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Hyperbolic dimension of metric spaces

Sergei Buyalo* & Viktor Schroeder†

Abstract

We introduce a new quasi-isometry invariant of metric spaces called the hyperbolic dimension, hypdim, which is a version of the Gromov's asymptotic dimension, asdim. One always has hypdim $\leq$ asdim, however, unlike the asymptotic dimension, $\operatorname{hypdim} \mathbb{R}^n = 0$ for every Euclidean space $\mathbb{R}^n$ (while $\operatorname{asdim} \mathbb{R}^n = n$). This invariant possesses usual properties of dimension like monotonicity and product theorems. Our main result says that the hyperbolic dimension of any Gromov hyperbolic space X (with mild restrictions) is at least the topological dimension of the boundary at infinity plus 1, $\operatorname{hypdim} X \geq \dim \partial_\infty X + 1$. As an application we obtain that there is no quasi-isometric embedding of the real hyperbolic space H^n into the metric product of $n-1$ metric trees stabilized by any Euclidean factor, $T_1 \times \cdots \times T_{n-1} \times \mathbb{R}^m$, $m \geq 0$.

1 Introduction

We introduce a new quasi-isometry invariant of metric spaces called the hyperbolic dimension, hypdim, which is a version of the Gromov's asymptotic dimension, asdim. One always has hypdim $\leq$ asdim, however, unlike the asymptotic dimension, $\operatorname{hypdim} \mathbb{R}^n = 0$ for every Euclidean space $\mathbb{R}^n$ (while $\operatorname{asdim} \mathbb{R}^n = n$). This invariant possesses usual properties of dimension like monotonicity and product theorems. To formulate our main result, we recall that a metric space X has *bounded growth at some scale*, if for some constants r, R with $R > r > 0$, and $N \in \mathbb{N}$ every ball of radius R in X can be covered by N balls of radius r, see [BoS].

Theorem 1.1. *Let X be a geodesic Gromov hyperbolic space, which has bounded growth at some scale and whose boundary at infinity $\partial_\infty X$ is infinite. Then*

$$\operatorname{hypdim} X \geq \dim \partial_\infty X + 1.$$

As an application we obtain.

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Corollary 1.2. *For every $n \geq 2$ there is no quasi-isometric embedding $\mathrm{H}^n \to T_1 \times \cdots \times T_{n-1} \times \mathbb{R}^m$ of the real n-dimensional hyperbolic space H^n into the product of $n-1$ trees stabilized by any Euclidean factor $\mathbb{R}^m$, $m \geq 0$.*

Remark 1.3. For $n = 2$ this result has been proved in [BS2] by a different method.

Remark 1.4. In [BS2] we have constructed for every $n \geq 2$ a quasi-isometric embedding of H^n into the n-fold product of homogeneous trees whose vertices have an infinite (countable) degree and whose edges have length 1. By Corollary 1.2, that embedding is optimal with respect to the number of tree-factors even if we allow the stabilization by Euclidean factors. Furthermore, it follows that $\operatorname{hypdim} \mathrm{H}^n = n$ for every $n \geq 2$.

In [JS] it was shown that for every n there exists a right angled Gromov hyperbolic Coxeter group Γ_n with virtual cohomological dimension and coloring number equal to n. In [DS], a bilipschitz embedding $f_n : X_n \to T \times \cdots \times T$ of the Cayley graph X_n of Γ_n into the n-fold product of an arbitrary exponentially branching tree T has been constructed. The Cayley graph of any Gromov hyperbolic group satisfies the conditions of Theorem 1.1, and $\dim \partial_\infty X_n = \dim \partial_\infty \Gamma_n = n - 1$ by a result from [BM](every finitely generated Coxeter group is virtually torsion free). Thus Theorem 1.1 implies (see Theorem 5.6 below) that the embedding f_n is optimal w.r.t. the number of tree-factors even if we allow the stabilization by Euclidean factors.

Hyperbolic dimension versus subexponential corank

In our earlier paper [BS1] we introduced another quasi-isometry invariant of metric spaces called subexponential corank. This invariant gives an upper bound for the topological dimension of a Gromov hyperbolic space which can be quasi-isometrically embedded into a given metric space X, $\operatorname{rank}_h(X) \leq \operatorname{corank}(X)$. Thus corank is a useful tool for finding obstacles to such embeddings, and it works perfectly well in many cases, see [BS1] for details. However, not in all, e.g., for quasi-isometric embeddings $\mathrm{H}^n \to T_1 \times \cdots \times T_k \times \mathbb{R}^m$ it gives only $k \geq n - 2$, while hypdim gives optimal $k \geq n - 1$ by Corollary 1.2. This drawback of corank is closely related to that $\operatorname{corank}(T) = 1 > 0 = \dim \partial_\infty T$ for every metric tree T, while $\operatorname{corank}(X) = \dim \partial_\infty X$ for every $\mathrm{CAT}(-1)$ Hadamard manifold X.

On the other hand, hypdim, which is perfect for product of trees, is much harder to compute than corank. For example, we do not even know the precise value of $\operatorname{hypdim}(\mathrm{H}^2 \times \mathrm{H}^2)$ (it must be 3 or 4). We also do not see any direct way to compare corank and hypdim. At the present stage of knowledge, it looks like that these two invariants are in a sense independent, and each of them works perfectly well in its own range while failing in the other.

Structure of the paper. In section 2 we introduce and discuss properties of a class of metric spaces with uniformly bounded growth rate, UBG-spaces, which is a key ingredient of the hyperbolic dimension. The main result here is Proposition 2.10, which is an important step in the proof of Theorem 1.1.

In section 3 we give three definitions of the hyperbolic dimension following the standard line of the topological dimension theory, and prove their equivalence. It is convenient to use different definitions in different situations. In section 4 we discuss properties of the hyperbolic dimension and prove monotonicity and product theorems. Section 5 is devoted to the proof of Theorem 1.1 and Corollary 1.2.

Here we briefly recall some notions which are used in the body of the paper. We denote by $|x - x'|$ the distance in a metric space X between x, $x' \in X$. A map $f : X \to Y$ between metric spaces is quasi-isometric if for some $\Lambda \geq 1$, $\lambda \geq 0$ the estimates

$$\frac{1}{\Lambda}|x - x'| - \lambda \leq |f(x) - f(x')| \leq \Lambda|x - x'| + \lambda,$$

hold for every x, $x' \in X$. In this case we also say that f is (Λ, λ)-quasi-isometric. A metric space is geodesic if every two its points are connected by a geodesic. By a CAT(-1)-space we mean a complete, geodesic space whose triangles are thinner than the comparison triangles in the real hyperbolic plane H^2.

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2 Spaces with uniformly bounded growth rate

Here we introduce a class of metric spaces which is a key ingredient of the notion of the hyperbolic dimension.

2.1 Definition and properties

We say that a metric space X has uniformly bounded growth rate (or is an UBG-space) if for every $\rho > 0$ there exist $N \in \mathbb{N}$ and $r_0 > 0$ so that every ball of radius $r \geq r_0$ in X contains at most N points which are ρr-separated.

Equivalently, X is UBG if for every $\sigma > 1$ there exist N and r_0 such that every ball of radius σr in X with $r \geq r_0$ can be covered by N balls of radius r. Or X is UBG if for every $\rho > 0$ there is N such that every ball of radius 1 in the scaled λX for all sufficiently small $\lambda > 0$ contains at most N points which are ρ-separated.

Remark 2.1. The notion of UBG-spaces is close to the notion of *asymptotically doubling* spaces, where a metric space X is asymptotically doubling if for some positive constants N and r_0 every ball in X of radius $r \geq r_0$ can be covered by at most N balls of radius $r/2$. One can show essentially by the same argument as in [BoS, p. 295] that if a geodesic space X is asymptotically doubling then it is UBG. However, the assumption that X is geodesic is too restrictive for our purposes. UBG-spaces typically appear in our work as preimages of some sets under quasi-isometric maps, and there is no reason for them to be geodesic. In other words, we study and use UBG-spaces as tools rather than in their own right.

For technical reason, it is convenient to characterize an UBG-space by two functions. We let $\mathcal{N}$ be the set of functions $N : (0,1) \to \mathbb{N}$ and $\mathcal{R}$ be the set of functions $R : (0,1) \to [0,\infty)$. Then the definition above says that X is UBG if for some functions $N \in \mathcal{N}$ and $R \in \mathcal{R}$ and for every $\rho \in (0,1)$ every ball of radius $r \geq R(\rho)$ in X contains at most $N(\rho)$ points which are ρr-separated. In this case we say that X is (N,R)-bounded. We also say that X is N-bounded if it is (N,R)-bounded for some $R \in \mathcal{R}$.

Examples 2.2. 1. Any Euclidean space $\mathbb{R}^n$ is (N,R)-bounded for $N(\rho) \asymp \rho^{-n}$ and $R(\rho) \equiv 0$.

The basic example of UBG-spaces is this.

2. Let B be a bounded metric space. Then the metric product $X = B \times \mathbb{R}^n$ is (N,R)-bounded for $N(\rho) \asymp \rho^{-n}$ and $R(\rho) \geq \frac{2\operatorname{diam} B}{\rho}$. We emphasize that in this example the function N counting the number of separated points is actually independent of B, while the function R describing the corresponding scales tends to infinity as $\operatorname{diam} B \to \infty$ if one takes as B, say, an $\mathbb{R}$-tree.

Two functions $N = N(\rho)$ and $R = R(\rho)$ are included in the definition of UBG-spaces instead of fixing some $\rho \in (0,1)$ for the purpose to make this notion quasi-isometry invariant.

Lemma 2.3. *Let $f : X \to Y$ be a (Λ,λ)-quasi-isometric map, where $\Lambda \geq 1$, $\lambda \geq 0$. Assume that Y is (N,R)-bounded for some $N \in \mathcal{N}$ and $R \in \mathcal{R}$. Then X is (N',R')-bounded for $N'(\rho) = N(\rho/2\Lambda^2)$ and $R'(\rho) = \max\{\frac{1}{\Lambda}(R(\rho) - \lambda), \frac{\lambda}{\Lambda}(1 + \frac{1}{\rho})\}$.*

Proof. Fix $\rho \in (0, 1/2\Lambda^2)$. Then for all $r \geq R(\rho)$ every ball $B_r \subset Y$ contains at most $N(\rho)$ ρr-separated points. We have $f(B_{r'}) \subset B_{\Lambda r' + \lambda}$ for every ball $B_{r'} \subset X$, hence, for $r' \geq \frac{1}{\Lambda}(R(\rho) - \lambda)$ the set $f(B_{r'})$ contains at most $N(\rho)$ σ-separated points, $\sigma = \rho(\Lambda r' + \lambda)$. Take $\sigma' = \Lambda(\sigma + \lambda)$. If x, $x' \in X$ are σ'-separated, then $f(x)$, $f(x') \in Y$ are σ-separated. It follows that the ball $B_{r'}$ itself contains at most $N(\rho)$ σ'-separated points. We put $R'(\rho) = \max\{\frac{1}{\Lambda}(R(\rho) - \lambda), \frac{\lambda}{\Lambda}(1 + \frac{1}{\rho})\}$, $\rho' = 2\Lambda^2\rho$. Hence, for $r' \geq R'(\rho)$ we have $\sigma' \leq 2\Lambda^2\rho r' = \rho' r'$, and $\rho' r'$-separated points are certainly σ'-separated. Then for every $r' \geq R'(\rho)$ every ball $B_{r'} \subset X$ contains at most $N(\rho)$ $\rho' r'$-separated points, and the space X is (N',R')-bounded, where $N'(\rho) = N(\rho/2\Lambda^2)$ and R' as above. □

Corollary 2.4. *The property of a metric space to have a uniformly bounded growth rate is a quasi-isometry invariant.* □

Lemma 2.5. *Every UBG space Y has a polynomial growth rate, that is, there exists $k = k(Y) > 0$ such that for every (sufficiently large) δ every ball of radius r in Y contains at most $d \cdot r^k$ δ-separated points provided r is sufficiently large, where the constant $d > 0$ depends only on Y and δ.*

Proof. We fix $\rho \in (1/2, 1)$ and take δ so that $\delta' = \frac{2\rho-1}{\rho}\delta \geq R(\rho)$. Then every ball $B_{\delta'} \subset Y$ of radius δ' contains at most $N = N(\rho)$ points which are δ'-separated.

Take $r \geq \delta$. Then $r > R(\rho)$, and any ball $B_r \subset Y$ contains at most N points which are ρr-separated. Take a maximal ρr-separated subset in B_r. Then the balls $B_{\rho r}$ centered at its points cover B_r, and their number is at most N. Applying this argument to every $B_{\rho r}$ from the covering of a fixed B_r and proceeding by induction, we obtain that B_r contains at most N^q points which are $\rho^q r$-separated, provided $\rho^q r \geq \delta'$. There is $q \in \mathbb{N}$ with $\rho^q r < \delta \leq \rho^{q-1} r$. For this q the condition $\rho^q r \geq \delta'$ is fulfilled, thus every B_r contains at most $N^q \leq d \cdot r^k$ points which are δ-separated, where $k = \ln N / \ln \frac{1}{\rho}$, $d = (\rho\delta)^{-k}$. □

Lemma 2.6. *Let X, Y be UBG-spaces. Then $Z = X \times Y$ is an UBG-space.*

Proof. The proof is straightforward, if one uses the covering definition of UBG. □

2.2 UBG-subsets in a CAT(−1)-space

Let X be a CAT(-1)-space. We fix a base point $x_0 \in X$ and define the angle metric $\angle_\infty$ in the boundary at infinity $\partial_\infty X$ as follows. Given ξ, $\xi' \in \partial_\infty X$, we consider the unit speed geodesic rays c_ξ, $c_{\xi'}$ from x_0 to ξ, ξ' respectively, and put

$$\angle_\infty(\xi, \xi') = \lim_{s\to\infty} \angle(\overline{c}_\xi(s)\overline{o}\ \overline{c}_{\xi'}(s)),$$

where $\angle(\overline{c}_\xi(s)\overline{o}\ \overline{c}_{\xi'}(s))$ is the angle at $\overline{o}$ of the comparison triangle in H^2 for the triangle $c_\xi(s)x_0c_{\xi'}(s)$. From the hyperbolic geometry we know that

$$\tan\left(\frac{1}{4}\angle_\infty(\xi, \xi')\right) = e^{-\operatorname{dist}(\overline{o}, \overline{\xi}\,\overline{\xi}')},$$

where $\overline{\xi}$, $\overline{\xi}' \in \partial_\infty \mathrm{H}^2$ satisfy $\angle_{\overline{o}}(\overline{\xi}, \overline{\xi}') = \angle_\infty(\xi, \xi')$, and $\overline{\xi}\,\overline{\xi}'$ is the geodesic in H^2 with the end points at infinity $\overline{\xi}$, $\overline{\xi}'$. Thus $\angle_\infty(\xi, \xi') \leq 4e^{-\operatorname{dist}(\overline{o}, \overline{\xi}\,\overline{\xi}')}$.

The shadow of a set $A \subset X$ is a subset $\mathrm{sh}(A) \subset \partial_\infty X$ which consists of the ends ξ of all rays $x_0\xi$ intersecting A (so $\mathrm{sh}(x_0) = \partial_\infty X$). Given $\delta > 0$ we define *the angle δ-measure* of A, $\angle_\delta A$, as

$$\angle_\delta A = \inf_{\mathcal{C}} \sum_{B\in\mathcal{C}} \operatorname{diam}(\mathrm{sh}(B)),$$

where the infimum is taken over all coverings $\mathcal{C}$ of A by balls of radius $\geq \delta$ in X.

Lemma 2.7. *Given functions $N \in \mathcal{N}$, $R \in \mathcal{R}$, for every sufficiently large δ there is a positive constant C depending only on $\rho \in (1/2, 1)$, $N(\rho)$, $R(\rho)$ and δ such that if a subset $A \subset X$ is (N, R)-bounded and* $\operatorname{dist}(x_0, A) \geq c > \delta$, *then*

$$\angle_\delta A \leq C \cdot e^{-c/2},$$

Proof. As in the proof of Lemma 2.5, we fix $\rho \in (1/2, 1)$ and take $\delta \geq \frac{\rho}{2\rho - 1} R(\rho)$. Then, by Lemma 2.5, every ball $B_r \subset X$ with $r > \delta$ contains at most dr^k points of A which are δ separated, where k depends on ρ, $N(\rho)$, and $d = (\rho\delta)^{-k}$. Furthermore, since k is independent of δ, we can assume that $e^{c/2} \geq (c+1)^k$ for each $c > \delta$.

Take a maximal δ-separated subset $A' \subset A$. Then $A \subset \cup_{a \in A'} B_\delta(a)$. For any ball $B_\delta(a)$, $a \in A'$, consider ξ, $\xi' \in \operatorname{sh}(B_\delta(a))$ with $\angle_\infty(\xi, \xi') = \operatorname{diam}(\operatorname{sh}(B_\delta(a)))$. Then $\operatorname{diam}(\operatorname{sh}(B_\delta(a))) \leq 4e^{-\operatorname{dist}(\overline{o}, \overline{\xi}\,\overline{\xi}')}$ in notation introduced above. We take $x \in x_0\xi \cap B_\delta(a)$, $x' \in x_0\xi' \cap B_\delta(a)$ and consider the piecewise geodesic curve γ in X, which consists of the geodesic rays $x\xi$, $x'\xi'$ and the segment xx'. The curve γ, as well as the geodesic $\xi\xi'$, connects in X the points ξ, ξ', and $\operatorname{dist}(x_0, \gamma) \geq \operatorname{dist}(x_0, B_\delta(a)) = |x_0 - a| - \delta$. Furthermore, $\operatorname{dist}(x_0, \gamma) \leq \operatorname{dist}(\overline{o}, \overline{\xi}\,\overline{\xi}')$ by comparison with H^2. Thus

$$\operatorname{diam}(\operatorname{sh}(B_\delta(a))) \leq 4e^{\delta - |x_0 - a|}$$

and

$$\angle_\delta A \leq \sum_{a \in A'} \operatorname{diam}(\operatorname{sh}(B_\delta(a))).$$

Since $c > \delta$, for each $\tau \geq c+1$, the number of points from A' whose distances to x_0 lie in the interval $[\tau - 1, \tau)$ is $\leq d \cdot \tau^k$. Thus we have

$$\begin{aligned} \angle_\delta A &\leq 4e^\delta \sum_{a \in A'} e^{-|x_0 - a|} \leq 4de^\delta \left(\sum_{q=0}^{\infty} (c+q+1)^k e^{-q} \right) e^{-c} \\ &\leq 4de^\delta \left(\sum_{q=0}^{\infty} (q+1)^k e^{-q} \right) (c+1)^k e^{-c} \leq C \cdot e^{-c/2} \end{aligned}$$

by the choice of c. □

2.3 A cut property of UBG-subsets

A subset A of a metric space X is said to be roughly connected if for some $\sigma > 0$ and for every a, $a' \in A$ there is a sequence $a_0 = a, \dots, a_k = a'$ in A with $|a_i - a_{i-1}| \leq \sigma$, $i = 1, \dots, k$. Such a sequence is called a rough or a σ-path between a and a', and we also say that A is σ-connected. One says that a roughly connected subset A of a geodesic space X is *cut-quasi-convex,* if there is $c > 0$ such that for every a, $a' \in A$ and every $x \in aa'$, every rough path in A between a, a' intersects the ball $B_c(x) \subset X$ of radius c centered at x. In this case we also say that A is c-cut-convex, and c is called *the cut radius* of A. This property, obviously, implies that every geodesic segment $aa' \subset X$ with the end points a, $a' \in A$ lies in the c-neighborhood of A, i.e., the set A is in particular quasi-convex. This justifies our terminology.

Proposition 2.8. *Assume that a σ-connected subset A of CAT(-1)-space X is (N, R)-bounded for some functions $N \in \mathcal{N}$, $R \in \mathcal{R}$. Then A is cut-quasi-convex, and the cut radius c depends only on $\rho \in (1/2, 1)$, $N(\rho)$, $R(\rho)$ and σ, $c = c(\rho, N(\rho), R(\rho), \sigma)$.*

Proof. Fix $\rho \in (1/2, 1)$ and take a sufficiently large δ provided by Lemma 2.7. Furthermore, we can assume that $\delta \geq \sigma$. Next, we take $c' > \delta$ such that $C \cdot e^{-c'/2} < \pi$, where C is the constant from Lemma 2.7, and put $c = c' + \delta$. Then $c = c(\rho, N(\rho), R(\rho), \sigma)$.

Assume that for some a, $a' \in A$ there is a σ-path γ in A between a, a' which misses the ball $B_c(x_0) \subset X$ for some $x_0 \in aa'$. Then $A' = \cup_{b \in \gamma} B_\delta(b)$ is a connected subset in X containing the points a, a', thus $\angle_\delta A' \geq \pi$ for every $\delta > 0$.

On the other hand, $\text{dist}(x_0, A') \geq c'$, and we have $\angle_\delta A \leq C \cdot e^{-c'/2} < \pi$ by Lemma 2.7. This is a contradiction, and hence A is cut-quasi-convex with the cut radius c. □

Corollary 2.9. *If a ball B_r of radius r in a CAT(-1)-space X is (N, R)-bounded for some $N \in \mathcal{N}$, $R \in \mathcal{R}$, then $r \leq c$ for some constant $c = c(\rho, N(\rho), R(\rho))$, $\rho \in (1/2, 1)$.* □

Discussing the cut-quasi-convex property, we have used so far only the fact that any UBG-space has a polynomial growth rate (actually, a subexponential growth rate suffices). In what follows, the next Proposition plays a key role, and we use in it the whole power of the definition of UBG-spaces.

Proposition 2.10. *Let A be an N-bounded (for some function $N \in \mathcal{N}$), subset in a CAT(-1)-space X. Then for every $\sigma > 0$ the union $\partial_\infty A_\sigma$ of the boundaries at infinity of σ-connected components of A contains at most M points, where $M < \infty$ depends only on the function N.*

Proof. Fix $x_0 \in X$. It follows from the cut-quasi-convex property of any roughly connected component of A that a tail of every geodesic ray $x_0\xi \subset X$, $\xi \in \partial_\infty A_\sigma$, lies in the c-neighborhood of A for some $c > 0$ depending only on the bounding parameters for A and σ. Thus for every $\rho \in (0, 1)$ and for all sufficiently large r the ball $B_r(x_0) \subset X$ contains at least as much ρr-separated points from A as the cardinality of $\partial_\infty A_\sigma$. Hence, the claim. □

3 Three definitions of the hyperbolic dimension

The hyperbolic dimension is a variation of the Gromov's asymptotic dimension (see [Gr, 1.E]) with only difference that we take as "small" the UBG-sets. Here we give three equivalent definitions of the hyperbolic dimension following the standard line of topological dimension theory. As in [BD], we find convenient to use the Lebesgue number of a covering in our definitions instead of d-multiplicity as in [Gr, 1.E].

Recall that the Lebesgue number of a covering $\mathcal{U}$ of a metric space X is the maximal radius $L(\mathcal{U})$ such that any (open) ball in X of that radius is contained in some element of the covering,

$$L(\mathcal{U}) = \inf_{x \in X} \max\{\operatorname{dist}(x, X \setminus U) : U \in \mathcal{U}\}.$$

Given $N \in \mathcal{N}$, a covering $\mathcal{U}$ of a metric space X is said to be *uniformly N-bounded,* if

- there is a function $R \in \mathcal{R}$ such that every element of the covering is (N, R)-bounded;
- any finite union of elements of the covering is N-bounded.

Remark 3.1. To get a better grip on the second property, which is rather strong, assume that X is a CAT(-1)-space. Then by Proposition 2.10, the boundary at infinity $\partial_\infty A_\sigma \subset \partial_\infty X$ of roughly connected components of any finite union $A = \cup_i U_i$ of elements $U_i \in \mathcal{U}$ has the cardinality bounded above independently of the number of the elements.

3.1 First definition

The hyperbolic dimension of a metric space X, $\operatorname{hypdim}_1 X$, is the minimal n such that for every $d > 0$ there are a function $N \in \mathcal{N}$ and a covering of X by $n+1$ subsets $X_j = \cup_\alpha X_{j\alpha}$, $j = 1, \dots, n+1$ such that

- $X_{j\alpha} \cap X_{j\alpha'} = \emptyset$ for every $j = 1, \dots, n+1$ and all $\alpha \neq \alpha'$;
- the covering $\{X_{j\alpha}\}$ of X is uniformly N-bounded and its Lebesgue number is $\geq d$.

3.2 Second definition

The hyperbolic dimension of a metric space X, $\operatorname{hypdim}_2 X$, is the minimal n such that for every $d > 0$ there are a function $N \in \mathcal{N}$ and a covering $\mathcal{U}$ of X with Lebesgue number $L(\mathcal{U}) \geq d$ and multiplicity $\leq n+1$, which is uniformly N-bounded.

Clearly, $\operatorname{hypdim}_2 X \leq \operatorname{hypdim}_1 X$.

3.3 Third definition

Given a index set J, we let R^J be the Euclidean space of functions $J \to \mathbb{R}$ with finite support, i.e., $x \in \mathbb{R}^J$ iff only finitely many coordinates $x_j = x(j)$ are not zero. Then the distance is well defined by

$$|x - x'|^2 = \sum_{j \in J} (x_j - x'_j)^2.$$

Let $\Delta^J \subset \mathbb{R}^J$ be the standard simplex, i.e., $x \in \Delta^J$ iff $x_j \geq 0$ for all $j \in J$ and $\sum_{j \in J} x_j = 1$.

A metric in n-dimensional simplicial complex P is said to be *uniform* if P is isometric to a subcomplex of $\Delta^J \subset \mathbb{R}^J$ for some index set J. Every simplex $\sigma \subset P$ is then isometric to the standard k-simplex $\Delta^k \subset \mathbb{R}^{k+1}$, $k = \dim \sigma$ (so, for a finite J, $\dim \Delta^J = |J| - 1$). For every simplicial polyhedron P there is the canonical embedding $u : P \to \Delta^J$, where J is the vertex set of P, which is affine on every simplex. Its image $P' = u(P)$ is called *the uniformization* of P, and u is the uniformization map.

The hyperbolic dimension of a metric space X, $\operatorname{hypdim}_3 X$, is the minimal n such that for every $\lambda > 0$ there are a function $N \in \mathcal{N}$ and a λ-Lipschitz map $p : X \to P$ into a uniform n-dimensional simplicial polyhedron P, for which the covering $\{p^{-1}(\mathrm{st}_v) : v \in P\}$ of X by preimages of the open stars $\mathrm{st}_v \subset P$ of the vertices of P is uniformly N-bounded.

3.4 Equivalence of the definitions

Proposition 3.2. *For every metric space X we have*

$$\operatorname{hypdim}_1 X = \operatorname{hypdim}_2 X = \operatorname{hypdim}_3 X.$$

Proof. We have already mentioned that $\operatorname{hypdim}_2 X \leq \operatorname{hypdim}_1 X$ easily follows from the definitions. The proof of $\operatorname{hypdim}_3 X \leq \operatorname{hypdim}_2 X$ is fairly standard (see [Gr, 1.E$_1$], [BD, Propositions 1,2]). Denote $n = \operatorname{hypdim}_2 X$. Given $d > 0$, we have a function $N \in \mathcal{N}$ and a uniformly N-bounded covering $\mathcal{U} = \{U_j\}_{j \in J}$ of X with multiplicity $\leq n+1$ and Lebesgue number $L(\mathcal{U}) \geq d$. Using these data we construct a λ-Lipschitz map $p : X \to \Delta^J$ with $\lambda \leq \frac{(n+2)^2}{d}$, whose image lies in a n-dimensional subpolyhedron $P \subset \Delta^J$, as follows.

Given $j \in J$, we define $q_j : X \to \mathbb{R}$ by $q_j(x) = \min\{d, \operatorname{dist}(x, X \setminus U_j)\}$. Then $\sum_{j \in J} q_j(x) \geq d$ for every $x \in X$. Furthermore, $q_j(x') \leq q_j(x) + |x - x'|$ and

$$\sum_{j \in J} q_j(x') \leq \sum_{j \in J} q_j(x) + (n+1)|x - x'|,$$

because in each sum there are at most $n+1$ nonzero summands. Using this one obtains

$$\frac{1}{\sum_{j \in J} q_j(x')} \leq \frac{1}{\sum_{j \in J} q_j(x)} + \frac{(n+1)|x - x'|}{d \cdot \sum_{j \in J} q_j(x)}.$$

Now, we put $p_j(x) = q_j(x)/\sum_{j\in J} q_j(x)$. Then abbreviating $\Sigma = \sum_{j\in J} q_j(x)$, $\Sigma' = \sum_{j\in J} q_j(x')$ we obtain

$$p_j(x') - p_j(x) = \frac{q_j(x') - q_j(x)}{\Sigma'} + \left(\frac{1}{\Sigma'} - \frac{1}{\Sigma}\right) q_j(x) \le \frac{n+2}{d}|x - x'|.$$

Finally, for $p = \{p_j\}_{j\in J}$ we have

$$|p(x') - p(x)|^2 = \sum_{j\in J}(p_j(x') - p_j(x))^2 \le \frac{(n+2)^2(2n+2)}{d^2}|x - x'|^2,$$

hence $p : X \to \Delta^J$ is λ-Lipschitz with $\lambda \le \frac{(n+2)^2}{d}$. Since for every $x \in X$ there are at most $n+1$ nonzero coordinates $p_j(x)$, $j \in J$, the image $p(X)$ lies in a n-dimensional subpolyhedron $P \subset \Delta^J$. For each vertex $v \in P$ the preimage of its open star $p^{-1}(\mathrm{st}_v) \subset X$ is contained in some element of the covering $\mathcal{U}$. Thus the covering $\{p^{-1}(\mathrm{st}_v) : v \in P\}$ of X is uniformly N-bounded. It follows that $\operatorname{hypdim}_3 X \le \operatorname{hypdim}_2 X$.

To prove that $\operatorname{hypdim}_1 X \le \operatorname{hypdim}_3 X$, we assume that for every $\lambda > 0$ there are a function $N \in \mathcal{N}$ and a λ-Lipschitz map $p : X \to P$ into a uniform n-dimensional simplicial polyhedron P, for which the covering $\{p^{-1}(\mathrm{st}_v) : v \in P\}$ of X is uniformly N-bounded.

Every j-dimensional simplex $\sigma \subset P$ is matched by its barycenter, which is the vertex v_σ in the first barycentric subdivision $\operatorname{ba} P$ of P. Let P_j be the union of the open starts $P_{j\sigma}$ of $\operatorname{ba} P$ of all v_σ with $\dim \sigma = j$,

$$P_j = \bigcup_{\dim \sigma = j} P_{j\sigma}.$$

Then $P_{j\sigma} \cap P_{j\sigma'} = \emptyset$ for $\sigma \ne \sigma'$ and $P = \cup_{j=0}^n P_j$. Now, we put $X_{j\sigma} = p^{-1}(P_{j\sigma})$, $X_j = p^{-1}(P_j)$. Then $X = \cup_{j=0}^n X_j$ and $X_{j\sigma} \cap X_{j\sigma'} = \emptyset$ for every $j = 0, \dots, n$ and all $\sigma \ne \sigma'$. Furthermore, the stars $P_{j\sigma}$ of $\operatorname{ba} P$ are contained in appropriate open stars of P, thus the covering $\{X_{j\sigma}\}$ of X is uniformly N-bounded. Since the polyhedron P is uniform, there is a lower bound $l_n > 0$ for the Lebesgue number of the covering $\{P_{j\sigma}\}$. Therefore, the Lebesgue number of $\{X_{j\sigma}\}$ is at least l_n/λ. This shows that $\operatorname{hypdim}_1 X \le \operatorname{hypdim}_3 X$. □

From now on, we use notation $\operatorname{hypdim} X$ for the common value

$$\operatorname{hypdim}_1 X = \operatorname{hypdim}_2 X = \operatorname{hypdim}_3 X.$$

4 Properties of the hyperbolic dimension

The following two properties of the hyperbolic dimension are obvious from the definitions:

- hypdim $X = 0$ for every UBG-space X.
- hypdim $X \leq$ asdim X for every metric space X.

(The asymptotic dimension of a metric space X, asdim X, is defined precisely in the same way as the hyperbolic dimension taking instead of uniformly N-bounded coverings, coverings by sets with uniformly bounded diameter, see [Gr], [BD]). The last inequality can be strong. For example, asdim $\mathbb{R}^n = n$ for every $n \geq 0$, while hypdim $\mathbb{R}^n = 0$ since $\mathbb{R}^n$ is an UBG-space.

4.1 Monotonicity of the hyperbolic dimension

The following simple but important monotonicity theorem implies, in particular, that the hyperbolic dimension is a quasi-isometry invariant of a metric space.

Theorem 4.1. *Let $f : X \to X'$ be a quasi-isometric map between metric spaces X, X'. Then*

$$\operatorname{hypdim} X \leq \operatorname{hypdim} X'.$$

Proof. The proof is straightforward using the second definition of the hyperbolic dimension and Lemma 2.3. □

Corollary 4.2. *The hyperbolic dimension is a quasi-isometry invariant of metric spaces.* □

4.2 Product Theorem for the hyperbolic dimension

While simplicial complexes appear as nerves of coverings, they are not convenient for the proof of the Product Theorem for the hyperbolic dimension. Instead, we use cubical complexes. Thus we first describe relation between simplicial and cubical complexes.

Given a index set J, one defines $Q^J = \{x \in \mathbb{R}^J : \max_{j\in J} x_j = 1, x_j \geq 0\}$. There is a canonical homeomorphism $\pi_J : Q^J \to \Delta^J$ (the radial projection) given by

$$\pi_J(x) = \frac{x}{\sum_{j\in J} x_j}$$

for every $x \in Q^J$.

Lemma 4.3. *Restricted to any n-dimensional coordinate subspace $\mathbb{R}^n \subset \mathbb{R}^J$ the map π_J is Lipschitz with the Lipschitz constant $\leq n+1$.*

Proof. For x, $x' \in Q^J \cap \mathbb{R}^n$ we have $|x_j - x'_j| \leq |x - x'|$ for every $j \in J$ and $\sum_{j\in J} |x_j - x'_j| \leq n|x - x'|$, $\sum_{j\in J} x_j \geq 1$. Using this, one easily obtains $|\pi_J(x) - \pi_J(x')| \leq (n+1)|x - x'|$, and the claim follows. □

The inverse homeomorphism $\omega_J = \pi_J^{-1} : \Delta^J \to Q^J$ is given by $\omega_J(x) = \lambda(x)x$, where $\lambda(x) = (\max_{j\in J} x_j)^{-1}$.

Lemma 4.4. *Restricted to any n-dimensional coordinate subspace $\mathbb{R}^n \subset \mathbb{R}^J$ the map ω_J is Lipschitz with the Lipschitz constant $\le n(1+\sqrt{n})$.*

Proof. For x, $x' \in \Delta^J \cap \mathbb{R}^n$ we have $|x_j - x'_j| \le |x - x'|$ for every $j \in J$ and

$$|\max_{j\in J} x_j - \max_{j\in J} x'_j| \le \max_{j\in J} |x_j - x'_j| \le |x - x'|,$$

$\max_{j\in J} x_j$, $\max_{j\in J} x'_j \ge 1/n$. Using this, we easily obtain $|\omega_J(x) - \omega_J(x')| \le (n + n\sqrt{n})|x - x'|$, and the claim follows. □

Faces of dimension $k \ge 0$ of Q^J are defined as the closures of its subsets where exactly k coordinates have values in $(0,1)$. They are isometric to the unit cube in $\mathbb{R}^k$, and every such a face is mapped by π_J onto the union of simplices of the first barycentric subdivision $\mathrm{ba}\,\Delta^J$. Vice versa, for every vertex of a k-dimensional simplex $\Delta^k \subset \Delta^J$ its (closed) star in the first barycentric subdivision $\mathrm{ba}\,\Delta^k$ is mapped by ω_J homeomorphically onto a k-dimensional face of Q^J. Therefore, the image $\omega_J(\Delta^k) \subset Q^J$ consists of $k+1$ cubical k-faces. As a corollary of this and Lemmas 4.3, 4.4, we obtain.

Lemma 4.5. *For any n-dimensional subcomplex $P \subset \Delta^J$, the image $P' = \omega_J(P)$ is a n-dimensional cubical subcomplex of Q^J, and $\omega_J : P \to P'$ is a Lipschitz homeomorphism with the Lipschitz constant depending only on n.*

Conversely, for any n-dimensional subcomplex $P' \subset Q^J$, the image $P = \pi_J(P') \subset \Delta^J$ is a n-dimensional simplicial subcomplex of $\mathrm{ba}\,\Delta^J$*, and $\pi_J : P' \to P$ is a Lipschitz homeomorphism with the Lipschitz constant depending only on n.* □

Now, we prove the product Theorem for the hyperbolic dimension.

Theorem 4.6. *For any metric spaces X_1, X_2, we have*

$$\operatorname{hypdim}(X_1 \times X_2) \le \operatorname{hypdim} X_1 + \operatorname{hypdim} X_2.$$

Proof. We use the third definition of the hyperbolic dimension. Given $\lambda_k > 0$ there are a function $N_k \in \mathcal{N}$ and a λ_k-Lipschitz map $p_k : X_k \to P_k \subset \Delta^{J_k}$ into n_k-dimensional uniform simplicial polyhedron P_k, where $n_k = \operatorname{hypdim} X_k$, such that the covering $\{p_k^{-1}(\mathrm{st}_v) : v \in P_k\}$ of X_k is uniformly N_k-bounded, $k = 1, 2$. We assume that n_1, $n_2 < \infty$, since otherwise there is nothing to prove.

By Lemma 4.5, $P'_k = \omega_k(P_k) \subset Q^{J_k}$ is a n_k-dimensional cubical subcomplex of Q^{J_k}, and the homeomorphism $\omega_k = \omega_{J_k} : P_k \to P'_k$ is Lipschitz with the Lipschitz constant depending only on n_k, $k = 1, 2$. Then $\omega_k \circ p_k : X_k \to P'_k$ is Lipschitz with Lipschitz constant $\le \mathrm{const}(n_k) \cdot \lambda_k$. Furthermore, the covering $\{(\omega_k \circ p_k)^{-1}(\mathrm{st}_w) : w \in P'_k\}$ of X_k by preimages of the open cubical stars st_w of the vertices of P'_k is uniformly N_k-bounded, because every such a star $\mathrm{st}_w \subset P'_k$ lies in $\omega_k(\mathrm{st}_v)$ for an appropriate vertex $v \in P_k$, $k = 1, 2$.

Let $J = J_1 \cup J_2$ be the disjoint union. We define $p : X_1 \times X_2 \to \mathbb{R}^J$ by $p(x_1, x_2) = (\omega_1 \circ p_1(x_1), \omega_2 \circ p_2(x_2))$ for every $(x_1, x_2) \in X_1 \times X_2$. Then $p(X_1 \times X_2) \subset P'$, where $P' = P_1' \times P_2' \subset Q^J$ is a cubical subcomplex of dimension $n_1 + n_2$, and the map p is Lipschitz with Lipschitz constant $\mathrm{Lip}(p) \le \mathrm{const}(n_1, n_2) \cdot \max\{\lambda_1, \lambda_2\}$. Using Lemma 2.6 one easily checks that the covering $\{p^{-1}(\mathrm{st}_w) : w \in P'\}$ of $X_1 \times X_2$ by preimages of the open cubical stars is uniformly N-bounded for some function $N \in \mathcal{N}$ depending only on N_1, N_2. Applying the homeomorphism $\pi_J : Q^J \to \Delta^J$, we obtain a simplicial subcomplex $P = \pi_J(P') \subset \mathrm{ba}\,\Delta^J$ and a μ-Lipschitz map $\pi_J \circ p : X_1 \times X_2 \to P$, $\mu \le \mathrm{const}(n_1, n_2) \cdot \max\{\lambda_1, \lambda_2\}$, with required N-boundedness property. It remains to compose this map with a Lipschitz simplicial homeomorphism sending P into a simplicial subcomplex of $\Delta^{J'}$, where J' is the set of all nonempty finite subsets in J. □

5 Proof of Theorem 1.1

5.1 d-multiplicity of a covering

Fix $d > 0$. Recall that the d-multiplicity of a covering $\mathcal{U}$ of a metric space X is $\le n$, if no ball of radius d in X meets more than n elements of the covering (see [Gr, 1.E]). One can define an auxiliary hyperbolic dimension $\mathrm{hypdim}' X$ as a minimal n such that for every $d > 0$ there are a function $N \in \mathcal{N}$ and a uniformly N-bounded covering $\mathcal{U}$ of X with d-multiplicity $\le n + 1$.

Lemma 5.1. *For every metric space X we have* $\mathrm{hypdim}' X \le \mathrm{hypdim}\, X$*.*

Proof. Let $\mathcal{U} = \{U_j\}_{j \in J}$ be a uniformly N-bounded covering of X with multiplicity $\le n + 1$, $n = \mathrm{hypdim}\, X$, and Lebesgue number $L(\mathcal{U}) \ge 2d$ for some function $N \in \mathcal{N}$ and some $d > 0$. Following [BD, Assertion 1], we define $V_j = U_j \setminus D_d(X \setminus U_j)$, $j \in J$, where $D_d(A)$ is the metric d-neighborhood of $A \subset X$. Since $L(\mathcal{U}) \ge 2d$, the collection $\mathcal{V} = \{V_j\}_{j \in J}$ is still a covering of X. Moreover, $\mathcal{V}$ is uniformly N-bounded because $V_j \subset U_j$ for every $j \in J$. Furthermore, if a ball $B_d(x) \subset X$ meets some V_j, then $x \in U_j$. Thus the d-multiplicity of $\mathcal{V}$ is $\le n + 1$. Hence, the claim. □

5.2 Ends of uniformly N-bounded coverings of X

Recall that a metric space X has bounded geometry if there are $\rho_X > 0$ and a function $M_X : (0, \infty) \to (0, \infty)$ such that every ball $B_r \subset X$ of radius $r > 0$ contains at most $M_X(r)$ points which are ρ_X-separated.

Lemma 5.2. *Let X be a CAT(-1)-space with bounded geometry, and let $\mathcal{U}$ be a uniformly N-bounded covering of X with finite d-multiplicity for a function $N \in \mathcal{N}$ and a sufficiently large d. Furthermore, assume that the* *elements of $\mathcal{U}$ are σ-connected for some $\sigma \geq 10d$. Then $\partial_\infty U \subset \partial_\infty X$ is finite for every $U \in \mathcal{U}$ and the number of different elements of $\mathcal{U}$ with infinite diameter is finite.*

Proof. Since the covering $\mathcal{U}$ is uniformly N-bounded, there is a function $R \in \mathcal{R}$ such that every element of the covering is (N, R)-bounded. Then by Proposition 2.8 every $U \in \mathcal{U}$ is cut-quasi-convex with cut radius $c > 0$ depending only on N, R and σ. By Proposition 2.10 there is an upper bound $M < \infty$ for the cardinality of the union $\partial_\infty A_\sigma$ of the boundaries at infinity of σ-connected components of any N-bounded $A \subset X$. In particular, $|\partial_\infty U| \leq M$ for every $U \in \mathcal{U}$.

We assume that X has (ρ_X, M_X)-bounded geometry, and that d-multiplicity of $\mathcal{U}$ is $\leq n$ for $d \geq \rho = \rho_X$. Then we put $M_0 = (n+1) \cdot M \cdot M_X(2c)$ and assume that there are $\geq M_0$ different elements of the covering $\mathcal{U}$ with infinite diameter. Since every finite union of elements of $\mathcal{U}$ is N-bounded, there is $\xi \in \partial_\infty X$ which is a common point of $\partial_\infty U$ for at least M_0/M different elements $U \in \mathcal{U}$.

We fix $x_0 \in X$ and consider the geodesic ray $x_0\xi \subset X$. By the cut-quasi-convex property of U, a tail $x_1\xi \subset x_0\xi$ lies in the c-neighborhood of every selected above U. Thus for every $x \in x_1\xi$ the ball $B_{2c}(x)$ intersects at least $(n+1) \cdot M_X(2c)$ different elements of the covering. On the other hand, $B_{2c}(x)$ can be covered by $\leq M_X(2c)$ balls of radius ρ. Hence, there is a ball $B_\rho(x') \subset X$ which intersects at least $n+1$ different elements of the covering. This is a contradiction, because $d \geq \rho$ and d-multiplicity of the covering is $\leq n$. Thus there is at most $M_0 - 1$ different elements of $\mathcal{U}$ with infinite diameter. □

5.3 Radial contraction

Lemma 5.3. *Let $\mathcal{W}$ be a locally finite collection of subsets of finite diameter in the ray $[0, \infty)$. Then there is a 1-Lipschitz homeomorphism $f : [0, \infty) \to [0, \infty)$ such that every set $f(W)$, $W \in \mathcal{W}$ has diameter at most 1.*

Proof. We let $r_0 = [0, \infty)$ and V_1 be the set of all $W \in \mathcal{W}_0 = \mathcal{W}$ which intersect the initial segment of length 1 of r_0. Since the collection $\mathcal{W}$ is locally finite, V_1 is finite. Then $t_1 = \sup\{t \in r_0 : t \in W \in V_1\}$ is finite, and we define $\varphi_1 : r_0 \to r_0$ by $\varphi_1(t) = t$ for $t \in [0, 1]$, and for $t \geq 1$ by $\varphi_1(t) = t$ if $t_1 \leq 2$, $\varphi_1(t) = \frac{1}{t_1 - 1}(t - 1) + 1$ otherwise. Then φ_1 is a 1-Lipschitz homeomorphism, and the diameter $\mathrm{diam}(\varphi_1(W)) \leq 2$ for every $W \in V_1$.

We put $f_1 = \varphi_1$ and note that $\mathcal{W}_1 = f_1(\mathcal{W}) \cap r_1$ is a locally finite collection of subsets in the ray $r_1 = [1, \infty)$. Now, we apply the same procedure to the ray r_1 and the collection $\mathcal{W}_1$ extending the resulting 1-Lipschitz homeomorphism $\varphi_2 : r_1 \to r_1$ to r_0 by the identity and putting $f_2 = \varphi_2 \circ f_1$.

Repeating we obtain a stabilizing sequence of 1-Lipschitz homeomorphisms $f_n = \varphi_n \circ f_{n-1} : r_0 \to r_0$, $f_n = f_{n-1}$ on $[0, n]$, and for which

$\operatorname{diam}(f_n(W)) \le 2$ for all $W \in \mathcal{W}$ intersecting $[0, n]$. Composing the limit homeomorphism $\lim_{n\to\infty} f_n$ with the homothety $\lambda(t) = t/2$, we obtain a required homeomorphism f. □

5.4 Cone over the boundary at infinity

The following estimate from below is the major step in the proof of Theorem 1.1.

Proposition 5.4. *Let $X \subset \mathrm{H}^n$, $n \ge 2$, be the convex hull of a compact infinite $Z \subset \partial_\infty \mathrm{H}^n$. Then* $\operatorname{hypdim} X \ge \dim Z + 1$.

For the proof we need some preparations. We fix a base point $o \in X$ and define the cone over Z, $\mathrm{Co}(Z) \subset X$, as the union of all geodesic rays emanating from o towards Z. Note that $\mathrm{Co}(Z)$ with the metric induced from H^n in general is neither $\mathrm{CAT}(-1)$ nor even geodesic space as for example in the case $\dim Z = 0$. On the other hand, $\mathrm{Co}(Z)$ is cobounded in X (see [BoS, Proposition 10.1(2)]), that is $\operatorname{dist}(x, \mathrm{Co}(Z)) \le \sigma_0$ for some $\sigma_0 > 0$ and every $x \in X$. In what follows, we also use the angle metric in $\partial_\infty \mathrm{H}^n$ based at o.

Next, we consider the annulus $\mathrm{An}(Z) \subset \mathrm{Co}(Z)$, which consists of all $x \in \mathrm{Co}(Z)$ with $1 \le |x - o| \le 2$. Clearly, $\mathrm{An}(Z)$ is homeomorphic to $Z \times I$, $I = [0, 1]$. According to a well known result from the dimension theory (see [Al]), the topological dimension

$$\dim \mathrm{An}(Z) = \dim Z + 1. \qquad (*)$$

We also need the following simple fact from the dimension theory.

Lemma 5.5. *Let Z be an infinite compact metric space. Then for every finite $A \subset Z$ and all sufficiently small $\varepsilon > 0$ we have* $\dim(Z \setminus D_\varepsilon(A)) = \dim Z$, *where $D_\varepsilon(A)$ is the open ε-neighborhood of A.*

Proof. Since Z is infinite, one can assume that $n = \dim Z > 0$. Let $Y \subset Z$ be the subset which consists of all points $z \in Z$ at which Z has dimension n. It is well known (see [HW, Ch.IV.5]) that $\dim Y = n$, thus Y cannot be finite, and $Y \not\subset D_\varepsilon(A)$ for all sufficiently small $\varepsilon > 0$. The claim follows. □

Proof of Proposition 5.4. The space H^n and hence its subspace X certainly have a bounded geometry. Let $\rho_{\mathrm{H}^n} > 0$ and $M_{\mathrm{H}^n} : (0, \infty) \to (0, \infty)$ be the corresponding bounding parameters. Assume that $\operatorname{hypdim} X < \dim Z + 1$. Then, moreover, $\operatorname{hypdim}' X < \dim Z + 1$ (see sect. 5.1). Thus for $d \ge \rho_{\mathrm{H}^n}$ there are a function $N \in \mathcal{N}$ and a uniformly N-bounded covering $\mathcal{U}$ of X with d-multiplicity $\le \dim Z + 1$. We fix $\sigma \ge 10d$ and note that taking σ-connected components of $\mathcal{U}$ changes neither N-boundedness nor

d-multiplicity. Thus we can assume W.L.G. that the elements of $\mathcal{U}$ are σ-connected. By Lemma 5.2 there are only finitely many elements $U \in \mathcal{U}$ with infinite diameter, and the boundary at infinity each of them is finite.

We let $\mathcal{V}$ be the set of σ-connected components of the induced covering $\mathcal{U} \cap \mathrm{Co}(Z)$ of $\mathrm{Co}(Z)$. Then $\mathcal{V}$ is uniformly N-bounded and its d-multiplicity is $\leq \dim Z + 1$. The covering $\mathcal{V}$ of $\mathrm{Co}(Z)$ can be represented as the disjoint union $\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_\infty$, where $\mathcal{V}_0$ consists of all $V \in \mathcal{V}$ with finite diameter, and respectively $\mathcal{V}_\infty$ consists of all $V \in \mathcal{V}$ with infinite diameter.

There are natural polar coordinates $x = (z, t)$, $z \in Z$, $t \geq 0$, in $\mathrm{Co}(Z)$. Every 1-Lipschitz homeomorphism $f : [0, \infty) \to [0, \infty)$ induces a 1-Lipschitz homeomorphism $F : \mathrm{Co}(Z) \to \mathrm{Co}(Z)$ by $F(z, t) = (z, f(t))$, which does not change the visual diameter $\mathrm{diam}(\mathrm{sh}(A))$ of any $A \subset \mathrm{Co}(Z)$.

Given $V \in \mathcal{V}_0$ let $W = W_V \subset [0, \infty)$ be the ray projection of V, i.e.,

$$W = \{t \geq 0 : (z, t) \in V \text{ for some } z \in Z\}.$$

Then $\mathrm{diam}\, W < \infty$ and the collection $\mathcal{W} = \{W_V : V \in \mathcal{V}_0\}$ is locally finite in $[0, \infty)$ since Z is compact and $\mathcal{V}_0$ is locally finite. By radial contraction Lemma 5.3 there is 1-Lipschitz homeomorphism $f : [0, \infty) \to [0, \infty)$ such that every set $f(W) \subset [0, \infty)$, $W \in \mathcal{W}$, has diameter at most 1. For the induced homeomorphism $F : \mathrm{Co}(Z) \to \mathrm{Co}(Z)$ we denote by $\mathcal{V}^1$ the covering $F(\mathcal{V}) = \{F(V) : V \in \mathcal{V}\}$ of $\mathrm{Co}(Z)$. Then $\mathcal{V}^1 = \mathcal{V}^1_0 \cup \mathcal{V}^1_\infty$ for $\mathcal{V}^1_0 = F(\mathcal{V}_0)$, $\mathcal{V}^1_\infty = F(\mathcal{V}_\infty)$, and every $V \in \mathcal{V}^1_0$ lies in some annulus of width 1 centered at o.

Consider the sequence of contracting homeomorphisms $F_k : \mathrm{Co}(Z) \to \mathrm{Co}(Z)$ given by $F_k(z, t) = (z, \frac{1}{k}t)$, $(z, t) \in \mathrm{Co}(Z)$, $k \in \mathbb{N}$, and the corresponding sequence of coverings $\mathcal{V}^k = \mathcal{V}^k_0 \cup \mathcal{V}^k_\infty$, where $\mathcal{V}^k = F_k(\mathcal{V}^1)$. Using Lemma 2.7 and the fact that F_k does not change the visual diameter, one easily obtains that the diameter of the elements from $\mathrm{An}(Z) \cap \mathcal{V}^k_0$ vanishes as $k \to \infty$.

On the other hand, the angle measure of $U \setminus B_c(o)$, $U \in \mathcal{U}$, is exponentially small in c by Lemma 2.7. It follows that for every $\varepsilon > 0$ the shadow of $U \setminus B_c(o)$ is contained in the ε-neighborhood of the finite set $\partial_\infty U$, if c is chosen sufficiently large, $\mathrm{sh}(U \setminus B_c(o)) \subset D_\varepsilon(\partial_\infty U) \subset \partial_\infty \mathrm{H}^n$. Thus for each $V \in \mathcal{V}_\infty$ the sequence $F_k \circ F(V) \cap \mathrm{An}(Z)$ converges to a finite union of segments $(z, [1, 2]) \subset \mathrm{An}(Z)$, $z \in Z_V$, as $k \to \infty$, where $Z_V \subset \partial_\infty U$ for $U \in \mathcal{U}$, $U \supset V$. Since there are only finitely many elements $U \in \mathcal{U}$ with infinite diameter, and every $V \in \mathcal{V}_\infty$ is contained in one of them, there is a finite $A \subset Z$ such that $Z_V \subset A$ for every $V \in \mathcal{V}_\infty$. Moreover, for every $\varepsilon > 0$ there is $k_\varepsilon \in \mathbb{N}$ such that the compact set $\mathrm{An}_\varepsilon(Z) = \mathrm{An}(Z \setminus D_\varepsilon(A))$ misses any $V \in \mathcal{V}^k_\infty$, and thus it is covered by elements of $\mathcal{V}^k_0$, $k \geq k_\varepsilon$. The multiplicity of the covering $\mathcal{V}^k_0$ of $\mathrm{An}_\varepsilon(Z)$ is $\leq \dim Z + 1$, and the diameter of its elements vanishes as $k \to \infty$. Thus $\dim \mathrm{An}_\varepsilon(Z) \leq \dim Z$. However, by $(*)$ and Lemma 5.5, $\dim \mathrm{An}_\varepsilon(Z) = \dim Z + 1$ for all sufficiently small $\varepsilon > 0$. This is a contradiction. □

5.5 Proofs of Theorem 1.1 and Corollary 1.2

Proof of Theorem 1.1. By [BoS, Theorem 1.1], the space X is roughly similar to a convex subset of H^n for some integer n. Actually, it is proved there that $\partial_\infty X$ is homeomorphic to a compact $Z \subset \mathrm{H}^n$ such that X is roughly similar to the convex hull of Z in H^n. Thus we can assume that X is the convex hull of some compact, infinite $Z \subset \mathrm{H}^n$. Now, Proposition 5.4 completes the proof. □

Proof of Corollary 1.2. We actually prove a more general result.

Theorem 5.6. *Assume that there is a quasi-isometric embedding*

$$f : X \to T_1 \times \cdots \times T_k \times Y,$$

of a Gromov hyperbolic space X, satisfying the conditions of Theorem 1.1, into the k-fold product of metric trees $T_1, \dots, T_k$ stabilized by an UBG-factor Y. Then $k \geq \dim \partial_\infty X + 1$.

Proof. First, we note that $\operatorname{hypdim} Y = 0$. Next, the asymptotic dimension of every metric tree T is at most 1, $\operatorname{asdim} T \leq 1$, see [DJ, Proposition 4]. Therefore, $\operatorname{hypdim} T \leq 1$, and by the Product Theorem the hyperbolic dimension of the target space is at most k. On the other hand, $\operatorname{hypdim} X \geq \dim \partial_\infty X + 1$ by Theorem 1.1. Now, the required estimate follows from mononicity of hypdim, see Theorem 4.1. □

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