frac{d}{dt}\left\langle \psi, \widetilde{\mathscr{U}}^*(t;0)\left(\widetilde{\mathscr{L}_N}(t) + De^{K|t|}(\mathscr{N}+1)\right)\widetilde{\mathscr{U}}(t;0)\psi\right\rangle \\ &\quad \leq \left\langle \psi, \widetilde{\mathscr{U}}^*(t;0)\left[(2 + 2De^{K|t|})\widetilde{\mathscr{L}_N}(t)\right.\right. \\ &\qquad\qquad \left.\left. + (De^{K|t|} + DKe^{K|t|} + D^2 e^{2K|t|})(\mathscr{N}+1)\right]\widetilde{\mathscr{U}}(t;0)\psi\right\rangle \\ &\quad \leq 4De^{K|t|}\left\langle \psi, \widetilde{\mathscr{U}}^*(t;0)\left(\widetilde{\mathscr{L}_N}(t) + De^{K|t|}(\mathscr{N}+1)\right)\widetilde{\mathscr{U}}(t;0)\psi\right\rangle. \end{aligned}$$

By Gronwall's lemma, we conclude that

$$\begin{aligned}&\left\langle \psi, \widetilde{\mathscr{U}}^*(t;0)\left(\widetilde{\mathscr{L}}_N(t)+De^{K|t|}(\mathscr{N}+1)\right)\widetilde{\mathscr{U}}(t;0)\psi\right\rangle\\ &\leq e^{4\frac{D}{K}e^{K|t|}}\left\langle \psi, \left(\widetilde{\mathscr{L}}_N(0)+D(\mathscr{N}+1)\right)\psi\right\rangle\\ &\leq e^{4\frac{D}{K}e^{K|t|}}\left\langle \psi, \left(\frac{3}{2}\mathscr{H}_N+C\frac{\mathscr{N}^2}{N}+C(\mathscr{N}+1)\right)\psi\right\rangle\end{aligned}$$

where in the last inequality we used the upper bound (3.12). From assumption (4.1) we then obtain

$$\begin{aligned}&\left\langle \psi, \widetilde{\mathscr{U}}^*(t;0)\left(\widetilde{\mathscr{L}}_N(t)+De^{K|t|}(\mathscr{N}+1)\right)\widetilde{\mathscr{U}}(t;0)\psi\right\rangle\\ &\quad\leq C\exp(c_1\exp(c_2|t|)).\end{aligned}\tag{4.9}$$

Furthermore, from (4.8) we have

$$\left\langle \psi, \widetilde{\mathscr{U}}^*(t;0)\widetilde{\mathscr{L}}_N(t)\widetilde{\mathscr{U}}(t;0)\psi\right\rangle \geq -\frac{C_1}{2}e^{K|t|}\left\langle \psi, \widetilde{\mathscr{U}}^*(t;0)(\mathscr{N}+1)\widetilde{\mathscr{U}}(t;0)\psi\right\rangle.$$

Since $D-C_1\geq 1$, the last equation, combined with (4.9), implies the desired bound for the expectation of $\mathscr{N}$. □

5 Proof of the main theorem

Using the bounds established in Theorem 4.1, we proceed now to prove our main result.

Proof of Theorem 1.1. Let $\Psi_{N,t}=e^{-i\mathscr{H}_Nt}W(\sqrt{N}\varphi)T(k_0)\psi$. The one-particle reduced density of $\Psi_{N,t}$ has the integral kernel

$$\Gamma^{(1)}_{N,t}(x;y)=\frac{1}{\langle\Psi_{N,t},\mathscr{N}\Psi_{N,t}\rangle}\langle\Psi_{N,t},a_y^*a_x\Psi_{N,t}\rangle.\tag{5.1}$$

We start by computing the denominator. Since $\mathscr{H}_N$ commutes with the number of particles operator, we find

$$\begin{aligned}\left\langle\Psi_{N,t},\mathscr{N}\Psi_{N,t}\right\rangle &= \left\langle\psi, T^*(k_0)W^*(\sqrt{N}\varphi)\mathscr{N}W(\sqrt{N}\varphi)T(k_0)\psi\right\rangle\\ &= \left\langle\psi, T^*(k_0)\left(\mathscr{N}-\sqrt{N}\phi(\varphi)+N\right)T(k_0)\psi\right\rangle\\ &= N+\left\langle\psi, T^*(k_0)\left(\mathscr{N}-\sqrt{N}\phi(\varphi)\right)T(k_0)\psi\right\rangle.\end{aligned}$$

By Lemma 4.3

$$\begin{aligned}\langle\psi, T^*(k_0)\mathscr{N}T(k_0)\psi\rangle &\leq C\langle\psi,(\mathscr{N}+1)\,\psi\rangle\\ &\leq C\end{aligned}$$

and

$$\begin{aligned}|\langle\psi, T^*(k_0)\phi(\varphi)T(k_0)\psi\rangle| &\leq C\langle\psi, T^*(k_0)(\mathscr{N}+1)^{1/2}T(k_0)\psi\rangle\\ &\leq C\langle\psi,(\mathscr{N}+1)\psi\rangle\\ &\leq C.\end{aligned}$$

Hence, there exists $C > 0$ with

$$|\langle \Psi_{N,t}, \mathcal{N} \Psi_{N,t}\rangle - N| \leq C N^{1/2}. \tag{5.2}$$

On the other hand, with $\varphi_t^{(N)}$ denoting the solution of the modified Gross-Pitaevskii equation (3.1), the numerator of (5.1) can be written as

$$\begin{aligned}
&\left\langle \Psi_{N,t}, a_y^* a_x \Psi_{N,t}\right\rangle \\
&= \left\langle \psi, T(k_0) W(\sqrt{N}\varphi) e^{it\mathcal{H}_N} a_y^* a_x e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi) T(k_0)\psi\right\rangle \\
&= N\varphi_t^{(N)}(x)\overline{\varphi}_t^{(N)}(y) \\
&\quad + \sqrt{N}\,\varphi_t(x)\left\langle \psi, T^*(k_0) W^*(\sqrt{N}\varphi) e^{it\mathcal{H}_N} \right. \\
&\qquad\qquad \left. \times \left(a_y^* - \sqrt{N}\overline{\varphi}_t^{(N)}(y)\right) e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi) T(k_0)\psi\right\rangle \\
&\quad + \sqrt{N}\,\overline{\varphi}_t(y)\left\langle \psi, T^*(k_0) W^*(\sqrt{N}\varphi) e^{it\mathcal{H}_N} \right. \\
&\qquad\qquad \left. \times \left(a_x - \sqrt{N}\varphi_t^{(N)}(x)\right) e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi) T(k_0)\psi\right\rangle \\
&\quad + \left\langle \psi, T^*(k_0) W^*(\sqrt{N}\varphi) e^{it\mathcal{H}_N} \left(a_y^* - \sqrt{N}\overline{\varphi}_t^{(N)}(y)\right) \right. \\
&\qquad\qquad \left. \times \left(a_x - \sqrt{N}\varphi_t^{(N)}(x)\right) e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi) T(k_0)\psi\right\rangle.
\end{aligned}$$

Recognizing that

$$(a_y^* - \sqrt{N}\overline{\varphi}_t^{(N)}(y)) = W(\sqrt{N}\varphi_t^{(N)}) a_y^* W^*(\sqrt{N}\varphi_t^{(N)})$$

and

$$(a_x - \sqrt{N}\varphi_t^{(N)}(x)) = W(\sqrt{N}\varphi_t^{(N)}) a_x W^*(\sqrt{N}\varphi_t^{(N)})$$

we obtain

$$\begin{aligned}
&\left\langle \Psi_{N,t}, a_y^* a_x \Psi_{N,t}\right\rangle \\
&= N\varphi_t^{(N)}(x)\overline{\varphi}_t^{(N)}(y) \\
&\quad + \sqrt{N}\,\varphi_t^{(N)}(x)\left\langle \psi, T^*(k_0) W^*(\sqrt{N}\varphi) e^{it\mathcal{H}_N} W(\sqrt{N}\varphi_t^{(N)}) \right. \\
&\qquad\qquad \left. \times a_y^* W^*(\sqrt{N}\varphi_t^{(N)}) e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi) T(k_0)\psi\right\rangle \\
&\quad + \sqrt{N}\,\overline{\varphi}_t^{(N)}(y)\left\langle \psi, T^*(k_0) W^*(\sqrt{N}\varphi) e^{it\mathcal{H}_N} W(\sqrt{N}\varphi_t^{(N)}) \right. \\
&\qquad\qquad \left. \times a_x W^*(\sqrt{N}\varphi_t^{(N)}) e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi) T(k_0)\psi\right\rangle \\
&\quad + \langle \psi, T^*(k_0) W^*(\sqrt{N}\varphi) e^{it\mathcal{H}_N} W(\sqrt{N}\varphi_t^{(N)}) \\
&\qquad\qquad \times a_y^* a_x W^*(\sqrt{N}\varphi_t^{(N)}) e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi) T(k_0)\psi\big\rangle.
\end{aligned}$$

Combining the last equation with (5.2) and inserting in (5.1), we have that

$$\begin{aligned}
&\left|\Gamma^{(1)}_{N,t}(x;y) - \overline{\varphi}^{(N)}_t(y)\varphi^{(N)}_t(x)\right| \\
&\quad\le \frac{C}{\sqrt{N}}|\varphi^{(N)}_t(x)||\varphi^{(N)}_t(y)| \\
&\qquad + \frac{1}{\sqrt{N}}|\varphi^{(N)}_t(y)|\,\|a_x W^*(\sqrt{N}\varphi^{(N)}_t)e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)T(k_0)\psi\| \\
&\qquad + \frac{1}{\sqrt{N}}|\varphi^{(N)}_t(x)|\,\|a_y W^*(\sqrt{N}\varphi^{(N)}_t)e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)T(k_0)\psi\| \\
&\qquad + \frac{1}{N}\|a_y W^*(\sqrt{N}\varphi^{(N)}_t)e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)T(k_0)\psi\| \\
&\qquad\qquad \times \|a_x W^*(\sqrt{N}\varphi^{(N)}_t)e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)T(k_0)\psi\|.
\end{aligned}$$

Taking the square and integrating over x, y, we find

$$\begin{aligned}
&\left\|\Gamma^{(1)}_{N,t} - |\varphi^{(N)}_t\rangle\langle\varphi^{(N)}_t|\right\|_{\text{HS}} \\
&\quad\le \frac{C}{\sqrt{N}}\|(\mathcal{N}+1)^{1/2}W^*(\sqrt{N}\varphi^{(N)}_t)e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)T(k_0)\psi\|^2.
\end{aligned}$$

Lemma 4.3 implies that

$$\begin{aligned}
&\left\|\Gamma^{(1)}_{N,t} - |\varphi^{(N)}_t\rangle\langle\varphi^{(N)}_t|\right\|_{\text{HS}} \\
&\quad\le \frac{C}{\sqrt{N}}\|(\mathcal{N}+1)^{1/2}T^*(k_t)W^*(\sqrt{N}\varphi^{(N)}_t)e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)T(k_0)\psi\|^2 \\
&\quad= \frac{C}{\sqrt{N}}\|(\mathcal{N}+1)^{1/2}\,\mathcal{U}(t;0)\psi\|^2
\end{aligned}$$

with the fluctuation dynamics $\mathcal{U}(t;s)$ defined in (3.8). From Theorem 4.1 we conclude that

$$\left\|\Gamma^{(1)}_{N,t} - |\varphi^{(N)}_t\rangle\langle\varphi^{(N)}_t|\right\|_{\text{HS}} \le \frac{C\exp(c_1\exp(c_2|t|))}{\sqrt{N}}.$$

Since $|\varphi^{(N)}_t\rangle\langle\varphi^{(N)}_t|$ is a rank-one projection, and $\Gamma^{(1)}_{N,t} \ge 0$, it follows that the difference $\Gamma^{(1)}_{N,t} - |\varphi^{(N)}_t\rangle\langle\varphi^{(N)}_t|$ can have only one negative eigenvalue. Since the trace of $\Gamma^{(1)}_{N,t} - |\varphi^{(N)}_t\rangle\langle\varphi^{(N)}_t|$ vanishes, it must have one negative eigenvalue, with absolute value equal to the sum of all positive eigenvalues. As a consequence, the trace norm of the difference is controlled by the operator norm (given by the absolute value of the negative eigenvalue) and therefore also by the Hilbert-Schmidt norm. This shows that

$$\operatorname{Tr}\left|\Gamma^{(1)}_{N,t} - |\varphi^{(N)}_t\rangle\langle\varphi^{(N)}_t|\right| \le \frac{C\exp(c_1\exp(c_2|t|))}{\sqrt{N}}.$$

Theorem 1.1 now follows because, if φ_t denotes the solution of the Gross-Pitaevskii equation (1.37), Proposition 3.1 implies that

$$\operatorname{Tr}\left| |\varphi_t\rangle\langle\varphi_t| - |\varphi_t^{(N)}\rangle\langle\varphi_t^{(N)}| \right| \leq 2\|\varphi_t - \varphi_t^{(N)}\|_2 \leq \frac{Ce^{c_1|t|}}{N}. \qquad \square$$

6 Key bounds on the generator of the fluctuation dynamics

In this section, we prove Theorem 3.5, concerning the generator $\mathscr{L}_N(t)$ of the fluctuation dynamics

$$\mathscr{U}(t;s) = T^*(k_t)W^*(\sqrt{N}\varphi_t^{(N)})e^{-i\mathscr{H}_N(t-s)}W(\sqrt{N}\varphi_s^{(N)})T(k_s)$$

as defined in (3.8). We write

$$\begin{aligned}\mathscr{L}_N(t) &= T^*(k_t)\big(i\partial_t W^*(\sqrt{N}\varphi_t^{(N)})\big)W(\sqrt{N}\varphi_t^{(N)})T(k_t)\\ &\quad + T^*(k_t)W^*(\sqrt{N}\varphi_t^{(N)})\mathscr{H}_N W(\sqrt{N}\varphi_t^{(N)})T(k_t) + \big(i\partial_t T^*(k_t)\big)T(k_t)\\ &= T^*(k_t)\mathscr{L}_N^{(0)}(t)T(k_t) + \big(i\partial_t T^*(k_t)\big)T(k_t)\end{aligned} \tag{6.1}$$

with

$$\mathscr{L}_N^{(0)}(t) = \big(i\partial_t W^*(\sqrt{N}\varphi_t^{(N)})\big)W(\sqrt{N}\varphi_t^{(N)}) + W^*(\sqrt{N}\varphi_t^{(N)})\mathscr{H}_N W(\sqrt{N}\varphi_t^{(N)}).$$

A simple computation shows that

$$\begin{aligned}&\big(i\partial_t W^*(\sqrt{N}\varphi_t^{(N)})\big)W(\sqrt{N}\varphi_t^{(N)})\\ &\qquad = -a(\sqrt{N}i\partial_t\varphi_t^{(N)}) - a^*(\sqrt{N}i\partial_t\varphi_t^{(N)}) - N\big\langle\varphi_t^{(N)}, i\partial_t\varphi_t^{(N)}\big\rangle.\end{aligned}$$

On the other hand, using (2.4), we find

$$\begin{aligned}&W^*(\sqrt{N}\varphi_t^{(N)})\mathscr{H}_N W(\sqrt{N}\varphi_t^{(N)})\\ &= N\left[\|\nabla\varphi_t^{(N)}\|_2^2 + \frac{1}{2}\int dx\,(N^3V(N\cdot)*|\varphi_t^{(N)}|^2)(x)|\varphi_t^{(N)}(x)|^2\right]\\ &+ \sqrt{N}\big[a^*\big((N^3V(N\cdot)*|\varphi_t^{(N)}|^2)\varphi_t^{(N)}\big) + a\big((N^3V(N\cdot)*|\varphi_t^{(N)}|^2)\varphi_t^{(N)}\big)\big]\\ &+ \int dx\,\nabla_x a_x^*\nabla_x a_x + \int dx\,(N^3V(N\cdot)*|\varphi_t^{(N)}|^2)(x)a_x^*a_x\\ &+ \int dxdy N^3V(N(x-y))\varphi_t^{(N)}(x)\overline{\varphi}_t^{(N)}(y)a_x^*a_y\\ &+ \frac{1}{2}\int dxdy N^3V(N(x-y))\big(\varphi_t^{(N)}(x)\varphi_t^{(N)}(y)a_x^*a_y^* + \overline{\varphi}_t^{(N)}(x)\overline{\varphi}_t^{(N)}(y)a_xa_y\big)\\ &+ \frac{1}{\sqrt{N}}\int dxdy N^3V(N(x-y))a_x^*\big(\varphi_t^{(N)}(y)a_y^* + \overline{\varphi}_t^{(N)}(y)a_y\big)a_x\\ &+ \frac{1}{2N}\int dxdy N^3V(N(x-y))a_x^*a_y^*a_ya_x.\end{aligned}$$

Combining the last two equations and using (3.1), we conclude that

$$\begin{aligned}
\mathscr{L}_N^{(0)}(t) &= N\int dx\Big(N^3V(N\cdot)\Big(\frac{1}{2}-f(N\cdot)\Big)*|\varphi_t^{(N)}|^2\Big)(x)|\varphi_t^{(N)}(x)|^2\\
&+\sqrt{N}\big[a^*\big((N^3w(N\cdot)V(N\cdot)*|\varphi_t^{(N)}|^2)\varphi_t^{(N)}\big)\\
&\qquad+a\big((N^3w(N\cdot)V(N\cdot)*|\varphi_t^{(N)}|^2)\varphi_t^{(N)}\big)\big]\\
&+\int dx\,\nabla_x a_x^*\nabla_x a_x+\int dx\,(N^3V(N\cdot)*|\varphi_t^{(N)}|^2)(x)a_x^*a_x\\
&+\int dxdyN^3V(N(x-y))\varphi_t^{(N)}(x)\overline{\varphi}_t^{(N)}(y)a_x^*a_y\\
&+\frac{1}{2}\int dxdyN^3V(N(x-y))\left(\varphi_t^{(N)}(x)\varphi_t^{(N)}(y)a_x^*a_y^*+\overline{\varphi}_t^{(N)}(x)\overline{\varphi}_t^{(N)}(y)a_xa_y\right)\\
&+\frac{1}{\sqrt{N}}\int dxdyN^3V(N(x-y))a_x^*\left(\varphi_t^{(N)}(y)a_y^*+\overline{\varphi}_t^{(N)}(y)a_y\right)a_x\\
&+\frac{1}{2N}\int dxdyN^3V(N(x-y))a_x^*a_y^*a_ya_x\\
&=:\mathscr{L}_{0,N}^{(0)}(t)+\mathscr{L}_{1,N}^{(0)}(t)+\mathscr{L}_{2,N}^{(0)}(t)+\mathscr{L}_{3,N}^{(0)}(t)+\mathscr{L}_{4,N}^{(0)}
\end{aligned}$$

where $\mathscr{L}_{j,N}^{(0)}(t)$, for $j=0,1,2,3,4$, is the part of $\mathscr{L}_N^{(0)}(t)$ containing the products of exactly j creation and annihilation operators. Recall here that $w(x)=1-f(x)$, as defined in (3.3).

From (6.1), we find that the generator of the fluctuation dynamics is given by

$$\mathscr{L}_N(t)=\mathscr{L}_{0,N}^{(0)}(t)+\sum_{j=1}^{4}T^*(k_t)\mathscr{L}_{j,N}^{(0)}(t)T(k_t)+\big(i\partial_tT^*(k_t)\big)T(k_t). \tag{6.2}$$

In the next subsections, we study separately the different terms on the r.h.s. of (6.2). The final goal of this analysis, a proof of Theorem 3.5, will be reached in Subsection 6.6.

Notation. In the rest of this section, we will use the shorthand notation

$$\begin{aligned}
c_x(y)&=\mathrm{ch}(k_t)(y,x), & s_x(y)&=\mathrm{sh}(k_t)(y,x),\\
p_x(y)&=p(k_t)(y,x), & r_x(y)&=r(k_t)(y,x).
\end{aligned}\tag{6.3}$$

Moreover, $\|p\|_2$, $\|r\|_2$ and $\|\mathrm{sh}\|_2$ will denote the L^2-norms of the kernels $p(k_t)(x,y)$, $r(k_t)(x,y)$ and $\mathrm{sh}(k_t)(x,y)$ over $\mathbb{R}^3\times\mathbb{R}^3$ (in other words, they denote the Hilbert-Schmidt norms of the corresponding operators). The norms $\|p_x\|_2, \|r_x\|_2, \|\mathrm{sh}_x\|_2$, on the other hand, indicate norms over $\mathbb{R}^3$. Finally, the notation $\langle\cdot,\cdot\rangle$ will denote the L^2 inner product. We will abbreviate $T(k_t)$ by T.

6.1 Analysis of $T^*\mathscr{L}^{(0)}_{1,N}(t)T$

Conjugating the linear term $\mathscr{L}^{(0)}_{N,1}(t)$ with T produces again linear terms. From Lemma 2.3, we obtain

$$\begin{aligned}
&T^*\mathscr{L}^{(0)}_{1,N}(t)T \\
&= \sqrt{N}\int dxdy N^3V(N(x-y))w(N(x-y))|\varphi_t^{(N)}(y)|^2 \\
&\qquad\qquad \times \left(\varphi_t^{(N)}(x)T^*a_x^*T + \overline{\varphi}_t^{(N)}(x)T^*a_xT\right) \\
&= \sqrt{N}\int dxdy N^3V(N(x-y))w(N(x-y))|\varphi_t^{(N)}(y)|^2\,\varphi_t^{(N)}(x) \\
&\qquad\qquad \times (a^*(c_x)+a(s_x)) \\
&\quad + \sqrt{N}\int dxdy N^3V(N(x-y))w(N(x-y))|\varphi_t^{(N)}(y)|^2\,\overline{\varphi}_t^{(N)}(x) \\
&\qquad\qquad \times (a(c_x)+a^*(s_x)).
\end{aligned} \tag{6.4}$$

These terms are potentially dangerous because they are large (of order $\sqrt{N}$) and do not commute with the number of particles. We will see however that they cancel with contributions arising from the cubic part $\mathscr{L}^{(0)}_{3,N}(t)$.

6.2 Analysis of $T^*\mathscr{L}^{(0)}_{2,N}(t)T$

We split $\mathscr{L}^{(0)}_{2,N}(t)$ into two terms, writing

$$\mathscr{L}^{(0)}_{2,N}(t) = \mathscr{K} + \widehat{\mathscr{L}}^{(0)}_{2,N}(t),$$

with $\mathscr{K} = \int dx \nabla_x a_x^* \nabla_x a_x$ being the kinetic energy operator.

Properties of $T^*\mathscr{K}T$

We have

$$\begin{aligned}
T^*\mathscr{K}T &= \int dx \nabla_x(a^*(c_x)+a(s_x))\,\nabla_x(a(c_x)+a^*(s_x)) \\
&= \int dx\nabla_x a^*(c_x)\nabla_x a(c_x) + \int dx\nabla_x a^*(c_x)\nabla_x a^*(s_x) \\
&\quad + \int dx\nabla_x a(s_x)\nabla_x a(c_x) + \int dx\nabla a^*(s_x)\nabla_x a(s_x) \\
&\quad + \int dx\|\nabla_x s_x\|_2^2.
\end{aligned}$$

Following (2.13), we decompose

$$c_x(y) = \delta(x-y)+p_x(y) \qquad \text{and} \qquad s_x(y) = k(x,y)+r_x(y).$$

Hence

$$
\begin{aligned}
T^*\mathscr{K}T &= \mathscr{K} + \int dxdy\,|\nabla_x \mathrm{sh}_{k_t}(y,x)|^2 \\
&+ \int dx \nabla_x a_x^* a(\nabla_x p_x) + \int dx a^*(\nabla_x p_x)\nabla_x a_x + \int dx a^*(\nabla_x p_x) a(\nabla_x p_x) \\
&+ \int dx \nabla_x a_x^* a^*(\nabla_x k_x) + \int dx a^*(\nabla_x p_x) a^*(\nabla_x k_x) + \int dx \nabla_x a_x^* a^*(\nabla_x r_x) \\
&+ \int dx a^*(\nabla_x p_x) a^*(\nabla_x r_x) + \int dx\, a(\nabla_x k_x)\nabla_x a_x + \int dx a(\nabla_x r_x)\nabla_x a_x \\
&+ \int dx a(\nabla_x k_x) a(\nabla_x p_x) + \int dx a(\nabla_x r_x) a(\nabla_x p_x) \\
&+ \int dx\, a^*(\nabla_x k_x) a(\nabla_x k_x) + \int dx\, a^*(\nabla_x r_x) a(\nabla_x k_x) \\
&+ \int dx\, a^*(\nabla_x k_x) a(\nabla_x r_x) + \int dx\, a^*(\nabla_x r_x) a(\nabla_x r_x).
\end{aligned} \tag{6.5}
$$

The properties of $T^*\mathscr{K}T$ are summarized in the next proposition.

Proposition 6.1. *We have*

$$
\begin{aligned}
&T^*\mathscr{K}T = \int dxdy |\nabla_x sh_{k_t}(y,x)|^2 + \mathscr{K} \\
&+ N^3 \int dxdy (\Delta w)(N(x-y)) \left(\varphi_t^{(N)}(x)\varphi_t^{(N)}(y) a_x^* a_y^* + \overline{\varphi}_t^{(N)}(x)\overline{\varphi}_t^{(N)}(y) a_x a_y\right) \\
&+ \mathscr{E}_K(t)
\end{aligned} \tag{6.6}
$$

where the error $\mathscr{E}_K(t)$ is an operator such that for every $\delta > 0$ there exists a constant $C_\delta > 0$ with

$$
\begin{aligned}
\pm\, \mathscr{E}_K(t) &\le \delta \mathscr{K} + C_\delta e^{K|t|}(\mathscr{N}+1), \\
\pm\, [\mathscr{N}, \mathscr{E}_K(t)] &\le \delta \mathscr{K} + C_\delta e^{K|t|}(\mathscr{N}+1), \\
\pm\, \dot{\mathscr{E}}_K(t) &\le \delta \mathscr{K} + C_\delta e^{K|t|}(\mathscr{N}+1).
\end{aligned} \tag{6.7}
$$

To prove Proposition 6.1, we will use the next lemma.

Lemma 6.2. *Let $j_1, j_2 \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$. Let $j_{i,x}(z) := j_i(z,x)$ for $i = 1,2$. Then we have*

$$
\left| \int dx \langle \psi, a^\sharp(j_{1,x}) a^\sharp(j_{2,x})\psi\rangle \right| \le C\|j_1\|_2 \|j_2\|_2 \,\|(\mathscr{N}+1)^{1/2}\psi\|^2. \tag{6.8}
$$

Here and in the following $a^\sharp$ can be either the annihilation operator a or the creation operator a^*. Moreover, for every $\delta > 0$, there exists $C_\delta > 0$ such that

$$
\begin{aligned}
\left| \int dx \langle \psi, \nabla_x a_x^* a^\sharp(j_{1,x})\psi\rangle \right| &\le \delta \mathscr{K} + C_\delta \|j_1\|_2^2 \|(\mathscr{N}+1)^{1/2}\psi\|^2, \\
\left| \int dx \langle \psi, a^\sharp(j_{1,x}) \nabla_x a_x \psi\rangle \right| &\le \delta \mathscr{K} + C_\delta \|j_1\|_2^2 \,\|(\mathscr{N}+1)^{1/2}\psi\|^2.
\end{aligned} \tag{6.9}
$$

Terms where the argument of a creation and/or annihilation operator is the kernel $\nabla_x k_x$ (whose L^2-norm diverges as $N \to \infty$) can be handled with the following bounds. For every $\delta > 0$ there exists $C_\delta > 0$ such that

$$\begin{aligned} \left| \int dx \langle \psi, a^*(\nabla_x k_x) a^\sharp(j_{1,x}) \psi \rangle \right| &\leq \delta \mathcal{K} + C_\delta (1 + \|j_1\|_2^2) \|(\mathcal{N}+1)^{1/2} \psi\|^2, \\ \left| \int dx \langle \psi, a^\sharp(j_{1,x}) a(\nabla_x k_x) \psi \rangle \right| &\leq \delta \mathcal{K} + C_\delta (1 + \|j_1\|_2^2) \|(\mathcal{N}+1)^{1/2} \psi\|^2. \end{aligned} \tag{6.10}$$

Moreover, we have

$$\left| \int dx \langle \psi, a^*(\nabla_x k_x) a(\nabla_x k_x) \psi \rangle \right| \leq C \|\mathcal{N}^{1/2} \psi\|^2. \tag{6.11}$$

To control the time derivative of $\mathcal{E}_K(t)$, we will also use the following bounds. For every $\delta > 0$ there exists $C_\delta > 0$ such that

$$\left| \int dx \langle \psi, a^*(\nabla_x \dot{k}_x) a^\sharp(j_{1,x}) \psi \rangle \right| \leq \delta \mathcal{K} + C_\delta e^{K|t|} (1 + \|j_1\|_2^2) \, \|(\mathcal{N}+1)^{1/2} \psi\|^2. \tag{6.12}$$

Moreover,

$$\left| \int dx \langle \psi, a^*(\nabla_x \dot{k}_x) a(\nabla_x k_x) \psi \rangle \right| \leq C e^{K|t|} \|\mathcal{N}^{1/2} \psi\|^2. \tag{6.13}$$

Proof. To prove (6.8), we compute

$$\begin{aligned} \left| \int dx \langle \psi, a^\sharp(j_{1,x}) a^\sharp(j_{2,x}) \psi \rangle \right| &\leq \int dx \|a^\sharp(j_{1,x}) \psi\| \, \|a^\sharp(j_{2,x}) \psi\| \\ &\leq \left(\int dx \|j_{1,x}\|_2 \|j_{2,x}\|_2 \right) \|(\mathcal{N}+1)^{1/2} \psi\|^2 \\ &\leq \|j_1\|_2 \|j_2\|_2 \, \|(\mathcal{N}+1)^{1/2} \psi\|^2. \end{aligned}$$

Eq. (6.9), on the other hand, follows by

$$\begin{aligned} \left| \int dx \langle \psi, \nabla_x a_x^* a^\sharp(j_{1,x}) \psi \rangle \right| &\leq \int dx \|\nabla_x a_x \psi\| \, \|a^\sharp(j_{1,x}) \psi\| \\ &\leq \delta \langle \psi, \mathcal{K} \psi \rangle + C_\delta \int dx \|j_{1,x}\|_2^2 \, \|(\mathcal{N}+1)^{1/2} \psi\|^2 \\ &\leq \delta \langle \psi, \mathcal{K} \psi \rangle + C_\delta \|j_1\|_2^2 \, \|(\mathcal{N}+1)^{1/2} \psi\|^2. \end{aligned} \tag{6.14}$$

To show (6.10), we need to integrate by parts. We write

$$\int dx a^*(\nabla_x k_x) a^\sharp(j_{1,x}) = \int dx dy \, \nabla_x k(y,x) \, a_y^* a^\sharp(j_{1,x})$$

and we observe that

$$\begin{aligned}
&\nabla_x k(y,x) \\
&\quad = -\nabla_y k(y,x) - Nw(N(y-x))\big(\nabla\varphi_t^{(N)}(x)\varphi_t^{(N)}(y) + \varphi_t^{(N)}(x)\nabla\varphi_t^{(N)}(y)\big).
\end{aligned}$$

Hence

$$\begin{aligned}
&\int dx a^*(\nabla_x k_x) a^\sharp(j_{1,x}) \\
&\quad = \int dxdy k(x,y)\,\nabla_y a_y^* a^\sharp(j_{1,x}) \\
&\qquad - \int dxdy Nw(N(x-y))\nabla\varphi_t^{(N)}(x)\varphi_t^{(N)}(y)\, a_y^*\, a^\sharp(j_{1,x}) \\
&\qquad - \int dxdy Nw(N(x-y))\nabla\varphi_t^{(N)}(y)\varphi_t^{(N)}(x)\, a_y^*\, a^\sharp(j_{1,x}).
\end{aligned}$$

This implies, using Lemma 3.2 to bound $Nw(N(x-y))$,

$$\begin{aligned}
&\Big|\int dx \langle\psi, a^*(\nabla_x k_x) a^\sharp(j_{1,x})\psi\rangle\Big| \\
&\leq \int dxdy\, |k(x,y)| \|\nabla_y a_y \psi\|\, \|a^\sharp(j_{1,x})\psi\| \\
&\quad + C\int dxdy\, \frac{1}{|x-y|}\big(|\nabla\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)| + |\nabla\varphi_t^{(N)}(y)||\varphi_t^{(N)}(x)|\big) \\
&\qquad\qquad \times \|a_y\psi\|\, \|a^\sharp(j_{1,x})\psi\| \\
&\leq \delta \int dxdy |\varphi_t^{(N)}(x)|^2 \|\nabla_y a_y\psi\|^2 \\
&\quad + C_\delta \int dxdy \frac{1}{|x-y|^2} |\varphi_t^{(N)}(y)|^2 \|j_{1,x}\|_2^2 \|(\mathcal{N}+1)^{1/2}\psi\|^2 \\
&\quad + C\int dxdy \frac{1}{|x-y|^2} |\varphi_t^{(N)}(y)|^2 \|j_{1,x}\|_2^2 \|(\mathcal{N}+1)^{1/2}\psi\|^2 \\
&\quad + C\int dxdy |\nabla\varphi_t^{(N)}(x)|^2 \|a_y\psi\|^2 + C\int dxdy \frac{1}{|x-y|^2} |\varphi_t^{(N)}(x)|^2 \|a_y\psi\|^2 \\
&\quad + C\int dxdy |\nabla\varphi_t^{(N)}(y)|^2 \|j_{1,x}\|_2^2 \|(\mathcal{N}+1)^{1/2}\psi\|^2.
\end{aligned}$$

Using Hardy's inequality, we conclude that for every $\delta > 0$ there exists $C_\delta > 0$ (depending on $\|j_1\|_2$, δ, $\|\varphi_t^{(N)}\|_{H^1}$) such that

$$\Big|\langle\psi, \int dx\, a^*(\nabla_x k_x) a^\sharp(j_{1,x})\psi\rangle\Big| \leq \delta\langle\psi, \mathcal{K}\psi\rangle + C_\delta(1+\|j_1\|_2^2)\,\langle\psi, (\mathcal{N}+1)\psi\rangle.$$

To show (6.11), we write

$$\begin{aligned}\int dx\, a^*(\nabla_x k_x) a(\nabla_x k_x) &= \int dx dy_1 dy_2\, \nabla_x k(y_1,x) \nabla_x \overline{k}(y_2,x)\, a^*_{y_1} a_{y_2} \\ &=: \int dy_1 dy_2\, g(y_1,y_2) a^*_{y_1} a_{y_2}.\end{aligned}$$

We have

$$\begin{aligned}\Big| \int dy_1 dy_2\, g(y_1,y_2) \langle \psi, a^*_{y_1} a_{y_2} \psi \rangle \Big| &\leq \int dy_1 dy_2 |g(y_1,y_2)| \|a_{y_1}\psi\| \, \|a_{y_2}\psi\| \\ &\leq \left(\int dy_1 dy_2 |g(y_1,y_2)|^2 \right)^{1/2} \left(\int dy_1 dy_2 \, \|a_{y_1}\psi\|^2 \, \|a_{y_2}\psi\|^2 \right)^{1/2} \\ &= \|g\|_2 \, \|\mathcal{N}^{1/2}\psi\|^2.\end{aligned}$$

Moreover, from the definition (3.6) of the kernel k_t and from the bounds of Lemma 3.2, we find

$$\begin{aligned}\|g\|_2^2 \leq\; & C \int dy_1 dy_2 dx_1 dx_2 \frac{|\varphi_t^{(N)}(x_1)|^2 |\varphi_t^{(N)}(x_2)|^2 |\varphi_t^{(N)}(y_1)|^2 |\varphi_t^{(N)}(y_2)|^2}{|x_1-y_1|^2 |x_1-y_2|^2 |x_2-y_1|^2 |x_2-y_2|^2} \\ & + C \int dy_1 dy_2 dx_1 dx_2 \frac{|\nabla\varphi_t^{(N)}(x_1)|^2 |\nabla\varphi_t^{(N)}(x_2)|^2 |\varphi_t^{(N)}(y_1)|^2 |\varphi_t^{(N)}(y_2)|^2}{|x_1-y_1| |x_1-y_2| |x_2-y_1| |x_2-y_2|} \\ \leq\; & C \int dy_1 dy_2 dx_1 dx_2 \frac{|\varphi_t^{(N)}(x_1)|^6 |\varphi_t^{(N)}(x_2)|^2}{|x_1-y_1|^2 |x_1-y_2|^2 |x_2-y_1|^2 |x_2-y_2|^2} \\ & + C \int dy_1 dy_2 dx_1 dx_2 \frac{|\nabla\varphi_t^{(N)}(x_1)|^2 |\nabla\varphi_t^{(N)}(x_2)|^2 |\varphi_t^{(N)}(y_1)|^2 |\varphi_t^{(N)}(y_2)|^2}{|x_1-y_1|^2 |x_2-y_2|^2} \\ \leq\; & C \int dx_1 dx_2 \frac{1}{|x_1-x_2|^2} |\varphi_t^{(N)}(x_1)|^6 |\varphi_t^{(N)}(x_2)|^2 \\ & + C \left(\sup_x \int dy \frac{1}{|x-y|^2} |\varphi_t^{(N)}(y)|^2 \right)^2 \|\varphi_t^{(N)}\|_{H^1}^4 \\ \leq\; & C\end{aligned}$$

for a constant C depending only on the H^1-norm of $\varphi_t^{(N)}$. The last two bounds prove (6.11).

The inequalities (6.12) and (6.13) can be proven similarly to (6.10) and (6.11); this time, however, the bounds will contain the norm $\|\dot{\varphi}_t^{(N)}\|_{H^1}$, which is bounded by $Ce^{K|t|}$, as proven in Proposition 3.1. □

Proof of Proposition 6.1. We prove the first bound in (6.7). To this end, we observe that Lemma 6.2 can be used to bound all factors on the r.h.s. of (6.5) (using the uniform estimates for $\|\nabla_1 p(k_t)\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}$, $\|\nabla_1 r_{k_t}\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}$ from Lemma 3.3), with two exceptions, given by the term

$$\int dx \nabla_x a_x^* a^*(\nabla_x k_x) \tag{6.15}$$

and its hermitian conjugate. To control (6.15), we use that, from (3.6),

$$\begin{aligned}\nabla_x k_t(y,x) = &-N^2 \nabla w(N(x-y))\varphi_t^{(N)}(x)\varphi_t^{(N)}(y) \\ &- Nw(N(x-y))\nabla\varphi_t^{(N)}(x)\varphi_t^{(N)}(y).\end{aligned}$$

Hence

$$\begin{aligned}&\int dx \nabla_x a_x^* a^*(\nabla_x k_x) \\ &\quad = -N^2 \int dxdy \nabla w(N(x-y))\varphi_t^{(N)}(x)\varphi_t^{(N)}(y)\nabla_x a_x^* a_y^* \\ &\qquad - N\int dxdy\, w(N(x-y))\nabla\varphi_t^{(N)}(x)\varphi_t^{(N)}(y)\nabla_x a_x^* a_y^*.\end{aligned} \tag{6.16}$$

The last term can be written as

$$\int dx \nabla_x a_x^* a^*(j_x) \tag{6.17}$$

with $j(y,x) = -Nw(N(x-y))\nabla\varphi_t^{(N)}(x)\varphi_t^{(N)}(y)$. Combining Lemma 3.2 and Proposition 3.1, we find that $j \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, with uniformly bounded norm. Hence, Lemma 6.2 implies that, for every $\delta > 0$, there exists $C_\delta > 0$ with

$$\left| \int dx \langle \psi, \nabla_x a_x^* a^*(j_x)\psi\rangle \right| \le \delta \|\mathscr{K}^{1/2}\psi\|^2 + C_\delta \|(\mathscr{N}+1)^{1/2}\psi\|^2. \tag{6.18}$$

The first term on the r.h.s. of (6.16), on the other hand, can be written as

$$\begin{aligned}&-N^2 \int dxdy \nabla w(N(x-y))\varphi_t^{(N)}(x)\varphi_t^{(N)}(y)\nabla_x a_x^* a_y^* \\ &\qquad = N^3 \int dxdy (\Delta w)(N(x-y))\varphi_t^{(N)}(x)\varphi_t^{(N)}(y) a_x^* a_y^* \\ &\qquad\quad + N^2 \int dxdy \nabla w(N(x-y))\nabla\varphi_t^{(N)}(x)\varphi_t^{(N)}(y) a_x^* a_y^*.\end{aligned}$$

The first contribution on the r.h.s. of the last equation is large and appears explicitly on the r.h.s. of (6.6) (it will cancel later, when combined with other terms arising from $\widehat{\mathscr{L}}_{2,N}^{(0)}$ and $\mathscr{L}_{4,N}^{(0)}$). The second term, on the other hand, is an error; integrating by parts, it can be expressed as

$$\begin{aligned}
N^2 \int dxdy \nabla w(N(x-y)) \nabla \varphi_t^{(N)}(x) \varphi_t^{(N)}(y) a_x^* a_y^* \\
= -N \int dxdy w(N(x-y)) \nabla \varphi_t^{(N)}(x) \nabla \varphi_t^{(N)}(y) a_x^* a_y^* \\
- N \int dxdy w(N(x-y)) \nabla \varphi_t^{(N)}(x) \varphi_t^{(N)}(y) a_x^* \nabla_y a_y^* \\
= - \int dx \nabla \varphi_t^{(N)}(x) a_x^* a^*(Nw(N(x-.) \nabla \varphi_t^{(N)}) + \int dy \nabla_y a_y^* a^*(j_y)
\end{aligned}$$

with

$$j(x,y) = -Nw(N(x-y)) \nabla \varphi_t^{(N)}(x) \varphi_t^{(N)}(y).$$

The second term is bounded as in (6.18). The first term, on the other hand, is estimated by

$$\begin{aligned}
\left| \int dx \nabla \varphi_t^{(N)}(x) \langle a_x \psi, a^*(Nw(N(x-\cdot)) \nabla \varphi_t^{(N)}) \psi \rangle \right| \\
\leq C \sup_x \|Nw(N(x-.)) \nabla \varphi_t^{(N)}\|_2 \|(\mathcal{N}+1)^{1/2} \psi\|^2 \\
\leq C \|\varphi_t^{(N)}\|_{H^2} \|(\mathcal{N}+1)^{1/2} \psi\|^2
\end{aligned} \tag{6.19}$$

and (3.2).

Also the second bound in (6.7) follows from Lemma 6.2. In fact, when one takes the commutator of $\mathcal{N}$ with the terms on the r.h.s. of (6.5) one either finds zero (for all terms with one creation and one annihilation operators, which therefore preserve the number of particles), or one finds again the same terms (up to a possible change of sign). This follows because, by the canonical commutation relations

$$[\mathcal{N}, a(f)] = -a(f)$$

and

$$[\mathcal{N}, a^*(f)] = a^*(f)$$

for every $f \in L^2(\mathbb{R}^3)$. Finally, the third bound in (6.7) is a consequence of Lemma 6.2 as well. In fact, the time derivative $\dot{\mathcal{E}}_K(t)$ is a sum of terms very similar to the terms appearing on the r.h.s. of (6.5), with the difference that one of the appearing kernels contains a time-derivative. Combining the estimates from Lemma 6.2 (including, in this case, also (6.12) and (6.13)) with the bounds for $\|\nabla_1 \dot{p}_{k_t}\|_2$, $\|\nabla_1 \dot{r}_{k_t}\|_2$ from Lemma 3.4 and with the bound for $\|\dot{\varphi}_t^{(N)}\|_{H^1}$ from Proposition 3.1 (needed to control terms similar to (6.16) and (6.19), with a factor of $\varphi_t^{(N)}$ replaced by $\dot{\varphi}_t^{(N)}$), we obtain the last inequality in (6.7). □

Properties of $T^* \widehat{\mathscr{L}}^{(0)}_{2,N}(t) T$

We consider now the other quadratic terms, collected in $\widehat{\mathscr{L}}^{(0)}_{2,N}(t)$, defined by $\mathscr{L}^{(0)}_{2,N}(t) = \mathscr{K} + \widehat{\mathscr{L}}^{(0)}_{2,N}(t)$. We have

$$\begin{aligned}
&T^* \widehat{\mathscr{L}}^{(0)}_{2,N}(t) T \\
&= \int dx \big(N^3 V(N\cdot) * |\varphi_t^{(N)}|^2\big)(x) \\
&\qquad \times (a^*(c_x) + a(s_x))(a(c_x) + a^*(s_x)) \\
&\quad + \int dxdy N^3 V(N(x-y)) \varphi_t^{(N)}(x) \overline{\varphi}_t^{(N)}(y) \\
&\qquad \times (a^*(c_x) + a(s_x))(a(c_y) + a^*(s_y)) \\
&\quad + \frac{1}{2} \int dxdy N^3 V(N(x-y)) \\
&\qquad \times \big[\varphi_t^{(N)}(x) \varphi_t^{(N)}(y) (a^*(c_x) + a(s_x))(a^*(c_y) + a(s_y)) + \text{h.c.}\big].
\end{aligned}$$

Expanding the products, and bringing all terms to normal order, we find

$$\begin{aligned}
&T^* \widehat{\mathscr{L}}^{(0)}_{2,N}(t) T \\
&= \int dx \big(N^3 V(N\cdot) * |\varphi_t^{(N)}|^2\big)(x) \\
&\quad \times [a^*(c_x)a(c_x) + a^*(s_x)a(s_x) + a^*(c_x)a^*(s_x) + a(s_x)a(c_x) + \langle s_x, s_x \rangle] \\
&+ \int dxdy N^3 V(N(x-y)) \varphi_t^{(N)}(x) \overline{\varphi}_t^{(N)}(y) \\
&\quad \times [a^*(c_x)a(c_y) + a^*(s_y)a(s_x) + a^*(c_x)a^*(s_y) + a(s_x)a(c_y) + \langle s_x, s_y \rangle] \\
&+ \frac{1}{2} \int dxdy N^3 V(N(x-y)) \varphi_t^{(N)}(x) \varphi_t^{(N)}(y) \\
&\quad \times [a^*(c_x)a^*(c_y) + a^*(c_x)a(s_y) + a^*(c_y)a(s_x) + a(s_x)a(s_y) + \langle s_x, c_y \rangle] \\
&+ \frac{1}{2} \int dxdy N^3 V(N(x-y)) \overline{\varphi}_t^{(N)}(x) \overline{\varphi}_t^{(N)}(y) \\
&\quad \times [a(c_x)a(c_y) + a^*(s_y)a(c_x) + a^*(s_x)a(c_y) + a^*(s_x)a^*(s_y) + \langle c_y, s_x \rangle] .
\end{aligned} \tag{6.20}$$

The properties of $T^* \widehat{\mathscr{L}}^{(0)}_{2,N}(t) T$ are summarized in the following proposition.

Proposition 6.3. *We have*

$$
\begin{aligned}
&T^* \widehat{\mathscr{L}}^{(0)}_{2,N}(t) T \\
&= \int dx (N^3 V(N\cdot) * |\varphi_t^{(N)}|^2)(x) \langle s_x, s_x \rangle \\
&\quad + \int dxdy N^3 V(N(x-y)) \varphi_t^{(N)}(x) \overline{\varphi}_t^{(N)}(y) \langle s_x, s_y \rangle \\
&\quad + \mathrm{Re} \int dxdy N^3 V(N(x-y)) \varphi_t^{(N)}(x) \varphi_t^{(N)}(y) \langle s_x, c_y \rangle \\
&\quad + \frac{1}{2} \int dxdy N^3 V(N(x-y)) \\
&\qquad \times \left(\varphi_t^{(N)}(x) \varphi_t^{(N)}(y)\, a_x^* a_y^* + \overline{\varphi}_t^{(N)}(x) \overline{\varphi}_t^{(N)}(y)\, a_x a_y \right) \\
&\quad + \mathscr{E}_2(t)
\end{aligned}
\tag{6.21}
$$

where the error $\mathscr{E}_2(t)$ *is such that, for appropriate constants* $C, K > 0$*,*

$$
\begin{aligned}
\pm \mathscr{E}_2(t) &\leq C e^{K|t|} (\mathscr{N} + 1), \\
\pm [\mathscr{N}, \mathscr{E}_2(t)] &\leq C e^{K|t|} (\mathscr{N} + 1), \\
\pm \dot{\mathscr{E}}_2(t) &\leq C e^{K|t|} (\mathscr{N} + 1).
\end{aligned}
\tag{6.22}
$$

To show Proposition 6.3, we will make use of the next lemma.

Lemma 6.4. *Let* $j_1, j_2 \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$*. Let* $j_{i,x}(z) := j_i(z,x)$ *for* $i = 1,2$*. Then there exists a constant* C *such that*

$$
\begin{aligned}
&\int dxdy N^3 V(N(x-y)) |\varphi_t^{(N)}(x)| |\varphi_t^{(N)}(y)| \| a^\sharp(j_{1,x}) \psi \| \, \| a^\sharp(j_{2,y}) \psi \| \\
&\qquad \leq C \| \varphi_t^{(N)} \|_{H^2}^2 \, \| j_1 \|_2 \, \| j_2 \|_2 \, \| (\mathscr{N} + 1)^{1/2} \psi \|^2.
\end{aligned}
\tag{6.23}
$$

Moreover,

$$
\begin{aligned}
&\int dxdy N^3 V(N(x-y)) |\varphi_t^{(N)}(x)| |\varphi_t^{(N)}(y)| \| a^\sharp(j_{1,x}) \psi \| \| a_y \psi \| \\
&\qquad \leq C \| \varphi_t^{(N)} \|_{H^2}^2 \, \| j_1 \|_2 \| (\mathscr{N} + 1)^{1/2} \psi \|^2
\end{aligned}
\tag{6.24}
$$

and

$$
\begin{aligned}
&\int dxdy N^3 V(N(x-y)) |\varphi_t^{(N)}(x)| |\varphi_t^{(N)}(y)| \, \| a_x \psi \| \, \| a_y \psi \| \\
&\qquad \leq C \| \varphi_t^{(N)} \|_{H^2}^2 \, \| (\mathscr{N} + 1)^{1/2} \psi \|^2.
\end{aligned}
\tag{6.25}
$$

The bounds remain true if both creation and/or annihilation operators act on the same variable, in the sense that

$$\begin{aligned}
&\int dxdyN^3V(N(x-y))|\varphi_t^{(N)}(x)|^2\|a^\sharp(j_{1,y})\psi\|\,\|a^\sharp(j_{2,y})\psi\| \\
&\qquad \leq C\|\varphi_t^{(N)}\|_{H^2}^2\|j_1\|_2\|j_2\|_2\,\|(\mathcal{N}+1)^{1/2}\psi\|^2, \\
&\int dxdyN^3V(N(x-y))|\varphi_t^{(N)}(x)|^2\|a^\sharp(j_{1,y})\psi\|\,\|a_y\psi\| \\
&\qquad \leq C\|\varphi_t^{(N)}\|_{H^2}^2\|j_1\|_2\|(\mathcal{N}+1)^{1/2}\psi\|^2, \\
&\int dxdyN^3V(N(x-y))|\varphi_t^{(N)}(x)|^2\|a_y\psi\|^2 \\
&\qquad \leq C\|\varphi_t^{(N)}\|_{H^2}^2\|\mathcal{N}^{1/2}\psi\|^2.
\end{aligned} \tag{6.26}$$

Proof. To prove (6.23), we notice that, for any $\alpha > 0$,

$$\begin{aligned}
&\int dxdyN^3V(N(x-y))|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)|\|a^\sharp(j_{1,x})\psi\|\,\|a^\sharp(j_{2,y})\psi\| \\
&\quad \leq \alpha\int dxdyN^3V(N(x-y))|\varphi_t^{(N)}(x)|^2\|j_{2,y}\|_2^2\|(\mathcal{N}+1)^{1/2}\psi\|^2 \\
&\qquad + \alpha^{-1}\int dxdyN^3V(N(x-y))|\varphi_t^{(N)}(y)|^2\|j_{1,x}\|_2^2\|(\mathcal{N}+1)^{1/2}\psi\|^2 \\
&\quad \leq \|N^3V(N\cdot)*|\varphi_t^{(N)}|^2\|_\infty\left(\alpha\|j_1\|_2^2+\alpha^{-1}\|j_2\|_2^2\right)\,\|(\mathcal{N}+1)^{1/2}\psi\|^2.
\end{aligned}$$

Using $\|N^3V(N\cdot)*|\varphi_t^{(N)}|^2\|_\infty \leq C\|\varphi_t^{(N)}\|_{H^2}^2$, and optimizing over $\alpha > 0$, we obtain (6.23). Analogously, (6.24) follows from

$$\begin{aligned}
&\int dxdyN^3V(N(x-y))|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)|\|a^\sharp(j_{1,x})\psi\|\|a_y\psi\| \\
&\leq \alpha\int dxdyN^3V(N(x-y))|\varphi_t^{(N)}(x)|^2\|a_y\psi\|^2 \\
&\quad + \alpha^{-1}\int dxdyN^3V(N(x-y))|\varphi_t^{(N)}(y)|^2\|j_{1,x}\|_2^2\|(\mathcal{N}+1)^{1/2}\psi\|^2 \\
&\leq \|N^3V(N\cdot)*|\varphi_t^{(N)}|^2\|_\infty(\alpha+\alpha^{-1}\|j_1\|_2^2)\|(\mathcal{N}+1)^{1/2}\psi\|^2
\end{aligned}$$

for any $\alpha > 0$. Optimizing over α gives (6.24). Eq. (6.25) follows from Cauchy-Schwarz. The bounds in (6.26) can be shown similarly. □

Proof of Proposition 6.3. To prove the first bound in (6.22), we notice that the quadratic terms on the r.h.s. of (6.20) can be controlled with Lemma 6.4, decomposing, if needed, $a(c_x) = a_x + a(p_x)$ and then applying (6.23), (6.24), or (6.25). There are two exceptions, given by the terms proportional to $a^*(c_x)a^*(c_y)$ and its hermitian conjugate, proportional to $a(c_x)a(c_y)$. For these two terms the bounds from Lemma

6.4 do not apply. Instead, using $a^*(c_x) = a^*_x + a^*(p_x)$, we write

$$\begin{aligned} \int dxdy N^3 V(N(x-y))\varphi_t^{(N)}(x)\varphi_t^{(N)}(y) a^*(c_x)a^*(c_y) \\ = \int dxdy N^3 V(N(x-y))\varphi_t^{(N)}(x)\varphi_t^{(N)}(y) a^*_x a^*_y \\ + \int dxdy N^3 V(N(x-y))\varphi_t^{(N)}(x)\varphi_t^{(N)}(y) a^*(p_x)a^*_y \\ + \int dxdy N^3 V(N(x-y))\varphi_t^{(N)}(x)\varphi_t^{(N)}(y) a^*(c_x)a^*(p_y). \end{aligned} \tag{6.27}$$

The contribution of the last two terms can be bounded by Lemma 6.4, because one of the arguments of the creation operators is square integrable. In fact

$$\begin{aligned} \Big| \int dxdy N^3 V(N(x-y))\varphi_t^{(N)}(x)\varphi_t^{(N)}(y) \langle \psi, a^*(p_x)a^*_y \psi\rangle \Big| \\ \leq \int dxdy N^3 V(N(x-y))|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)| \|a_y\psi\| \, \|a^*(p_x)\psi\| \\ \leq C\|\varphi_t^{(N)}\|^2_{H^2} \|(\mathcal{N}+1)^{1/2}\psi\|^2 \\ \leq Ce^{K|t|}\|(\mathcal{N}+1)^{1/2}\psi\|^2 \end{aligned}$$

by (6.24) and (3.2), and similarly for the last term on the r.h.s. of (6.27). The hermitian conjugate of (6.27), proportional to $a(c_x)a(c_y)$, can be handled identically.

The second bound in (6.22) follows similarly, using the fact that the commutator of $\mathcal{N}$ with the terms on the r.h.s. of (6.20) leaves their form unchanged (apart from the constant terms and the quadratic terms with one creation and one annihilation operators, whose contribution to the commutator $[\mathcal{N}, \mathcal{E}_2(t)]$ vanishes).

Also the third bound in (6.22) can be proven analogously, using the bounds for $\|\dot{\text{sh}}_{k_t}\|_2$ and $\|\dot{p}_{k_t}\|_2$, as proven in Lemma 3.4 . When the time derivative hits the factor $\varphi_t^{(N)}(x)$ or $\varphi_t^{(N)}(y)$, it generates a contribution which is bounded by $\|\dot{\varphi}_t^{(N)}\|_{H^2} \leq C\|\varphi_t^{(N)}\|^4_{H^4} \leq Ce^{K|t|}$ (for some K depending only on $\|\varphi\|_{H^1}$; here we used Proposition 3.1). □

6.3 Analysis of $T^*\mathscr{L}_{3,N}^{(0)}(t)T$

We consider now the contributions arising from cubic terms in $\mathscr{L}_{3,N}^{(0)}(t)$. We have

$$\begin{aligned}
&T^*\mathscr{L}_{3,N}^{(0)}(t)T\\
&=\frac{1}{\sqrt{N}}\int dxdy N^3 V(N(x-y))\\
&\qquad\times\left[\varphi_t^{(N)}(y)\,(a^*(c_x)+a(s_x))(a^*(c_y)+a(s_y))(a(c_x)+a^*(s_x))+\text{h.c.}\right]\\
&=\frac{1}{\sqrt{N}}\int dxdy N^3 V(N(x-y))\varphi_t^{(N)}(y)\\
&\qquad\times\left[a^*(c_x)a^*(c_y)a^*(s_x)+a^*(c_x)a^*(c_y)a(c_x)+a^*(c_x)a(s_y)a^*(s_x)\right.\\
&\qquad\quad+a^*(c_x)a(s_y)a(c_x)+a(s_x)a^*(c_y)a^*(s_x)+a(s_x)a^*(c_y)a(c_x)\\
&\qquad\quad\left.+a(s_x)a(s_y)a^*(s_x)+a(s_x)a(s_y)a(c_x)\right]\\
&\quad+\text{h.c.}
\end{aligned}$$

Writing the terms in normal order, we find

$$\begin{aligned}
&T^*\mathscr{L}_{3,N}^{(0)}(t)T\\
&=\frac{1}{\sqrt{N}}\int dxdy N^3 V(N(x-y))\varphi_t^{(N)}(y)\\
&\qquad\times\left[a^*(c_x)a^*(c_y)a^*(s_x)+a^*(c_x)a^*(c_y)a(c_x)+a^*(c_x)a^*(s_x)a(s_y)\right.\\
&\qquad\quad+a^*(c_x)a(s_y)a(c_x)+a^*(c_y)a^*(s_x)a(s_x)+a^*(c_y)a(s_x)a(c_x)\\
&\qquad\quad\left.+a^*(s_x)a(s_x)a(s_y)+a(s_x)a(s_y)a(c_x)\right]\\
&\quad+\frac{1}{\sqrt{N}}\int dxdy N^3 V(N(x-y))\varphi_t^{(N)}(y)\\
&\qquad\times\left[\langle s_y,s_x\rangle(a^*(c_x)+a(s_x))+\langle s_x,c_y\rangle\,(a(c_x)+a^*(s_x))\right.\\
&\qquad\quad\left.+\langle s_x,s_x\rangle\,(a^*(c_y)+a(s_y))\right]\\
&\quad+\text{h.c.}
\end{aligned}\tag{6.28}$$

The properties of $T^*\mathscr{L}_{3,N}^{(0)}(t)T$ are summarized in the following proposition.

Proposition 6.5. *We have*

(6.29)

$$\begin{aligned} &T^* \mathscr{L}_{3,N}^{(0)} T \\ &= \frac{1}{\sqrt{N}} \int dxdy N^3 V(N(x-y)) \\ &\qquad \times \left[\varphi_t^{(N)}(y) \bar{k}_t(x,y) \left(a(c_x) + a^*(s_x) \right) + \overline{\varphi}_t^{(N)}(y) k_t(x,y) \left(a^*(c_x) + a(s_x) \right) \right] \\ &\quad + \mathscr{E}_3(t) \\ &= -\sqrt{N} \int dxdy N^3 V(N(x-y)) \\ &\qquad \times w(N(x-y)) |\varphi_t^{(N)}(y)|^2 \varphi_t^{(N)}(x) (a(c_x) + a^*(s_x)) + h.c. \\ &\quad + \mathscr{E}_3(t) \end{aligned}$$

where we used the definition (3.6) of the kernel k_t and where the error term $\mathscr{E}_3(t)$ is such that for every $\delta > 0$ there exists a constant $C_\delta > 0$ with

(6.30)

$$\begin{aligned} \pm \mathscr{E}_3(t) &\leq \delta \int dxdy N^2 V(N(x-y)) a_x^* a_y^* a_x a_y + \delta \frac{\mathscr{N}^2}{N} + C_\delta e^{K|t|} (\mathscr{N}+1), \\ \pm \left[\mathscr{N}, \mathscr{E}_3(t) \right] &\leq \delta \int dxdy N^2 V(N(x-y)) a_x^* a_y^* a_x a_y + \delta \frac{\mathscr{N}^2}{N} + C_\delta e^{K|t|} (\mathscr{N}+1), \\ \pm \dot{\mathscr{E}}_3(t) &\leq \delta \int dxdy N^2 V(N(x-y)) a_x^* a_y^* a_x a_y + \delta \frac{\mathscr{N}^2}{N} + C_\delta e^{K|t|} (\mathscr{N}+1). \end{aligned}$$

Notice here that the first term on the r.h.s. of (6.29) cancels exactly with the contribution (6.4); we will make use of this crucial observation in the proof of Theorem 3.5 below.

Proof. To bound the cubic terms on the r.h.s. of (6.28), we systematically apply Cauchy-Schwarz. This way, we control cubic terms by quartic and quadratic contributions, which are then estimated making use of Lemma 6.4 (the quadratic part) and Lemmas 6.7 and 6.8 (the quartic part). For example,

$$\begin{aligned} &\left| \frac{1}{\sqrt{N}} \int dxdy N^3 V(N(x-y)) \varphi_t^{(N)}(y) \langle \psi, a^*(c_x) a^*(c_y) a^*(s_x) \psi \rangle \right| \\ &\qquad \leq \frac{1}{\sqrt{N}} \int dxdy N^3 V(N(x-y)) |\varphi_t^{(N)}(y)| \|a(c_x) a(c_y) \psi\| \, \|a^*(s_x) \psi\| \\ &\qquad \leq \frac{\delta}{N} \int dxdy N^3 V(N(x-y)) \|a(c_x) a(c_y) \psi\|^2 \\ &\qquad\quad + C_\delta \int dxdy N^3 V(N(x-y)) |\varphi_t^{(N)}(y)|^2 \|a^*(s_x) \psi\|^2 \\ &\qquad \leq \frac{C\delta}{N} \int dxdy N^3 V(N(x-y)) \|a_y a_x \psi\|^2 + C_\delta \|(\mathscr{N}+1)^{1/2} \psi\|^2 \end{aligned}$$

where, in the last line, we used (6.38) (from Lemma 6.8) and (6.26) (from Lemma 6.4). All other cubic terms can be bounded similarly. We always separate the three creation and/or annihilation operators putting a small weight δ in front of the quartic term and in such a way that, in the resulting quartic contribution, two operators depend on the x and two on the y variable. The corresponding quadratic term depends on x and can always be bounded by (6.26). It should be noted that the quartic contribution has either the form $\|a(c_x)a(c_y)\psi\|^2$ or $\|a(c_x)a^\sharp(j_y)\psi\|^2$ or $\|a^\sharp(j_{1,x})a^\sharp(j_{2,y})\psi\|^2$, with square-integrable arguments j_1, j_2 (here $a^\sharp$ is either a or a^*). These terms can always be controlled using Lemma 6.7 or Lemma 6.8. As for the linear contributions on the r.h.s. of (6.28), the first and third can simply be bounded by $N^{-1/2}(\mathcal{N}+1)^{1/2}$, since

$$|\langle s_y, s_x\rangle| \leq C|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)|.$$

To bound the second linear term, we write

$$\begin{aligned}\frac{1}{\sqrt{N}}\int dxdyN^3V(N(x-y))\varphi_t^{(N)}(y)\langle s_x, c_y\rangle(a(c_x)+a^*(s_x))\\ = \frac{1}{\sqrt{N}}\int dxdyN^3V(N(x-y))\varphi_t^{(N)}(y)\overline{k}_t(x,y)\,(a(c_x)+a^*(s_x))+\widetilde{\mathcal{E}}(t)\end{aligned}$$

where $\pm\widetilde{\mathcal{E}}(t) \leq N^{-1/2}(\mathcal{N}+1)^{1/2}$ because, using Lemma 3.3,

$$|\langle s_x, c_y\rangle - \overline{k}_t(x,y)| \leq C|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)|.$$

From (3.2), this concludes the proof of the first estimate in (6.30). The other two estimates are proven analogously, using the fact that the commutators of $\mathcal{N}$ with the terms on the r.h.s. of (6.28) have the same form as the terms on the r.h.s. of (6.28) (with possibly just a different sign), and using the bounds for $\|\dot{\text{sh}}_{k_t}\|_2$ and $\|\dot{p}_{k_t}\|_2$ from Lemma 3.4, and the bound for $\|\dot{\varphi}_t^{(N)}\|_{H^2}$ from Proposition 3.1. □

6.4 Analysis of $T^*\mathcal{L}_{4,N}^{(0)}T$

We consider next the contributions arising from the quartic part $\mathcal{L}_{4,N}^{(0)}$ of $\mathcal{L}_N^{(0)}$. We have

$$\begin{aligned}T^*\mathcal{L}_{4,N}^{(0)}(t)T =& \frac{1}{2N}\int dxdyN^3V(N(x-y))\\ &\times(a^*(c_x)+a(s_x))(a^*(c_y)+a(s_y))(a(c_y)+a^*(s_y))(a(c_x)+a^*(s_x)).\end{aligned}$$

Expanding the products, we find

$$\begin{aligned}
&2T^*\mathscr{L}^{(0)}_{4,N}(t)T \\
&= \int dxdy N^2 V(N(x-y))\big[a^*(c_x)a^*(c_y)a^*(s_y)a^*(s_x) + a^*(c_x)a^*(c_y)a^*(s_y)a(c_x) \\
&\quad + a^*(c_x)a^*(c_y)a(c_y)a^*(s_x) + a^*(c_x)a^*(c_y)a(c_y)a(c_x) + a^*(c_x)a(s_y)a^*(s_y)a^*(s_x) \\
&\quad + a^*(c_x)a(s_y)a^*(s_y)a(c_x) + a^*(c_x)a(s_y)a(c_y)a^*(s_x) + a^*(c_x)a(s_y)a(c_y)a(c_x) \\
&\quad + a(s_x)a^*(c_y)a^*(s_y)a^*(s_x) + a(s_x)a^*(c_y)a^*(s_y)a(c_x) + a(s_x)a^*(c_y)a(c_y)a^*(s_x) \\
&\quad + a(s_x)a^*(c_y)a(c_y)a(c_x) + a(s_x)a(s_y)a^*(s_y)a^*(s_x) + a(s_x)a(s_y)a^*(s_y)a(c_x) \\
&\quad + a(s_x)a(s_y)a(c_y)a^*(s_x) + a(s_x)a(s_y)a(c_y)a(c_x)\big].
\end{aligned}$$

Writing all terms in normal order, we obtain

(6.31)

$$\begin{aligned}
&2T^*\mathscr{L}^{(0)}_{4,N}(t)T \\
&= \int dxdy N^2 V(N(x-y))\big[a^*(c_x)a^*(c_y)a^*(s_y)a^*(s_x) + a^*(c_x)a^*(c_y)a^*(s_y)a(c_x) \\
&\quad + a^*(c_x)a^*(c_y)a^*(s_x)a(c_y) + a^*(c_x)a^*(c_y)a(c_y)a(c_x) + a^*(c_x)a^*(s_y)a^*(s_x)a(s_y) \\
&\quad + a^*(c_x)a^*(s_y)a(s_y)a(c_x) + a^*(c_x)a^*(s_x)a(s_y)a(c_y) + a^*(c_x)a(s_y)a(c_y)a(c_x) \\
&\quad + a^*(c_y)a^*(s_y)a^*(s_x)a(s_x) + a^*(c_y)a^*(s_y)a(s_x)a(c_x) + a^*(c_y)a^*(s_x)a(s_x)a(c_y) \\
&\quad + a^*(c_y)a(s_x)a(c_y)a(c_x) + a^*(s_y)a^*(s_x)a(s_x)a(s_y) + a^*(s_y)a(s_x)a(s_y)a(c_x) \\
&\quad + a^*(s_x)a(s_x)a(s_y)a(c_y) + a(s_x)a(s_y)a(c_y)a(c_x)\big] \\
&+ \int dxdy N^2 V(N(x-y))\big[\langle c_y, s_x\rangle a^*(c_x)a^*(c_y) + \langle s_x, c_y\rangle a(c_y)a(c_x) \\
&\quad + 2\langle s_y, s_y\rangle a^*(c_x)a^*(s_x) + 2\langle s_y, s_y\rangle a(s_x)a(c_x) + 2\langle s_y, s_x\rangle a^*(c_x)a^*(s_y) \\
&\quad + 2\langle s_x, s_y\rangle a(s_y)a(c_x) + 2\langle s_y, s_y\rangle a^*(c_x)a(c_x) + 2\langle s_y, s_x\rangle a^*(c_x)a(c_y) \\
&\quad + 2\langle s_x, s_y\rangle a^*(s_x)a(s_y) + 2\langle s_y, s_y\rangle a^*(s_x)a(s_x) + \langle c_y, s_x\rangle a^*(c_x)a(s_y) \\
&\quad + \langle s_x, c_y\rangle a^*(s_y)a(c_x) + \langle s_x, c_y\rangle a^*(s_y)a^*(s_x) + \langle c_y, s_x\rangle a(s_x)a(s_y) \\
&\quad + \langle s_x, c_y\rangle a^*(s_x)a(c_y) + \langle c_y, s_x\rangle a^*(c_y)a(s_x)\big] \\
&+ \int dxdy N^2 V(N(x-y))\big[|\langle s_x, c_y\rangle|^2 + |\langle s_x, s_y\rangle|^2 + \langle s_y, s_y\rangle\langle s_x, s_x\rangle\big].
\end{aligned}$$

The properties of $T^*\mathscr{L}^{(0)}_{4,N}T$ are summarized in the next proposition.

Proposition 6.6. *We have*

$$\begin{aligned}
&2T^*\mathscr{L}^{(0)}_{4,N}(t)T \\
&= \int dxdyN^2V(N(x-y))\left[|\langle s_x, c_y\rangle|^2 + |\langle s_x, s_y\rangle|^2 + \langle s_y, s_y\rangle\langle s_x, s_x\rangle\right] \\
&\quad + \int dxdyN^2V(N(x-y))a_x^*a_y^*a_ya_x \\
&\quad + \int dxdyN^2V(N(x-y))\left(k(x,y)a_x^*a_y^* + \bar{k}(x,y)a_xa_y\right) + \mathscr{E}_4(t)
\end{aligned} \tag{6.32}$$

where the error $\mathscr{E}_4(t)$ is such that, for every $\delta > 0$, there exists a constant $C_\delta > 0$ with

$$\begin{aligned}
&\pm\mathscr{E}_4(t) \le \delta\int dxdyN^2V(N(x-y))a_x^*a_y^*a_ya_x + C_\delta\frac{\mathscr{N}^2}{N} + C_\delta e^{K|t|}(\mathscr{N}+1), \\
&\pm[\mathscr{N}, \mathscr{E}_4(t)] \\
&\qquad \le \delta\int dxdyN^2V(N(x-y))a_x^*a_y^*a_ya_x + C_\delta\frac{\mathscr{N}^2}{N} + C_\delta e^{K|t|}(\mathscr{N}+1), \\
&\pm\dot{\mathscr{E}}_4(t) \le \delta\int dxdyN^2V(N(x-y))a_x^*a_y^*a_ya_x + C_\delta e^{K|t|}\left(\frac{\mathscr{N}^2}{N} + \mathscr{N} + 1\right).
\end{aligned} \tag{6.33}$$

To prove Proposition 6.6, we will make use of the following two lemmas.

Lemma 6.7. *Suppose $j_1, j_2 \in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ are kernels with the property that*

$$M_i := \max\left(\sup_x\int dy\,|j_i(x,y)|^2, \sup_y\int dx\,|j_i(x,y)|^2\right) < \infty$$

for $i = 1, 2$. Let $j_{i,x}(z) := j_i(z,x)$ and recall the definition $c_x(z) = ch_{k_t}(z,x)$ from (6.3). Then there exists a constant C depending only on M_1, M_2 and on the L^2-norms $\|j_1\|_2$, $\|j_2\|_2$ such that

$$\int dxdyN^3V(N(x-y))\|a^\sharp(j_{1,x})a_y\psi\|^2 \le CM_1\|(\mathscr{N}+1)\psi\|^2 \tag{6.34}$$

and

$$\begin{aligned}
&\int dxdyN^3V(N(x-y))\|a^\sharp(j_{1,x})a^\sharp(j_{2,y})\psi\|^2 \\
&\qquad\qquad \le C\min\left(M_1\|j_2\|_2^2, M_2\|j_1\|_2^2\right)\|(\mathscr{N}+1)\psi\|^2.
\end{aligned} \tag{6.35}$$

As a consequence

$$\int dxdyN^3V(N(x-y))\|a^\sharp(j_{1,x})a(c_y)\psi\|^2 \le CM_1\|(\mathscr{N}+1)\psi\|^2. \tag{6.36}$$

The inequalities remain true (and are easier to prove) if both operators act on the same variable. In other words

$$\begin{aligned}
&\int dxdy N^3 V(N(x-y)) \|a^\sharp(j_{1,x}) a_x \psi\|^2 \leq C M_1 \|(\mathcal{N}+1)\psi\|^2, \\
&\int dxdy N^3 V(N(x-y)) \|a^\sharp(j_{1,x}) a^\sharp(j_{2,x}) \psi\|^2 \\
&\qquad \leq C \min(M_1 \|j_2\|_2^2, M_2 \|j_1\|_2^2) \|(\mathcal{N}+1)\psi\|^2, \\
&\int dxdy N^3 V(N(x-y)) \|a^\sharp(j_{1,x}) a(c_y) \psi\|^2 \leq C M_1 \|(\mathcal{N}+1)\psi\|^2.
\end{aligned} \tag{6.37}$$

Here $a^\sharp$ is either the annihilation operator a or the creation operator a^*.

Proof. To prove (6.34), we observe that

$$\begin{aligned}
&\int dxdy N^3 V(N(x-y)) \|a^\sharp(j_{1,x}) a_y \psi\|^2 \\
&\qquad \leq \int dxdy N^3 V(N(x-y)) \|j_{1,x}\|_2^2 \|(\mathcal{N}+1)^{1/2} a_y \psi\|^2 \\
&\qquad \leq M_1 \int dxdy N^3 V(N(x-y)) \|a_y \mathcal{N}^{1/2} \psi\|^2 \\
&\qquad = C M_1 \|\mathcal{N}\psi\|^2.
\end{aligned}$$

As for (6.35), we notice that (considering for example the case $a^\sharp(j_{2,y}) = a^*(j_{2,y})$)

$$\begin{aligned}
&\int dxdy N^3 V(N(x-y)) \|a^\sharp(j_{1,x}) a^*(j_{2,y}) \psi\|^2 \\
&\qquad \leq \int dxdy N^3 V(N(x-y)) \|j_{1,x}\|_2 \|(\mathcal{N}+1)^{1/2} a^*(j_{2,y}) \psi\|^2, \\
&\int dxdy N^3 V(N(x-y)) \|j_{1,x}\|_2^2 \|a^*(j_{2,y}) (\mathcal{N}+2)^{1/2} \psi\|^2 \\
&\qquad \leq \int dxdy N^3 V(N(x-y)) \|j_{1,x}\|_2^2 \|j_{2,y}\|_2^2 \|(\mathcal{N}+1)^{1/2} (\mathcal{N}+2)^{1/2} \psi\|^2 \\
&\qquad \leq C M_1 \|j_2\|_2^2 \|(\mathcal{N}+1)\psi\|^2.
\end{aligned}$$

Eq. (6.36) follows from the first two, by writing $a(c_y) = a_y + a(p_y)$ (recall here that we are using the notation $p_y(z) = p(k_t)(z,y)$ with the kernel $p(k_t) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ defined in Lemma 3.3). Eq. (6.37) follows similarly; in this case, however, one can immediately integrate over the variable y, simplifying the proof. □

Terms of the form (6.36), but with $j_{1,x}$ replaced by c_x (which is not in L^2) are treated differently.

Lemma 6.8. *Recall the definition $c_x(z) = ch_{k_t}(z,x)$ from (6.3). Then there exists a constant $C > 0$ with*

$$\int dxdyN^3V(N(x-y))\|a(c_x)a(c_y)\psi\|^2 \leq C\int dxdyN^3V(N(x-y))\|a_xa_y\psi\|^2 + C\|(\mathcal{N}+1)\psi\|^2. \tag{6.38}$$

More precisely, we have

$$\int dxdyN^3V(N(x-y))\|a(c_x)a(c_y)\psi\|^2 = \int dxdyN^3V(N(x-y))\|a_xa_y\psi\|^2 + \widetilde{\mathcal{E}}(t) \tag{6.39}$$

where the error $\widetilde{\mathcal{E}}(t)$ is such that, for every $\delta > 0$, there exists a constant C_δ with

$$\pm\widetilde{\mathcal{E}}(t) \leq \delta\int dxdyN^3V(N(x-y))\|a_xa_y\psi\|^2 + C_\delta\|(\mathcal{N}+1)\psi\|^2.$$

Proof. We write $a(c_x) = a_x + a(p_x)$, using the notation $p_x(z) = p(k_t)(z,x)$ introduced in (6.3). We have

$$\|a(c_x)a(c_y)\psi\| \leq \|a_xa_y\psi\| + \|a_xa(p_y)\psi\| + \|a(p_x)a(c_y)\psi\|.$$

Therefore, using (6.34) and (6.36), we immediately find (using Lemma 3.3 to bound $\|p\|_2$ and $\sup_x\|p_x\|_2$)

$$\int dxdyN^3V(N(x-y))\|a(c_x)a(c_y)\psi\|^2 \leq C\int dxdyN^3V(N(x-y))\|a_xa_y\psi\|^2 + C\|(\mathcal{N}+1)\psi\|^2.$$

To prove (6.39), we notice that

$$\begin{aligned}
&\int dxdyN^3V(N(x-y))\|a(c_x)a(c_y)\psi\|^2 \\
&= \int dxdyN^3V(N(x-y))\langle\psi, a^*(c_x)a^*(c_y)a(c_y)a(c_x)\psi\rangle \\
&= \int dxdyN^3V(N(x-y))\langle\psi, a_x^*a_y^*a_ya_x\psi\rangle \\
&\quad + \int dxdyN^3V(N(x-y))\big\langle\psi, \big[a^*(p_x)a_y^*a_ya_x + a^*(c_x)a^*(p_y)a_ya_x \\
&\qquad + a^*(c_x)a^*(c_y)a(p_y)a_x + a^*(c_x)a^*(c_y)a(c_y)a(p_x)\big]\psi\big\rangle \\
&=: \int dxdyN^3V(N(x-y))\|a_xa_y\psi\|^2 + \widetilde{\mathcal{E}}(t)
\end{aligned}$$

where

$$\begin{aligned}
|\widetilde{\mathscr{E}}(t)| \leq & \int dxdyN^3V(N(x-y))\left[\|a(p_x)a_y\psi\|\|a_ya_x\psi\|\right.\\
& + \|a(c_x)a(p_y)\psi\|\|a_ya_x\psi\| + \|a(c_x)a(c_y)\psi\|\|a(p_y)a_x\psi\|\\
& \left. + \|a(c_x)a(c_y)\psi\|\|a(c_y)a(p_x)\psi\|\right]\\
\leq & \, \delta\int dxdyN^3V(N(x-y))\left[\|a_xa_y\psi\|^2 + \|a(c_x)a(c_y)\psi\|^2\right]\\
& + C_\delta\int dxdyN^3V(N(x-y))\left[\|a(p_x)a_y\psi\|^2 + \|a(c_x)a(p_y)\psi\|^2\right]\\
\leq & \, \delta\int dxdyN^3V(N(x-y))\|a_xa_y\psi\|^2 + C_\delta\|(\mathcal{N}+1)\psi\|^2.
\end{aligned}$$

Here, in the last inequality, we used (6.34), (6.36) from Lemma 6.7 and (6.38). □

Proof of Proposition 6.6. To prove the first bound in (6.33) we observe that all quartic terms on the r.h.s. of (6.31) can be bounded using Lemmas 6.7 and 6.8. For example, the contribution arising from the first term on the r.h.s. of (6.31) is bounded by

$$\begin{aligned}
& \left|\int dxdyN^2V(N(x-y))\langle\psi, a^*(c_x)a^*(c_y)a^*(s_y)a^*(s_x)\psi\rangle\right|\\
& \leq \int dxdyN^2V(N(x-y))\|a(c_x)a(c_y)\psi\|\|a^*(s_y)a^*(s_x)\psi\|\\
& \leq \delta\int dxdyN^2V(N(x-y))\|a(c_x)a(c_y)\psi\|^2\\
& \quad + C_\delta\int dxdyN^2V(N(x-y))\|a^*(s_y)a^*(s_x)\psi\|^2\\
& \leq C\delta\int dxdyN^2V(N(x-y))\|a_xa_y\psi\|^2 + \frac{C_\delta}{N}\|(\mathcal{N}+1)\psi\|^2
\end{aligned}$$

where, in the last inequality, we used (6.38) and (6.35). All the other quartic terms on the r.h.s. of (6.31), with the exception of the fourth term (the one containing only c_x or c_y as arguments of the creation and annihilation operators), can be bounded similarly; the key observation here is that all these terms have at least one creation or annihilation operator with square integrable argument (this allow us to apply Lemma 6.7). Moreover, in all these terms, the quartic expression does not contain the annihilation operators $a(c_x)$ and $a(c_y)$ in the two factors on the left, nor the creation operators $a^*(c_x)$ and $a^*(c_y)$ in the two factors on the right (in Lemma 6.7, in particular in (6.36) it is of course important that the factor $a(c_y)$ in the norm appears as an annihilation and not as a creation operator). To bound the fourth term on the r.h.s. of (6.31), where all the arguments of the creation and annihilation operators are not integrable, we cannot apply Lemma 6.7. Instead, we use (6.39) from Lemma 6.8. We obtain

$$\int dxdyN^2V(N(x-y))\langle\psi, a^*(c_x)a^*(c_y)a(c_y)a(c_x)\psi\rangle$$
$$= \int dxdyN^2V(N(x-y))\langle\psi, a_x^*a_y^*a_ya_x\psi\rangle + \widetilde{\mathscr{E}}(t)$$

where the error $\widetilde{\mathscr{E}}(t)$ is such that, for every $\delta > 0$, there exists $C_\delta > 0$ with

$$|\widetilde{\mathscr{E}}(t)| \le \delta \int dxdyN^2V(N(x-y))\|a_xa_y\psi\|^2 + \frac{C_\delta}{N}\|(\mathscr{N}+1)\psi\|^2.$$

The quadratic terms on the r.h.s. of (6.31) can be bounded using Lemma 6.4. To this end, we observe that

$$|\langle s_x, s_y\rangle|, |\langle s_x, c_y\rangle| \le CN|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)|.$$

It is therefore easy to check that all the quadratic terms, with the exception of the first two (the quadratic terms appearing on the eighth line of (6.31)), have a form suitable to apply one of the bounds in Lemma 6.4. More precisely, we apply (6.23), if the arguments of the two creation and/or annihilation operators are either s_x or s_y. If, on the other hand, one of the two arguments is c_x or c_y and the other one is s_x or s_y, we write $a^\sharp(c_x) = a_x^\sharp + a^\sharp(p_x)$ and then we apply (6.23) (to bound the contribution proportional to $a^\sharp(p_x)$) and (6.24) (to bound the contribution proportional to $a_x^\sharp$). Finally, if both arguments are either c_x or c_y (and we have exactly one creation and one annihilation operators), we write $a^\sharp(c_x) = a_x^\sharp + a^\sharp(p_x)$ and we apply (6.23), (6.24) and (6.25). To control the two remaining quadratic contributions, we observe that, writing $a^*(c_x) = a_x^* + a^*(p_x)$,

$$\begin{aligned}
&\int dxdyN^2V(N(x-y))\langle c_y, s_x\rangle\langle\psi, a^*(c_x)a^*(c_y)\psi\rangle \\
&\quad = \int dxdyN^2V(N(x-y))\langle c_y, s_x\rangle\langle\psi, a_x^*a_y^*\psi\rangle \\
&\quad\quad + \int dxdyN^2V(N(x-y))\langle c_y, s_x\rangle\langle\psi, a^*(p_x)a_y^*\psi\rangle \\
&\quad\quad + \int dxdyN^2V(N(x-y))\langle c_y, s_x\rangle\langle\psi, a^*(c_x)a^*(p_y)\psi\rangle.
\end{aligned} \tag{6.40}$$

Since $|\langle c_y, s_x\rangle| \le CN|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)|$, the last two terms can be bounded (in absolute value) using (6.24) and (6.25), respectively. We find

$$\left|\int dxdyN^2V(N(x-y))\langle c_y, s_x\rangle\langle\psi, a^*(p_x)a_y^*\psi\rangle\right|$$
$$\le C\|\varphi_t^{(N)}\|_{H^2}^2\|(\mathscr{N}+1)^{1/2}\psi\|^2,$$

and

$$\left| \int dxdy N^2 V(N(x-y)) \langle c_y, s_x \rangle \langle \psi, a^*(c_x) a^*(p_y) \psi \rangle \right| \\ \leq C \|\varphi_t^{(N)}\|_{H^2}^2 \|(\mathcal{N}+1)^{1/2} \psi\|^2.$$

As for the first term on the r.h.s. of (6.40), we notice that

$$\langle c_y, s_x \rangle = k(x,y) + g(x,y)$$

where $g(x,y) = r(x,y) + \langle p_y, s_x \rangle$ is such that

$$|g(x,y)| \leq C |\varphi_t^{(N)}(x)| \, |\varphi_t^{(N)}(y)|.$$

Therefore,

$$\begin{aligned} \int dxdy N^2 V(N(x-y)) \langle c_y, s_x \rangle \langle \psi, a_x^* a_y^* \psi \rangle \\ = \int dxdy N^2 V(N(x-y)) k(x,y) \langle \psi, a_x^* a_y^* \psi \rangle \\ + \int dxdy N^2 V(N(x-y)) g(x,y) \langle \psi, a_x^* a_y^* \psi \rangle. \end{aligned}$$

The second term can be bounded by

$$\begin{aligned} &\left| \int dxdy N^2 V(N(x-y)) g(x,y) \langle \psi, a_x^* a_y^* \psi \rangle \right| \\ &\leq C \int dxdy N^2 V(N(x-y)) |\varphi_t^{(N)}(x)| |\varphi_t^{(N)}(y)| \|a_x a_y \psi\| \, \|\psi\| \\ &\leq \delta \int dxdy N^2 V(N(x-y)) \|a_x a_y \psi\|^2 \\ &\quad + C_\delta \int dxdy N^2 V(N(x-y)) |\varphi_t^{(N)}(x)|^2 |\varphi_t^{(N)}(y)|^2 \\ &\leq \delta \int dxdy N^2 V(N(x-y)) \|a_x a_y \psi\|^2 + C_\delta \|\varphi_t^{(N)}\|_{H^2}^2. \end{aligned}$$

Proceeding analogously to control the second quadratic term on the r.h.s. of (6.31), we conclude that

$$\begin{aligned} \int dxdy N^2 V(N(x-y)) \left[\langle c_y, s_x \rangle a^*(c_x) a^*(c_y) + \langle s_x, c_y \rangle a(c_y) a(c_x) \right] \\ = \int dxdy N^2 V(N(x-y)) \left[k(x,y) a_x^* a_y^* + \overline{k}(x,y) a_x a_y \right] + \widetilde{\mathcal{E}}(t) \end{aligned}$$

where the error $\widetilde{\mathcal{E}}(t)$ is such that, for every $\delta > 0$ there exists C_δ with

$$\pm \widetilde{\mathcal{E}}(t) \leq \delta \int dxdy N^2 V(N(x-y)) a_x^* a_y^* a_y a_x + C_\delta \|\varphi_t^{(N)}\|_{H^2}^2 (\mathcal{N}+1).$$

We then use (3.2) to conclude the proof of the first bound in (6.33).

The proof of the second inequality in (6.33) is analogous, because commuting the terms contributing to $\mathcal{E}_4(t)$ with the number of particles operator $\mathcal{N}$ either gives zero or leaves the terms essentially invariant (up to a constant and a possible sign change). Finally, also the third estimate in (6.33) can be proven similarly, because the time derivative of the terms contributing to $\mathscr{E}_4(t)$ can be expressed as linear combination of terms having the same form, just with one argument c_x, c_y, s_x or s_y replaced by its time-derivative. These terms can then be handled as above, using however the bounds for $\|\dot{\mathrm{sh}}_{k_t}\|_2$ and $\|\dot{p}_{k_t}\|_2$ from Lemma 3.4 and the bound for $\|\dot{\varphi}_t^{(N)}\|_{H^2}$ from Proposition 3.1. □

6.5 Analysis of $(i\partial_t T^*)T$

In this subsection we prove that the term $(i\partial_t T^*)T$ can be controlled in terms of the number of particles operator. We set

$$B = \frac{1}{2}\int dxdy \left(k_t(x,y)a_x^* a_y^* - \overline{k_t(x,y)}a_x a_y\right)$$

and

$$\dot{B} = \frac{1}{2}\int dxdy \left(\dot{k}_t(x,y)a_x^* a_y^* - \overline{\dot{k}_t(x,y)}a_x a_y\right)$$

with

$$k_t(x,y) = -Nw(N(x-y))\varphi_t^{(N)}(x)\varphi_t^{(N)}(y)$$

and

$$\dot{k}_t(x,y) = -Nw(N(x-y))\left(\dot{\varphi}_t^{(N)}(x)\varphi_t^{(N)}(y) + \varphi_t^{(N)}(x)\dot{\varphi}_t^{(N)}(y)\right).$$

Then $T = \exp(B)$. Using (2.15), we find

$$(\partial_t T^*)T = -\int_0^1 d\lambda\, e^{-\lambda B(t)}\dot{B}(t)e^{\lambda B(t)} = \sum_{n\geq 0}\frac{(-1)^{n+1}}{(n+1)!}\,\mathrm{ad}_B^n(\dot{B}). \tag{6.41}$$

More precisely, the integral can first be expanded as a finite sum and an error term; the error term however converges to zero in expectation values on the domain $D(\mathscr{N})$ of the number of particles operator (this can be shown as in Lemma 2.3). Notice that, by the estimates in Prop. 6.10, the series is absolutely convergent in expectation values. Since $D(\mathscr{N})$ is invariant w. r. t. the fluctuation dynamics $\mathscr{U}(t;s)$ (this is proven similarly to Prop. 4.2), we can use (6.41) to compute the expectation of $(\partial_t T^*)T$ in the state $\mathscr{U}(t;0)\psi$ for any $\psi \in D(\mathscr{N})$.

Next, we compute the terms on the r.h.s. of (6.41).

Lemma 6.9. *For each $n\in\mathbb{N}$ there exist $f_{n,1}, f_{n,2} \in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ such that*

$$\begin{aligned}\mathrm{ad}_B^n(\dot{B}) &= \frac{1}{2}\int dxdy \left(f_{n,1}(x,y)a_y^* a_x^* + f_{n,2}(x,y)a_x a_y\right) \quad \textit{for all even } n \textit{ and}\\ \mathrm{ad}_B^n(\dot{B}) &= \frac{1}{2}\int dxdy \left(f_{n,1}(x,y)a_x^* a_y + f_{n,2}(x,y)a_x a_y^*\right) \quad \textit{for all odd } n\end{aligned} \tag{6.42}$$

where

$$\|f_{n,i}\|_2 \leq 2^n \|k_t\|_2^n \|\dot{k}_t\|_2, \tag{6.43}$$

for all $n \geq 0$ *and* $i = 1,2$,

$$\|\dot{f}_{n,i}\|_2 \leq \begin{cases} \|\ddot{k}_t\|_2 & \text{if } n = 0 \\ 4^n \|k_t\|_2^{n-1} \left(\|\ddot{k}_t\|_2 \|k_t\|_2 + \|\dot{k}_t\|_2^2\right) & \text{if } n \geq 1 \end{cases} \tag{6.44}$$

and

$$\begin{aligned} \int dx\, |f_{n,i}(x,x)| &\leq 2^n \|k_t\|_2^n \|\dot{k}_t\|_2, \\ \int dx\, |\dot{f}_{n,i}(x,x)| &\leq 4^n \|k_t\|_2^{n-1} \left(\|\dot{k}_t\|_2^2 + \|\ddot{k}_t\|_2 \|k_t\|_2\right) \end{aligned} \tag{6.45}$$

for all $n \geq 1$.

Proof. The proof is by induction in n. For $n = 0$,

$$\mathrm{ad}_B^0(\dot{B}) = \dot{B} = \frac{1}{2} \int dxdy \big(\dot{k}_t(x,y) a_x^* a_y^* - \overline{\dot{k}_t(x,y)} a_x a_y\big).$$

Hence $f_{0,1}(x,y) = \dot{k}_t(x,y)$ and $f_{0,2}(x,y) = \overline{\dot{k}_t(x,y)}$, and the estimates (6.43) and (6.44) are clearly satisfied. Suppose now the statement holds for some $n \in \mathbb{N}$. We prove them for $(n+1)$. We assume first that n is even. Then, using the canonical commutation relations (2.1), we find

$$\begin{aligned} \mathrm{ad}_B^{n+1}(\dot{B}) &= [B, \mathrm{ad}_B^n(\dot{B})] \\ &= \Big[\frac{1}{2} \int dxdy \big(k_t(x,y) a_x^* a_y^* - \overline{k_t(x,y)} a_x a_y\big), \\ &\qquad \frac{1}{2} \int dxdy \left(f_{n,1}(x,y) a_x^* a_y^* + f_{n,2}(x,y) a_x a_y\right)\Big] \\ &= \frac{1}{2} \int dxdz \left(f_{n+1,1}(x,z) a_x^* a_z + f_{n+1,2}(x,z) a_x a_z^*\right), \end{aligned}$$

where

$$\begin{aligned} f_{n+1,1}(x,z) &= -\frac{1}{2} \int dy \big(k_t(x,y) \big(f_{n,2}(z,y) + f_{n,2}(y,z)\big) \\ &\qquad + \overline{k_t(y,z)} \big(f_{n,2}(x,y) + f_{n,2}(y,x)\big)\big), \\ f_{n+1,2}(x,z) &= -\frac{1}{2} \int dy \big(k_t(y,z) \big(f_{n,1}(x,y) + f_{n,1}(y,x)\big) \\ &\qquad + \overline{k_t(x,y)} \big(f_{n,1}(z,y) + f_{n,1}(y,z)\big)\big). \end{aligned} \tag{6.46}$$

By Cauchy-Schwarz (similarly to (2.17)), we have

$$\begin{aligned} \|f_{n+1,1}\|_2 &\leq 2\|k_t\|_2 \|f_{n,2}\|_2 \leq 2^{n+1} \|k_t\|_2^{n+1} \|\dot{k}_t\|_2, \\ \|f_{n+1,2}\|_2 &\leq 2\|k_t\|_2 \|f_{n,1}\|_2 \leq 2^{n+1} \|k_t\|_2^{n+1} \|\dot{k}_t\|_2, \end{aligned} \tag{6.47}$$

where we used the induction assumption. Moreover, again by Cauchy-Schwarz,

$$\int dx\, |f_{n+1,1}(x,x)| \leq 2\|k_t\|_2 \|f_{n,2}\|_2 \leq 2^{n+1} \|k_t\|_2^{n+1} \|\dot{k}_t\|_2 .$$

As for the time-derivative of $f_{n+1,i}$, we find

$$\begin{aligned}&\dot f_{n+1,1}(x,z)\\&=-\frac{1}{2}\int dy\big[\dot k_t(x,y)\big(f_{n,2}(z,y)+f_{n,2}(y,z)\big)+k_t(x,y)\big(\dot f_{n,2}(z,y)+\dot f_{n,2}(y,z)\big)\\&\qquad+\overline{\dot k_t(y,z)}\big(f_{n,2}(x,y)+f_{n,2}(y,x)\big)+\overline{k_t(y,z)}\big(\dot f_{n,2}(x,y)+\dot f_{n,2}(y,x)\big)\big]\end{aligned}$$

and similarly for $\dot f_{n+1,2}$. Hence, we find

$$\begin{aligned}\|\dot f_{n+1,1}\|_2&\leq 2\big(\|\dot k_t\|_2\|f_{n,2}\|_2+\|k_t\|_2\|\dot f_{n,2}\|_2\big)\\&\leq 2\big(2^n\|k_t\|_2^n\|\dot k_t\|_2^2+4^n\|k_t\|_2^n\big(\|\ddot k_t\|_2\|k_t\|_2+\|\dot k_t\|_2^2\big)\big)\\&\leq (2^{n+1}+4^n)\|k_t\|_2^n\|\dot k_t\|_2^2+2\cdot 4^n\|k_t\|_2^{n+1}\|\ddot k_t\|_2\\&\leq 4^{n+1}\|k_t\|_2^n\big(\|\ddot k_t\|_2\|k_t\|_2+\|\dot k_t\|_2^2\big),\end{aligned}$$

proving (6.44) for $i=1$. The same bound for $i=2$ and the second bound in (6.45) for $i=1,2$ can be proven similarly.

If n is odd, we have, using again the canonical commutation relations,

$$\begin{aligned}\mathrm{ad}_B^{n+1}(\dot B)&=\Big[\frac{1}{2}\int dxdy\,\Big(k_t(x,y)a_x^*a_y^*-\overline{k_t(x,y)}a_xa_y\Big),\\&\qquad\frac{1}{2}\int dxdy\,\big(f_{n,1}(x,y)a_x^*a_y+f_{n,2}(x,y)a_xa_y^*\big)\Big]\\&=\frac{1}{2}\int dxdz\big(a_x^*a_z^*f_{n+1,1}(x,z)+a_xa_zf_{n+1,2}(x,z)\big)\end{aligned}$$

where

$$\begin{aligned}f_{n+1,1}(x,z)&=-\int dy\,k_t(x,y)\,(f_{n,1}(z,y)+f_{n,2}(y,z))\,,\\f_{n+1,2}(x,z)&=-\int dy\,\overline{k_t(x,y)}\,(f_{n,1}(y,z)+f_{n,2}(z,y))\,.\end{aligned}\tag{6.48}$$

The bounds (6.43), (6.44), (6.45) follow as above. □

Using Lemma 6.9, we obtain the following properties of $(\partial_t T^*)T$.

Proposition 6.10. *There exists a constant $C>0$ with*

$$\begin{aligned}\pm\,(i\partial_t T^*)T&\leq Ce^{K|t|}\,(\mathcal{N}+1),\\\pm\,[\mathcal{N},(i\partial_t T^*)T]&\leq Ce^{K|t|}\,(\mathcal{N}+1),\\\pm\,\partial_t\,[(i\partial_t T^*)T]&\leq Ce^{K|t|}\,(\mathcal{N}+1).\end{aligned}\tag{6.49}$$

Proof. We first note that, for $f_1,f_2\in L^2(\mathbb{R}^3\times\mathbb{R}^3)$ with $\int dx\,|f_2(x,x)|<\infty$, we have

$$\begin{aligned}&\Big|\Big\langle\psi,\int dxdy\,\big(f_1(x,y)a_x^*a_y^*+f_2(x,y)a_xa_y\big)\,\psi\Big\rangle\Big|\\&\qquad\qquad\leq(\|f_1\|_2+\|f_2\|_2)\,\big\langle\psi,(\mathcal{N}+1)\psi\big\rangle\end{aligned}\tag{6.50}$$

and

$$\Big|\Big\langle \psi, \int dxdy\,(f_1(x,y)a_x^* a_y + f_2(x,y)a_x a_y^*)\psi\Big\rangle\Big| \leq (\|f_1\|_2 + \|f_2\|_2)\langle \psi, \mathcal{N}\psi\rangle + \int dx\,|f_2(x,x)|\,\|\psi\|^2. \tag{6.51}$$

In fact, (6.50) follows because

$$\begin{aligned}
&\Big|\langle \psi, \int dxdy\,(f_1(x,y)a_x^* a_y^* + f_2(x,y)a_x a_y)\psi\rangle\Big| \\
&\quad\leq \int dx\,(\|a_x\psi\|\|a^*(f_1(x,.))\psi\| + \|a^*(f_2(x,.))\psi\|\|a_x\psi\|) \\
&\quad\leq \|(\mathcal{N}+1)^{1/2}\psi\| \int dx\,(\|f_1(x,.)\|_2 + \|f_2(x,.)\|_2)\,\|a_x\psi\| \\
&\quad\leq (\|f_1\|_2 + \|f_2\|_2)\|(\mathcal{N}+1)^{1/2}\psi\|^2.
\end{aligned}$$

Eq. (6.51) can be proven similarly. Combining the last estimates with Lemma 6.9 and with (6.41), we find

$$\begin{aligned}
&|\langle \psi, (\partial_t T^*)T\psi\rangle| \\
&\leq \sum_{n\geq 0} \frac{1}{(n+1)!}\big|\langle \psi, \text{ad}_B^n(\dot{B})\psi\rangle\big| \\
&\leq \sum_{n\geq 0} \frac{1}{(n+1)!}\,(\|f_{n,1}\|_2 + \|f_{n,2}\|_2)\,\|(\mathcal{N}+1)^{1/2}\psi\|^2 \\
&\quad + \sum_{n\geq 1} \frac{1}{(2n)!}\|\psi\|^2 \int dx\,|f_{2n-1,2}(x,x)| \\
&\leq C\sum_{n\geq 0} \frac{(2\|k_t\|_2)^n}{(n+1)!}\,\|\dot{k}_t\|_2\,\|(\mathcal{N}+1)^{1/2}\psi\|^2 + \sum_{n\geq 1} \frac{(2\|k_t\|_2)^n}{(2n)!}\|\dot{k}_t\|_2\|\psi\|^2 \\
&\leq Ce^{2\|k_t\|_2}\|\dot{k}_t\|_2\,\|(\mathcal{N}+1)^{1/2}\psi\|^2 \\
&\leq Ce^{K|t|}\,\|(\mathcal{N}+1)^{1/2}\psi\|^2
\end{aligned}$$

using also Lemma 3.4. The second inequality in (6.49) follows similarly because, essentially, the only consequence of taking the commutator with $\mathcal{N}$ is to eliminate the terms $\text{ad}_B^n(\dot{B})$ for all odd n. Also the third bound in (6.49) can be proven analogously, taking the time derivative of the expressions for $\text{ad}_B^n(\dot{B})$ given in (6.42), using the bounds for $\|\dot{f}_{n,i}\|_2$ in (6.44) and (6.45) and, finally, using the estimate for $\|\ddot{k}_t\|_2$ proven in Lemma 3.4. □

6.6 Proof of Theorem 3.5

We combine the results of the previous subsections to obtain a proof of Theorem 3.5. From (6.4) and Propositions 6.1, 6.3, 6.5, 6.6 it follows that

$$\begin{aligned}\mathscr{L}_N(t) &= C_N(t) + \mathscr{K} + \frac{1}{2N}\int dxdy N^3 V(N(x-y)) a_x^* a_y^* a_y a_x \\ &+ \Big[N^3 \int dxdy (\Delta w)(N(x-y)) \varphi_t^{(N)}(x)\varphi_t^{(N)}(y) a_x^* a_y^* \\ &+ \frac{1}{2}\int dxdy N^3 V(N(x-y))(1-w(N(x-y)))\varphi_t^{(N)}(x)\varphi_t^{(N)}(y)\, a_x^* a_y^* + \text{h.c.}\Big] \\ &+ \mathscr{E}(t)\end{aligned} \tag{6.52}$$

where the time-dependent constant $C_N(t)$ is defined in (3.9) and the error $\mathscr{E}(t)$ is such that, for every $\delta > 0$, there exists $C_\delta > 0$ with

$$\begin{aligned}\pm\mathscr{E}(t) &\leq \delta\left(\mathscr{K} + \int dxdy N^2 V(N(x-y)) a_x^* a_y^* a_y a_x\right) \\ &\quad + C_\delta \frac{\mathscr{N}^2}{N} + C_\delta e^{K|t|}\left(\mathscr{N}+1\right), \\ \pm\left[\mathscr{N}, \mathscr{E}(t)\right] &\leq \delta\left(\mathscr{K} + \int dxdy N^2 V(N(x-y)) a_x^* a_y^* a_y a_x\right) \\ &\quad + C_\delta \frac{\mathscr{N}^2}{N} + C_\delta e^{K|t|}\left(\mathscr{N}+1\right), \\ \pm\dot{\mathscr{E}}(t) &\leq \delta\left(\mathscr{K} + \int dxdy N^2 V(N(x-y)) a_x^* a_y^* a_y a_x\right) \\ &\quad + C_\delta e^{K|t|}\left(\frac{\mathscr{N}^2}{N} + \mathscr{N} + 1\right).\end{aligned} \tag{6.53}$$

In deriving (6.52), we made use of the crucial cancellation between the linear contributions in (6.4) and the linear terms in (6.29). Next, we notice another crucial cancellation. The terms on the second and third line in (6.52) can be written as

$$N^3 \int dxdy\, a_x^* a_y^* \Big[\Big(-\Delta + \frac{1}{2}V\Big)(1-w)\Big](N(x-y))\varphi_t^{(N)}(x)\varphi_t^{(N)}(y) = 0$$

since $f = 1-w$ is a solution of the zero-energy scattering equation $(-\Delta + \frac{1}{2}V)f = 0$.

We conclude that

$$\mathscr{L}_N(t) = C_N(t) + \mathscr{K} + \frac{1}{2N}\int dxdy N^3 V(N(x-y)) a_x^* a_y^* a_y a_x + \mathscr{E}(t) \tag{6.54}$$

where the error $\mathscr{E}(t)$ satisfies (6.53). Then (3.11) follows from the first bound in (6.53), taking $\delta = 1/2$. Also (3.13) and (3.14) follow from the second and third bounds in (6.53), since both $\mathscr{K}$ and the quartic term on the r.h.s. of (6.54) commute with $\mathscr{N}$ and are time-independent.

This concludes the proof of Theorem 3.5.

Appendix A: Properties of the solution of the Gross-Pitaevskii equation

Proof of Proposition 3.1. (i) This part of the proposition is standard. One proves first local well-posedness of the two equations in $H^1(\mathbb{R}^3)$. The time of existence depends only on the H^1-norm of the initial data. Since $V, f \geq 0$, the H^1-norm is bounded by the energy, which is conserved. Hence one obtains global existence and a uniform bound on the H^1-norm.

(ii) Also this part is rather standard, but since the non-linearity in (3.1) depends on N, and we need bounds uniform in N, we sketch the proof of the bound (3.2) for $\|\varphi_t^{(N)}\|_{H^n}$ (the bound for $\|\varphi_t\|_{H^n}$ can be proven analogously). We present the proof for the case $t > 0$. We claim, first of all, that there exists $T > 0$ depending only on $\|\varphi\|_{H^1}$ and $n \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]} \|\varphi_t^{(N)}\|_{H^n} \leq 2\|\varphi_0\|_{H^n} + \sup_{t\in[0,T]} \|\varphi_t^{(N)}\|^3_{H^{n-1}}. \tag{A.1}$$

Introducing the short-hand notation $U_N(x) = N^3 V(Nx) f(Nx)$, we write the solution $\varphi_t^{(N)}$ of (3.1) as

$$\varphi_t^{(N)} = e^{it\Delta}\varphi - i\int_0^t ds\, e^{i(t-s)\Delta}(U_N * |\varphi_s^{(N)}|^2)\varphi_s^{(N)}.$$

Differentiating this equation w.r.t. the spatial variables we find that

$$\begin{aligned}\partial^\alpha \varphi_t^{(N)} = e^{it\Delta}\partial^\alpha \varphi - i\int_0^t ds\, e^{i(t-s)\Delta} &\sum_{\beta\leq\alpha}\sum_{\nu\leq\beta} \binom{\alpha}{\beta}\binom{\beta}{\nu} \\ &\times \left(U_N * (\partial^\nu \overline{\varphi}_s^{(N)} \partial^{\beta-\nu}\varphi_s^{(N)})\right)\partial^{\alpha-\beta}\varphi_s^{(N)}.\end{aligned}$$

Here α is a three-dimensional multi-index of non-negative integers, with $|\alpha| \leq n$.

The $L_t^\infty([0,T], L_x^2)$-norm of the above expression can be controlled by using Strichartz estimates for the free Schrödinger evolution $e^{it\Delta}$ (see [25, Theorem 1.2]). We find

$$\begin{aligned}\|\partial^\alpha \varphi_{(\cdot)}^{(N)}\|_{L_t^\infty L_x^2} \leq \|\partial^\alpha\varphi\|_{L^2} &+ \sum_{\beta\leq\alpha}\sum_{\nu\leq\beta}\binom{\alpha}{\beta}\binom{\beta}{\nu} \\ &\times \left\|\left(U_N * (\partial^\nu \overline{\varphi}_{(\cdot)}^{(N)}\partial^{\beta-\nu}\varphi_{(\cdot)}^{(N)})\right)\partial^{\alpha-\beta}\varphi_{(\cdot)}^{(N)}\right\|_{L_t^2 L_x^{6/5}} \\ \leq \|\partial^\alpha\varphi\|_{L^2} &+ T^{1/2}\sum_{\beta\leq\alpha}\sum_{\nu\leq\beta}\binom{\alpha}{\beta}\binom{\beta}{\nu} \\ &\times \sup_{t\in[0,T]}\left\|\left(U_N * (\partial^\nu\overline{\varphi}_t^{(N)}\partial^{\beta-\nu}\varphi_t^{(N)})\right)\partial^{\alpha-\beta}\varphi_t^{(N)}\right\|_{L_x^{6/5}}.\end{aligned}$$

By Hölder and Young inequality, we find

$$\|\partial^\alpha \varphi^{(N)}_{(\cdot)}\|_{L^\infty_t L^2_x} \leq \|\partial^\alpha \varphi\|_{L^2} + CT^{1/2} \sum_{\beta \leq \alpha} \sum_{\nu \leq \beta} \binom{\alpha}{\beta} \binom{\beta}{\nu} \times \sup_{t \in [0,T]} \|\partial^\nu \varphi^{(N)}_t\|_{L^{p_1}} \|\partial^{\beta-\nu} \varphi^{(N)}_t\|_{L^{p_2}} \|\partial^{\alpha-\beta} \varphi^{(N)}_t\|_{L^{p_3}}$$

for $p_1, p_2, p_3 \geq 1$ with $p_1^{-1} + p_2^{-1} + p_3^{-1} = 5/6$. It is important to note that the indices (p_1, p_2, p_3) can be chosen differently for each term in the summation. In some of the terms with $|\alpha| = n$, all n derivatives hit the same $\varphi^{(N)}_t$. Since $1/6 + 1/6 + 1/2 = 5/6$, these terms can be bounded by

$$\|\varphi^{(N)}_t\|^2_{L^6} \|\partial^\alpha \varphi^{(N)}_t\|_{L^2} \leq C \|\varphi^{(N)}_t\|_{H^n} \tag{A.2}$$

for C depending only on $\|\varphi\|_{H^1}$ (recall here that the H^1-norm is bounded uniformly in t, by part (i)). In some of the other terms, one $\varphi^{(N)}_t$ has $n-1$ derivatives, one has at most one derivative and the last one has no derivatives. Since $\|\partial^\gamma \varphi^{(N)}_t\|_{L^6} \leq \|\varphi^{(N)}_t\|_{H^n}$, if $|\gamma| \leq n-1$, these terms are bounded by the r.h.s. of (A.2). In all other terms, the three copies of $\varphi^{(N)}_t$ have at most $n-2$ derivatives. These terms are bounded by

$$\|\partial^{\gamma_1} \varphi^{(N)}_t\|_{L^6} \|\partial^{\gamma_2} \varphi^{(N)}_t\|_{L^6} \|\partial^{\gamma_3} \varphi^{(N)}_t\|_{L^2} \leq C \|\varphi^{(N)}_t\|^3_{H^{n-1}}$$

for all multi-indices $\gamma_1, \gamma_2, \gamma_3$ with $|\gamma_i| \leq n-2$, for $i = 1, 2, 3$. We conclude that

$$\|\partial^\alpha \varphi^{(N)}_{(\cdot)}\|_{L^\infty_t L^2_x} \leq \|\partial^\alpha \varphi\|_{L^2} + CT^{1/2} \sup_{t \in [0,T]} \|\varphi^{(N)}_t\|_{H^n} + CT^{1/2} \sup_{t \in [0,T]} \|\varphi^{(N)}_t\|^3_{H^{n-1}}.$$

Summing over all α with $|\alpha| \leq n$, we find

$$\sup_{t \in [0,T]} \|\varphi^{(N)}_t\|_{H^n} \leq \|\varphi\|_{H^n} + CT^{1/2} \sup_{t \in [0,T]} \|\varphi^{(N)}_t\|_{H^n} + CT^{1/2} \sup_{t \in [0,T]} \|\varphi^{(N)}_t\|^3_{H^{n-1}}.$$

Choosing $T > 0$ so small that $CT^{1/2} \leq 1/2$, we find (A.1).

To show (3.2), we iterate now (A.1). We proceed by induction over n. For $n = 1$, the claim follows from part (i). Suppose now that $\|\varphi^{(N)}_t\|_{H^{(n-1)}} \leq C_{n-1} \exp(K_{n-1}|t|)$, for constants C_{n-1}, K_{n-1} depending on $\|\varphi\|_{H^{(n-1)}}$ and, respectively, on $\|\varphi\|_{H^1}$. Let T be as in (A.1). For an arbitrary $t > 0$, there exists an integer $j \in \mathbb{N}$ such that

$(j-1)T < t \leq jT$. Then

$$\begin{aligned}\|\varphi_t^{(N)}\|_{H^n} &\leq \sup_{s\in[(j-1)T,jT]} \|\varphi_s^{(N)}\|_{H^n} \\ &\leq 2\|\varphi_{(j-1)T}^{(N)}\|_{H^n} + 2 \sup_{s\in[(j-1)T,jT]} \|\varphi_s^{(N)}\|_{H^{n-1}}^3 \\ &\leq 2\|\varphi_{(j-1)T}^{(N)}\|_{H^n} + 2C_{n-1}^3 e^{3K_{n-1}jT}.\end{aligned}$$

Similarly we have

$$\|\varphi_{(j-1)T}^{(N)}\|_{H^n} \leq 2\|\varphi_{(j-2)T}^{(N)}\|_{H^n} + 2C_{n-1}^3 e^{3K_{n-1}(j-1)T}.$$

Iterating j-times, we obtain

$$\|\varphi_t^{(N)}\|_{H^n} \leq 2^j\|\varphi\|_{H^n} + 2C_{n-1}^3 \sum_{\ell=0}^{j} 2^\ell e^{3K_{n-1}(j-\ell)T} \leq C_n e^{K_n t},$$

for some constant C_n depending on $\|\varphi\|_{H^n}$ and K_n depending only on $\|\varphi\|_{H^1}$.

(iii) From the modified Gross-Pitaevskii equation (3.1), letting

$$U_N(x) = N^3 V(Nx) f(Nx),$$

we find

$$\begin{aligned}\|\dot{\varphi}_t^{(N)}\|_2 &\leq \|\varphi_t^{(N)}\|_{H^2} + \left\|\left(U_N * |\varphi_t^{(N)}|^2\right)\varphi_t^{(N)}\right\|_2 \\ &\leq \|\varphi_t^{(N)}\|_{H^2} + \left\|U_N * |\varphi_t^{(N)}|^2\right\|_2 \|\varphi_t^{(N)}\|_\infty \\ &\leq \|\varphi_t^{(N)}\|_{H^2} + C\|U_N\|_1 \|\varphi_t^{(N)}\|_4^2 \|\varphi_t^{(N)}\|_\infty \\ &\leq C\|\varphi_t^{(N)}\|_{H^2}^3 \leq Ce^{K|t|}\end{aligned}$$

for a constant C depending only on $\|\varphi\|_{H^2}$ and $\|U_N\|_1$, and for $K > 0$ depending only on $\|\varphi\|_{H^1}$. Here we used part (ii). Applying the gradient to (3.1), we find

$$\begin{aligned} i\nabla\dot{\varphi}_t^{(N)} &= -\nabla\Delta\varphi_t^{(N)} + \left(U_N * |\varphi_t^{(N)}|^2\right)\nabla\varphi_t^{(N)} \\ &\quad + \left(U_N * \overline{\varphi}_t^{(N)}\nabla\varphi_t^{(N)}\right)\varphi_t^{(N)} + \left(U_N * \nabla\overline{\varphi}_t^{(N)}\varphi_t^{(N)}\right)\varphi_t^{(N)}.\end{aligned} \tag{A.3}$$

Clearly, $\|\nabla\Delta\varphi_t^{(N)}\|_2 \leq \|\varphi_t^{(N)}\|_{H^3}$. The second term on the first line is bounded in norm by

$$\begin{aligned}\left\|(U_N * |\varphi_t^{(N)}|^2)\nabla\varphi_t^{(N)}\right\|_2 &\leq \|(U_N * |\varphi_t^{(N)}|^2)\|_\infty \|\nabla\varphi_t^{(N)}\|_2 \\ &\leq \|U_N\|_1 \|\varphi_t^{(N)}\|_\infty^2 \|\nabla\varphi_t^{(N)}\|_2 \leq C\|\varphi_t^{(N)}\|_{H^2}^3.\end{aligned}$$

The terms on the second line of (A.3) can be bounded similarly. From part (ii), we conclude that $\|\nabla\dot{\varphi}_t^{(N)}\|_2 \leq C\exp(K|t|)$. Analogously, we can also show that

$\|\nabla^2 \dot{\varphi}_t^{(N)}\|_2 \leq Ce^{K|t|}$. We conclude that $\|\dot{\varphi}_t^{(N)}\|_{H^2} \leq Ce^{K|t|}$. Finally, (3.1) implies

$$-\ddot{\varphi}_t^{(N)} = -\Delta i\dot{\varphi}_t^{(N)} + \big(U_N * (\overline{\varphi}_t^{(N)} i\dot{\varphi}_t^{(N)})\big)\varphi_t^{(N)} + \big(U_N * (i\dot{\overline{\varphi}}_t^{(N)} \varphi_t^{(N)})\big)\varphi_t^{(N)} + \big(U_N * |\varphi_t^{(N)}|^2\big) i\dot{\varphi}_t^{(N)}.$$

Plugging in the r.h.s. of (3.1) for $i\dot{\varphi}_t^{(N)}$, we arrive at

$$\begin{aligned}-\ddot{\varphi}_t^{(N)} = {} & \Delta^2 \varphi_t^{(N)} - \Delta\big((U_N * |\varphi_t^{(N)}|^2)\varphi_t^{(N)}\big) \\ & + \big(U_N * |\varphi_t^{(N)}|^2\big)^2 \varphi_t^{(N)} + (U_N * \overline{\varphi}_t^{(N)}(-\Delta\varphi_t^{(N)}))\varphi_t^{(N)} \\ & + 2\big[U_N * \big(|\varphi_t^{(N)}|^2 (U_N * |\varphi_t^{(N)}|^2)\big)\big]\varphi_t^{(N)} \\ & + (U_N * (-\Delta\overline{\varphi}_t^{(N)})\varphi_t^{(N)})\varphi_t^{(N)} + \big(U_N * |\varphi_t^{(N)}|^2\big)(-\Delta\varphi_t^{(N)}).\end{aligned}$$

Proceeding similarly as above, we find that

$$\|\ddot{\varphi}_t^{(N)}\|_2 \leq C\|\varphi_t^{(N)}\|_{H^4} \leq C\exp(K|t|).$$

(iv) Using (1.37) and (3.1), we find

$$\begin{aligned}\partial_t \|\varphi_t - \varphi_t^{(N)}\|_2^2 &= -2\,\mathrm{Im}\,\big\langle \varphi_t, \big(U_N * |\varphi_t^{(N)}|^2 - 8\pi a_0|\varphi_t|^2\big)\varphi_t^{(N)}\big\rangle \\ &= -2\,\mathrm{Im}\,\big\langle \varphi_t, \big(U_N * |\varphi_t|^2 - 8\pi a_0|\varphi_t|^2\big)\varphi_t^{(N)}\big\rangle \\ &\quad - 2\,\mathrm{Im}\,\big\langle \varphi_t, \big(U_N * \big(|\varphi_t^{(N)}|^2 - |\varphi_t|^2\big)\big)\varphi_t^{(N)}\big\rangle.\end{aligned} \tag{A.4}$$

The second term on the r.h.s. can be written as

$$\begin{aligned}&\mathrm{Im}\,\big\langle \varphi_t, \big(U_N * \big(|\varphi_t^{(N)}|^2 - |\varphi_t|^2\big)\big)\varphi_t^{(N)}\big\rangle \\ &\qquad = \mathrm{Im}\,\big\langle \varphi_t, \big(U_N * \big(|\varphi_t^{(N)}|^2 - |\varphi_t|^2\big)\big)(\varphi_t - \varphi_t^{(N)})\big\rangle.\end{aligned}$$

Hence, by Hölder's and triangle's inequality,

$$\begin{aligned}&\big|\mathrm{Im}\,\big\langle \varphi_t, \big(U_N * \big(|\varphi_t^{(N)}|^2 - |\varphi_t|^2\big)\big)\varphi_t^{(N)}\big\rangle\big| \\ &\quad \leq \|\varphi_t\|_\infty \big\|\big(U_N * \big(|\varphi_t^{(N)}|^2 - |\varphi_t|^2\big)\big)(\varphi_t - \varphi_t^{(N)})\big\|_1 \\ &\quad \leq \|\varphi_t\|_\infty \big\|U_N * (|\varphi_t^{(N)}|^2 - |\varphi_t|^2)\big\|_2 \|\varphi_t - \varphi_t^{(N)}\|_2 \\ &\quad \leq \|U_N\|_1 \|\varphi_t\|_\infty \|\varphi_t - \varphi_t^{(N)}\|_2 \big\||\varphi_t^{(N)}|^2 - |\varphi_t|^2\big\|_2 \\ &\quad \leq \|U_N\|_1 \|\varphi_t\|_\infty \big(\|\varphi_t\|_\infty + \|\varphi_t^{(N)}\|_\infty\big)\, \|\varphi_t - \varphi_t^{(N)}\|_2^2 \\ &\quad \leq C\big(\|\varphi_t\|_{H^2}^2 + \|\varphi_t^{(N)}\|_{H^2}^2\big)\, \|\varphi_t - \varphi_t^{(N)}\|_2^2.\end{aligned} \tag{A.5}$$

As for the first term on the r.h.s. of (A.4), we find (since $\int U_N(y)dy = 8\pi a_0$)

$$\begin{aligned}\big|\big\langle \varphi_t, \big(U_N * |\varphi_t|^2 - 8\pi a_0|\varphi_t|^2\big)\varphi_t^{(N)}\big\rangle\big| \\ &= \left|\int dx\, \overline{\varphi}_t(x)\varphi_t^{(N)}(x)\int dy\, U_N(y)\left(|\varphi_t(x-y)|^2 - |\varphi_t(x)|^2\right)\right| \\ &\leq \int dxdy\, U_N(y)\,|\varphi_t(x)||\varphi_t^{(N)}(x)|\,\big||\varphi_t(x-y)|^2 - |\varphi_t(x)|^2\big|.\end{aligned}$$

Writing $U_N(x) = N^3U(Nx)$, with $U(x) = V(x)f(x)$ and changing integration variables, we find

$$\begin{aligned}\big|\big\langle \varphi_t, \big(U_N*|\varphi_t|^2 - 8\pi a_0|\varphi_t|^2\big)\varphi_t^{(N)}\big\rangle\big| \\ &\leq \int dxdyU(y)|\varphi_t(x)||\varphi_t^{(N)}(x)|\,\big||\varphi_t(x-y/N)|^2 - |\varphi_t(x)|^2\big|.\end{aligned}$$

Using

$$\begin{aligned}\big||\varphi_t(x-y/N)|^2 - |\varphi_t(x)|^2\big| &= \left|\int_0^1 ds\frac{d}{ds}|\varphi_t(x-sy/N)|^2\right| \\ &\leq 2|y|N^{-1}\int_0^1 ds\,|\nabla\varphi_t(x-sy/N)||\varphi_t(x-sy/N)|\end{aligned}$$

we conclude that

$$\begin{aligned}&\left|\left\langle \varphi_t, \big(U_N * |\varphi_t|^2 - 8\pi a_0|\varphi_t|^2\big)\varphi_t^{(N)}\right\rangle\right| \\ &\leq 2N^{-1}\int dxdy\int_0^1 ds\, U(y)|y||\varphi_t(x)||\varphi_t^{(N)}(x)| \\ &\qquad\qquad \times|\nabla\varphi_t(x-sy/N)||\varphi_t(x-sy/N)| \\ &\leq 2N^{-1}\|\varphi_t\|_\infty^2\int dxdy\int_0^1 ds\, U(y)|y|\big(|\varphi_t^{(N)}(x)|^2 + |\nabla\varphi_t(x-sy/N)|^2\big) \\ &\leq CN^{-1}\|\varphi_t\|_\infty^2\big(\|\varphi_t^{(N)}\|_2^2 + \|\nabla\varphi_t\|_2^2\big) \leq CN^{-1}\|\varphi_t\|_{H^2}^2\end{aligned} \tag{A.6}$$

where the constant C depends on $\int dyU(y)|y|$ and on $\|\varphi\|_{H^1}$. Inserting (A.6) and (A.5) into (A.4), and using the estimate from part (ii) for $\|\varphi_t\|_{H^2}$, we find

$$\partial_t\|\varphi_t^{(N)} - \varphi_t\|_2^2 \leq Ce^{K|t|}\|\varphi_t^{(N)} - \varphi_t\|_2^2 + \frac{C}{N}e^{K|t|}.$$

The claim now follows from Gronwall's inequality, since $\varphi_{t=0} = \varphi_{t=0}^{(N)}$. □

Appendix B: Properties of the kernel k_t

This section is devoted to the proof of Lemma 3.3 and Lemma 3.4.

Proof of Lemma 3.3. Recall that C here may depend on $\|\varphi_t^{(N)}\|_{H^1}$.

(i) We will make use of the bounds (3.5). The first bound implies immediately that

$$|k_t(x,y)| \leq \min\left(N|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)|, \frac{1}{|x-y|}|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)|\right) \tag{B.1}$$

and therefore, by Hardy's inequality,

$$\|k_t\|_2^2 \leq C\int dxdy \frac{1}{|x-y|^2}|\varphi_t^{(N)}(x)|^2|\varphi_t^{(N)}(y)|^2 \leq C\|\varphi_t^{(N)}\|_{H^1}^2\|\varphi_t^{(N)}\|_2^2 \leq C.$$

As for the gradient of k_t, we have

$$\begin{aligned}\nabla_1 k_t(x,y) = &-N^2\nabla w(N(x-y))\varphi_t^{(N)}(x)\varphi_t^{(N)}(y)\\ &-Nw(N(x-y))\nabla\varphi_t^{(N)}(x)\varphi_t^{(N)}(y)\end{aligned}$$

and thus, from the second bound in (3.5),

$$\begin{aligned}\|\nabla_1 k_t\|_2^2 &\leq C\int dxdy\, \frac{N^4}{(N^2|x-y|^2+1)^2}|\varphi_t^{(N)}(x)|^2|\varphi_t^{(N)}(y)|^2\\ &\leq CN\int dx \frac{N^3}{(N^2|x|^2+1)^2}\|\varphi_t^{(N)}\|_{H^1}^4 \leq CN\end{aligned}$$

where we used Young and then Sobolev inequalities. Next we compute

$$\begin{aligned}\nabla_1(k_t\bar{k}_t)(x,y) &= \nabla_x\int dz k_t(x,z)\bar{k}_t(z,y)\\ &= \nabla_x\left[\varphi_t^{(N)}(x)\varphi_t^{(N)}(y)\int dz N^2 w(N(x-z))w(N(z-y))|\varphi_t^{(N)}(z)|^2\right]\\ &= \nabla\varphi_t^{(N)}(x)\varphi_t^{(N)}(y)\int dz N^2 w(N(x-z))w(N(z-y))|\varphi_t^{(N)}(z)|^2\\ &\quad+\varphi_t^{(N)}(x)\varphi_t^{(N)}(y)\int dz N^3\nabla w(N(x-z))w(N(z-y))|\varphi_t^{(N)}(z)|^2.\end{aligned}$$

Using (3.5), we find

$$\begin{aligned}&\|\nabla_1(k_t\bar{k}_t)\|_2^2\\ &\leq C\int dxdydz_1dz_2\, \frac{|\nabla\varphi_t^{(N)}(x)|^2|\varphi_t^{(N)}(y)|^2|\varphi_t^{(N)}(z_1)|^2|\varphi_t^{(N)}(z_2)|^2}{|x-z_1||z_1-y||x-z_2||z_2-y|}\\ &\quad+C\int dxdydz_1dz_2\, \frac{|\varphi_t^{(N)}(x)|^2|\varphi_t^{(N)}(y)|^2|\varphi_t^{(N)}(z_1)|^2|\varphi_t^{(N)}(z_2)|^2}{|x-z_1|^2|z_1-y||x-z_2|^2|z_2-y|}\\ &\leq C\int dxdydz_1dz_2\, \frac{|\nabla\varphi_t^{(N)}(x)|^2|\varphi_t^{(N)}(y)|^2|\varphi_t^{(N)}(z_1)|^2|\varphi_t^{(N)}(z_2)|^2}{|x-z_1|^2|z_2-y|^2}\\ &\quad+C\int dxdydz_1dz_2\, \frac{|\varphi_t^{(N)}(x)|^2|\varphi_t^{(N)}(y)|^2|\varphi_t^{(N)}(z_1)|^2|\varphi_t^{(N)}(z_2)|^2}{|x-z_1|^2|z_1-y|^2|x-z_2|^2}\\ &\leq C\|\varphi_t^{(N)}\|_{H^1}^3\|\varphi_t^{(N)}\|_2^2.\end{aligned}$$

(ii) The pointwise bound for $k_t(x,y)$ follows directly from (3.5), as noticed in (B.1). To bound $|r(k_t)(x,y)|$, we observe that, by Hölder's inequality, (3.5) and part (i),

$$\begin{aligned}
&|(k_t\bar{k}_t)^n k_t(x,y)| = |k_t\bar{k}_t(k_t\bar{k}_t)^{n-1}k_t(x,y)| \\
&= |\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)| \\
&\qquad \times \left| \int dz_1 dz_2 N^2 w(N(x-z_1)) w(N(z_2-y)) \right. \\
&\qquad\qquad \left. \times \varphi_t^{(N)}(z_1)\varphi_t^{(N)}(z_2)\bar{k}_t(k_t\bar{k}_t)^{n-1}(z_1,z_2) \right| \\
&\leq |\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)| \, \|\bar{k}_t(k_t\bar{k}_t)^{n-1}\|_2 \left(\int dz_1 N^2 w(N(x-z_1))^2 |\varphi_t^{(N)}(z_1)|^2 \right. \\
&\qquad\qquad \left. \times \int dz_2 N^2 w(N(z_2-y))^2 |\varphi_t^{(N)}(z_2)|^2 \right)^{1/2} \\
&\leq C|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)| \|\nabla\varphi_t^{(N)}\|_2^2 \|k\|_2^{2n-1}.
\end{aligned}$$

Thus

$$|r(k_t)(x,y)| \leq \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} |(k_t\bar{k}_t)^n k(x,y)| \leq C|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)| e^{\|k\|_2}.$$

The pointwise estimate for $p(k_t)(x,y)$ can be proven similarly. This completes the proof of part (ii). Part (iii) follows easily from the pointwise bounds in part (ii). □

Proof of Lemma 3.4. In the following proof we will use the bounds

$$\|\dot{\varphi}_t^{(N)}\|_{H^2}, \|\ddot{\varphi}_t^{(N)}\|_2 \leq Ce^{K|t|}$$

from Proposition 3.1.

(i) From (3.7), we find

$$\begin{aligned}
\|\dot{k}_t\|^2 &\leq 4\int dxdy \frac{1}{|x-y|^2} |\dot{\varphi}_t^{(N)}(x)|^2 |\varphi_t^{(N)}(y)|^2 \\
&\leq C\|\dot{\varphi}_t^{(N)}\|_2^2 \|\varphi_t^{(N)}\|_{H^1}^2 \leq Ce^{K|t|}
\end{aligned} \tag{B.2}$$

by Hardy's inequality, and from Proposition 3.1, part (iii). Similarly,

$$\|\ddot{k}_t\|_2 \leq C\|\ddot{\varphi}_t^{(N)}\|_2 + C\|\dot{\varphi}_t^{(N)}\|_2 \|\nabla\dot{\varphi}_t^{(N)}\|_2 \leq Ce^{K|t|}$$

by Proposition 3.1. Writing $p(k_t) = \sum_{n\geq 0}(k_t\bar{k}_t)^n/(2n!)$, we find immediately

$$\|\dot{p}(k_t)\|_2 \leq \sum_{n\geq 0} \frac{1}{(2n)!} n \|\dot{k}_t\|_2 \|k_t\|_2^{2n-1} \leq \|\dot{k}_t\|_2 \, e^{\|k_t\|_2} \leq Ce^{K|t|}$$

applying (B.2). The bound for $\dot{r}(k_t)$ can be proven analogously.

(ii) From 2.3, part (v), we have

$$\|\nabla_1 \dot{p}(k_t)\|_2 \leq Ce^{\|k_t\|_2} \big(\|\dot{k}_t\|_2 \, \|\nabla_1(k_t\bar{k}_t)\|_2 + C\|\nabla_1(k_t\dot{\bar{k}}_t)\|_2 + C\|\nabla_1(\dot{k}_t\bar{k}_t)\|_2 \big).$$

We are left with the task of estimating $\|\nabla_1(k_t\dot{\overline{k}}_t)\|_2$ and $\|\nabla_1(\dot{k}_t\overline{k}_t)\|_2$. We start by applying the product rule:

$$\begin{aligned}
&\|\nabla_1(k_t\dot{\overline{k}}_t)\|_2^2\\
&= \int dxdy \Big|\nabla_x \int dz\big[Nw(N(x-z))\dot{\overline{\varphi}}_t^{(N)}(x)\overline{\varphi}_t^{(N)}(z) \qquad \text{(B.3)}\\
&+Nw(N(x-z))\,\overline{\varphi}_t^{(N)}(x)\dot{\overline{\varphi}}_t^{(N)}(z)\big]Nw(N(z-y))\varphi_t^{(N)}(z)\varphi_t^{(N)}(y)\Big|^2\\
&\le 4\int dxdy\Big|\int dzN^2\nabla w(N(z-x)) \qquad \text{(B.4)}\\
&\qquad \times\dot{\overline{\varphi}}_t^{(N)}(x)|\varphi_t^{(N)}(z)|^2\varphi_t^{(N)}(y)Nw(N(y-z))\Big|^2\\
&+4\int dxdy\Big|\int dzNw(N(z-x)) \qquad \text{(B.5)}\\
&\qquad \times\nabla\dot{\overline{\varphi}}_t^{(N)}(x)|\varphi_t^{(N)}(z)|^2\varphi_t^{(N)}(y)Nw(N(y-z))\Big|^2\\
&+4\int dxdy\Big|\int dzN^2\nabla w(N(z-x))\overline{\varphi}_t^{(N)}(x)\dot{\overline{\varphi}}_t^{(N)}(z) \qquad \text{(B.6)}\\
&\qquad \times\varphi_t^{(N)}(z)\varphi_t^{(N)}(y)\,Nw(N(y-z))\Big|^2\\
&+4\int dxdy\Big|\int dzNw(N(z-y)) \qquad \text{(B.7)}\\
&\qquad \times\nabla\overline{\varphi}_t^{(N)}(x)\dot{\overline{\varphi}}_t^{(N)}(z)\varphi_t^{(N)}(z)\varphi_t^{(N)}(y)Nw(N(y-z))\Big|^2.
\end{aligned}$$

Next, we estimate the four terms on the r.h.s. of the last equation. For the summands (B.5) and (B.7) we use that $Nw(Nx)\le C|x|^{-1}$ from (3.5). By Hardy's inequality, both terms are bounded by $C\|\nabla\dot{\varphi}_t^{(N)}\|_2^2\le C\exp(K|t|)$, using Proposition 3.1. Since, again by (3.5), $N^2\nabla w(Nx)\le C|x|^{-2}$, the contribution (B.4) is bounded by

$$\begin{aligned}
&C\int dxdy\left(\int dz\frac{1}{|x-z|^2|z-y|}|\dot{\varphi}_t^{(N)}(x)||\varphi_t^{(N)}(z)|^2|\varphi_t^{(N)}(y)|\right)^2\\
&=C\int dxdydz_1dz_2\frac{|\dot{\varphi}_t^{(N)}(x)|^2|\varphi_t^{(N)}(y)|^2|\varphi_t^{(N)}(z_1)|^2|\varphi_t^{(N)}(z_2)|^2}{|z_1-y||z_2-y||x-z_1|^2|x-z_2|^2}\\
&=C\int dx\,|\dot{\varphi}_t^{(N)}(x)|^2\int dz_1dz_2\frac{|\varphi_t^{(N)}(z_1)|^2|\varphi_t^{(N)}(z_2)|^2}{|x-z_1|^2|x-z_2|^2}\int dy\,\frac{|\varphi_t^{(N)}(y)|^2}{|z_1-y||z_2-y|}\\
&\le C\|\dot{\varphi}_t^{(N)}\|_2^2\|\varphi_t^{(N)}\|_{H^1}^6\le Ce^{K|t|}
\end{aligned}$$

by Proposition 3.1. Analogously, we can also bound the contribution (B.6). This shows the bound for $\|\nabla_1 \dot{p}(k_t)\|_2$. The bounds for $\|\nabla_2 \dot{p}(k_t)\|_2$ and $\|\nabla_2 \dot{p}(k_t)\|_2$, and $\|\nabla_2 \dot{p}(k_t)\|_2$ are proven similarly.

(iii) From (3.7), using $Nw(Nx) \leq C|x|^{-1}$, we find immediately that

$$\sup_x \|\dot{k}_t(\cdot,x)\|_2 \leq C\big(\|\nabla \dot{\varphi}_t^{(N)}\|_2 \|\varphi_t^{(N)}\|_\infty + \|\nabla \varphi_t^{(N)}\|_2 \|\dot{\varphi}_t^{(N)}\|_\infty\big) \leq Ce^{K|t|}$$

by Proposition 3.1. To show the bound for $\dot{p}(k_t)$, we observe that

$$\begin{aligned} \dot{p}(k_t)(x,y) &= \partial_t \sum_{n\geq 0} \frac{1}{(2n)!} \int dz_1 dz_2\, k_t(x,z_1)(\bar{k}_t k_t)^{(n-1)}(z_1,z_2)\bar{k}_t(z_2,y) \\ &= \sum_{n\geq 0} \frac{1}{(2n)!} \Big[\int dz_1 dz_2\, \dot{k}_t(x,z_1)(\bar{k}_t k_t)^{(n-1)}(z_1,z_2)\bar{k}_t(z_2,y) \\ &\qquad + \int dz_1 dz_2\, k_t(x,z_1)(\partial_t (\bar{k}_t k_t)^{(n-1)})(z_1,z_2)\bar{k}_t(z_2,y) \\ &\qquad + \int dz_1 dz_2\, k_t(x,z_1)(\bar{k}_t k_t)^{(n-1)}(z_1,z_2)\dot{\bar{k}}_t(z_2,y)\Big]. \end{aligned} \tag{B.8}$$

The first term in the parenthesis can be bounded in absolute value by

$$\begin{aligned} &\Big|\int dz_1 dz_2\, \dot{k}_t(x,z_1)(\bar{k}_t k_t)^{(n-1)}(z_1,z_2)\bar{k}_t(z_2,y)\Big| \\ &\leq C\int dz_1 dz_2\, \frac{1}{|x-z_1|}\big(|\dot{\varphi}_t^{(N)}(x)||\varphi_t^{(N)}(z_1)| + |\varphi_t^{(N)}(x)||\dot{\varphi}_t^{(N)}(z_1)|\big) \\ &\qquad \times |(\bar{k}_t k_t)^{(n-1)}(z_1,z_2)|\,\frac{1}{|y-z_2|}|\varphi_t^{(N)}(z_2)||\varphi_t^{(N)}(y)| \\ &\leq C|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)|\int dz_1 dz_2\, \frac{1}{|x-z_1||y-z_2|} \\ &\qquad \times |\dot{\varphi}_t^{(N)}(z_1)||\varphi_t^{(N)}(z_2)|\,|(\bar{k}_t k_t)^{(n-1)}(z_1,z_2)| \\ &\quad + C|\dot{\varphi}_t^{(N)}(x)||\varphi_t^{(N)}(y)|\int dz_1 dz_2\, \frac{1}{|x-z_1||y-z_2|} \\ &\qquad \times |\varphi_t^{(N)}(z_1)||\varphi_t^{(N)}(z_2)|\,|(\bar{k}_t k_t)^{(n-1)}(z_1,z_2)| \\ &\leq C\|(\bar{k}_t k_t)^{n-1}\|_2 \big(|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)|\|\nabla\dot{\varphi}_t^{(N)}\|\|\nabla\varphi_t^{(N)}\| \\ &\qquad + |\dot{\varphi}_t^{(N)}(x)||\varphi_t^{(N)}(y)|\|\nabla\varphi_t^{(N)}\|^2\big). \\ &\leq C\|k_t\|_2^{2(n-1)}(|\dot{\varphi}_t^{(N)}(x)| + |\varphi_t^{(N)}(x)|)|\varphi_t^{(N)}(y)|. \end{aligned}$$

The last term in the parenthesis on the r.h.s. of (B.8) can be bounded analogously. The middle term, on the other hand is bounded in absolute value by

$$\begin{aligned}
&\Big|\int dz_1 dz_2\, k_t(x,z_1)(\partial_t(\bar{k}_t k_t)^{(n-1)})(z_1,z_2)\bar{k}_t(z_2,y)\Big| \\
&\quad \le C|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)| \\
&\qquad \times \int dz_1 dz_2 \frac{|\varphi_t^{(N)}(z_1)||\varphi_t^{(N)}(z_2)|}{|x-z_1||y-z_2|}|\partial_t(\bar{k}_t k_t)^{n-1}(z_1,z_2)| \\
&\quad \le C\|\partial_t(\bar{k}_t k_t)^{n-1}\|_2\, \|\nabla\varphi_t^{(N)}\|_2^2\, |\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)| \\
&\quad \le C\|\dot{k}_t\|_2\, \|k_t\|_2^{2n-3}|\varphi_t^{(N)}(x)||\varphi_t^{(N)}(y)|.
\end{aligned}$$

Inserting the last bounds in (B.8), we find

$$\begin{aligned}
|\dot{p}(k_t)(x,y)| \le Ce^{K|t|}e^{\|k_t\|_2}\,(|\varphi_t^{(N)}(x)|+|\dot{\varphi}_t^{(N)}(x)|) \\
\times(|\varphi_t^{(N)}(y)|+|\dot{\varphi}_t^{(N)}(y)|).
\end{aligned}$$

Integrating over x and taking the supremum over y gives, as before using (3.2),

$$\sup_y \|\dot{p}(k_t)(\cdot,y)\|_2 \le Ce^{K|t|}.$$

The bound for $\dot{r}(k_t)$ can be proven analogously. Combining the bound for $\dot{r}(k_t)$ with the one for $\dot{k}_t$, we also obtain the bound for $\dot{\text{sh}}(k_t)$. □

Appendix C: Convergence for N-particle states

In this section, we show how the result of Theorem 1.1, stated there for initial data of the form $W(\sqrt{N}\varphi)T(k_0)\psi$ can be extended to a certain class of data with number of particles fixed to N.

Theorem C.1. *Let $\varphi \in H^4(\mathbb{R}^3)$ and suppose $\psi \in \mathscr{F}$ (possibly depending on N) with $\|\psi\| = 1$ is such that*

(C.1) $$\left\langle \psi, \left(\frac{\mathcal{N}^2}{N} + \mathcal{N} + \mathscr{H}_N\right)\psi\right\rangle \le D$$

for a constant $D > 0$ (independent of N). Let P_N denote the projection onto the N-particle sector of the Fock space and assume that

(C.2) $$\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\| \ge CN^{-1/4}$$

for all $N \in \mathbb{N}$ large enough. We consider the time evolution

$$\psi_{N,t} = e^{-i\mathscr{H}_N t}\frac{P_N W(\sqrt{N}\varphi)T(k_0)\psi}{\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|}$$

and we denote by $\gamma^{(1)}_{N,t}$ *the one-particle reduced density associated with the* N*-particle state* $\psi_{N,t}$*. Then there exist constants* $C, c_1, c_2 > 0$ *with*

$$\mathrm{Tr}\left|\gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\right| \leq \frac{C\exp\left(c_1 \exp(c_2|t|)\right)}{N^{1/4}}$$

for all $t \in \mathbb{R}$ *and all* N *large enough. Here* φ_t *denotes the solution of the time-dependent Gross-Pitaevskii equation (1.37), with initial data* $\varphi_{t=0} = \varphi$*.*

Remark 1. If we relax (C.2) to the weaker condition

$$\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\| \geq CN^{-\alpha},$$

for some $1/4 \leq \alpha < 1/2$, the proof below still implies the convergence $\gamma^{(1)}_{N,t} \to |\varphi_t\rangle\langle\varphi_t|$ but only with the slower rate $N^{-1/2+\alpha}$.

Remark 2. The assumption (C.2) and its weaker versions mentioned in the previous remark are very reasonable; let us explain why. The expected number of particles in the Fock state $W(\sqrt{N}\varphi)T(k_0)\psi$ is given by

$$\begin{aligned}\text{(C.3)}\quad &\left\langle W(\sqrt{N}\varphi)T(k_0)\psi, \mathcal{N} W(\sqrt{N}\varphi)T(k_0)\psi\right\rangle \\ &\qquad = N + \sqrt{N}\langle T(k_0)\psi, \phi(\varphi)T(k_0)\psi\rangle + \langle T(k_0)\psi, \mathcal{N} T(k_0)\psi\rangle\end{aligned}$$

with the notation $\phi(\varphi) = a(\varphi) + a^*(\varphi)$. Let us introduce the shorthand notation

$$\langle \mathcal{N}\rangle := \left\langle W(\sqrt{N}\varphi)T(k_0)\psi, \mathcal{N} W(\sqrt{N}\varphi)T(k_0)\psi\right\rangle.$$

From Lemma 2.1, Lemma 4.3 and the assumption (C.1) we conclude that there exists a constant $C > 0$ with

$$N - CN^{1/2} \leq \langle \mathcal{N}\rangle \leq N + CN^{1/2}.$$

The expectation of $\mathcal{N}^2$, on the other hand, is given by

$$\begin{aligned}&\left\langle W(\sqrt{N}\varphi)T(k_0)\psi, \mathcal{N}^2 W(\sqrt{N}\varphi)T(k_0)\psi\right\rangle \\ &= \langle T(k_0)\psi, (\mathcal{N} + \sqrt{N}\phi(\varphi) + N)^2 T(k_0)\psi\rangle \\ &= N^2 + 2N^{3/2}\langle T(k_0)\psi, \phi(\varphi)T(k_0)\psi\rangle + 2N\langle T(k_0)\psi, \mathcal{N} T(k_0)\psi\rangle \\ &\quad + N\langle T(k_0)\psi, \phi(\varphi)^2 T(k_0)\psi\rangle + \sqrt{N}\langle T(k_0)\psi, (\mathcal{N}\phi(\varphi) + \phi(\varphi)\mathcal{N})T(k_0)\psi\rangle \\ &\quad + \langle T(k_0)\psi, \mathcal{N}^2 T(k_0)\psi\rangle.\end{aligned}$$

Subtracting the square of (C.3), and applying again Lemma 2.1, Lemma 4.3 and the assumption (C.1), we estimate the variance of the number of particles in the state $W(\sqrt{N}\varphi)T(k_0)\psi$ by

$$\left\langle W(\sqrt{N}\varphi)T(k_0)\psi, (\mathcal{N} - \langle\mathcal{N}\rangle)^2 W(\sqrt{N}\varphi)T(k_0)\psi\right\rangle \leq CN$$

for an appropriate constant $C > 0$. We conclude that

$$\left\langle W(\sqrt{N}\varphi)T(k_0)\psi, \mathbf{1}\left(|\mathcal{N} - \langle\mathcal{N}\rangle| \geq K\sqrt{N}\right) W(\sqrt{N}\varphi)T(k_0)\psi\right\rangle \leq CK^{-2}.$$

Choosing $K > 0$ sufficiently large, we find

$$\left\langle W(\sqrt{N}\varphi)T(k_0)\psi, \mathbf{1}\left(|\mathcal{N} - \langle \mathcal{N} \rangle| \leq K\sqrt{N}\right) W(\sqrt{N}\varphi)T(k_0)\psi \right\rangle \geq 1/2.$$

From (C.3), adjusting the value of K, we obtain

$$\left\langle W(\sqrt{N}\varphi)T(k_0)\psi, \mathbf{1}\left(|\mathcal{N} - N| \leq K\sqrt{N}\right) W(\sqrt{N}\varphi)T(k_0)\psi \right\rangle \geq 1/2.$$

This means that

$$\sum_{j=N-K\sqrt{N}}^{N+K\sqrt{N}} \left\| P_j W(\sqrt{N}\varphi)T(k_0)\psi \right\|^2 \geq 1/2.$$

The average value of $\|P_j W(\sqrt{N}\varphi)T(k_0)\psi\|^2$ for j between $N - K\sqrt{N}$ and $N + K\sqrt{N}$ is therefore larger or equal to $N^{-1/2}$, in accordance with the assumption (C.2). In fact, this argument shows that for every N there exists an $M \in [N - K\sqrt{N}, N + K\sqrt{N}]$ with $\|P_M W(\sqrt{N}\varphi)T(k_0)\psi\| \geq N^{-1/4}$. Letting

$$\psi_{N,M,t} = e^{-i\mathcal{H}_N t} \frac{P_M W(\sqrt{N}\varphi)T(k_0)\psi}{\|P_M W(\sqrt{N}\varphi)T(k_0)\psi\|}$$

and denoting by $\gamma^{(1)}_{N,M,t}$ the one-particle reduced density associated with $\psi_{N,M,t}$, one can show, similarly to Theorem C.1, that

$$\operatorname{Tr} \left| \gamma^{(1)}_{N,M,t} - |\varphi_t\rangle\langle\varphi_t| \right| \leq \frac{C \exp\left(c_1 \exp(c_2|t|)\right)}{N^{1/4}}$$

The fact that the number of particles M does not exactly match the parameter N entering the Hamiltonian and the Weyl operator $W(\sqrt{N}\varphi)$ does not affect the analysis in any substantial way, since $|M - N| \leq CN^{1/2} \ll N$.

Proof of Theorem C.1. We write the integral kernel of $\gamma^{(1)}_{N,t}$ as

$$\begin{aligned}
&\gamma^{(1)}_{N,t}(x;y) \\
&= \frac{1}{N\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|^2} \\
&\qquad \times \left\langle e^{-i\mathcal{H}_N t} P_N W(\sqrt{N}\varphi)T(k_0)\psi, a^*_y a_x e^{-i\mathcal{H}_N t} P_N W(\sqrt{N}\varphi)T(k_0)\psi \right\rangle \\
&= \frac{1}{N\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|^2} \\
&\qquad \times \left\langle e^{-i\mathcal{H}_N t} P_N W(\sqrt{N}\varphi)T(k_0)\psi, W(\sqrt{N}\varphi^{(N)}_t)\left(a^*_y + \sqrt{N}\overline{\varphi}^{(N)}_t(y)\right)\right. \\
&\qquad\quad \left. \times \left(a_x + \sqrt{N}\varphi^{(N)}_t(x)\right) W^*(\sqrt{N}\varphi^{(N)}_t) e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi)T(k_0)\psi \right\rangle.
\end{aligned}$$

Hence, we find

$$\begin{aligned}
&\gamma^{(1)}_{N,t}(x;y) - \overline{\varphi}^{(N)}_t(y)\varphi^{(N)}_t(x) \\
&= \frac{\varphi^{(N)}_t(x)}{\sqrt{N}\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|^2}\big\langle e^{-i\mathcal{H}_N t}P_N W(\sqrt{N}\varphi)T(k_0)\psi, \\
&\qquad\qquad \times W(\sqrt{N}\varphi^{(N)}_t)a^*_y W^*(\sqrt{N}\varphi^{(N)}_t)e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)T(k_0)\psi\big\rangle \\
&\quad + \frac{\overline{\varphi}^{(N)}_t(y)}{\sqrt{N}\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|^2}\big\langle e^{-i\mathcal{H}_N t}P_N W(\sqrt{N}\varphi)T(k_0)\psi, \\
&\qquad\qquad \times W(\sqrt{N}\varphi^{(N)}_t)a_x W^*(\sqrt{N}\varphi^{(N)}_t)e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)T(k_0)\psi\big\rangle \\
&\quad + \frac{1}{N\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|^2}\big\langle e^{-i\mathcal{H}_N t}P_N W(\sqrt{N}\varphi)T(k_0)\psi, W(\sqrt{N}\varphi^{(N)}_t) \\
&\qquad\qquad \times a^*_y a_x W^*(\sqrt{N}\varphi^{(N)}_t)e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)T(k_0)\psi\big\rangle.
\end{aligned}$$

Therefore, for any compact operator J on $L^2(\mathbb{R}^3)$ we find

$$\begin{aligned}
&\mathrm{Tr}\, J\left(\gamma^{(1)}_{N,t} - |\varphi^{(N)}_t\rangle\langle\varphi^{(N)}_t|\right) \\
&= \frac{1}{\sqrt{N}\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|^2}\big\langle e^{-i\mathcal{H}_N t}P_N W(\sqrt{N}\varphi)T(k_0)\psi, W(\sqrt{N}\varphi^{(N)}_t) \\
&\quad \times \left(a(J\varphi^{(N)}_t) + a^*(J\varphi^{(N)}_t)\right)W^*(\sqrt{N}\varphi^{(N)}_t)e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)T(k_0)\psi\big\rangle \\
&\quad + \frac{1}{N\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|^2}\big\langle e^{-i\mathcal{H}_N t}P_N W(\sqrt{N}\varphi)T(k_0)\psi, W(\sqrt{N}\varphi^{(N)}_t) \\
&\qquad\qquad \times d\Gamma(J)\, W^*(\sqrt{N}\varphi^{(N)}_t)e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)T(k_0)\psi\big\rangle
\end{aligned}$$

where $d\Gamma(J)$ denotes the second quantization of the operator J, defined by

$$(d\Gamma(J)\psi)^{(n)} = \sum_{i=1}^n J^{(i)}\psi^{(n)}$$

for every $\psi = \{\psi^{(n)}\}_{n\geq 0} \in \mathscr{F}$ (here $J^{(i)}$ denotes the operator acting as J on the i-th particle and as the identity on the other $(n-1)$ particles). Since $\|d\Gamma(J)\psi\| \leq \|J\|\,\|\mathcal{N}\psi\|$, we find, applying Lemma 2.1,

$$\begin{aligned}
&\left|\mathrm{Tr}\, J\left(\gamma^{(1)}_{N,t} - |\varphi^{(N)}_t\rangle\langle\varphi^{(N)}_t|\right)\right| \\
&\leq \frac{\|J\|}{\sqrt{N}\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|} \\
&\quad \times \|(\mathcal{N}+1)^{1/2}W^*(\sqrt{N}\varphi^{(N)}_t)e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)T(k_0)\psi\| \\
&\quad + \frac{\|J\|}{N\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|}\|\mathcal{N}\, W^*(\sqrt{N}\varphi^{(N)}_t)e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)T(k_0)\psi\|
\end{aligned}$$

where $\|J\|$ denotes the operator norm of J. From Lemma 4.3 and recalling the definition (3.8) of the fluctuation dynamics, we find

$$\begin{aligned}
&\left|\operatorname{Tr} J\left(\gamma_{N,t}^{(1)} - |\varphi_t^{(N)}\rangle\langle\varphi_t^{(N)}|\right)\right| \\
&\leq \frac{C\|J\|}{\sqrt{N}\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|} \\
&\qquad \times \|(\mathcal{N}+1)^{1/2} T^*(k_t) W^*(\sqrt{N}\varphi_t^{(N)}) e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi)T(k_0)\psi\| \\
&\quad + \frac{C}{N\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|} \\
&\qquad \times \|(\mathcal{N}+1) T^*(k_t) W^*(\sqrt{N}\varphi_t^{(N)}) e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi)T(k_0)\psi\| \\
&\leq \frac{C\|(\mathcal{N}+1)^{1/2}\mathcal{U}(t;0)\psi\|}{\sqrt{N}\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|} + \frac{C\|(\mathcal{N}+1)\mathcal{U}(t;0)\psi\|}{N\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|}.
\end{aligned}$$

Using Proposition 4.2, we conclude that

$$\begin{aligned}
\left|\operatorname{Tr} J\left(\gamma_{N,t}^{(1)} - |\varphi_t^{(N)}\rangle\langle\varphi_t^{(N)}|\right)\right| &\leq \frac{C\|J\|\,\|(\mathcal{N}+1)^{1/2}\mathcal{U}(t;0)\psi\|}{\sqrt{N}\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|} \\
&\quad + \frac{C\|(\mathcal{N}+1)^{1/2}\psi\|}{\sqrt{N}\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|} + \frac{C\|(\mathcal{N}+1)\psi\|}{N\|P_N W(\sqrt{N}\varphi)T(k_0)\psi\|}.
\end{aligned}$$

From the assumptions (C.1) and (C.2), we obtain

$$\left|\operatorname{Tr} J\left(\gamma_{N,t}^{(1)} - |\varphi_t^{(N)}\rangle\langle\varphi_t^{(N)}|\right)\right| \leq \frac{C\|J\|}{N^{1/4}}\|(\mathcal{N}+1)^{1/2}\mathcal{U}(t;0)\psi\| + \frac{C}{N^{1/4}}.$$

Finally, Theorem 4.1 implies that

$$\left|\operatorname{Tr} J\left(\gamma_{N,t}^{(1)} - |\varphi_t^{(N)}\rangle\langle\varphi_t^{(N)}|\right)\right| \leq \frac{C\|J\|\exp(c_1\exp(c_2|t|))}{N^{1/4}}. \tag{C.4}$$

Since the Banach space $\mathcal{L}^1(L^2(\mathbb{R}^3))$ is the dual space to the space of compact operators, equipped with the operator norm, (C.4) (together with Prop. 3.1(iv)) implies the claim. □

Acknowledgment.

G. de Oliveira and B. Schlein gratefully acknowledge support by the European Research Council Grant MAQD-240518.

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Received Month 200X.