Interconnectedness as a Source of Uncertainty in Systemic Risk [☆]

Tarik Roukny[a], Stefano Battiston[b], Joseph E. Stiglitz[c]

[a]*Université Libre de Bruxelles, E-Mail: troukny@ulb.ac.be*
[b]*Dept. Banking and Finance, University of Zurich*
[c]*Columbia University*

Abstract

Financial networks have shown to be important in understanding systemic events in credit markets. In this paper, we investigate how the structure of those networks can affect the capacity of regulators to assess the level of systemic risk. We introduce a model to compute the individual and systemic probability of default in a system of banks connected in a generic network of credit contracts and exposed to external shocks with a generic correlation structure. Even in the presence of complete knowledge, we identify conditions on the network for the emergence of multiple equilibria. Multiple equilibria give rise to uncertainty in the determination of the default probability. We show how this uncertainty can affect the estimation of systemic risk in terms of expected losses. We further quantify the effects of cyclicality, leverage, volatility and correlations. Our results are relevant to the current policy discussions on new regulatory framework to deal with systemic events of distress as well as on the desirable level of regulatory data disclosure.

Keywords: financial networks, systemic risk, uncertainty, regulatory framework, contagion

---

1. Introduction

The emergence of systemic risk in financial networks is receiving increasing attention in the literature (Stiglitz, 2008; Allen and Babus, 2009; Acemoglu et al., 2015a) and among

---

---

regulators (IMF, 2015; Yellen, 2013). Network effects matter for financial stability because shocks can be amplified along various channels: common funding sources can lead to a spreading of bank runs (e.g. Diamond and Dybvig, 1983; Goldstein and Pauzner, 2004); balance sheet interlocks (e.g. loans, repurchasing agreement, derivatives, etc.) can lead to cascades of defaults, (e.g. Allen and Gale, 2000; Eisenberg and Noe, 2001) or propagation of distress (Battiston et al., 2012); exposures to common assets can lead to a spiral of fire sales and deleveraging across banks (e.g. Caballero and Simsek, 2013; Caccioli et al., 2014).

The main focus of the literature has been so far on understanding how the structure of the financial network (along the various aforementioned channels) can mitigate or amplify systemic risk (Elsinger et al., 2006; Gai and Kapadia, 2010; Gai et al., 2011; Georg, 2013; Cont et al., 2013; Roukny et al., 2013; Elliott et al., 2014; Acemoglu et al., 2015b; Glasserman and Young, 2015). Fewer works have been devoted to understanding how the structure of the financial network can instead affect the very ability to assess systemic risk. At the regulatory level, forecasting limitations is a source of uncertainty that, in turn, can make any decision potentially harmful to the system despite the apparent need for action (Bernanke, 2009). At the individual level, uncertainty about cross exposures and losses of counterparties can trigger panic and fire sales even from healthy banks (Caballero and Simsek, 2013; Alvarez and Barlevy, 2014).

In this paper, we develop a model of a financial network that includes an interbank market as well as external assets and external sources of funds. In spirit, our model is closest to (Elsinger et al., 2006; Rogers and Veraart, 2013) who build on the frameworks introduced by (Eisenberg and Noe, 2001) in which a clearing vector of payments and a recovery rate is determined after the maturity of the contracts (i.e., *ex-post*) assuming different levels of liquidation efficiency. Instead, we are here interested in *ex-ante* valuation of contracts in order to determine probabilities of default. Furthermore, as in practice asset liquidation implies lengthy legal settlements, we assume that, in the short run, recovery rates are limited (Cont et al., 2013).

Our first contribution is thus to provide a framework that allows to compute analytically the probability of default of any subset of banks for a generic structure of the interbank network and a generic structure of shocks correlation among external assets. To our knowledge, few works have provided a simple solution to this problem (Gouriéroux et al., 2013; Glasserman and Young, 2015). The simplicity of our result stems from the analysis of the set of default conditions as a system of coupled equations in the space of shocks.

Our second contribution is to show how, even in the case of complete knowledge of the web of contracts and distributions of shocks, multiple equilibria can exist depending on the network structure. Those co-existing equilibria may include the case in which all banks default and the case in which no bank defaults, as well as intermediate cases. More precisely, we show that, under mild conditions on the balance sheets across banks, a sufficient condition for multiple equilibria to exist is that the interbank network architecture

exhibits cyclical structures, i.e. at least one closed chain of lending ties. The reason is that a closed chain of lending ties implies a mutual dependence of the interbank asset values[1].

A common way to deal with multiple equilibria is to add to the model some mechanism of equilibrium selection, in order to be able to focus only on one equilibrium[2]. However, making assumptions on the equilibrium selection process rules out by construction an intrinsic uncertainty that may be valuable to assess. Indeed, both from the point of view of a firm in a network of contracts as well as for the regulator, uncertainty is fundamental for decision making (Caballero and Simsek, 2013; De Grauwe and Ji, 2015). In this paper, our goal is precisely to characterise the uncertainty stemming from equilibrium multiplicity.

Accordingly, our third contribution is the quantification of this uncertainty, that is, the extent of the area of multiplicity of equilibria in the space of the shocks. While the realisation of each equilibrium is mathematically consistent with the set of conditions, there might be different mechanisms at work leading the system to one equilibrium rather than another. In fact, each equilibrium is the result of a coordination of actions at the agent level which, in turn, depends on each agent's belief. In this paper, we do not model explicitly the mechanisms that could lead to coordination. We focus instead on the relation between the structure of the financial network and the existence of the multiplicity[3].

We further quantify how the uncertainty is affected by the network structure and potential correlations across banks' portfolio returns. In particular, we determine the difference between the probability of default in an optimistic scenario (i.e. the equilibrium with the least number of defaulting banks) and in a pessimistic scenario (i.e. the equilibrium with the highest number of defaulting banks) and we introduce a method to measure the cost of equilibrium selection in terms of expected losses. For instance, we show that a market structure in which banks are arranged in a ring of obligations bears less uncertainty then a centralized structure in which one bank lends to and borrows from

---

[1]The existence of multiple equilibria due to mutual dependence between agents also relates to previous work on specific ring structures reported in (see chap.7 in Stiglitz and Greenwald, 2003).

[2]For example, both in (Rogers and Veraart, 2013) and in (Elliott et al., 2014), the authors rank the equilibria according to a systemic risk criteria (e.g., the number of defaults) and select the first equilibrium.

[3]There are several mechanisms that could make agents coordinate on socially good or bad equilibria. Such a coordination could be rational, if the equilibrium is consistent with the beliefs, or not. The rational case further includes sunspot equilibria as described in (Stiglitz and Greenwald, 2003). While the mechanisms leading to the good equilibrium can be very simple (i.e., rational incentives for each bank to survive), the bad equilibrium case can result from information asymmetry, interest rate dynamics, and the combination of liquidity hoarding and asset fire sales. In the latter, if agents come to believe that external assets held by their counterparties are overpriced they will hoard liquidity and sell assets, thus inducing counterparties to do the same, effectively causing a coordination on socially bad equilibria.

all others mainly due to an increase of number of cycles in the network. We also find that correlations have an ambiguous effect on uncertainty. Correlations across shocks increase uncertainty when banks balance sheets are homogenous, but decrease uncertainty for certain heterogeneous allocations of assets across banks.

The insights from this work are relevant to three current policy discussions. A first discussion concerns the lack of a satisfactory framework to deal with too-big-to-fail institutions and with systemic events of distress in the financial system (Haldane and May, 2011; BoE, 2013). In this respect, our work makes a contribution to the stream of work aimed a estimating the systemic impact of financial institutions in a network context. A second discussion concerns the level of financial data disclosure on the side of individual institutions that would be desirable for the regulator to properly assess systemic risk (Abbe et al., 2012). Our results show that the knowledge of the structure is crucial to assess systemic risk but that some level of uncertainty is intrinsic to more interconnected systems. A third discussion concerns the role of the regulator (Draghi, 2012; Miller and Zhang, 2014; De Grauwe and Ji, 2015): in the presence of multiple equilibria, the actions of the regulator can affect, voluntarily or not, the equilibrium selection. On the one hand, our model helps to quantify the expected monetary loss due to a misassessment of systemic risk. On the other hand, our model also allows to identify when instead the outcomes are very close and, thus, regulatory decisions would have limited impact.

The structure of the paper is as follows. Section (2) describes the model. Section (3) analyses necessary and sufficient conditions for the existence of multiple equilibria. Section (4) shows how to compute default probability in different scenarios and analyses the effect of the structure of the network and the effect of correlations across shocks. Section (5) introduces a method to compute the expected losses . Finally, section (6) concludes.

2. The Model

We consider a financial network with over-the-counter (OTC) credit contracts among n agents or, for simplicity, banks. We distinguish between secured contracts (i.e. banks have to post a collateral in order to receive a loan) within the banking system itself ("interbank") and contracts of banks on securities outside the banking system ("external"). Formally, we define the interbank financial network as a directed graph as follows.

Definition 1. *The network or graph G is the pair (N, E) where N is a set of nodes representing the banks and a set of edges E representing directed credit contracts between two banks going from the lender to the borrower.*

2.1. Timing of the model

The timing of the model is as follows. At time 1, banks raise funds and make investments in external and interbank assets. At time 2, the values of the external assets are shocked and updated. While the shock distribution is known at time 1, shocks are
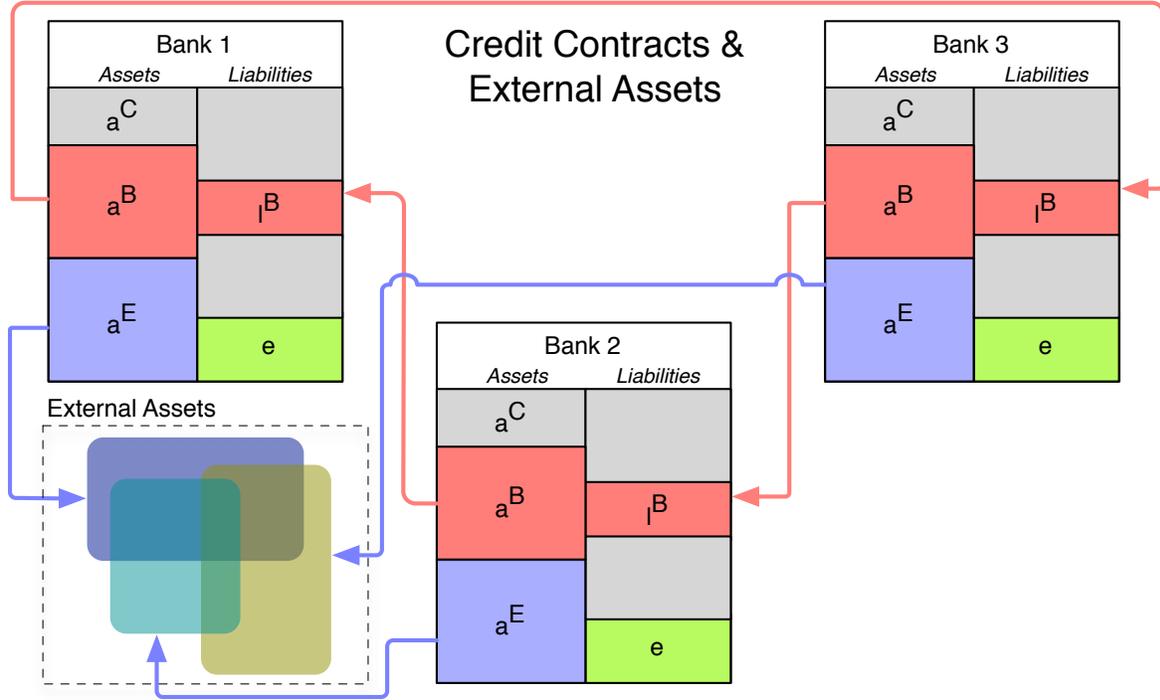
Figure 1: An illustration of an interbank market of 3 banks borrowing and lending from each other and also investing in some external assets.

only observed at time 2. At time 2 the interbank contracts mature and their value is also updated depending on the shocks that have occurred. For each bank i, the main quantities are detailed in the following section.

2.2. Balance Sheets

Assets and liabilities of i on the external markets are denoted as a_i^E and ℓ_i^E. Assets and liabilities of i on the interbank credit market are denoted as a_i^B and ℓ_i^B. Additionally, banks also hold other assets (e.g., treasury bonds) used as collateral for their interbank liabilities, denoted as a_i^C. Total assets and liabilities are respectively denoted as a_i and ℓ_i. Below we detail the different types of assets. Figure (3) illustrates the model.

2.2.1. External Assets

At time 1, each bank i allocates its external assets in a portfolio of securities on the external markets. Let E_{ik} denote the fraction of i's external assets invested at time 1 in the security k. The unitary value of the external security k is x_k^E. Without loss of generality: at time 1, $x_k^E(1) = 1$ for all k, while $x_k^E(2)$ is a random variable drawn from a given distribution. At time 2, then the external assets of bank i, is a sum of random

variables:

$$a_i^E(2) = a_i^E(1) \sum_k E_{ik} x_k^E(2). \tag{1}$$

For our purposes, it is sufficient to assume that we can express the external assets of bank i as follows:

$$a_i^E(2) = a_i^E(1)(1 + \mu_i + \sigma_i u_i), \tag{2}$$

where u_i is a random variable drawn from a given distribution with mean zero and variance one, the parameter μ_i is the expected net return of the portfolio and σ_i its standard deviation. We also assume to know the joint probability distribution $p(u_1, ..., u_n)$. Let us define $e_i(1)$ as the equity of bank i at time 1. It is then convenient to use the parameter $\varepsilon_i = \frac{a_i^E(1)}{e_i(1)}$ which measures the magnitude, per unit of initial equity of bank i, of the investments of bank i in external assets. We thus obtain:

$$\frac{a_i^E(2)}{e_i(1)} = \varepsilon_i(1 + \mu_i + \sigma_i u_i). \tag{3}$$

2.2.2. Interbank Assets

At time 1, each bank i allocates its interbank assets among the other banks, B_{ij} denoting the fraction of i's interbank assets invested at time 1 in the liability of bank j. Let us define a default indicator $\chi_j(t)$, with $\chi_j(t) = 1$ in case of default of bank i at time t and $\chi_j(t) = 0$ otherwise. The unitary value of the interbank liability of bank j to bank i is $x_{ij}^B(\chi_j(t))$.

If all banks are assume to not default at time 1, we have without loss of generality: $x_{ij}^B(\chi_j(1)) = 1$ for all i and j. The liabilities of bank j are constant in value from the perspective of bank i, i.e. the debt agreed upon in the contract at time 1. However, from the point of view of counterparties of j, $x_{ij}^B(\chi_j(2)) = 1$ if bank j honors its obligation, $x_{ij}^B(\chi_j(2)) = R_{ij}$ otherwise, where R_{ij} is the recovery rate, i.e. the fraction of assets that the lender can recover after the default of j. In formulas:

$$\forall i, j \begin{cases} x_{ij}^B(\chi_j(2) = 0) = 1 \\ x_{ij}^B(\chi_j(2) = 1) = R_{ij} \end{cases} \tag{4}$$

Accordingly, at time 2, the interbank assets of bank i, is

$$a_i^B(2) = a_i^B(1) \sum_j B_{ij} x_{ij}^B(\chi_j(2)). \tag{5}$$

Similar to the external assets, we introduce the parameter $\beta_i = \frac{a_i^B(1)}{e_i(1)}$, which measures the magnitude, per unit of initial equity, of i's investments in interbank assets. We thus

obtain:

$$\frac{a_i^B(2)}{e_i(1)} = \beta_i \sum_j x_{ij}^B(\chi_j(2)). \qquad (6)$$

2.2.3. Risk-free assets used as collateral

Finally, banks hold some assets a_i^C to be used as collateral for their borrowing. We assume that these assets are risk-free (e.g., treasury bonds) and that their value does not change across the two periods (i.e., $a_i^C(2) = a_i^C(1)$):

$$a_i^C(2) = \sum_j R_{ij} l_{ij}^B, \qquad (7)$$

where R_{ij} is the fraction of the interbank liability l_{ij}^B that is secured by the collateral. Similar to the previous cases, we introduce the parameter γ_i, which measures the magnitude, per unit of initial equity, of i's collateralised assets:

$$\frac{a_i^C(2)}{e_i(1)} = \gamma_i. \qquad (8)$$

2.3. Default Condition

As it is standard in financial accounting, we consider that an agent i defaults when its equity at time (2) is negative (i.e., $a_i(2) - l_i(2) < 0$). Assuming that the funding side remains constant between time (1) and (2) (i.e. no liability shock), we can re-express such condition in equity relative terms, that is, we divide by the value of equity at time (1) using Equations (3), (6) and (8). Note that, in equity relative terms, the liability side is equal to the value of the asset side relative to the equity at time (1) minus the equity, we obtain :

$$\varepsilon_i(1 + \mu_i + \sigma_i u_i) + \beta_i(\sum_j B_{ij} x_{ij}^B(\chi_j)) + \gamma - (\varepsilon_i + \beta_i + \gamma_i - 1) < 0. \qquad (9)$$

Finally, we express the above condition as a function of the stochastic shock variable on the external assets u_i. We obtain a condition such that if the external shock is below a threshold θ_i, this leads to the default of bank i:

$$u_i < \theta_i = \frac{-\varepsilon_i \mu_i + \beta_i(1 - \sum B_{ij} x_{ij}^B(\chi_j)) - 1}{\varepsilon_i \sigma_i}. \qquad (10)$$

Where θ_i is the threshold value below which u_i would cause the default of i. Notice the we have dropped the time in the notation. Thus, depending on the magnitude and the sign of the shock u_i on each bank, some can default on their obligations, potentially pushing other banks to default.

We can now express the default indicators χ_i of all banks as a system of equations

$$\forall i \quad \chi_i = \Theta(u_i - \theta_i(\chi_1, ..., \chi_n)), \tag{11}$$

where Θ denotes the step function (or Heaviside function, i.e. equals one if the argument is positive, zero otherwise). A solution of the system above is denoted as χ^*. We define an equilibrium as follows:

Definition 2. *An equilibrium is a vector of default indicators $\chi = \{\chi_1, ..., \chi_n\}$ that is a solution to Equation (11).*

Because Equation (11) is a system of non-linear equations, in general there can be multiple equilibria. This aspect will be analysed in Section (3).

2.4. Threshold Values

From the above definitions, it follows that the threshold value of every bank i can take a finite amount of different values depending on the identity of i's counterparties that default at time 2. Note that, in case of homogeneity, only the number of defaulting counterparties matters and not their identity. Let V_i denote the subset of banks borrowing from bank i and $|V_i|$ the cardinality of such a set. Let Λ_i be the discrete set of all values that θ_i can take, sorted by ascending order, $\Lambda_i = \{\theta_i^1, ..., \theta_i^{\lambda_i}\}$ with $\theta_i^s \leq \theta_i^t$ if $0 < s < t < \lambda_i$ and $s, t \in \mathbb{N}$. As every counterparty can have 2 states (i.e., $\chi_j = 1$ or $\chi_j = 0$), we denote the number of values that θ_i can take as λ_i such that:

$$\lambda_i = |\Lambda_i| \leq 2^{|V_i|}. \tag{12}$$

We can characterize the minimum and maximum values that the threshold can take, respectively θ_i^1 and $\theta_i^{\lambda_i}$. For convenience, we will refer to those values as θ_i^- and θ_i^+. Intuitively, the former corresponds to the case where all counterparties do not default (i.e., $x_{ij}^B = 1 \quad \forall j \in V_i$) while the latter corresponds to the case where all counterparties default (i.e., $x_{ij}^B = R_{ij} \quad \forall j \in V_i$). From Equation (10) it follows:

$$\begin{cases} \theta_i^- = \frac{-\varepsilon_i \mu_i - 1}{\varepsilon_i \sigma_i} \\ \theta_i^+ = \frac{-\varepsilon_i \mu_i + \beta_i(1 - \sum_j B_{ij} R_{ij}) - 1}{\varepsilon_i \sigma_i} = \theta_i^- + \frac{\beta_i(1 - R_i)}{\varepsilon_i \sigma_i}, \end{cases} \tag{13}$$

where for convenience we denote by $R_i = \sum_j B_{ij} R_{ij}$ the total amount of collateral that bank i recovers from the default of its counterparties.

The equation above shows that as long as $R_i < 1$ and the parameters $\varepsilon_i, \sigma_i, \beta_i$ are positive, it follows that $\theta_i^- < \theta_i^+$. This reflects the fact that it is easier to default when all counterparties have defaulted than when none has defaulted. Indeed, a larger value of the threshold implies that more shocks will fall below the threshold.

An equivalent way to interpret θ_i^- is to view it as the threshold condition below which a shock would lead bank i to default irrespective of its counterparties' default state (recall

8

that in our model shocks on the external assets can be positive or negative). Indeed, if the shock is below θ_i^-, bank i defaults even if none of its counterparties defaults. Hence, it does not matter whether there are some default or not, the shock will always lead to the default of bank i. In contrast, θ_i^+ represents the minimal shock needed to sustain the worse case scenario, i.e. when bank i has lost all its investments in the interbank market, apart from the collateral.

In the remainder of this section, we discuss the parameter's value of interest for our study. Given that the shock variable u_i is random and assumed to be between -1 and 1, we identify conditions under which the value of u_i has an effect on the default of bank i. Let us start with the extreme cases:

- when $\theta_i \leq -1$. In this case, no matter what the value of the shock u_i is, the bank will never default.

- when $\theta_i \geq 1$. In this case, no matter what the value of the shock u_i is, the bank will always default.

- when $-1 < \theta_i < 1$. In this case, the default is a function of the shock u_i. Hence, we are interested in this range of values.

As our study will focus on thresholds whose values belong to the third point, let us analyse what are the underlying parameter conditions for that condition to hold. In order to do so, we can focus on the two extreme thresholds values, θ_i^- and θ_i^+, as any other $\theta_i \in \Lambda_i$ will be comprised between those two extremes.

For the case of θ_i^-, using Equation (13) on the above condition, we have:

$$-1 < \theta_i^- < 1 \leftrightarrow \begin{cases} -\varepsilon_i(\mu_i - \sigma_i) > 1 \\ -\varepsilon_i(\mu_i + \sigma_i) < 1 \end{cases} \qquad (14)$$

To interpret the elements $\varepsilon_i(\mu_i - \sigma_i)$ and $\varepsilon_i(\mu_i + \sigma_i)$, recall the expression of the volatility of the external portfolio at time 2 and extract the changes with time 1: $\varepsilon_i(\mu_i + u_i\sigma_i)$. Hence, the two expressions result from the extreme cases where u_i is equal to -1 and 1, respectivelly. As we are framing our accounting in terms relative to the equity, we have that the equity is equal to 1. It is thus possible to read the conditions of Equation (14) as a set of 2 conditions on the effect of the worse shock and best shock. The worse shock, that is, when $u_i = -1$, must yield a return on the external asset that is a loss larger than the equity. The best shock, that is, when $u_i = 1$, must yield a return on the external asset that can be a loss but not larger than the equity.

Similarly, for the case of θ_i^+, we have:

$$-1 < \theta_i^+ < 1 \leftrightarrow \begin{cases} -\varepsilon_i(\mu_i + \sigma_i) < 1 - \beta_i(1 - R_i) \\ -\varepsilon_i(\mu_i - \sigma_i) > 1 - \beta_i(1 - R_i) \end{cases} \qquad (15)$$

Following the same reasoning as for the interpretation on the conditions for θ_i^-, similar results hold with the subtraction of the element $\beta_i(1 - R_i)$ from the equity. Indeed, this element represents the fraction of assets covered by collateral in terms relative to the equity. Hence, $1 - \beta_i(1 - R_i)$ represents the equity of bank i after all its borrowers have defaulted and all the collateral has been transfered to bank i.

It is clear that all values of θ_i do not have to be constrained between -1 and 1. Nevertheless, those ranges are the ones of interest when focusing on default scenarios. The above exercise is thus an illustration of how to interpret the conditions under which the network of contracts and the distribution of external shocks will have an effect on the system's equilibrium.

For convenience and without loss of generality, we can thus use the following expression:

$$\hat{\theta}_i = min\{max\{\theta_i, -1\}, 1\} \tag{16}$$

where $\hat{\theta}_i$ is the result of a cut-off on both sides of the threshold variable. Indeed, as shown above, when values of θ_i exceed the range $[-1, 1]$, the results become independent of the rest of the system conditions.

3. Multiple Equilibria

We now analyse under which conditions the system of equations characterizing the default (i.e., Equation (11)) leads to multiple equilibria. In particular, we are interested in identifying the conditions on the network structure leading to unique or multiple solutions. Indeed, the structure of the network of contracts enters in the default conditions of Equation (10) via the matrix B_{ij} of interbank assets. For each bank i, the associated equation has a default thresholds θ_i that is a function of the bank's borrowing counterparties' default status. Such subset of banks is, in turn, determined by the network of contracts.

Let us define the notions of walks, paths and simple cycles, that will then be used to address the multiplicity of equilibria.

Definition 3. *A walk W_{i_1, i_k} connecting bank i_1 and i_k is a sequence of banks $(i_1, i_2, ..., i_k)$ such that the ordered pairs $(i_1 i_2)$, $(i_2 i_3)$,...,$(i_{k-1} i_k) \in E$, i.e. are edges in the network. A walk is closed if the first and last bank in the sequence are the same, and open if they are different. The length of the walk is given by the number of edges it contains, i.e. any walk $W(i_1, i_k)$ has length $k1$. A simple cycle, denoted by C_n, is a closed walk encompassing n different banks and $n - 1$ edges.*

In simple terms, in an interbank financial network, a *cycle* is thus an arrangement of contracts that can be displayed on a circle such that a bank at a position i is borrowing from its left neighbour and lending to its right neighbour. Using this definition, we can now state the necessary and sufficient condition for the system to generate multiple equilibria.

Proposition 1 (**Multiple Equilibria**). *Consider the case of N banks, with: recovery rate $R_i < 1$; interbank leverage $\beta_i > 0$; external leverage ε_i and shock variance σ_i positive and finite; shock mean μ_i finite. Multiple equilibria exist if and only if:*

1. *there exists a simple cycle C_k of credit contracts along $k \geq 2$ banks*
2. *for each bank i and its borrowing counterparty $i+1$ along the cycle C_k, it holds that $\hat{\theta}_i(\chi_1, ..., \chi_i, 0, ..., \chi_k) \neq \hat{\theta}_i(\chi_1, ..., \chi_i, 1, ..., \chi_k)$*

Proof. See Appendix A. $\qquad\square$

The necessary and sufficient conditions in the proposition above relate to two related but different aspects. The first is the network structure: there can only be multiple solutions to the Equation (11) if the network contains at least 2 banks whose default condition depend upon each other's default status. More in general, a cycle is a chain of dependencies that goes back to the first node, so that the banks involved in such a structure are indirectly lending from and borrowing to all others. Therefore their default conditions depend on all others' defaults. In contrast, an acyclic structure, implies that some banks are not lending to any other. It is thus possible to compute univocally the default state of those banks independently of any other bank. Once we know their status, the default status of their lender can be determined univocally. The same process is iterated recursively to the lenders of the lenders until a unique vector of all default states is reached (i.e., a unique equilibrium). This result is in line with the finding of previous model in a more specific context (Stiglitz and Greenwald, 2003)[Chapt. 7].

The second aspect is related to the dependencies among the default states. If, for a given shock on the external assets, the default condition of bank i does not change whether i's borrowing counterparties default or not because, for instance, both thresholds are above 1 or below -1 or equal, then there are no multiple equilibria for bank i. Accordingly, the second condition in the above Proposition, states that the default of a borrower of bank i in the cycle must imply a difference on the default condition of bank i. Note that in the Proposition we use the cutoff expression (see Equation (16)) for the thresholds in order to account for the fact that the interval between the thresholds must intersect the shock domain $[-1, 1]$.

While it might be intuitive to think that the different structure will lead to different profiles of multiplicity in the shock space, Proposition (1) allows to gains precise insights regarding specific structures. For example, several network structures are represented in Figure (2).

For the tree structure, which is acyclical, we can deduce that, for all combinations of shocks, the equilibrium will be unique. The same holds for the star-out structure. On the contrary, the complete and ring structure will display multiple equilibria as long as condition 2 of Proposition (1) holds. From condition 1 in Proposition (1), we can also state a sufficient condition on the interbank market to ensure unicity of the equilibrium.

Corollary 1. *An interbank market where banks only act as borrowers or lenders always lead to a unique equilibrium for the default state.*

This result is obtained from the fact that, in a directed graph, if the nodes have only out-going or in-coming links, for sure there cannot be a directed cycle. In other words, in a market where bank play only one role (borrowing or lending), there are no cyclical interdependencies and thus no multiplicity of equilibria.
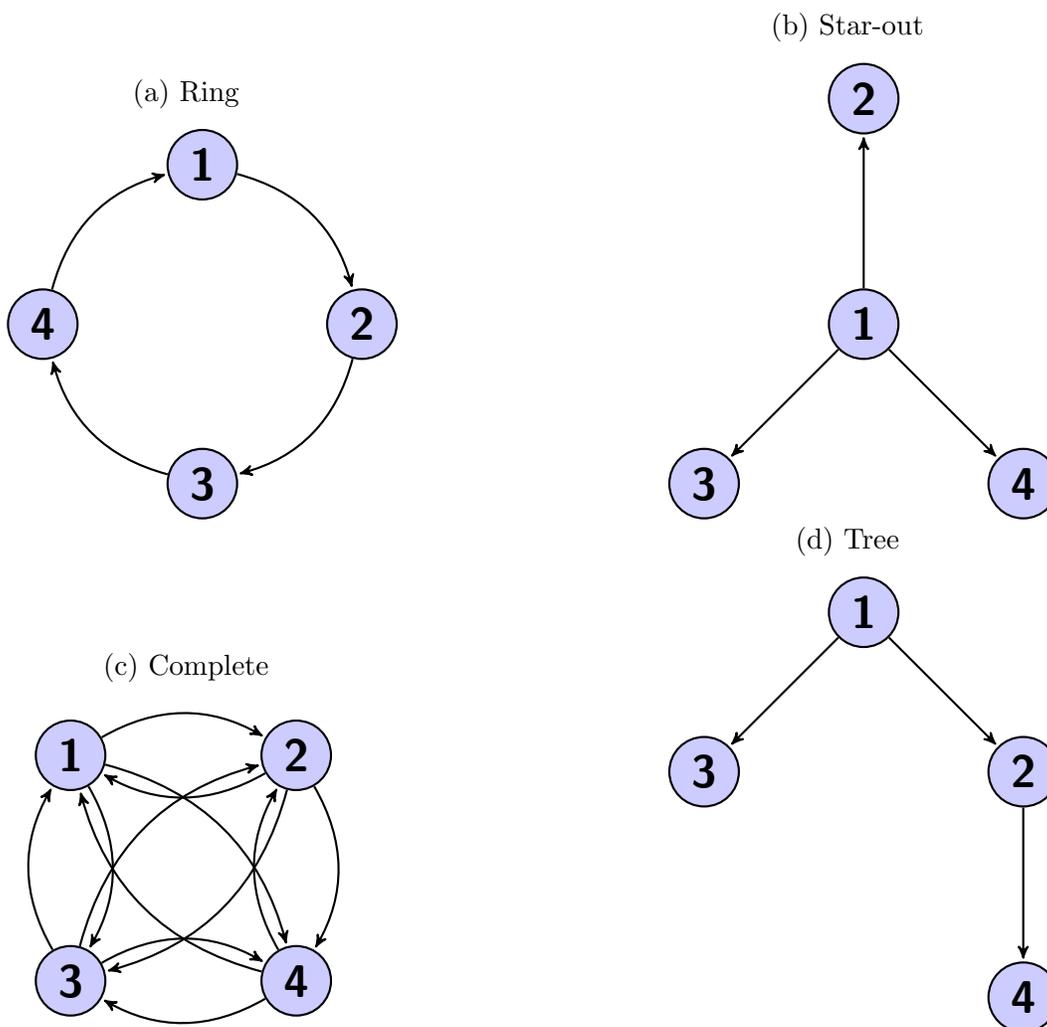

Figure 2: Example of network structures

4. Probability of Default

We now show how our model can be used to compute the individual and systemic probability of default, given a network of contracts and a joint probability density function

of shocks on external assets. In order to compute the systemic probability of default, in analogy to the individual default indicator, we define a systemic default indicator χ^{sys} as follows:

$$\chi^{sys} = \Pi_i \chi_i. \tag{17}$$

The definition implies that $\chi^{sys} = 1$ only when all banks default, i.e. $\chi_i = 1 \ \forall \ i$ and $\chi^{sys} = 0$ otherwise. Note that this definition can be modified to account for a more granular definition of systemic default. For example, if we consider as systemic default a situation in which more than half of the banks default, we have

$$\chi^{sys} = \begin{cases} 1 & \text{if } \sum_i \chi_i > \frac{n}{2} \\ 0 & \text{else} \end{cases}$$

Let us first consider the case in which Equation (11) has a unique solution for every combination of shocks. Given the individual and systemic indicators of default, we can compute the individual (resp. systemic) probability of default by integrating the individual (resp. systemic) default indicator across the whole space of shocks, weighted by the joint probability associated with each shock combination. The result is stated in the following proposition.

Proposition 2 (The expression of default probability). *Consider a market of N banks with a unique equilibrium for the default state χ. The expression of default probability of bank i P_i and the systemic default probability P^{sys} are as follows:*

$$\forall i \quad P_i = \int \chi_i(u)\, p(u)\, du, \tag{18}$$

$$P^{sys} = \int \chi^{sys}(u)\, p(u)\, du, \tag{19}$$

where χ_i is a solution of Equation (11) and $p(u)$ denotes the joint probability distribution of the shocks.

Proof. See Appendix B. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Notice that by weighting the elements of the shock space by the joint probability distribution of the shocks we account for possible correlations across shocks. Any correlation structure can be embedded in the function $p(u)$.

4.1. Multiple equilibria and probability of default

In the presence of multiple equilibria of the default state χ for a given combination of shocks, the formula of Proposition (2) cannot be applied anymore, because, when integrating over the shock space, the integrand can take several values for the same shock.

In order to overcome this problem, we resort to the idea of scenarios. Depending on the scenario, for any given shock for which multiple equilibria exist, one equilibrium is

selected according to a predefined rule. More precisely, we define the worst and the best equilibrium as the function that for any given shock u, selects the solution in which the largest (resp. the smallest) number of banks default.

Definition 4. *Given the set $\{\chi_k^{sys}(u)\}$ of all the possible solutions for the default conditions of Equation (11) for any given shock u, the worst equilibrium is the function of the shock u: $\chi^{sys\ +}(u) = max_k\{\chi_k^{sys}(u)\}$. The best equilibrium is the function of the shock u: $\chi^{sys\ -}(u) = min_k\{\chi_k^{sys}(u)\}$*

In the following we will refer to the best (worst) scenario as the one in which the best (worst) equilibrium is selected. Accordingly, we define as P^+ (respectively P^-) the systemic default probabilities in the two scenarios.

Definition 5. *The **systemic default probability** in the worst (+) ("pessimistic") scenario and best (−) ("optimistic") is defined as:*

$$P^\pm = \int \chi^{sys\pm}(u)\, p(u)\, du. \tag{20}$$

*We define as **uncertainty on the systemic default probability** the difference between the systemic default probability in the two scenarios:*

$$\Delta P = P^+ - P^-. \tag{21}$$

*We define as **uncertainty area**, the portion of the shock space in which the worst and the best equilibria are different.*

$$\Delta U = \int (\chi^{sys+}(u) - \chi^{sys-}(u))\ du. \tag{22}$$

Note that the following inequality always holds, and it holds strictly in the case of multiple equilibria.

$$\Delta P = P^+ - P^- \geq 0, \tag{23}$$

By measuring the difference between the probabilities of systemic default in the best and the worst scenario, the quantity ΔP is also a measure of the uncertainty in the assessment of P^{sys}, due to the presence of multiple equilibria.

In the following, we obtain analytical expressions for this measure and we provide results on how it depends on the network structure and the balance sheet of the banks. To make the presentation more intuitive, we start with a simple example.

4.2. Example: A Market of 2 Banks

Let us take a market composed of only 2 banks that lend to and borrow from each other. For the sake of simplicity, let us assume the distribution of shocks to be uniform

and uncorrelated. In this simple case, bank 1 and bank 2 experience respectively the shocks u_1 and u_2 on their external assets, and their default state depends on each other's default state. More precisely, their default condition threshold θ_i can take 2 values. In formulas, we have:

$$\theta_i = \begin{cases} \theta_i^- = \frac{-\varepsilon_i \mu_i - 1}{\varepsilon_i \sigma_i} & \text{if } x_{ij} = 1 \\ \theta_i^+ = \frac{-\varepsilon_i \mu_i - 1}{\varepsilon_i \sigma_i} + \frac{\beta_i(1 - R_{ij})}{\varepsilon_i \sigma_i} & \text{if } x_{ij} = R_{ij} \end{cases} \tag{24}$$

The case of 2 banks can be easily illustrated on the 2-dimensional shock space of the 2 banks, as shown in Figure (3). Recall that θ_i^- defines the threshold below which the bank defaults unconditionally while the case of θ_i^+ defines the threshold below which the bank defaults conditional on the other bank's default. The space can be divided into 4 different regions characterized by the ordered pair of default states. Recall that the value of interbank liability of bank i and its default state are related as follows: $x_{ij}^B = 1$ iff $\chi_j = 0$ (no default) and $x_{ij}^B = R_{ij}$ iff $\chi_j = 1$, see Equation (4). Therefore, we have:

$$(x_1^B, x_2^B) = \begin{cases} (1, 1) & \text{when } u_i > \theta_i^- \ \forall i; \\ (1, R) & \text{when } u_1 > \theta_1^+ \text{ and } u_2 < \theta_2^-; \\ (R, 1) & \text{when } u_1 < \theta_1^- \text{ and } u_2 > \theta_2^+; \\ (R, R) & \text{when } u_i < \theta_i^+ \ \forall i; \end{cases} \tag{25}$$

Notice that the first and fourth conditions in the list above hold simultaneously for certain values of u_1, u_2, implying that there is a multiplicity of equilibria. In particular the equilibrium in which both banks default and the one in which no bank defaults coexist in the following region of shocks, indicated in blue in Figure (3).

$$(1, 1) \text{ AND } (R, R) \text{ when } \theta_1^- < u_1 < \theta_1^+ \text{ and } \theta_2^- < u_2 < \theta_2^+. \tag{26}$$

Following Equation (20), and recalling we have assumed in this example a uniform distribution of shocks in the space $[-1, 1]$ (hence $p(u) = 1$), we can provide the analytical expression of P^+ and P^-. Indeed, the region where the default indicator equals 1 is the region defined by the fourth condition in Equation (25). We thus obtain:

$$P^+ = \frac{(1 + \hat{\theta}_1^+)(1 + \hat{\theta}_2^+)}{4} \tag{27}$$

$$P^- = \frac{(1 + \hat{\theta}_1^+)(1 + \hat{\theta}_2^+)}{4} - \frac{(\hat{\theta}_1^+ - \hat{\theta}_1^-)(\hat{\theta}_2^+ - \hat{\theta}_2^-)}{4} \tag{28}$$

$$\Delta P = \frac{(\hat{\theta}_1^+ - \hat{\theta}_1^-)(\hat{\theta}_2^+ - \hat{\theta}_2^-)}{4} \tag{29}$$

Assuming that banks are homogenous and that $\hat{\theta}_i = \theta_i$, we obtain an expression of ΔP
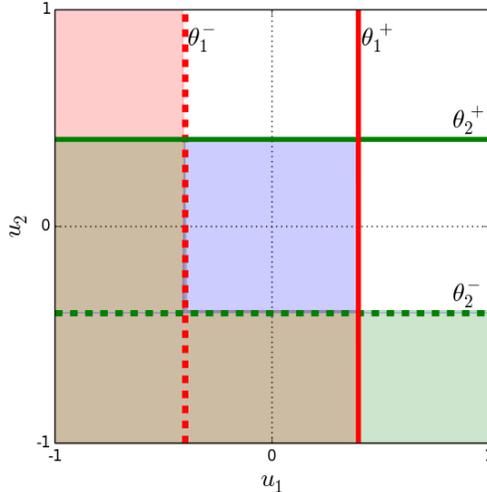
Figure 3: The shock space and the regions where the various configurations of (x_1^B, x_2^B) are equilibria. Region to the left of dotted red line when Bank 1 default all the time, region below dotted green line when Bank 2 defaults all the time, region below red and green line when the default of a bank is conditional on the other bank.

as a function of the banks' exposure to the external assets and to the interbank:

$$\Delta P = \frac{(\theta^+ - \theta^-)^2}{4} = (\frac{\beta(1-R)}{2\varepsilon\sigma})^2 \tag{30}$$

Note that, with the uniform distribution of the shocks, the portion of the shock space that is affected by multiple equilibria is simply $\Delta U = \Delta P$.

The above example highlights how mutual interdependencies leads to multiple outcomes of the default probability. Moreover, in the homogeneous case, the difference between the probability of default in the best and the worse equilibrium as well as the area of the region of multiplicity, increases with the exposure to losses on the interbank market (i.e., measured by $\beta_i(1-R)$). In contrast, they both decrease with volatility on the external markets (i.e., measured by σ_i).

4.3. A Ring Market of n Banks

We now generalise the results illustrated above in the simplest case of two banks to a market of n banks arranged in two types of benchmark structures: a *ring* and a *star*. Let us start with the ring structure, defined as follows.

Definition 6. *A ring market is a network composed of contracts arranged in one cycle* C_n

16

In light of the discussion of Proposition (1), in this market there is a closed chain of dependencies from any bank i through the whole set of other $(n-1)$ banks. For the sake of simplicity, we postpone the analysis of correlation across shocks to Section (4.5) and we assume a uniform distribution of shocks. We can state the following proposition.

Proposition 3 (Effect of Multiplicity in a Ring Market). *Consider n banks with interbank credit, arranged in a ring market. Assume: recovery rate $R_i < 1$ and interbank leverage $\beta_i > 0$; external leverage ε_i and shock variance σ_i positive and finite; shock average μ_i finite; the joint probability distribution of shocks $p(u)$ is uniform and with no correlation across shocks. Then:*

1. *The **uncertainty** ΔP on the default probabilities in the worst and best scenario, P^+ and P^- increases with the **interbank leverage** β_i of the banks; it inversely increases with **the fraction of collateral** R_i; it decreases with the **external asset leverage** ε_i; it decreases with the **variance** on the shocks σ_i. Its expression reads:*

$$\Delta P = \Pi_{i=1}^n (\frac{\hat{\theta}_i^+ - \hat{\theta}_i^-}{2}) \tag{31}$$

2. *if $\hat{\theta}_i = \theta_i \ \forall i$, the **uncertainty** ΔP decreases with the **length** n **of the ring** market and the expression reads*

$$\Delta P = \Pi_{i=1}^n (\frac{\beta_i(1 - R_i)}{2\varepsilon_i\sigma_i}) \tag{32}$$

Proof. See Appendix C. □

The proof is analogous to the computation illustrated above in the example of two banks. Note that the expression of the portion of the shock space subject to multiple equilibria ΔU follows the same equation as in Equation (32) under the uniform distribution assumption. Hence similar results can be inferred.

The above proposition shows that uncertainty increases with the leverage on the interbank, proportionally to the loss exposure (taking into account the collateral, i.e., with $\beta_i(1-R_i)$). This result stems from the fact that the probability of default in the optimistic scenario (i.e., selecting the best equilibrium) decreases with respect to the interbank exposure (see proof). This shows that increase in the reliance on the interbank market increases the uncertainty in terms of systemic risk but improves the outcome in case the optimistic scenario is realised.

Moreover, uncertainty decreases with the leverage on the external market and its volatility (i.e. with $\varepsilon_i\sigma_i$). This means that an increase in diversification in a bank's portfolio of external assets increases uncertainty. The result seems counterintuitive but it simply stems from the fact that, in relative terms, the smaller the variance of the shocks on the external assets, the smaller the role of the shocks in determining whether banks default or not and thus the larger the role played by the interbank assets. In particular,

if the variance of the shocks is very small and the interbank assets of banks exceed their equity, then, no matter what is the value of the shocks, in the scenario in which interbank assets are lost, default would occur with probability one.

Finally, the product in the expression of the uncertainty in the above proposition implies that the uncertainty decreases with the length n of the cycle as long as $\frac{\beta_i(1-R_i)}{2\varepsilon_i\sigma_i} < 1$, which is guaranteed if $\hat{\theta} = \theta$ (see proof). This means that the uncertainty of a ring market decreases exponentially with its size only if the relative weight of the interbank assets over the external assets is small enough.

4.4. A Star Market of n Banks

Let us now move to the case of a *star* market. In a star market, we have a central counterparty for all the banks in the periphery, both for their borrowing and lending relationships. We illustrate a case of 5 banks in Figure (4).

Definition 7. *A star market is a network composed of one bank in the center, denoted by c, and $n-1$ banks in the periphery, denoted by the index j. All banks in the periphery only lend to and borrow from the bank at the center.*
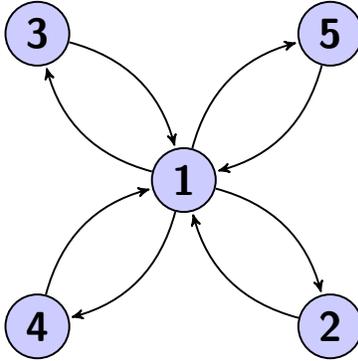

Figure 4: Star Market

For the sake of simplicity, in the following we assume that the bank at the center lends uniformly across its borrowing counterparties. We thus have that: $B_{cj} = \frac{1}{|V_c|}$ where V_c is the set of borrowers of the bank at the center. Note that we now have a system made of multiple cycles. As a result, we have the following proposition.

Proposition 4 (Effect of Multiplicity in a Star Market)**.** *Consider n banks with interbank credit arranged as a star network. Assume: recovery rate $R_i < 1$ and interbank leverage $\beta_i > 0$; external leverage ε_i and shock variance σ_i positive and finite; shock average μ_i finite; the joint probability distribution of shocks $p(u)$ is uniform and with no correlation across shocks. Then:*

The **uncertainty** ΔP on the default probabilities in the worst and best scenario, P^+ and P^-

$$\Delta P = \Pi_{i=1}^n \Big(\frac{\hat{\theta}_i^+ - \hat{\theta}_i^-}{2}\Big) + \frac{(\hat{\theta}_c^+ - \hat{\theta}_c(\sum_j \chi_j = 1))\Pi_{j=1}^{n-1}(1 + \hat{\theta}_j^+)}{2^n} \tag{33}$$

Proof. See Appendix C. □

Comparing Equation (33) for the star market with Equation (32) for the ring market, we observe that the first part of the star market expression is equal to the ring market. In fact, this term comes from the region of overlap between the cases in which all counterparties default and no one defaults. Such situation is thus similar to the ring case. There are however other equilibria overlap to account for in the case of the start market. Those equilibria correspond to cases where (1) the bank at the center does not default, (2) at least one peripheral bank default and (3) at least one peripheral bank survives.

Overall, we thus see that the uncertainty on the probability in the best and the worst equilibrium increases when we move from a ring market to a star market mainly because of the increase of cyclicality within the market structure.

4.5. Effect of Correlation

We now explore how the uncertainty (i.e., the distance between the best and worse possible probabilities) is affected when we introduce correlations between the shocks on the external assets. In order to illustrate the effect of correlation in the simplest terms, we focus on the case of a ring market structure and we assume a uniform distribution of shocks in $[-1, 1]$. Under the assumption of uniformity, the uncertainty on the default probability coincides with the area of the region of multiplicity. We further consider the case of fully correlated shock and we compare the results with those of uncorrelated shocks from Proposition (3).

Proposition 5 (Effect of Correlation)**.** *Consider a market of n banks with interbank credit arranged in a ring. Assume: recovery rate $R < 1$ and interbank leverage $\beta_i > 0$; external leverage ε_i and shock variance σ_i positive and finite; shock average μ finite. Shocks are distributed uniformly. Denote by ΔP^U the uncertainty in case of uncorrelated shocks and by ΔP^C the uncertainty in case of fully correlated shocks. Then, the following statements hold.*

1. *In the case of full shock correlation, the uncertainty on the default probability in the best and worst scenario is:*

$$\Delta P^C = max\Big\{\frac{min\{\theta_i^+\} - max\{\theta_i^-\}}{2}, 0\Big\}, \tag{34}$$

and if $min\{\theta_i^+\} \leq max\{\theta_i^-\}$, there in no uncertainty, $\Delta P^C = 0$.

2. $\Delta P^C > \Delta P^U$ if $\exists\ k \in N$ s.t. $min\{\theta_i^+\} = \theta_k^+$ and $max\{\theta_i^-\} = \theta_k^-$, i.e. complete correlation yields larger uncertainty than no correlation. As a special case, in a ring of identical banks, complete correlation implies larger uncertainty than in the uncorrelated case.

3. $\Delta P^C < \Delta P^U$ if $\sqrt[n]{min\{\theta_i^+\} - max\{\theta_i^-\}} < min\{\theta_i^+ - \theta_i^-\}$, i.e. complete correlation yields smaller uncertainty area than no correlation.

4. $\Delta P^C > \Delta P^U$ if $\sqrt[n]{min\{\theta_i^+\} - max\{\theta_i^-\}} > max\{\theta_i^+ - \theta_i^-\}$, i.e. then complete correlation yields larger uncertainty area than no correlation.

Proof. See Appendix D. □

In the proposition above, the expression of ΔP^C is obtained from a projecting the n-dimensional hypercuboid of the shock space onto its diagonal. Indeed, fully correlated shocks imply that all the shocks hitting all banks have the same value at a time.

From this projection, the determination of the uncertainty depends on the maximal threshold distance, i.e. the distance between the smallest θ_i^+ across all banks i and the largest θ_i^-, i.e. the smallest maximal and the largest minimal default thresholds across all the banks in the system.

The first point in the proposition above states that the uncertainty is completely removed if the system has a maximal threshold distance equal to 0, i.e. if there is a bank with a maximal default threshold that is smaller than the largest minimal default threshold.

Second, compared to the case of no correlation, full correlation brings in more uncertainty when the two threshold yielding the maximal threshold distance belong to the same bank. This means that there is a bank k for which the interval $[\theta_k^-, \theta_k^-]$ is comprised within the intervals $[\theta_i^-, \theta_i^-]$ of all the other banks. An important consequence is that in an homogenous ring, full correlation implies more uncertainty than in the uncorrelated case.

Third, correlation also brings more uncertainty when the $n - th$ root of the maximal threshold distance is higher then the maximum distance between the 2 extreme thresholds of a single bank in the system. On the contrary, correlation brings less uncertainty when the $n - th$ root of the maximal threshold distance is smaller than the minimum distance between the 2 extreme thresholds of a single bank in the system.

Overall, the above proposition shows that the relationship between correlation and uncertainty is in general non-monotonous.

5. Expected Individual and Systemic Losses

We are now interested in devising a measure delivering information on the expected monetary loss in the system at time 2.

Proposition 6 (Expected loss). *Consider a market of n banks with interbank credit. The expression of expected losses of bank i, $E_i[loss]$, and the total expected losses $E^{sys}[loss]$ are as follows:*

$$\forall i \quad E_i[loss] = \int e_i(\varepsilon_i + \beta_i - \gamma_i - 1)\chi_i(u)p(u)du, \tag{35}$$

$$E^{sys}[loss] = \int \sum_i e_i(\varepsilon_i + \beta_i - \gamma_i - 1)\chi_i(u)p(u)du, \tag{36}$$

where χ_i is a solution of Equation (11) and $p(u)$ denotes the joint probability distribution of the shocks.

Given the individual probabilities of default of each bank, we compute the expected loss that each bank causes to its creditors in case of default. Given a realisation of shocks, if bank i defaults, the aggregate amount of money that is lost from its interbank creditors and its external investors is equal to the total liability of bank i minus the amount posted as a collateral (i.e., $e_i(\varepsilon_i + \beta_i - \gamma_i - 1)$). Hence, in order to compute the expected loss from bank i, we integrate the aggregated monetary loss over the whole range of shocks accounting for the cases where bank i defaults (i.e., $\chi_i = 1$).

For the total expected loss, that is, the total amount of money expected to be lost due to the default of any bank at the system level, we simply aggregate the individual losses for each realisation of shocks (i.e., $\sum_i e_i(\varepsilon_i + \beta_i - \gamma_i - 1)\chi_i(u)$) and integrate over the whole range of shocks. Note that in this way, we can simply use the individual default indicator and do not need to identify any systemic default indicator.

Similar to the probabilities of default, several values of expected losses can be obtained as soon as we have multiple equilibria. In that case, we define $\Delta E^{sys}[loss]$ as the distance between the lowest and highest total expected losses $E^-[loss]$ and $E^+[loss]$, respectively. Using this metric, we can now measure how multiplicity affects the assessment on expected losses. Ultimately, we are thus able to formalize a method to compute a monetary price for the cost of uncertainty.

5.1. Example: Market of 2 Banks.

We illustrate the computation of expected losses with an example of a market of 2 banks lending and borrowing from each other. For the sake of clarity, let us assume that the distribution of shocks is homogenous and uncorrelated. Let us also define $e_i^* = e_i(\varepsilon_i + \beta_i - \gamma_i - 1)$. Losses occur each time at least one bank defaults. With 2 banks, we have 3 different combinations: bank 1 defaults while bank 2 does not; bank 2 defaults while bank 1 does not; both banks default. We thus have:

$$E^{sys}[loss] = e_1^* P(\{\chi_1 = 1, \chi_2 = 0\}) + e_2^* P(\{\chi_2 = 1, \chi_1 = 0\})$$
$$+ (e_1^* + e_2^*)P(\{\chi_1 = 1, \chi_2 = 1\})$$

The last component of the right hand side bears the multiple equilibria issues defined in the Example (4.2) (i.e., $P(\{\chi_1 = 1, \chi_2 = 1\})$). From Example (4.2) we can derive the expression of highest and lowest total expected losses:

$$E^{+\ sys}[loss] = \frac{e_1^*(1 - \theta_1^+)}{4} + \frac{e_2^*(1 - \theta_2^+)}{4} + (e_1^* + e_2^*)P^+$$

$$= \frac{e_1^*(1 - \theta_1^+) + e_2^*(1 - \theta_2^+) + (e_1^* + e_2^*)(1 + \theta_1^+)(1 + \theta_2^+)}{4}$$

$$E^{-\ sys}[loss] = \frac{e_1^*(1 - \theta_1^+)}{4} + \frac{e_2^*(1 - \theta_2^+)}{4} + (e_1^* + e_2^*)P^-$$

$$= \frac{e_1^*(1 - \theta_1^+) + e_2^*(1 - \theta_2^+)}{4}$$

$$+ \frac{(e_1^* + e_2^*)[(1 + \theta_1^+)(1 + \theta_2^+) - (\theta_1^+ - \theta_1^-)(\theta_2^+ - \theta_2^-)]}{4}$$

Finally, we can identify the effect of equilibrium choice on the expected loss assessment, which we identify by $\Delta E^{sys}[loss]$:

$$\Delta E^{sys}[loss] = \frac{(e_1^* + e_2^*)[(1 + \theta_1^+)(1 + \theta_2^+) - (\theta_1^+ - \theta_1^-)(\theta_2^+ - \theta_2^-)]}{4}$$

$$\Delta E^{sys}[loss] = (e_1^* + e_2^*)\Delta P$$

If we consider that banks are homogenous and that $\hat{\theta} = \theta$, we obtain:

$$\Delta E^{sys}[loss] = 2e(\epsilon + \beta - \gamma - 1)(\frac{\beta(1 - R)}{2\varepsilon\sigma})^2$$

6. Discussion

In this paper, we investigate how the network structure resulting from credit ties among financial agents (i.e., banks) can affect the capacity of a regulator to assess the level of systemic risk. We introduce a model to compute the individual and systemic probability of default in a system of banks connected in a generic interbank network and exposed to shocks with a generic correlation structure. We find that multiple equilibria can exist even in the presence of complete knowledge.

Our main contribution is to show that multiple equilibria can arise from the presence of *closed chains* of debt in the network (i.e., *cycles*). Note that this mechanism differs from the one described in previous works where multiple equilibria result from self-fulling expectations (e.g., (Diamond and Dybvig, 1983)). In its simplest form, our result states that, if the default conditions of a set of banks are mutually dependent along cycles of credit contracts, there exists a range of external shocks such that the equilibrium where

all those banks default and the equilibrium where none of them defaults co-exist. More generally, in any network structure the multiplicity of equilibria can arise in the presence of at least one closed chain of lending ties in the market. It is worth noting that, empirically, closed chains of lending ties are ubiquitous in financial markets. Indeed, a large portion of various national interbank markets is often found to be *strongly connected*, i.e. connected through at least one closed chain (Roukny et al., 2014). Furthermore, the core-periphery structure identified in many cases of inter-bank markets (Fricke and Lux, 2012; Craig and Von Peter, 2014; van Lelyveld et al., 2014) is characterized by an important level of cyclical dependencies between the core-banks (i.e., they form a fully connected network).

The importance of multiple equilibria and the fact that focusing only on the best or the worst equilibrium might be insufficient is witnessed by the growing interest not only in the academic literature but also in policy debates (Draghi, 2012; Miller and Zhang, 2014; De Grauwe and Ji, 2015).

Beyond the mere existence of multiple equilibria, the range of shocks where multiplicity occurs as well as the difference in expected losses across equilibria are also very important. Multiplicity is particularly relevant if both the range of shocks and the gap on the expected losses are large.

In this respect, in this paper we develop an analytical framework to formalize the problem and quantify the gap in relation to the network structure. We quantify this uncertainty by analytically computing the difference between the most extreme scenarios. The *optimistic* scenario is the one where, when multiple equilibria exist, the equilibrium with the minimum number of defaults is selected. Similarly, the *pessimistic* scenario is constructed by selecting the equilibrium with the maximum number of defaults. In addition, we also provide a method to quantify such difference in monetary terms (i.e. expected losses), thus allowing to assess the *cost of uncertainty*.

Furthermore, we investigate how such uncertainty depends on leverage, volatility, interbank market structures and correlation across external shocks. We find that leverage both on the interbank market and the external assets increases uncertainty. Volatility in external assets has ambiguous effects. Correlation across shocks can also have non monotonous effects on uncertainty. However, complete correlation in a set of homogenous banks univocally increases uncertainty with respect to the uncorrelated case. In terms of network structure, we show that the uncertainty decreases with the length of the credit chain. When analysing a market composed of multiple cycles (i.e., star market), we find that uncertainty increases compared to single-cycle structures.

By design, the model and the analysis could be applied to real data and parameters could be calibrated using information on assets portfolios, credit registers and balance sheets. The model can be used to assess both the level of (individual and systemic) risk and the uncertainty arising from the interconnectedness.

Finally, the work also contributes to several policy related discussions. As it offers a novel way to estimate the systemic impact of financial institutions in a network context, it can bring new insights in the discussion about too-big-to-fail institutions (Haldane and

May, 2011; BoE, 2013). By showing how cyclical structures in the network imply more uncertainty over default probability, we also contribute to the discussion on regulatory financial data disclosure (Abbe et al., 2012; Alvarez and Barlevy, 2014).

Bibliography

Abbe, E. A., Khandani, A. E., and Lo, A. W. (2012). Privacy-Preserving Methods for Sharing Financial Risk Exposures. *American Economic Review*, 102(3):65–70.

Acemoglu, D., Ozdaglar, A., and Tahbaz-Salehi, A. (2015a). Networks, shocks, and systemic risk. Technical report, National Bureau of Economic Research.

Acemoglu, D., Ozdaglar, A., and Tahbaz-Salehi, A. (2015b). Systemic Risk and Stability in Financial Networks. *American Economic Review*, 105(2):564–608.

Allen, F. and Babus, A. (2009). *The Network Challenge (Chapter 21): Networks in Finance.* Pearson Education.

Allen, F. and Gale, D. (2000). Financial contagion. *Journal of political economy*, 108(1):1–33.

Alvarez, F. and Barlevy, G. (2014). Mandatory disclosure and financial contagion. *Working Paper, Federal Reserve Bank of Chicago.*

Battiston, S., Puliga, M., Kaushik, R., Tasca, P., and Caldarelli, G. (2012). DebtRank: Too Central to Fail? Financial Networks, the FED and Systemic Risk. *Scientific Reports*, 2:541.

Bernanke, B. (2009). Financial reform to address systemic risk. *Speech at the Council on Foreign Relations*, 10.

BoE (2013). A framework for stress testing the UK banking system. Technical Report October.

Caballero, R. J. and Simsek, A. (2013). Fire sales in a model of complexity. *The Journal of Finance*, 68(6):2549–2587.

Caccioli, F., Shrestha, M., Moore, C., and Doyne Farmer, J. (2014). Stability analysis of financial contagion due to overlapping portfolios. *Journal of Banking & Finance.*

Cont, R., Moussa, A., Santos, E. B., et al. (2013). Network structure and systemic risk in banking systems. *Handbook of Systemic Risk*, pages 327–368.

Craig, B. and Von Peter, G. (2014). Interbank tiering and money center banks. *Journal of Financial Intermediation*, 23(3):322–347.

De Grauwe, P. and Ji, Y. (2015). Market sentiments and the sovereign debt crisis in the Eurozone. Technical report, FinMaP-Working Paper.

Diamond, D. W. and Dybvig, P. H. (1983). Bank Runs, Deposit Insurance, and Liquidity. *The Journal of Political Economy*, 91(3):401.

Draghi, M. (2012). Speech, Frankfurt 6 September 2012. Technical report.

Eisenberg, L. and Noe, T. H. (2001). Systemic Risk in Financial Systems. *Management Science*, 47(2):236–249.

Elliott, M., Golub, B., and Jackson, M. O. (2014). Financial Networks and Contagion. *American Economic Review*, 104(10):3115–3153.

Elsinger, H., Lehar, A., and Summer, M. (2006). Risk Assessment for Banking Systems. *Management Science*, 52(9):1301–1314.

Fricke, D. and Lux, T. (2012). Core-periphery structure in the overnight money market: Evidence from the e-mid trading platform. Technical report, Kiel Working Papers.

Gai, P., Haldane, A., and Kapadia, S. (2011). Complexity, concentration and contagion. *Journal of Monetary Economics*, 58(5):453–470.

Gai, P. and Kapadia, S. (2010). Contagion in financial networks. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, 466(2120):2401–2423.

Georg, C.-P. (2013). The effect of the interbank network structure on contagion and common shocks. *Journal of Banking & Finance*.

Glasserman, P. and Young, H. P. (2015). How likely is contagion in financial networks? *Journal of Banking & Finance*, 50:383–399.

Goldstein, I. and Pauzner, A. (2004). Contagion of self-fulfilling financial crises due to diversification of investment portfolios. *Journal of Economic Theory*, 119(1):151–183.

Gouriéroux, C., Héam, J.-C., and Monfort, A. (2013). Liquidation equilibrium with seniority and hidden CDO. *Journal of Banking & Finance*, 37(12):5261–5274.

Haldane, A. G. and May, R. M. (2011). Systemic risk in banking ecosystems. *Nature*, 469(7330):351–5.

IMF (2015). Global Financial Stability Report: Restoring Confidence and Pro- gressing on Reforms. Technical report. Technical report.

Miller, M. and Zhang, L. (2014). Saving the Euro: self-fulfilling crisis and the "Draghi Put". *CEPR DP997*.

Rogers, L. C. and Veraart, L. A. (2013). Failure and rescue in an interbank network. *Management Science*, 59(4):882–898.

Roukny, T., Bersini, H., Pirotte, H., Caldarelli, G., and Battiston, S. (2013). Default cascades in complex networks: Topology and systemic risk. *Scientific reports*, 3.

Roukny, T., George, C.-P., and Battiston, S. (2014). A Network Analysis of the Evolution of the German Interbank Market. *Deutsche Bundesbank Discussion Paper 22/2014*.

Stiglitz, J. E. (2008). Risk and Global Economic Architecture: Why Full Financial Integration May Be Undesirable. *American Economic Review*, 100(2):388–392.

Stiglitz, J. E. and Greenwald, B. C. N. (2003). *Towards a New Paradigm in Monetary Economics*. Cambridge Univ. Press, Cambridge.

van Lelyveld, I. et al. (2014). Finding the core: Network structure in interbank markets. *Journal of Banking & Finance*, 49:27–40.

Yellen, J. L. (2013). Interconnectedness and Systemic Risk: Lessons from the Financial Crisis and Policy Implications: a speech at the American Economic Association/American Finance Association Joint Luncheon, San Diego, California, January 4, 2013. Technical report.

Appendix A. Proof of Proposition 1

We separate the proof in two parts. In the first we show the necessity of cyclicality in the network of contract and in the second part we show that, once we have a cycle, the sufficient condition to have multiple equilibria is that there is an overlap between the space of default thresholds depending on the counterparties' status in the cycle and the space of shocks.

We demonstrate the first part by looking at graph that do not exhibit any cyclicality. If the graph of dependencies is an acyclic graph, then we can identify the leafs (banks who do not lend to anyone) and recursively the parents of every bank in the interbank network (the creditor of those banks). For every vector of shocks u, the state of default χ of the leafs is determined and unique. Recursively, the state of all parent nodes is also determined and unique. Therefore, in order to have multiple solutions, the graph must not be acyclic and thus contain at least one cycle. Notice that only the nodes in the cycle and those pointing directly or indirectly to the cycle may display multiple solutions. All the other nodes, e.g. those that are not part of a cycle or that cannot reach the cycle along a path, will have a unique default state.

We now move to the sufficient condition once we have a cyclical graph. Following the assumptions that $R < 1$ and $\beta > 0$, we have that $\theta_i^- < \theta_i^+$ for all $i \in N$. Assume that

26

there is a cycle. If $\hat{\theta}_i(\chi_{i+1} = 0) \neq \hat{\theta}_i(\chi_{i+1} = 1)$, we have that $[\theta_i(\chi_{i+1} = 0), \theta_i(\chi_{i+1} = 1)] \cap [-1, 1] \neq \emptyset$.

Then the fact that the threshold values are strictly different for every node in the cycle and include some values of the shock domain involves the fact that there is an overlap of equilibria with respect to the shock space. In fact, the following two sets of equilibrium conditions overlap because $\theta_i^- < \theta_i^+$ for all $i \in N$.

$$(x_1^B, ..., x_m^B) = (1, .., 1) \iff \begin{cases} u_1 > \theta_1(x_2^B = 1) = \theta_1^- \\ ... \\ u_m > \theta_m(x_1^B = 1) = \theta_m^- \end{cases}$$

$$(x_1^B, ..., x_m^B) = (R, .., R) \iff \begin{cases} u_1 < \theta_1(x_2^B = R) = \theta_1^+ \\ ... \\ u_m < \theta_m(x_1^B = R) = \theta_m^+ \end{cases}$$

(A.1)

In the system above there are m banks in a cycle, numbered in ascending order along the direction of lending. Bank m lends to bank 1, thus closing the cycle. We thus see that the two equilibria co-exist in the space of shocks defined by $\theta_i^- < u_i < \theta_i^+$.

Appendix B. Proof of Proposition 2

We show here that, once the default status of all the banks is known for every combination of shocks and that each combination of shocks is associated with a probability, we can compute the probability of default of every banks individually and the the probability of systemic default.

In fact, for any given default state χ, we can determine the values of the thresholds $\theta = f(\chi)$. For example, in a market of two banks, we have:

$$(\chi_1, \chi_2) = (1, 1)$$
$$\text{iff} \quad u_1 > \theta_1^1(\chi_2 = 1) \quad \text{and} \quad u_2 > \theta_2^1(\chi_1 = 1)$$

From the values of the thresholds in the space of shocks, we can determine the individual probability of default P_is.

Recall that the space of shocks is weighted by the probability distribution of the shocks, which account for possible correlations. Every default condition holds in a region of the space. As the default information is retrieved by the binary default indicator χ_i summing up the area where a given condition holds gives its probability of occurrence. Hence we

have that:

$$\forall i \quad P_i = \int \chi_i(u)\, p(u)\, du,$$

$$P^{sys} = \int \chi^{sys}(u)\, p(u)\, du,$$

where the systemic default indicator χ^{sys} is function of the individual one. In this paper, we mostly define is as $\chi^{sys} = \Pi_i \chi_i$.

Appendix C. Proof of Proposition 3 and 4

To compute the distance between the best and worst probabilities of systemic default, stemming from the optimistic and pessimistic scenarios respectively, we introduce an generic algorithm that we then apply to the network structure of interest.

First, recall that we are interested in identifying the portion of the shock space where multiple equilibria exist, with respect to our goal of assessing the level of systemic risk. We are thus primarily interested in the identifying the area of systemic default. With respect to our systemic default indicator $\chi^{sys} = \Pi_i \chi_i$, we are looking for the equilibrium where every bank defaults. It is equivalent to the vector of default states $\{\chi_i = 1\}\ \forall i$. Such equilibrium corresponds to a situation where, for every bank, the shock is $u_i < \theta_i^+$ because this threshold is the one accounting for the default of all the borrowing counterpaties of bank i. The shock space is thus $\{\theta_i < \theta_i^+\}\ \forall i$. We also know that all the other default threshold have a value inferior to θ_i^+ (see Section (2)).

Next, we need to find the equilibria with ranges of shock values that overlap with the above equilibrium. Those equilibria are the equilibria for which the ranges of shocks form a non-empty set of intersection with the systemic default equilibrium shock space. Formally, we need to the find the set of conditions with respect to u_i such that $\{\theta_i^* < \theta_i^+\}\ \forall i$. There can by several sets of condition (i.e., several equilibria that overlap with the systemic default equilibria). Note that the overlapping equilibria necessarily have their set of default threshold different then θ^+.

Finally, we sum the fraction of shock space that overlaps between the systemic default equilibria and all the candidate. This provides the value of ΔU. We then use the joint probability distribution of shocks to determine ΔP.

Appendix C.1. Application to the Ring Structure.

In a cycle, each agent has only 2 thresholds:

$$\theta_i \in \{\theta_i^- = \theta_i(x_{i+1} = 0), \theta_i^+ = (x_{i+1} = R)\}$$

As previously defined, our equilibrium of interest is $\{\chi_i = 1\}\ \forall i$ and the shock space is $\{u_i < \theta_i^+\}\ \forall i$. The overlapping equilibria are the ones where θ_i is different for all i. The

only case is thus: $\{u_i > \theta_i^-\}$ $\forall i$. This corresponds to the equilibrium $\{\chi_i = 0\}$ $\forall i$. For each agent, the level of overlap is $(\theta_i^+ - \theta_i^-)$. In the n-dimensional, we take the product to obtain the n-volume and then divide by the total n-volume:

$$\Delta U = \frac{\Pi_i(\hat{\theta}_i^+ - \hat{\theta}_i^-)}{2^n}$$

We then obtain the expression for the probabilities, as

$$\Delta P = \frac{\Pi_i(\hat{P}_i(\hat{\theta}_i^+) - \hat{P}_i(\hat{\theta}_i^-))}{2^n}$$

Where $\hat{P}_i(u)$ is the cumulative density function obtained from joint probability function $p(u)$ projected on bank i.

If we assume a uniform density function, we obtain a result equal the ΔU and we can develop furthermore the expression to highlight the role of the different parameter of the system:

$$\Delta P = \frac{\Pi_i(\hat{\theta}_i^+ - \hat{\theta}_i^-)}{2^n} = \Pi_i\left(\frac{\beta_i(1 - R_i)}{2\varepsilon_i\sigma_i}\right)$$

Appendix C.2. Application to the Star Structure.

We assume $B_{ik} = \frac{1}{|V_i|}$. We denote by index c the center agent in the star and j the agents in the periphery.

In a star, agents in the periphery have only 2 thresholds:

$$\theta_j \in \{\theta_j^- = \theta_j(x_c = 0), \theta_j^+ = (x_c = R)\}$$

While the agent at the center has n thresholds:

$$\theta_c \in \{\theta_c^-, \theta_c(\sum_j \chi_j = 1), \theta_c(\sum_j \chi_j = 2), ..., \theta_c(\sum_j \chi_j = n - 1), \theta_c^+\}$$

Again, the equilibrium of interest is $\{\chi_i = 1\}$ $\forall i$ and the shock space is $\{u_i < \theta_i^+\}$ $\forall i$. The overlapping equilibria are the ones where θ_i is different for all i.

The overlapping equilibria are thus all the combinations of shock ranges using the thresholds $\theta_c < \theta_c^+$ and $\theta_j = \theta_j^-$. The case where $\theta_c = \theta_c^-$ yields a result similar to the ring:

$$\frac{\Pi_i(\hat{\theta}_i^+ - \hat{\theta}_i^-)}{2^n}$$

and thus, we introduce the cumulative density function to yields the expression in terms of probabilities distance

$$\frac{\Pi_i(\hat{P}_i(\hat{\theta}_i^+) - \hat{P}_i(\hat{\theta}_i^-))}{2^n}$$

In all other cases the fraction of overlapping shock space can be determined as follows. For each θ_c, all ranges of values for the other shocks are fulfilled in aggregating the overlapping equilibria. Hence the conditions for the periphery agents are $\{u_j < \theta_j^+\}$ $\forall j$. The fraction of the shocks space is thus:

$$\frac{(\hat{\theta}_c^+ - \hat{\theta}_c(\sum_j \chi_j = 1)))\Pi_j(1 + \hat{\theta}_j^+)}{2^n}$$

Similarly translating in probabilities distance, we obtain

$$\frac{(\hat{P}_c(\hat{\theta}_c^+) - \hat{P}_c(\hat{\theta}_c(\sum_j \chi_j = 1)))\Pi_j(\hat{P}_j(\hat{\theta}_j^+))}{2^n}$$

Finally, we sum of the elements and obtain:

$$\rightarrow \Delta U = \frac{\Pi_i(\hat{\theta}_i^+ - \hat{\theta}_i^-)}{2^n} + \frac{(\hat{\theta}_c^+ - \hat{\theta}_c(\sum_j \chi_j = 1)))\Pi_j(1 + \hat{\theta}_j^+)}{2^n}$$

$$\rightarrow \Delta P = \frac{\Pi_i(\hat{P}_i(\hat{\theta}_i^+) - \hat{P}_i(\hat{\theta}_i^-))}{2^n} + \frac{(\hat{P}_c(\hat{\theta}_c^+) - \hat{P}_c(\hat{\theta}_c(\sum_j \chi_j = 1)))\Pi_j(\hat{P}_j(\hat{\theta}_j^+))}{2^n}$$

If we assume a uniform density function and that $\hat{\theta}_i = \theta_i$, we obtain a result equal the ΔU and we can develop furthermore the expression to highlight the role of the different parameter of the system:

$$\Delta P = \Pi_i(\frac{\beta_i(1 - R_i)}{2\varepsilon_i\sigma_i}) + \frac{(\theta_c^+ - \theta_c(\sum_j \chi_j = 1))\Pi_j(1 + \theta_j^+)}{2^n}$$

Appendix D. Proof of Proposition 5

In this proof, we analyse a situation where the returns on all banks' portfolio are completely correlated. Let us start with a case of 2 banks lending and borrowing from each other. From the 2-dimensional representation, a fully correlated situation can be obtained by projecting the results obtained in the Example in Section (4) on the diagonal such that $u_1 = u_2$ for all combinations of shocks. We then need to compute the length of the diagonal that is under the different areas. The fraction of this length on the total lengths of the diagonal will give us the probability value.

$$P^- = \frac{\sqrt{(1 + max\{\hat{\theta}_i^-\})^2 + (1 + max\{\hat{\theta}_i^-\})^2}}{2\sqrt{2}} = \frac{|1 + max\{\hat{\theta}_i^-\}|}{2}$$

$$P^+ = \frac{\sqrt{(1+min\{\hat{\theta}_i^+\})^2 + (1+min\{\hat{\theta}_i^+\})^2}}{2\sqrt{2}} = \frac{|1+min\{\hat{\theta}_i^+\}|}{2}$$

$$\Delta P = \frac{1+min\{\hat{\theta}_i^+\}}{2} - \frac{1+max\{\hat{\theta}_i^-\}}{2} = \frac{min\{\hat{\theta}_i^+\} - max\{\hat{\theta}_i^-\}}{2}$$

Note that we can remove the absolute values as $0 \leq 1 + \hat{\theta}_i$. The results are the same when we generalise to a system of n banks arranged in a ring, as shown in what follows:

$$P^- = \frac{\sqrt{\sum_i (1+max\{\hat{\theta}_i^-\})^2}}{2\sqrt{n}} = \frac{|1+max\{\hat{\theta}_i^-\}|}{2}$$

$$P^+ = \frac{\sqrt{\sum_i (1+min\{\hat{\theta}_i^+\})^2}}{2\sqrt{n}} = \frac{|1+min\{\hat{\theta}_i^+\}|}{2}$$

$$\Delta P = \frac{1+min\{\hat{\theta}_i^+\}}{2} - \frac{1+max\{\hat{\theta}_i^-\}}{2} = \frac{min\{\hat{\theta}_i^+\} - max\{\hat{\theta}_i^-\}}{2}$$

Appendix D.1. Comparative Statics

We are now interested in comparing the difference of best of worst probability in a ring market where shocks are completely correlated with the case where shocks are completely independent.

Given the expression for the uncorrelated case $(\Delta P^u = \frac{(\hat{\theta}_1^+ - \hat{\theta}_1^-)(\hat{\theta}_2^+ - \hat{\theta}_2^-)}{4})$ and for the correlated case $(\Delta P^c = \frac{min\{\hat{\theta}_i^+\} - max\{\hat{\theta}_i^-\}}{2})$, we can explore how increasing correlation affects the uncertainty area.

Note that in case $min\{\hat{\theta}_i^+\} \leq max\{\hat{\theta}_i^-\}$, there is no uncertainty in the correlation.

In case $min\{\hat{\theta}_i^+\} = \hat{\theta}_1^+$ and $max\{\hat{\theta}_i^-\} = \hat{\theta}_1^-$: $\Delta P^u = \Delta P^c \frac{\hat{\theta}_2^+ - \hat{\theta}_2^-}{2}$. Given that $-2 \leq \hat{\theta}_2^+ - \hat{\theta}_2^- \leq 2$:

$$\Delta P^u < \Delta P^c$$

The same stands if $min\{\hat{\theta}_i^+\} = \hat{\theta}_2^+$ and $max\{\hat{\theta}_i^-\} = \hat{\theta}_2^-$. In general: $min\{\hat{\theta}_i^+\}$ and $max\{\hat{\theta}_j^-\}$ have $i = j$.

In case $min\{\hat{\theta}_i^+\}$ and $max\{\hat{\theta}_j^-\}$ have $i \neq j$: Note that:

$$(\hat{\theta}_1^+ - \hat{\theta}_2^-) < (\hat{\theta}_i^+ - \hat{\theta}_i^-) \quad \forall i$$

We look at the 2 extreme cases:

$$\frac{min\{\hat{\theta}_i^+ - \hat{\theta}_i^-\}^n}{2^n} < \Delta P^u < \frac{max\{\hat{\theta}_i^+ - \hat{\theta}_i^-\}^n}{2^n}$$

31

For the upperbound, we have that:

$$\Delta P^u < \frac{max\{\hat{\theta}_i^+ - \hat{\theta}_i^-\}^n}{2^n}$$

if $\sqrt[n]{min\{\hat{\theta}_i^+\} - max\{\hat{\theta}_i^-\}} > max\{\hat{\theta}_i^+ - \hat{\theta}_i^-\}$, we thus have that:

$$\Delta P^u < \Delta P^c$$

For the lower bound, we have:

$$\Delta P^u > \frac{min\{\hat{\theta}_i^+ - \hat{\theta}_i^-\}^n}{2^n}$$

if $\sqrt[n]{min\{\hat{\theta}_i^+\} - max\{\hat{\theta}_i^-\}} < min\{\hat{\theta}_i^+ - \hat{\theta}_i^-\}$, we thus have that:

$$\Delta P^u > \Delta P^c$$