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A "Fractal" Solution to the Chopstick Auction

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Abstract The present paper constructs a novel solution to the chopstick auction, and thereby disproves a conjecture of Szentes and Rosenthal (*Games and Economic Behavior,* 2003a, 2003b). In contrast to the existing solution, the identified equilibrium strategy allows a simple and intuitive characterization. Moreover, its best-response set has the same Hausdorff dimension as its support, which may be seen as a robustness property. The analysis also reveals some new links to the literature on Blotto games.

Keywords Chopstick auction · Exposure problem · Self-similarity · Blotto games

JEL-Codes C02 (Mathematical Methods), C72 (Noncooperative Games), D44 (Auctions)

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1. Introduction

Despite being one of the main mechanism for allocating multiple objects, simultaneous auctions are notorious for exposing bidders to the risk of ending up overpaying for the combination of objects ultimately won.[2] Much of the basic intuition is captured by the example of the so-called first-price chopstick auction (Szentes and Rosenthal, 2003a; see also Postlewaite and Wilson, 2003). In that auction, two bidders simultaneously place bids on three identical objects. Moreover, the value of winning at most one object is zero, whereas the value of winning at least two objects is positive. For the first-price chopstick auction, Szentes and Rosenthal constructed a doubly symmetric mixed-strategy equilibrium, henceforth referred to as the *Szentes-Rosenthal equilibrium (SRE)*, in which bidders randomize uniformly over the surface of a tetrahedron.

The structure of the SRE is intriguing, in particular because it features a two-dimensional equilibrium support encapsulating a three-dimensional best-response set. As Szentes and Rosenthal (2003b) noted, however, their solution has two drawbacks. First, the equilibrium strategy is surprisingly complicated. Second, given that the best-response set has a higher dimension than the equilibrium support, it seems unlikely that the SRE would be the result of a process that is aligned with some kind of better or best-response dynamics.

The present paper documents the existence of a new type of equilibrium for the chopstick auction. For ease of reference, this equilibrium will be referred to as the *self-similar equilibrium (SSE).* To construct the SSE, a dynamic variant of the chopstick auction is considered, which extends an approach used by Sion and Wolfe (1957).[3] Intuitively, the reformulation fragments the bidder's decision in the simultaneous auction so that, at each stage of the dynamic auction, each object requires only a

[2] See Milgrom (2000). For real-life illustrations of such exposure risk see, e.g., van Damme (2000), Ewerhart and Moldovanu (2005), and Ewerhart et al. (2012). Also the findings of the experimental literature are consistent with the view that inexperienced subjects fall prey to exposure risk (Englmaier et al. 2009; Mago and Sheremata, 2012).

[3] While Sion and Wolfe (1957) is more commonly known for providing an example of a two-person zero-sum game without a value, later sections of that same paper describe a way to transcribe any game on the square into a dynamic game of pursuit.

choice between two options, which will be interpreted as either holding or raising the respective bid. The equilibrium bid distribution of the SSE will then be characterized as the measure-theoretic image of a simple stationary equilibrium in the dynamic auction, where the stationarity property in the dynamic auction translates into a self-similarity property in the simultaneous game.

Related literature. "Fractal" solutions to non-cooperative games of the Blotto type have been identified by Gross and Wagner (1950) and Kvasov (2007). However, neither of these papers considered the case of the chopstick auction.[4] Gross (1954) has constructed an example of a zero-sum game on the square with rational payoff functions and the Cantor distribution as the unique equilibrium.[5] Related is also the recent paper by Topolyan (2014) who uses the binary expansion of a uniformly distributed random bid in order to construct an equilibrium in an all-pay contest with additive contributions. Ok (2004) proves a fixed-point theorem for correspondences and applies it to rationalizability and to the theory of self-similar sets.

The remainder of the present paper is structured as follows. Section 2 describes the set-up. In Section 3, a dynamic variant of the chopstick auction is introduced, and the SSE is constructed. A proof of the equilibrium property is provided in Section 4. Section 5 deals with the dimension of the best-response set. In Section 6, the SSE is characterized as a self-similar probability measure. Section 7 concludes.

2. Set-up

In the chopstick auction considered by Szentes and Rosenthal (2003a), a seller offers three identical objects, A, B, and C, via simultaneous sealed-bid auctions to a given population of two bidders. Each of the two bidders $i = 1, 2$ submits a vector of bids,

$$X^i = (X^i_A, X^i_B, X^i_C) \in \mathbb{R}^3_+. \tag{1}$$

Moreover, for each object $\nu \in \{A, B, C\}$, the bidder $i \equiv i_\nu \in \{1, 2\}$ submitting the highest bid on object ν wins that object, and pays her bid X^i_ν to the seller, where

[4] The relationship to the literature on "fractal" solutions will be discussed more thoroughly in a separate section at the end of the present paper, where some new conjectures are formulated as well.

[5] See also Gross (1952) and Karlin (1959).

ties are broken randomly, fairly, and independently across objects.[6] The valuation of any bidder $i \in \{1, 2\}$ of winning a total of $q^i \in \{0, 1, 2, 3\}$ objects is

$$V(q^i) = \begin{cases} 0 & \text{if } q^i \leq 1 \\ 2 & \text{if } q^i \geq 2. \end{cases} \tag{2}$$

I.e., a bidder has a valuation of zero if she wins at most one object, and a valuation of two if she wins at least two objects. This specification of bidders' valuations corresponds to the so-called pure chopstick case discussed in Szentes and Rosenthal (2003a, Ch. 2). Each bidder maximizes her expected payoff, i.e., her valuation $V(q^i)$ less the sum of her winning bids. The resulting game will be referred to as the *first-price chopstick auction*. If the bidder winning any object ν (with $\nu = A, B, C$) pays only the second-highest bid on that object, while all other rules of the game remain unchanged, then the resulting game will be called the *second-price chopstick auction*. Finally, if the bidders pay their bids unconditionally rather than conditional on winning, then we will speak of the *all-pay chopstick auction*. The term chopstick auction, i.e., without qualification, will be reserved for the first-price format, however.

3. A dynamic variant of the chopstick auction

Modifying the set-up introduced in the previous section, it will be assumed now that the bidding proceeds in stages, which extends the approach used by Sion and Wolfe (1957, Sec. 3 and 4). Specifically, at any stage $t \in \mathbb{N} = \{1, 2, ...\}$, each bidder $i \in \{1, 2\}$ chooses, for each object $\nu \in \{A, B, C\}$ separately, whether to *hold* her bid ($x^i_\nu(t) = 0$) or to *raise* it ($x^i_\nu(t) = 1$). Thus, at every stage $t \in \mathbb{N}$, each bidder $i \in \{1, 2\}$ simultaneous and independently chooses a binary vector

$$x^i(t) = (x^i_A(t), x^i_B(t), x^i_C(t)) \tag{3}$$

from the set $D_t = \{0, 1\} \times \{0, 1\} \times \{0, 1\}$. Bidder $i \in \{1, 2\}$ wins object $\nu \in \{1, 2, 3\}$ against bidder $j \neq i$ if and only if there is a stage $T_\nu \in \mathbb{N}$ such that $x^i_\nu(t) = x^j_\nu(t)$ for all $t < T_\nu$, and $x^i_\nu(T_\nu) = 1 > 0 = x^j_\nu(T_\nu)$. If such $T_\nu < \infty$ does not exist, then

[6] The results of the present paper hold, however, for any tie-breaking rule.

we will say that there is a tie on object ν. Thus, the allocation of the three objects is determined either after finitely many stages at $T \equiv \max\{T_A, T_B, T_C\} < \infty$, or the bidding develops entirely in parallel on at least one object. The binary vector $x^i(t) \in D_t$ chosen by bidder $i \in \{1, 2\}$ at any stage $t \in \mathbb{N}$ is assumed to remain unobservable for bidder $j \neq i$ at any later stage $\widehat{t} > t$. By a *(reduced-form) pure strategy* for bidder $i \in \{1, 2\}$, we mean a sequence $x^i = \{x^i(t)\}_{t=1}^{\infty}$ consisting of choices $x^i(t) \in D_t$ for all $t \in \mathbb{N}$. Alternatively, a pure strategy may be written as a vector $x^i = (x^i_A, x^i_B, x^i_C)$, where $x^i_\nu = \{x^i_\nu(t)\}_{t=1}^{\infty}$ is a binary sequence for each object $\nu \in \{A, B, C\}$. The set of pure strategies will be denoted by $D = \prod_{t=1}^{\infty} D_t$. Payoffs in the dynamic game are derived from the simultaneous first-price chopstick auction, where bidder i's bid X^i_ν on object $\nu \in \{A, B, C\}$ is replaced by the convergent series

$$\rho(x^i_\nu) = \sum_{t=1}^{\infty} \frac{x^i_\nu(t)}{2^t} \in [0, 1]. \tag{4}$$

Thus, the binary choices made by a bidder for an object in the course of the dynamic bidding process are taken as digits in a dyadic expansion of the corresponding bid in the simultaneous auction. The thereby defined infinite-horizon game will be referred to as the *dynamic (first-price) chopstick auction*. It should be noted that the mapping ρ is continuous, so that the payoff functions in the dynamic game are measurable.

Next, we define two mixed extensions of the dynamic chopstick auction, following essentially Kuhn (1953).[7] A *mixed strategy* in the dynamic chopstick auction is a probability measure on D, where the σ-field of Borel sets on D is derived from the product of the discrete topologies on each factor D_t.[8] Expected payoffs resulting from a pair of mixed strategies are defined as usual in terms of the bilinear extension of the payoff function. This extension is well-defined because payoffs are bounded. Moreover, by the substitution rule (Kallenberg, 1997, Lemma 1.22), expected payoffs may be determined equivalently by considering the image measures of both mixed strategies and

[7]While Kuhn's (1953) original work is restricted to finite games, an extension to games with infinite play length has been accomplished in unpublished work by P. Wolfe (cf. Aumann, 1964).

[8]Since each D_t is a separable metric space, the Borel σ-field on D corresponds precisely to the measure-theoretic product of the Borel σ-fields on each D_t. See Kallenberg (1997, Lemma 1.2).

calculating expectations in the simultaneous auction. A *(path-independent) behavior strategy* $\xi = \{\xi(t)\}_{t=1}^{\infty}$ in the dynamic chopstick auction specifies an independent probability distribution over the finite choice set D_t at each stage $t \in \mathbb{N}$. By taking the product measure over the component distributions of a given behavior strategy ξ, we obtain a unique mixed strategy in the dynamic chopstick auction, which will be denoted as $\widetilde{\xi}$. In particular, expected payoffs resulting, say, from a pair of behavior strategies are well-defined. A behavior strategy ξ is a *symmetric Nash equilibrium* if the associated mixed strategy $\widetilde{\xi}$ maximizes any bidder i's expected payoffs, within the set of all mixed strategies, under the condition that bidder i's opponent $j \neq i$ adheres to $\widetilde{\xi}$.

A specific behavior strategy ξ^{SSE} in the dynamic chopstick auction is defined by the requirement that, at each stage $t \in \mathbb{N}$, the bidder samples her choices independently and according to the following probability law:

State $\omega(t)$	ω_0	ω_1	ω_2	ω_3	
Probability $\mathrm{pr}(\omega(t))$	$1/4$	$1/4$	$1/4$	$1/4$	(5)
Binary vector $x(t)$	$(0,0,0)$	$(0,1,1)$	$(1,0,1)$	$(1,1,0)$	

Thus, adhering to ξ^{SSE} means that, at each stage, the bidder in question either holds her bids on all three objects, or raises her bids on precisely two randomly selected objects. Moreover, each of these altogether four possibilities are selected with equal probability, and independently across stages.

The following observation is key to most of the results of the present paper.

Lemma 1. ξ^{SSE} *is a symmetric Nash equilibrium in the dynamic chopstick auction.*

A proof will be provided in the next section. Lemma 1 is useful because it allows to construct a new type of equilibrium in the simultaneous chopstick auction. To see this, start from the mixed strategy $\widetilde{\xi}^{\mathrm{SSE}}$ induced by the behavior strategy ξ^{SSE}. Note next that any pure strategy $x = (x_A, x_B, x_C) \in D$ in the dynamic chopstick auction may be transformed, by component-wise application of the mapping ρ, into to a bid vector

$$X = (X_A, X_B, X_C) = (\rho(x_A), \rho(x_B), \rho(x_C)) \in [0,1]^3 \tag{6}$$

in the simultaneous auction. Since this transformation is continuous, the measure-theoretic image of the mixed strategy $\widetilde{\xi}^{\text{SSE}}$ under the transformation is a well-defined mixed strategy μ^{SSE} in the simultaneous auction. Moreover, as stated in the following proposition, the image distribution inherits the equilibrium property from ξ^{SSE}.

Proposition 1. *μ^{SSE} is a symmetric mixed-strategy Nash equilibrium in the simultaneous first-prize chopstick auction.*

Proof. Suppose that both bidders adhere to μ^{SSE} in the simultaneous chopstick auction. To prove that μ^{SSE} is an equilibrium, it suffices to show that there is no pure strategy $X \in \mathbb{R}^3_+$ that outperforms the mixed strategy μ^{SSE}. Indeed, if any pure-strategy deviation yields a weakly negative expected payoff, then the same is true for any mixed-strategy deviation because of the monotonicity of the Lebesgue integral. Let, therefore, $X = (X_A, X_B, X_C) \in \mathbb{R}^3_+$ be an arbitrary pure-strategy deviation in the simultaneous chopstick auction. Then, bidding more than unity on any object $\nu \in \{A, B, C\}$ is weakly dominated, because the bid $X_\nu = 1$ wins object ν against μ^{SSE} with probability one. One may therefore, without loss of generality, focus on the case of pure-strategy deviations $(X_A, X_B, X_C) \in [0,1]^3$, i.e., such that all three bids are chosen from the unit interval. But since the component-wise application of the mapping ρ is surjective on $[0,1]^3$ and leaves, by construction of the dynamic auction, payoffs invariant, the assertion follows now directly from Lemma 1. □

Szentes and Rosenthal (2003b, p. 293) conjectured that the SRE is unique within the class of symmetric equilibria of the chopstick auction. Proposition 1 above shows that this is not the case. Instead, the chopstick auction admits at least one alternative symmetric solution, viz. the SSE. With the help of additional arguments that will be detailed elsewhere, one can even show that any convex combination of the SSE and the SRE is again a symmetric equilibrium. However, as will be explained in Section 6, there is no easy way to assemble additional equilibria in the chopstick auction using the established replacement techniques known from the literature on the Blotto game.

The remainder of this section discusses some immediate properties of the SSE. Denote by $F(X_A, X_B, X_C)$ the probability that μ^{SSE} is component-wise weakly smaller than $(X_A, X_B, X_C) \in \mathbb{R}^3_+$. Thus, F is the distribution function of μ^{SSE}. Then, any bivariate marginal distribution of F, i.e., the distribution of bids on any fixed pair of objects, is uniform. For example, when integrating out the last component,

$$F(X_A, X_B, 1) = X_A X_B, \tag{7}$$

as follows directly from considering the projection of the probability law (5) on the first two coordinates. However, except on the equilibrium support, there is no simple algebraic expression for $F(X_A, X_B, X_C)$.

The expected payoff from playing ξ^{SSE} against itself is zero. Indeed, by symmetry, each bidder wins two or more objects with probability $\frac{1}{2}$, and therefore has an expected valuation of the outcome of $\frac{1}{2} \cdot 2 = 1$. Moreover, each bidder wins a given object with probability $\frac{1}{2}$, and the winning bid is distributed like the maximum of two independent uniform distributions on the unit interval, i.e., with mean $\frac{2}{3}$, so that the expected payment for each bidder is $3 \cdot \frac{1}{2} \cdot \frac{2}{3} = 1$. It follows that the expected payoff from the symmetric equilibrium ξ^{SSE} is indeed zero. Thus, bidders' rents are entirely extracted not only in the SRE, but also in the SSE.

If all bid realizations in the mixed-strategy equilibrium μ^{SSE} are doubled, one obtains an equilibrium of the second-prize chopstick auction. This is so because, in a second-price auction with uniform bids, the losing bid is on average half the winning bid (cf. Szentes and Rosenthal, 2003a). Similarly, an equilibrium of the all-pay variant of the chopstick auction can be constructed by squaring all bids in μ^{SSE}. This is a consequence of more general results of Szentes (2005), which have been made quite explicit for this particular case by Kovenock and Roberson (2012).

4. Proof of Lemma 1

The idea of the proof is it to exploit the stationarity of the behavior strategy ξ^{SSE} as much as possible.

We start by deriving an explicit expression for the bidder's expected payoff from playing an arbitrary pure strategy $x \in D$ against the behavior strategy ξ^{SSE} in the dynamic chopstick auction. Given two pure strategies $x = \{x(t)\}_{t=1}^{\infty} \in D$ and $\widehat{x} = \{\widehat{x}(t)\}_{t=1}^{\infty} \in D$ in the dynamic chopstick auction, we shall say that x *weakly wins all the objects* against $\widehat{x}$, in short $x \geq \widehat{x}$, if for all three objects $v = A, B, C$, the bid $x_\nu = \{x_\nu(t)\}_{t=1}^{\infty}$ either wins or ties against $\widehat{x}_\nu = \{\widehat{x}_\nu(t)\}_{t=1}^{\infty}$. We may then define, for the behavior strategy ξ^{SSE} in the dynamic chopstick auction, its "distribution function"

$$\Phi(x) = \mathrm{pr}(\xi^{\mathrm{SSE}} \leq x). \tag{8}$$

Then, noting that ties occur with probability zero, a bidder's expected payoff from playing the pure strategy $x = (x_A, x_B, x_C) \in D$ against the behavior strategy ξ^{SSE} may be expressed as

$$\begin{aligned}\Pi(x) = {} & 2\{\Phi(x_A, x_B, 1) + \Phi(x_A, 1, x_C) + \Phi(1, x_B, x_C) - 2\Phi(x)\} \\ & - \rho(x_A)\Phi(x_A, 1, 1) - \rho(x_B)\Phi(1, x_B, 1) - \rho(x_C)\Phi(1, 1, x_C).\end{aligned} \tag{9}$$

Exploiting further the fact that all bivariate marginals of Φ are products of independent uniform distributions, as discussed in the previous section, equation (9) may be written alternatively as

$$\begin{aligned}\Pi(x) = {} & 2\rho(x_A)\rho(x_B) + 2\rho(x_A)\rho(x_C) + 2\rho(x_B)\rho(x_C) \\ & - \rho(x_A)^2 - \rho(x_B)^2 - \rho(x_C)^2 - 4\Phi(x).\end{aligned} \tag{10}$$

This is the desired explicit expression for a bidder's expected payoff resulting from a pure-strategy deviation $x \in D$ in the dynamic chopstick auction.

Next, to exploit the stationarity of the behavior strategy ξ^{SSE}, we note that any given pure strategy $x = \{x(t)\}_{t=1}^{\infty} \in D$ in the dynamic chopstick auction may be decomposed into a *first-stage choice*

$$x(1) \in D_1 = \{0,1\} \times \{0,1\} \times \{0,1\}, \tag{11}$$

and a *shifted* pure strategy

$$x^+ = \{x^+(t)\}_{t=1}^{\infty} = \{x(t+1)\}_{t=1}^{\infty} \in D. \tag{12}$$

This decomposition, which can analogously be accomplished for the bid on an individual object, proves very useful for all what follows. For instance, one can readily check that the mapping ρ satisfies the recursive relationship

$$\rho(x_\nu) = \frac{x_\nu(1) + \rho(x_\nu^+)}{2}, \tag{13}$$

for any object $\nu \in \{A, B, C\}$ and for any pure strategy $x \in D$.

Simple recursive relationships can be derived now for the function $\Phi = \Phi(x)$, which will enable us to evaluate the sign of $\Pi(x)$. The following lemma states those relationships, where the symmetry of the behavior strategy ξ^{SSE} across objects allows to restrict attention to a subset of values for the first-stage choice $x(1)$.

Lemma 2. *For any pure strategy $x \in D$ in the dynamic chopstick auction,*

$$\Phi(x) = \frac{1}{4} \cdot \begin{cases} \Phi(x^+) & \text{if } x(1) = (0,0,0) \\ \rho(x_A^+)\rho(x_B^+) & \text{if } x(1) = (0,0,1) \\ \rho(x_A^+) + \Phi(x^+) & \text{if } x(1) = (0,1,1) \\ 1 + \rho(x_A^+)\rho(x_B^+) + \rho(x_A^+)\rho(x_C^+) + \rho(x_B^+)\rho(x_C^+) & \text{if } x(1) = (1,1,1). \end{cases} \tag{14}$$

Proof. Let $x \in D$ be an arbitrary pure strategy in the dynamic chopstick auction. As explained above, we may decompose x into a first-stage choice $x(1) \in D_1$ and a shifted pure strategy $x^+ \in D$. The four cases in equation (14) are now dealt with one at a time:

Case 1. Suppose first that the first-period choice prescribed by the pure strategy x is $x(1) = (0,0,0)$. Intuitively, strategy x speculates on $\omega(1)$ realizing to ω_0, because in all other cases, it becomes impossible to weakly win all three objects. Put more formally, x weakly wins all the objects against a specific realization $\widehat{x} \in D$ of the behavior strategy ξ^{SSE} if and only if the following two conditions hold: (i) the first-period choice prescribed by the realized pure strategy $\widehat{x}$ is $\widehat{x}(1) = (0, 0, 0)$, and (ii) the shifted strategy x^+ weakly wins all objects against the shifted realization

$$\widehat{x}^+ = \{\widehat{x}^+(t)\}_{t=1}^{\infty} = \{\widehat{x}(t+1)\}_{t=1}^{\infty} \in D. \tag{15}$$

But, by definition, the behavior strategy ξ^{SSE} is stationary, i.e., its realizations $\widehat{x} = \{\widehat{x}(t)\}_{t=1}^{\infty} \in D$ are distributed independently across stages. Therefore, the shifted realization $\widehat{x}^+$ follows the same stochastic distribution as $\widehat{x}$, viz. ξ^{SSE}. Moreover, the shifted realizations $\widehat{x}^+$ are distributed independently from the first-period choice $\widehat{x}(1)$ prescribed by the realized pure strategy $\widehat{x}$. Hence, noting that $\widehat{x}(1)$ follows the distribution (5), we arrive at $\Phi(x) = \frac{1}{4} \cdot \Phi(x^+)$, as claimed.

Case 2. Next, suppose that the first-period choice prescribed by the pure strategy x is $x(1) = (0, 0, 1)$. This case is similar to the previous one insofar that strategy x loses the possibility to win all three objects unless $\omega(1)$ realizes to ω_0. But, in contrast to the previous case, if indeed $\omega(1) = \omega_0$, then strategy x has already won object C in stage $t = 1$, so that the allocation remains undetermined only for objects A and B. Hence, in this case, x weakly wins all three objects against a specific pure-strategy realization $\widehat{x} \in D$ from the distribution ξ^{SSE} if and only if the following two conditions hold: (i) the first-stage choice prescribed by $\widehat{x}$ satisfies $\widehat{x}(1) = (0, 0, 0)$, and (ii) the shifted realization $\widehat{x}^+$ satisfies $\widehat{x}^+ \leq (x_A^+, x_B^+, 1)$. Exploiting again the stationarity of ξ^{SSE}, it follows that

$$\Phi(x) = \frac{1}{4} \cdot \Phi(x_A^+, x_B^+, 1) = \frac{\rho(x_A^+)\rho(x_B^+)}{4}. \tag{16}$$

Case 3. Suppose now that $x(1) = (0, 1, 1)$. Then, following the same reasoning as above, it can be checked that strategy x weakly wins all the objects against a specific realization $\widehat{x} \in D$ of the behavior strategy ξ^{SSE} if either (i) $\omega(1) = \omega_0$ and $\widehat{x}^+ \leq$

$(x_A^+, 1, 1)$, or (ii) $\omega(1) = \omega_1$ and $\widehat{x}^+ \leq x^+$. Hence,

$$\Phi(x) = \frac{1}{4} \cdot \rho(x_A^+) + \frac{1}{4} \cdot \Phi(x^+), \tag{17}$$

as claimed.

Case 4. Finally, suppose that $x(1) = (1, 1, 1)$. In this case, it is obvious that strategy x weakly wins all the objects against a specific realization $\widehat{x} \in D$ of the behavior strategy ξ^{SSE} if $\omega(1) = \omega_0$. But strategy x likewise weakly wins all the objects if either (i) $\omega(1) = \omega_1$ and $\widehat{x}^+ \leq (1, x_B^+, x_C^+)$, or (ii) $\omega(1) = \omega_2$ and $\widehat{x}^+ \leq (x_A^+, 1, x_C^+)$, or (iii) $\omega(1) = \omega_3$ and $\widehat{x}^+ \leq (x_A^+, x_B^+, 1)$. The assertion follows now as before.

Since all cases have been covered, this proves relationship (14). □

Next, returning to the proof of Lemma 1, it is shown that there are no profitable deviations. As shown in the previous section, the expected payoff of ξ^{SSE} against itself is zero. Hence, a profitable deviation must yield a strictly positive expected payoff. One can check that the dynamic chopstick auction is not continuous at infinity (Fudenberg and Tirole, 1991, p. 110), so that the one-stage deviation principle cannot be invoked. Fortunately, however, the arguments remain manageable because of the stationarity of ξ^{SSE}. The following cases need to be considered.

Case A. Suppose first that a pure strategy $x \in D$ exists such that

$$x(1) \in S_1^* \equiv \{(0,0,0), (0,1,1), (1,0,1), (0,1,1)\}, \tag{18}$$

and such that $\Pi(x) > 0$. Thus, intuitively, there is a profitable deviation that does not start right away, but only at a later stage. Then, in any of these cases, a straightforward calculation using Lemma 2 as well as equations (10) and (13) delivers $\Pi(x) = \frac{1}{4} \cdot \Pi(x^+)$. For example, if $x(1) = (0, 1, 1)$, then

$$\begin{aligned}\Pi(x) = &\frac{\rho(x_A^+)(1+\rho(x_B^+))}{2} + \frac{\rho(x_A^+)(1+\rho(x_C^+))}{2} + \frac{(1+\rho(x_B^+))(1+\rho(x_C^+))}{2} \\ &- \frac{\rho(x_A^+)^2}{4} - \frac{(1+\rho(x_B^+))^2}{4} - \frac{(1+\rho(x_C^+))^2}{4} - \rho(x_A^+) - \Phi(x^+).\end{aligned} \tag{19}$$

Collecting terms, we find that, indeed,

$$\Pi(x) = \frac{\rho(x_A^+)\rho(x_B^+)}{2} + \frac{\rho(x_A^+)\rho(x_C^+)}{2} + \frac{\rho(x_B^+)\rho(x_C^+)}{2} \tag{20}$$
$$- \frac{\rho(x_A^+)^2}{4} - \frac{\rho(x_B^+)^2}{4} - \frac{\rho(x_C^+)^2}{4} - \Phi(x^+)$$
$$= \frac{1}{4} \cdot \Pi(x^+). \tag{21}$$

The other cases are similar. Thus, even though there is no discounting, delaying a profitable deviation would only lower expected payoffs. Conversely, this shows that the deviation $x^+ \in D$, if used from stage $t = 1$ onwards, would magnify the strictly positive expected payoff from strategy x by the factor 4. Iterating this argument, if necessary, and using the fact that expected payoffs in the chopstick auction are bounded, we find after finitely many applications of the shift operator that there necessarily exists also a profitable pure-strategy deviation $\widehat{x} \in D$ that does not satisfy (18). Thus, it suffices to consider the remaining cases.

Case B. Suppose next that

$$x(1) \in \{(1,0,0),(0,1,0),(0,0,1)\}. \tag{22}$$

By renaming the objects, if necessary, one may assume without loss of generality that $x(1) = (0,0,1)$. But then, using equations (10) and (13), as well as Lemma 2, one obtains

$$\Pi(x) = \frac{\rho(x_A^+)\rho(x_B^+)}{2} + \frac{\rho(x_A^+)(1+\rho(x_C^+))}{2} + \frac{\rho(x_B^+)(1+\rho(x_C^+))}{2} \tag{23}$$
$$- \frac{\rho(x_A^+)^2}{4} - \frac{\rho(x_B^+)^2}{4} - \frac{\left(1+\rho(x_C^+)\right)^2}{4} - \rho(x_A^+)\rho(x_B^+)$$
$$= (-\frac{1}{4}) \cdot (\rho(x_A^+) + \rho(x_B^+) - \rho(x_C^+) - 1)^2 \tag{24}$$
$$\leq 0. \tag{25}$$

Thus, there is no profitable deviation x satisfying (22).[9]

[9] However, here and below, there are obvious indifference relationships, which will matter in the determination of the best-response set. See Section 5.

Case C. Finally, consider the case where

$$x(1) = (1, 1, 1). \tag{26}$$

In this case, one again makes use of equations (10) and (13) as well as of Lemma 2, and finds

$$\begin{aligned} \Pi(x) = {} & \frac{(1+\rho(x_A^+))(1+\rho(x_B^+))}{2} \\ & + \frac{(1+\rho(x_A^+))(1+\rho(x_C^+))}{2} \\ & + \frac{(1+\rho(x_B^+))(1+\rho(x_C^+))}{2} \\ & - \frac{(1+\rho(x_A^+))^2}{4} - \frac{(1+\rho(x_B^+))^2}{4} - \frac{(1+\rho(x_C^+))^2}{4} \\ & - 1 - \rho(x_A^+)\rho(x_B^+) - \rho(x_A^+)\rho(x_C^+) - \rho(x_B^+)\rho(x_C^+). \end{aligned} \tag{27}$$

Rearranging yields

$$\Pi(x) = (-\frac{1}{4}) \cdot (\rho(x_A^+) + \rho(x_B^+) + \rho(x_C^+) - 1)^2 \leq 0. \tag{28}$$

Hence, it is weakly suboptimal to use a pure strategy $x \in D$ satisfying (26) against ξ^{SSE}.

Since there is no profitable pure-strategy deviation, the behavior strategy ξ^{SSE} is a symmetric equilibrium in the dynamic chopstick auction. This completes the proof of Lemma 1.

5. Analysis of the best-response set

This section discusses issues related to the dimensionality of the SSE. More precisely, we will follow Szentes and Rosenthal (2003a) in comparing the respective dimensions of the equilibrium support and the best-response set. For the SRE, the equilibrium support has dimension 2, which is strictly smaller than the dimension of the corresponding best-response set, which is 3. To deal with the SSE, obviously, the traditional definition of dimensionality in terms of degrees of freedom, which is very suitable for smooth objects such as simplices and manifolds, needs to be extended. Below, we shall therefore make use of a more general notion of dimensionality.

We first recall the notion of the Hausdorff dimension.[10]. Given a bounded subset $S \subseteq \mathbb{R}^L$, with $L \geq 1$, and some nonnegative real number $d \geq 0$, the *d-dimensional Hausdorff content* of S is defined as the infimum of the set of numbers $\delta \geq 0$ such that there exist sequences $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{R}^L$ and $\{r_n\}_{n=1}^{\infty}$ in $\mathbb{R}_{++}$ such that (i) for any $z \in S$, there is some index n such that $|z - z_n| \leq r_n$, and (ii) $\sum_{n=1}^{\infty} r_n^d < \delta$. The *Hausdorff dimension* of S, denoted by $\dim_H(S)$, is the infimum of all d for which the d-dimensional Hausdorff content of S is zero.

Let S^* denote the support of the equilibrium bid distribution μ^{SSE}. One can convince oneself that the set S^* consists precisely of those bid vectors $X = (X_A, X_B, X_C) \in \mathbb{R}_+^3$ that are contained in the component-wise image of the support of $\widetilde{\xi}^{\text{SSE}}$ under the mapping ρ.[11] Thus, the set S^* is the popular self-similar structure known as the *Sierpinski tetrahedron.* Rather than offering a formal description, we will provide a geometric description of S^*. The Sierpinski tetrahedron may be constructed from its solid counterpart by first carving out a regular octahedron (see Figure 1), then repeating that task on each of the resulting four smaller tetrahedra, and finally iterating this step at infinitum.

The set S^* is compact and of the same cardinality as the unit cube $[0, 1]^3$, but its Lebesgue measure is zero, and it is not dense in any non-degenerate interval. The Hausdorff dimension of the Sierpinski tetrahedron S^* is

$$\dim_H(S^*) = \frac{\ln 4}{\ln 2} = 2, \tag{29}$$

as follows from standard results on the dimensionality of self-similar sets (see, e.g., Falconer, 2014, Theorem 9.3; cf. also Kvasov, 2007). Since the Sierpinski tetrahedron

[10] According to Falconer (2014), the Hausdorff dimension is the oldest and probably most important notion of fractal dimension. This notion is also consistent with the dimensionality notion used by Kvasov (2007, caption of Figure 2).

[11] Indeed, the support of $\widetilde{\xi}^{\text{SSE}}$ is compact by Tychonoff's theorem and, hence, its continuous image $S^{**} \subseteq [0, 1]^3$ under the component-wise application of the mapping ρ is likewise compact. It follows that S^{**} is closed and of measure one, so that $S^* \subseteq S^{**}$. Conversely, the pre-image of any non-empty set relative open in S^{**} is non-empty and relative open in the support of $\widetilde{\xi}^{\text{SSE}}$, and consequently has positive measure under $\widetilde{\xi}^{\text{SSE}}$. Hence, $S^* \supseteq S^{**}$, which implies $S^* = S^{**}$.

contains its four outer vertices, the convex hull of S^* is just the solid tetrahedron-shaped best-response set considered by Szentes and Rosenthal (2003a).

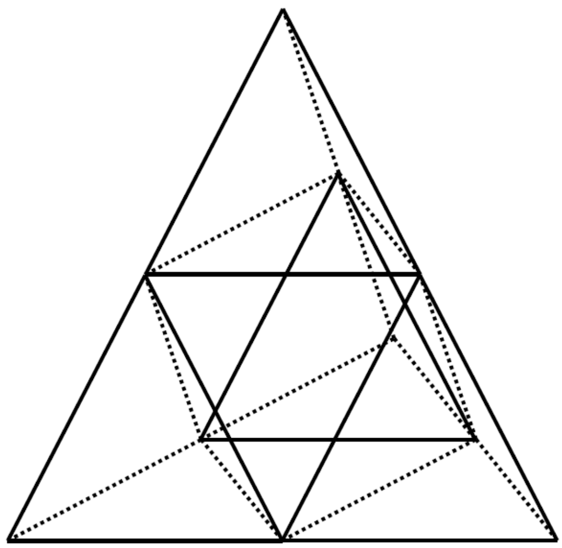

Figure 1

We can show now the following:

Proposition 2. *In the simultaneous chopstick auction, the set of pure best responses to* μ^{SSE} *has Hausdorff dimension two.*

Proof. As seen in Section 4, a pure best response $x \in D$ to ξ^{SSE} in the dynamic chopstick auction is either a pure strategy in the support of ξ^{SSE}, or a finite sequence in the set

$$S_1^* = \{(0,0,0),(0,1,1),(1,0,1),(0,1,1)\}, \tag{30}$$

followed by a pure strategy $\widehat{x} \in D$ such that *either*, up to a renaming of objects, $\widehat{x}(1) = (0,0,1)$ and

$$\rho(\widehat{x}_A^+) + \rho(\widehat{x}_B^+) - \rho(\widehat{x}_C^+) - 1 = 0, \tag{31}$$

or $\widehat{x}(1) = (1,1,1)$ and

$$\rho(\widehat{x}_A^+) + \rho(\widehat{x}_B^+) + \rho(\widehat{x}_C^+) - 1 = 0. \tag{32}$$

In the former case, relationship (13) implies

$$\rho(\widehat{x}_A) = \frac{\rho(\widehat{x}_A^+)}{2},\ \rho(\widehat{x}_B) = \frac{\rho(\widehat{x}_B^+)}{2},\ \text{and}\ \rho(\widehat{x}_C) = \frac{1+\rho(\widehat{x}_C^+)}{2}, \tag{33}$$

so that equation (31) becomes equivalent to

$$\rho(\widehat{x}_A) + \rho(\widehat{x}_B) = \rho(\widehat{x}_C), \tag{34}$$

where $\rho(\widehat{x}_C) \geq \frac{1}{2}$. In the latter case,

$$\rho(\widehat{x}_A) = \frac{1+\rho(\widehat{x}_A^+)}{2},\ \rho(\widehat{x}_B) = \frac{1+\rho(\widehat{x}_B^+)}{2},\ \text{and}\ \rho(\widehat{x}_C) = \frac{1+\rho(\widehat{x}_C^+)}{2}, \tag{35}$$

so that equation (32) is equivalent to

$$\rho(\widehat{x}_A) + \rho(\widehat{x}_B) + \rho(\widehat{x}_C) = 2, \tag{36}$$

where

$$\rho(\widehat{x}_A) \geq \frac{1}{2},\ \rho(\widehat{x}_B) \geq \frac{1}{2},\ \text{and}\ \rho(\widehat{x}_C) \geq \frac{1}{2}. \tag{37}$$

Moreover, the best-response set in the simultaneous chopstick auction is the image under the component-wise application of the mapping ρ of the best-response set in the dynamic auction. Thus, invoking some geometric intuition, the best-response set of the SSE may be thought of as adding to the equilibrium support the four faces of each of the tetrahedra considered during the iterative construction of the Sierpinski tetrahedron. The set of best responses to μ^{SSE} is, therefore, the denumerable union of two-dimensional sets, and as such, two-dimensional. □

Thus, in contrast to the SRE, the best-response set of the SSE is indeed of the same dimension as its equilibrium support. Still, it is hard to tell in the abstract if this property makes it more likely that a process favoring better or best responses would lead to the SSE rather than to the SRE.[12]

[12] However, the robustness of the SSE is supported by ongoing research on the numerical computation of the equilibrium.

6. Relationship to the literature on "fractal" solutions

In this section, it will be shown that the SSE may be characterized as a measure invariant under a simple replacement operation. This way, we can also explain how the SSE relates to "fractal" solutions considered in prior work on the Blotto game.

Consider the following four contraction mappings on the unit cube $[0, 1]^3$:

$$C_0(X) = \frac{1}{2}X \tag{38}$$

$$C_1(X) = \frac{1}{2}\{X + (0, 1, 1)\} \tag{39}$$

$$C_2(X) = \frac{1}{2}\{X + (1, 0, 1)\} \tag{40}$$

$$C_3(X) = \frac{1}{2}\{X + (1, 1, 0)\} \tag{41}$$

Then, by construction, μ^{SSE} is an *invariant measure* (Hutchinson, 1981) with respect to the above family of contractions $\{C_k\}_{k=1}^{4}$. In other words, the probability distribution μ^{SSE} is identical to the equally-weighted convex combination of the four image measures of μ^{SSE} with respect to the contractions (38)-(41). This property can actually be used to characterize the SSE.

Proposition 3. *The probability distribution μ^{SSE} is characterized by the property that it is invariant with respect to the family of contractions $\{C_k\}_{k=1}^{4}$.*

Proof. The claim follows immediately from the fact that the invariant measure is unique (Hutchinson, 1981, Sec. 4). □

The relationship to existing work on "fractal" solutions will be discussed now. Gross and Wagner (1950) constructed new classes of equilibria of Colonel Blotto games by replacing a given hexagon in the hexagonal solution by a convex combination of six smaller replicas. Iterating this replacement operation a finite number of times, absolutely continuous bid distributions of arbitrary geometric complexity (the so-called snowflake solutions) have be constructed. The main purpose of such constructions was it, however, to show that solutions exist with a support that has a Lebesgue measure smaller than any given $\varepsilon > 0$. It was also noted that the respective smallest building blocks in this construction, obtained after a finite number of operations, can be replaced by suitably demagnified disk solutions.[13]

More recently, the same approach of iteratively deriving new equilibria from existing ones has been used by Kvasov (2007) to construct "fractal" solutions to Colonel Blotto games with costly resources. Thereby, it has been shown that, similarly, solutions of Blotto games with costly resources exist the support of which has a Lebesgue measure smaller than any given $\varepsilon > 0$. The family of contractions (38)-(41) considered above for the chopstick auction is obviously a variant of the Gross-Wagner-Kvasov replacement operation for Blotto games.

However, it remains an open question whether a finite iteration of applying the replacement operation (38)-(41) to the SRE would similarly lead to new classes of equilibria in the chopstick auction. Indeed, while the equilibrium property in a Blotto game may be verified quite easily by checking that univariate marginals are uniform (Roberson, 2006), no such simple test is available in the case of the chopstick auction. Instead, geometric considerations of substantial complexity would be needed.[14] In particular, the results of the present paper do not easily follow from existing work.[15]

7. Conclusion

A new type of equilibrium has been identified for an important prototype model of the simultaneous auction, the so-called chopstick auction. A somewhat unusual aspect of the equilibrium bid distribution is that it may be characterized as a self-similar probability measure. Even though similar "fractal" solutions have been constructed before in the class of Blotto games and in games with rational payoff functions, this

[13] In fact, it is not hard to convince oneself that any equilibrium of the Blotto game, including those recently identified by Weinstein (2012), may be used as smallest-scale replacements in this sort of construction.

[14] Even using the methods developed in the present paper, the question is not be easy to answer. The reason is that, if the SRE is translated back into a behavior strategy in the dynamic auction using Kuhn's construction, it turns out to be not only non-stationary, but also path-dependent. This makes it quite difficult to decide the Nash property for candidate equilibria that are derived from the SRE through finite iterations of the replacement operation.

[15] Conversely, however, it is possible to construct entirely new equilibria in Blotto games using the methods developed in the present paper. See the Conclusion.

possibility was (to the author's knowledge) not a well-known feature of the class of simultaneous auctions. The observation must, therefore, be added, to the collection of perplexing properties of this interesting class of auctions.

It is tempting to repudiate any "fractal" solution on the grounds that it is too complicated. The analysis above has shown that this conclusion might be unwarranted. After all, using the dynamic transcription of the simultaneous auction, the self-similar equilibrium constructed in the body of the present paper may be described in simple and intuitive terms. Moreover, there is some evidence (work in progress) that the theoretical robustness property established in the present paper actually matters for numerical computations of the equilibrium strategy, which is also consistent with the prediction of Szentes and Rosenthal (2003b). Thus, the self-similar equilibrium may, paradoxically perhaps, be thought of as being both simpler and more robust than the known solution.

There are large classes of games in political economy, including in particular the interesting class of majority auctions, that may be seen as direct generalizations of the two-bidder three-object auction and for which, in some important special cases, essentially nothing is known about the equilibrium set (cf. Szentes and Rosenthal, 2003b). Unfortunately, a direct extension of the methods developed in the present paper is not fruitful. For example, in a majority auction with five objects, raising the bids dynamically on a randomly selected subset of, say, three objects does not constitute an equilibrium. Therefore, more refined methods are necessary to construct equilibria in those games. A more or less obvious case in which the methods developed in the present paper can actually be used to construct new and interesting equilibria is the class of continuous Colonel Blotto and General Lotto games. However, elaborating further on this extension would go beyond the scope of the present paper. We hope to be able to document these findings more explicitly in future research.

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