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Fictitious Play in Networks

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Abstract. This paper studies fictitious play in networks of noncooperative two-player games. We show that continuous-time fictitious play converges to Nash equilibrium provided that the overall game is zero-sum. Moreover, the rate of convergence is $1/\tau$, regardless of the size of the network. In contrast, arbitrary n-player zero-sum games do not possess the fictitious-play property. As an extension, we consider networks in which each bilateral game is strategically zero-sum, a weighted potential game, or a two-by-two game. In those cases, convergence requires either a condition on bilateral payoffs or that the underlying network structure is acyclic. The results are shown to hold also for the discrete-time variant of fictitious play, which entails a generalization of Robinson's theorem to arbitrary zero-sum networks. Applications include security games, conflict networks, and decentralized wireless channel selection.

Keywords. Fictitious play · networks · zero-sum games · conflicts · potential games · Miyasawa's theorem · Robinson's theorem

JEL-Codes C72 – Noncooperative Games; D83 – Search, Learning, Information and Knowledge, Communication, Belief, Unawareness; D85 – Network Formation and Analysis: Theory

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1. Introduction

Fictitious play (Brown, 1949, 1951; Robinson, 1951) refers to a class of simple and intuitive models of learning in games. The common element of such models is that a player is imagined to respond optimally to an evolving belief on the behavior of his opponents, where the player's belief at any point in time is formed on the basis of the empirical frequencies of strategy choices made by his opponents up to that point in time. Understanding the conditions under which fictitious play converges to Nash equilibrium is important because such results help to clarify the intuition that equilibrium play may be reached even if players are not perfectly rational.[1] While variants of fictitious play are known to converge in large classes of two-player games, the case of n-player games has been explored to a much lesser extent.[2]

This paper studies the dynamics of fictitious play in general classes of network games. We first consider what we call *zero-sum networks* (Bregman and Fokin, 1987, 1998; Daskalakis and Papadimitrou, 2009; Cai and Daskalakis, 2011; Cai et al., 2016). These are multiplayer zero-sum games that can be represented as a network of two-player games. This class of games is actually quite large and includes practically relevant examples of resource allocation games such as generalized Blotto and security games. We show that the Lyapunov methods of Hofbauer (1995) and Harris (1998) can be extended to the class of zero-sum networks. Specifically, the Lyapunov function considered in the present paper aggregates, across all players in the network, the maximum payoff that could be obtained by optimizing against the empirical frequency distribution of prior play. This function converges to zero at rate $1/t$ on any continuous-time fictitious-play (CTFP) path, and regardless of the size of the network. In particular, CTFP converges to equilibrium in any zero-sum network. However, as we also show with an example, arbitrary n-player zero-sum games do not possess the fictitious-play property, i.e., the network assumption is crucial.

To gauge the role of the zero-sum assumption, we consider three additional classes of network games. First, we look at networks of strategically zero-sum games, or *conflict networks*.[3] Thus,

[1] The literature on fictitious play is too large to be surveyed here. For an introduction to the theory of learning in games, see Fudenberg and Levine (1998). The literature on learning in social networks has recently been surveyed by Acemoglu and Ozdaglar (2011). For a concise discussion of epistemic vs. dynamic foundations of Nash equilibrium, see Krishna and Sjöström (1997).

[2] Positive convergence results for n-players have been established, in particular, for games solvable by iterated dominance (Milgrom and Roberts, 1991), games with identical interests (Monderer and Shapley 1996b; Harris, 1998), and various classes of star-shaped network games (Sela, 1999). Jordan (1993) has shown that fictitious play need not converge in a non-zero-sum three-player game. See also Gaunersdorfer and Hofbauer (1996).

[3] Recent papers that look at conflict networks include Bozbay and Vesperoni (2014), Dziubiński et al. (2016a),

each bilateral game in the network is assumed to be best-response equivalent in mixed strategies to a zero-sum game. Moulin and Vial (1978) noted that fictitious play converges in this class of two-player games. In networks of conflicts, CTFP converges as well, provided that valuations in the bilateral games satisfy a condition that we call *pairwise homogeneity of valuations*. This assumption is satisfied, for example, in transfer networks considered by Franke and Öztürk (2015). When valuations are heterogeneous, however, convergence need not hold in general. Intuitively, the aggregation of bilateral payoffs does not commute with the strategic equivalence because the payoff transformations that turn two different bilateral games into zero-sum games need not be identical. We illustrate this fact with a surprisingly simple example in the spirit of Jordan (1993). But convergence can still be obtained with heterogeneous valuations when the underlying network structure is *acyclic*, i.e., a disjoint union of trees.[4]

Next, we assume that bilateral games are weighted potential games. Applications include channel selection problems in wireless communication networks, and the spreading of ideas and technologies over social networks, for instance. Extending the analysis of Cai and Daskalakis (2011), it is shown that CTFP converges to equilibrium in any network of exact potential games. However, as we show with still another example, fictitious play need not converge in general networks of weighted potential games. Instead, in a somewhat unexpected analogy, the convergence result holds for weighted potential games under the condition that the underlying network structure is acyclic. Finally, by combining our findings for conflict networks with pairwise homogeneous valuations and for networks of exact potential games, we obtain a generalization of Miyasawa's (1961) theorem to network games on arbitrary graphs. As additional extensions, the paper looks at discrete-time fictitious play (DTFP), generalizing Robinson's (1951) famous result for two-person zero-sum games to arbitrary n-player zero-sum networks, and at the possibility of correlated beliefs which is a relevant aspect in multiplayer games.

As a contribution of potentially independent interest, we substantially simplify the Lyapunov approach to CTFP in two-player zero-sum games introduced by Hofbauer (1995) and Harris (1998). That approach has traditionally combined the envelope theorem with the theory of differential in-

Franke and Öztürk (2015), Huremovic (2016), Jackson and Nei (2015), König et al. (2015), Kovenock et al. (2015), amongst others. For a survey, see Dziubiński et al. (2016b).

[4] E.g., any star-shaped network considered by Sela (1999) is acyclic, but the network shown in Figure 1 below is not acyclic.

clusions, where the former part is fairly simple (cf., e.g., Krishna and Sjöström, 1997), while the latter part is quite hairy (cf. Harris, 1998, Sec. 5-6). Driesen (2009) proposed an alternative way to simplify the proof by relating to the general belief affirmation result of Monderer et al. (1997), yet at the cost of assuming that players choose pure strategies, which might interfere with existence (cf. Harris, 1998, p. 242).[5] In our derivation of the convergence result, however, no additional assumptions are imposed. In fact, the technical apparatus of differential inclusions is entirely dropped and replaced by elementary considerations. The brevity of the argument will become apparent from the proof of Proposition 1 below.[6]

Related literature. The first paper studying fictitious play in an environment similar to ours is Sela (1999). His observation was that some of the convergence results for two-player games generalize quite easily to n-player games with a "one-against-all" structure. In that setting, one player located in the center of the star-shaped network chooses a *compound* strategy that is the same in every bilateral interaction. The crucial point to note is then that the network game can be transformed into a two-player game in which the choices of the non-centered players are orchestrated by a single agent that maximizes the sum of the payoffs of all the non-centered players. Under a specific tie-breaking rule, the DTFP process in the reduced game turns out to be identical, for any given initial condition, to the DTFP process in the "one-against-all" game. Thereby, the fictitious-play property in the network game can be established as a corollary of results for two-person games provided that the bilateral games are all of the same type, like zero-sum, identical payoff, or two-by-two. It is, however, not immediate to see how this trick could be generalized to networks that are not star-shaped. The present paper extends the results of Sela (1999) to general network structures. We also drop the tie-breaking rule, and deal more explicitly with the case of CTFP.

An interesting recent strand of literature, related to the interdisciplinary field of algorithmic game theory, has taken up the study of networks of two-player games, where it is assumed that each player's payoff is the sum of payoffs obtained in the bilateral games with neighboring players.[7] Cai et al. (2016) have clarified how far results traditionally known only for two-person zero-sum games

[5] See also Shamma and Arslan (2004) for a unifying presentation of existing Lyapunov arguments in a set-up with a "soft-max" best-response function.

[6] However, differential inclusions will still be used to prove existence of a CTFP, as well as for the analysis of the discrete-time variant of fictitious play.

[7] Network games have, of course, a long tradition in game theory. See, e.g., the recent survey by Bramoullé and Kranton (2016), and references given therein.

(such as solvability by a linear program, existence of a value, equivalence of max-min and equilibrium strategies, exchangeability of Nash equilibria, and the relationship to coarse correlated equilibrium) can be extended to zero-sum network games. Moreover, in that class of games, discrete-time no-regret learning algorithms converge to Nash equilibrium (Daskalakis and Papadimitriou, 2009; Cai and Daskalakis, 2011).[8] Another natural class of network games is defined by the requirement that, in each bilateral game, both players have identical payoff functions (Cai and Daskalakis, 2011).[9] Under the condition that pairwise interactions are games with identical payoffs, the network game is shown to possess an exact potential, which implies that in that class of games, certain learning algorithms converge to equilibrium. In particular, the discrete dynamics of pure best responses converges to a Nash equilibrium. However, these contributions do not discuss any fictitious-play dynamics.

The remainder of this paper is structured as follows. Section 2 contains preliminaries. The CTFP property of zero-sum networks is established in Section 3. Section 4 deals with additional classes of games. DTFP is considered in Section 5. Section 6 discusses the case of correlated beliefs. Section 7 concludes.

2. Preliminaries

2.1 Network games

There is a finite set $V = \{1, ..., n\}$ of players (countries, firms, consumers, political institutions,...).[10] Let $E \subseteq V \times V$ be a set of bilateral relationships. Any two agents $i, j \in \{1, ..., n\}$ are either in interaction ($(i, j) \in E$) or not in interaction ($(i, j) \notin E$). Thus, the pair (V, E) is a graph, and we assume that it is (i) undirected ($\forall i, j : (i, j) \in E \Leftrightarrow (j, i) \in E$), and (ii) irreflexive ($\forall i : (i, i) \notin E$).[11] Each edge $(i, j) \in E$ represents a finite two-person game G_{ij} between players i and j, with strategy set S_{ij} for player i, strategy set S_{ji} for player j, payoff function $u_{ij} : S_{ij} \times S_{ji} \to \mathbb{R}$ for player i,

[8] In a *no-regret learning* process, aggregate historical payoffs are asymptotically not below optimal payoffs achievable against historical frequency distributions. An example is the multiplicative-weights adaptive learning algorithm of Freund and Schapire (1999). For additional background, see the monograph of Cesa-Bianchi and Lugosi (2006).

[9] This class includes, e.g., binary coordination games, as considered by Bramoullé and Kranton (2016, Prop. 3). See also Bramoullé et al. (2014) and Bourlès et al. (2015). For early uses of potential methods in network models, see Blume (1993) and Young (1993).

[10] Finiteness of the network looks essential to the convergence of fictitious play. However, we have not looked specifically into this issue. Blume (1993) studies the strategic interaction of players that are located on an infinite lattice. Morris (2000) considers best-response dynamics in locally finite networks of coordination games.

[11] While the underlying network structure is assumed to be exogenous, our set-up is consistent with the view that players strategically choose a subset of their neighbors as (potential) partners (see, e.g., Jackson, 2005). Along these lines, the dynamic evolution of a network is subsumed as well.

and payoff function $u_{ji} : S_{ji} \times S_{ij} \to \mathbb{R}$ for player j.[12] Choices made in the bilateral games (or edge games) will be referred to as *bilateral* strategies.

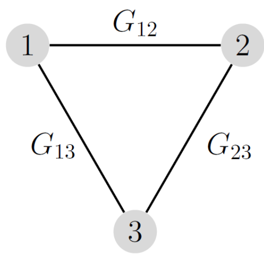

Figure 1. A network game.

Let $N(i) = \{j : (i,j) \in E\}$ denote the set of *neighbors* of player i. Figure 1 provides an illustration of a network with three players, each of them playing against two neighbors. Note that the number of neighbors of a player corresponds to the player's degree as a node in the network of interactions. In the example, the network is complete, but this is not assumed.[13] Denote by $\varnothing \neq X_i \subseteq \bigtimes_{j \in N(i)} S_{ij}$ the set of *multilateral* strategies of agent i (defense policies, trade quotas, promotional strategies, prices, invitation or acceptance of friendship, etc.). It is important to note that we allow for $X_i \subsetneq \bigtimes_{j \in N(i)} S_{ij}$, which will be the interesting case in most applications. For example, there could be budget constraints, limited resources (e.g., planes in a military conflict), or the need for price coherence across platforms.[14]

For a given multilateral strategy $x_i \in X_i$ of player i, we denote by $\sigma^j(x_i) = s_{ij} \in S_{ij}$ the corresponding bilateral strategy vis-a-vis player j. Going over all neighbors of player i, we see that any multilateral strategy $x_i \in X_i$ may be considered as a vector of bilateral strategies $x_i = \{s_{ij}\}_{j \in N(i)} = \{\sigma^j(x_i)\}_{j \in N(i)}$. Conversely, the set of profiles composed of multilateral strategies

[12] Thus, G_{ij} and G_{ji} refer to the same game, yet in the first case from player i's perspective, and in the second case from player j's perspective. Note also that the *first* argument in a bilateral payoff function u_{ij} always refers to player i's strategy. E.g., even in the expression $u_{21}(s_{21}, s_{12})$, strategy s_{21} is player 2's bilateral strategy vis-a-vis player 1, and s_{12} is player 1's bilateral strategy vis-a-vis player 2.

[13] We shall use the standard terminology of graph theory (see, e.g., Bollobás, 2013). Thus, a network (V, E) is *complete* if $N(i) = V \backslash \{i\}$ for all $i = 1, ..., n$; it is called *acyclic* when there is no finite sequence of pairwise distinct players $i_1, ..., i_\kappa \in V$ with $\kappa \geq 3$ such that $(i_1, i_2) \in E$, $(i_2, i_3) \in E$,..., $(i_{\kappa-1}, i_\kappa) \in E$, and $(i_\kappa, i_1) \in E$; a network is *star-shaped* if there is a player i such that $N(i) = V \backslash \{i\}$ and such that $N(j) = \{i\}$ for any $j \neq i$; finally, a network is called *connected* if, for any i and j with $i \neq j$, there is a finite sequence $i_1, i_2, ..., i_\kappa \in V$ with $i_1 = i$ and $i_\kappa = j$ such that $(i_1, i_2) \in E$, $(i_2, i_3) \in E$,..., and $(i_{\kappa-1}, i_\kappa) \in E$.

[14] A special case of our setting occurs if $S_{ij} \simeq X_i$ for any $i = 1, ..., n$ and any $j \in N(i)$. In that case, a player's multilateral strategy $x_i \in X_i$ is assumed to implement the same *compound* strategy $s_{ij} = x_i$ in each bilateral game with neighbor $j \in N(i)$, so that X_i corresponds to the diagonal in $\times_{j \in N(i)} S_{ij}$. Settings along these lines have been considered, in particular, by Sela (1999) and Cai et al. (2016), and will also be used in our examples. Clearly, our set-up is no less general than those settings.

chosen by the neighbors of player i is $X_{N(i)} = \bigtimes_{j \in N(i)} X_j$, with typical element $x_{N(i)} = \{x_j\}_{j \in N(i)}$. A *network game* (or *separable game* or *polymatrix game*) G is an n-player game in which each player $i = 1, ..., n$ has strategy set X_i and utility

$$u_i(x_i, x_{-i}) = u_i(x_i, x_{N(i)}) = \sum_{j \in N(i)} u_{ij}(s_{ij}, \sigma^i(x_j)). \tag{1}$$

Reflecting the local nature of interaction and payoffs, it will be assumed below that players form beliefs about the play of their neighbors only (rather than about the play of *all* the other players). While this assumption is not required for our results, it may be considered somewhat more plausible in a learning context.

We denote by $\Delta(X_i)$ the set of mixed multilateral strategies for player i, with typical element μ_i. Thus, μ_i is an arbitrary probability distribution on the finite set of pure multilateral strategies X_i. Player i's payoff function u_i in the network game extends to mixed multilateral strategies in the usual fashion. Specifically, if μ_{-i} denotes the profile of mixed multilateral strategies of all players except i, and if $\mu_{N(i)}$ denotes the profile of mixed multilateral strategies for the neighbors of player i, then player i's expected payoff from playing $x_i \in X_i$ is given by $u_i(\mu_i, \mu_{N(i)}) = E[u_i(x_i, x_{N(i)})] = E[u_i(x_i, x_{-i})]$, where the expectations are taken with respect to $\mu_{N(i)}$ and μ_{-i}, respectively. Player i's mixed best-response correspondence MBR_i assigns to any profile $\mu_{N(i)} \in \bigtimes_{j \in N(i)} \Delta(X_j)$ the set of mixed strategies $\mu_i^* \in \Delta(X_i)$ such that $u_i(\mu_i^*, \mu_{N(i)}) = \max_{\mu_i \in \Delta(X_i)} u_i(\mu_i, \mu_{N(i)})$. Further, the mixed best-response correspondence MBR of the game G assigns to any profile $\mu = (\mu_1, ..., \mu_n) \in \bigtimes_{i=1}^n \Delta(X_i)$ the Cartesian product $\mathrm{MBR}(\mu) = \bigtimes_{i=1}^n \mathrm{MBR}_i(\mu_{N(i)})$. Since strategy spaces are finite, a mixed-strategy Nash equilibrium $\mu = (\mu_1, ..., \mu_n)$ exists in the network game by Nash's theorem.

2.2 Continuous-time fictitious play with independent beliefs

In the main part of the analysis, we look at the fictitious-play process in continuous time (Rosenmüller, 1971). Moreover, consistent with a common interpretation of fictitious play, according to which players take it as given that their opponents adhere to some independently chosen mixed strategy, we assume that *all empirical frequencies are accounted for as marginal distributions only*, so that beliefs formed by a single player are independent across his opponents. Later, in Section 6, this assumption will be relaxed.

With continuous time, let $m : [0, \infty) \to \Delta(X_1) \times ... \times \Delta(X_n)$ be a (measurable) *path* specifying, for $i = 1, ..., n$, player i's mixed strategy at time τ, i.e., $m_i(\tau) \in \Delta(X_i)$. As time is continuous, averaging over time amounts to integrating each m_i over an interval $[0, \tau]$, for some $\tau > 0$. The *independent average* $\alpha : [1, \infty) \to \Delta(X_1) \times ... \times \Delta(X_n)$ of the path m at time $\tau \geq 1$ is consequently defined as

$$\alpha(\tau) \equiv \alpha(\tau, m) = \left(\frac{1}{\tau} \int_0^\tau m_1(\tau')d\tau', ..., \frac{1}{\tau} \int_0^\tau m_n(\tau')d\tau' \right). \tag{2}$$

The definition of a continuous-time path of fictitious play reads as follows.

Definition 1. (CTFP) *A continuous-time fictitious play (with independent beliefs) is a measurable mapping* $m : [0, \infty) \to \bigtimes_{i=1}^n \Delta(X_i)$ *such that* $m(\tau) \in \mathrm{MBR}(\alpha(\tau))$ *for all* $\tau \geq 1$.

The following lemma assures us of the existence of a CTFP learning process. The proof of the lemma checks that demanding optimality at all points in time, as this paper does, is equivalent to demanding optimality at almost any point in time (Harris, 1998).

Lemma 1. *A CTFP exists.*

Proof. From Harris (1998, p. 244), it is known that a path $\widehat{m} : [0, \infty) \to \bigtimes_{i=1}^n \Delta(X_i)$ exists such that $\widehat{m}(\tau) \in \mathrm{MBR}(\alpha(\tau, \widehat{m}))$ for almost all $\tau \geq 1$. Let $\mathcal{Z} \subseteq [1, \infty)$ be a null set such that $\widehat{m}(\tau) \in \mathrm{MBR}(\alpha(\tau, \widehat{m}))$ for all $\tau \in [1, \infty) \backslash \mathcal{Z}$. Define now the path $m : [0, \infty) \to \bigtimes_{i=1}^n \Delta(X_i)$ by letting $m(\tau) = \widehat{m}(\tau)$ for any $\tau \in [0, \infty) \backslash \mathcal{Z}$, and by letting $m(\tau) \in \mathrm{MBR}(\alpha(\tau, \widehat{m}))$ for any $\tau \in \mathcal{Z}$. Then, for any $\tau \geq 1$, clearly $\alpha(\tau, m) = \alpha(\tau, \widehat{m})$, so that $m(\tau) \in \mathrm{MBR}(\alpha(\tau, m))$. The claim follows. $\square$

Next, we define convergence of fictitious play in a given n-player network game G. Recall that, for an arbitrary path $m : [0, \infty) \to \Delta(X_1) \times ... \times \Delta(X_n)$, the corresponding averaging path $\tau \to \mu(\tau) \equiv \alpha(\tau, m)$ is a continuous curve in the space of mixed strategy profiles, $\Delta(X_1) \times ... \times \Delta(X_n)$. Denote by $\mathcal{A}(m)$ the set of all accumulation points of the curve $\mu(.)$.[15] Convergence of fictitious play is then defined by the requirement that $\mathcal{A}(m)$ is a subset of the set of Nash equilibria of G.

Definition 2. *A path* $m : [0, \infty) \to \Delta(X_1) \times ... \times \Delta(X_n)$ *is said to converge to Nash equilibrium if every limit distribution* $\mu^* \in \mathcal{A}(m)$ *is a Nash equilibrium in* G.

[15] Thus, $\mathcal{A}(m)$ consists of all strategy profiles that are limit points of some converging sequence of independent beliefs, $\{\mu(\tau_q)\}_{q=1}^\infty$, where $\{\tau_q\}_{q=1}^\infty$ is any sequence in $[1, \infty)$ such that $\lim_{q \to \infty} \tau_q = \infty$. Because the set of mixed strategy profiles $\Delta(X_1) \times ... \times \Delta(X_n)$ is bounded and closed, there will be at least one such limit point, i.e., $\mathcal{A}(m) \neq \varnothing$.

Clearly, a network player that optimizes *simultaneously* against several opponents behaves differently from an unconstrained player in a bilateral game. Sela (1999) has provided a couple of examples illustrating the fact that the fictitious-play property in the bilateral games is, indeed, not generally informative about the fictitious-play property in the network game. Specifically, even if two bilateral games have the fictitious-play property, this need not be the case for the network game (potentially after eliminating weakly dominated strategies). Conversely, even if neither of the bilateral games possesses the fictitious-play property, the network game may still possess the fictitious-play property.[16]

3. Zero-sum networks

Given two players i, j with $j \neq i$, the bilateral game G_{ij} is called *zero-sum* if $u_{ij} + u_{ji} \equiv 0$. By a *network of zero-sum games*, we mean a network game in which each bilateral game is zero-sum. More generally, by a *zero-sum network*, we mean a network game G that is zero-sum as an n-player game, i.e., a network game in which $u_1 + ... + u_n \equiv 0$.[17] Related classes of games have been studied, in particular, by Bregman and Fokin (1987, 1998), Daskalakis and Papadimitriou (2009), Cai and Daskalakis (2011), and Cai et al. (2016). Clearly, if each bilateral game in a given network is zero-sum, the network game G is an n-person zero-sum game. However, the converse is not generally true. Importantly, even if payoffs are given in the n-player normal form, there are efficient ways to check if the game is a zero-sum network (cf. Cai et al., 2016).

The following result is the first main result of the present paper.

Proposition 1. *Fictitious play converges to equilibrium in any zero-sum network.*

Proof. For a given independent profile of mixed strategy beliefs $\mu = (\mu_1, ..., \mu_n) \in \Delta(X_1) \times ... \times \Delta(X_n)$, let $\mu_{N(i)} = \{\mu_j\}_{j \in N(i)} \in \bigtimes_{j \in N(i)} \Delta(X_j)$ denote the corresponding independent profile of mixed strategies for the neighbors of player i. Define the Lyapunov function

$$\mathcal{L}(\mu) = \sum_{i=1}^{n} \max_{x_i \in X_i} \left\{ u_i(x_i, \mu_{N(i)}) - u_i(\mu_i, \mu_{N(i)}) \right\}. \tag{3}$$

Intuitively, this function measures first each player's scope for individual improvement relative to μ, and then aggregates the result across all n players in the network. Clearly, as a direct consequence

[16] For further details, we refer the reader to Sela (1999, Ex. 4 and 6).

[17] Alternative terminology includes *zero-sum polymatrix game* and *separable zero-sum multiplayer game.*

of the definition, $\mathcal{L}(\mu) \geq 0$. Note next that, because the network is zero-sum, an alternative way to write the Lyapunov function is

$$\mathcal{L}(\mu) = \sum_{i=1}^{n} \max_{x_i \in X_i} u_i(x_i, \mu_{N(i)}). \tag{4}$$

Consider now a continuous-time fictitious play $m : [0, \infty) \to \Delta(X_1) \times ... \times \Delta(X_n)$, with independent average $\mu(\tau) \equiv \alpha(\tau, m)$ at some fixed point in time $t \geq 1$. Then, because $m_i(\tau)$ is a best response to $\mu_{-i}(\tau)$ for $i \in \{1, ..., n\}$, and because interactions are bilateral,

$$\begin{aligned} \mathcal{L}(\mu(\tau)) &= \sum_{i=1}^{n} u_i(m_i(\tau), \mu_{N(i)}(\tau)) && (5) \\ &= \sum_{i=1}^{n} \sum_{j \in N(i)} u_{ij}(m_i(\tau), \mu_j(\tau)) && (6) \\ &= \sum_{i=1}^{n} \sum_{j \in N(i)} m_i(\tau) \cdot A_{ij} \mu_j(\tau), && (7) \end{aligned}$$

where A_{ij} is the matrix representing player i's payoffs in the bilateral game G_{ij}, i.e.,[18]

$$\mu_i \cdot A_{ij} \mu_j \equiv u_{ij}(\mu_i, \mu_j). \tag{8}$$

Consider now a player $i \in \{1, ..., n\}$. Then, given that $m_i(\tau)$ is a best response to $\mu_{N(i)}(\tau)$, we have

$$\textstyle\sum_{j \in N(i)} m_i(\tau) \cdot A_{ij} \mu_j(\tau) \geq \sum_{j \in N(i)} m_i(\widehat{\tau}) \cdot A_{ij} \mu_j(\tau), \tag{9}$$

for any $\widehat{\tau} \geq 1$. Adding up across players, and subsequently multiplying through with τ, one obtains

$$\tau \mathcal{L}(\mu(\tau)) \geq \textstyle\sum_{i=1}^{n} \sum_{j \in N(i)} m_i(\widehat{\tau}) \cdot A_{ij} F_j(\tau), \tag{10}$$

where

$$F_j(\tau) = \tau \cdot \mu_j(\tau) = \int_0^{\tau} m_j(\tau') d\tau'. \tag{11}$$

Subtracting inequality (10) from the equation

$$\widehat{\tau} \mathcal{L}(\mu(\widehat{\tau})) = \textstyle\sum_{i=1}^{n} \sum_{j \in N(i)} m_i(\widehat{\tau}) \cdot A_{ij} F_j(\widehat{\tau}), \tag{12}$$

and subsequently dividing by $\widehat{\tau} - \tau$, one arrives at the key inequality

$$\frac{\widehat{\tau} \mathcal{L}(\mu(\widehat{\tau})) - \tau \mathcal{L}(\mu(\tau))}{\widehat{\tau} - \tau} \leq \textstyle\sum_{i=1}^{n} \sum_{j \in N(i)} m_i(\widehat{\tau}) \cdot A_{ij} \left(\dfrac{F_j(\widehat{\tau}) - F_j(\tau)}{\widehat{\tau} - \tau} \right), \tag{13}$$

for any $\widehat{\tau} > \tau$. By the fundamental theorem of calculus, the differential quotient on the right-hand side converges for $\widehat{\tau} \to \tau$ to $m_j(\tau)$, for almost any $\tau \geq 1$. This shows that for almost any $\tau \geq 1$,

[18] Here and in the sequel, the dot $\cdot$ denotes the scalar product between two vectors.

$$\lim_{\widehat{\tau}\to\tau,\widehat{\tau}>\tau}\frac{\widehat{\tau}\mathcal{L}(\mu(\widehat{\tau}))-\tau\mathcal{L}(\mu(\tau))}{\widehat{\tau}-\tau}\leq\textstyle\sum_{i=1}^{n}\sum_{j\in N(i)}m_i(\tau)\cdot A_{ij}m_j(\tau)=0, \tag{14}$$

where the equality on the right-hand side reflects the zero-sum property of the network. Since $\mathcal{L}(\mu(\tau))$ is continuous, it follows that $\tau\mathcal{L}(\mu(\tau))$ is monotone decreasing. Hence, there is a constant $C\geq 0$ such that $\mathcal{L}(\mu(\tau))\leq C/\tau$ for any $\tau\geq 1$. Noting that the individual terms of the Lyapunov function (3) are all positive, we therefore obtain for any player $i=1,...,n$ and for any pure strategy $x_i\in X_i$ that

$$u_i(x_i,\mu_{N(i)}(\tau))-u_i(\mu_i(\tau),\mu_{N(i)}(\tau))\leq\frac{C}{\tau}. \tag{15}$$

Consider now any accumulation point $\mu^*\in\mathcal{A}(m)$ of the path $\mu(.)$. Then, taking the limit $\tau\to\infty$ shows that

$$u_i(x_i,\mu^*_{N(i)})-u_i(\mu^*_i,\mu^*_{N(i)})\leq 0, \tag{16}$$

i.e., x_i is not a profitable deviation for player i. Since i and x_i were arbitrary, μ^* is necessarily a Nash equilibrium. This proves convergence of CTFP in any zero-sum network. □

The proof uses the well-known combination of the Lyapunov method and the envelope theorem (Hofbauer, 1995; Harris, 1998). Indeed, as equation (12) shows, the expression $\tau\mathcal{L}(\mu(\tau))$ corresponds to the total (across all players in the network) of the maxima over "cumulative payoffs." But the network is zero-sum, so that the sum of instantaneous payoffs vanishes. Therefore, the first-order change to $\tau\mathcal{L}(\mu(\tau))$ must vanish as well. However, the proof provided above is much less technical than existing derivations (even in the case of two-player zero-sum games), because no reference is made to the approximation theorem nor to directional derivatives.[19]

Regarding the rate of convergence, we mention that a minor refinement of the proof shows that, as in the case of two-person zero-sum games considered by Harris (1998), the rate of convergence in payoffs, i.e., the rate by which the Lyapunov function approaches zero, is precisely $1/\tau$.

Corollary 1. *The rate of convergence of CTFP in any zero-sum network game is* $1/\tau$.

Proof. For $1<\widehat{\tau}<\tau$, the inequality sign in (13) is reversed. Therefore, following the steps of

[19] Notably, Proposition 1 provides a proof of existence of a Nash equilibrium in any zero-sum network. However, in contrast to the case of two-person zero-sum games, this observation does not imply a minmax theorem for zero-sum networks. For a generalization of the minmax theorem to zero-sum networks, the reader is referred to Cai et al. (2016).

the proof of Proposition 1, $\tau\mathcal{L}(\mu(\tau))$ is seen to be not only monotone decreasing, but also monotone increasing, hence constant. □

The observation that the rate of convergence does not depend on the size of the network may be surprising. However, it should be noted that convergence is measured here on the aggregate level. That is to say, when the network is large, then the value of the Lyapunov function provides little information about the scope of improvement that is feasible for an *individual* player. Therefore, it might indeed take longer in a larger network to reach, say, an ε-equilibrium.

One might conjecture that it is the zero-sum property that drives convergence also in n-player games with $n \geq 3$, and that the network structure of the game is not needed. However, as the following example illustrates, this is not the case. I.e., there are multiplayer zero-sum games in which CTFP need not converge.

Example 1. (Three-player zero-sum game) Consider the following game G^1 between three players:[20]

	x_3 = O		
	x_2 = L	x_2 = M	x_2 = R
x_1 = T	1, 0, -1	0, 0, 0	0, 1, -1
x_1 = M	0, 1, -1	1, 0, -1	0, 0, 0
x_1 = B	0, 0, 0	0, 1, -1	1, 0, -1

Figure 2. The game G^1.

Here, player 1 and player 2 each have three pure strategies, whereas player 3 has just one pure strategy. Therefore, to see what happens under either Nash assumptions or fictitious play, player 3 may be safely ignored. But with player 3 eliminated from the game, the two-player game between players 1 and 2 is of the Shapley (1964) type, so that nonconvergence obtains.[21]

[20] Here and elsewhere in the paper, payoff vectors are arranged diagonally in each box, starting with player 1's payoff in the respective upper-left corner.

[21] That conclusion does not depend on the fact that player 3 has only one strategy. In fact, we have constructed also an example of a $2 \times 2 \times 2$ zero-sum game with a Shapley hexagon, similar to the three-player non-zero-sum example of Jordan (1993). On a related note, we conjecture that Example 1 as well as the later examples of the present paper could be made robust by introducing additional strategies for all players.

The example shows that the network assumption in Proposition 1 cannot be easily dropped. On the other hand, the zero-sum assumption may be relaxed to a certain extent for convergence in network games, as will be discussed in the next section.

4. Additional classes of network games

4.1 Conflicts

A bilateral game G_{ij} will be called a *conflict* if there exist valuations $v_{ij} > 0$ and $v_{ji} > 0$, success functions $p_{ij} : S_{ij} \times S_{ji} \to [0,1]$ and $p_{ji} : S_{ji} \times S_{ij} \to [0,1]$, as well as cost functions $c_{ij} : S_{ij} \to \mathbb{R}$ and $c_{ji} : S_{ji} \to \mathbb{R}$ such that

$$u_{ij}(s_{ij}, s_{ji}) = p_{ij}(s_{ij}, s_{ji})v_{ij} - c_{ij}(s_{ij}), \tag{17}$$

$$u_{ji}(s_{ji}, s_{ij}) = p_{ji}(s_{ji}, s_{ij})v_{ji} - c_{ji}(s_{ji}), \tag{18}$$

and

$$p_{ij}(s_{ij}, s_{ji}) + p_{ji}(s_{ji}, s_{ij}) = 1 \tag{19}$$

hold for any $s_{ij} \in S_{ij}$ and any $s_{ji} \in S_{ji}$. Examples include probabilistic contests such as the Tullock (1980) contest, the Hirshleifer contest (1989), the Lazear-Rosen (1981) tournament, the first-price all-pay auction (Baye et al., 1996), and Colonel Blotto games (Roberson, 2006), to name a few. The class of conflicts defined above corresponds precisely to the class of *strategically zero-sum games* (Moulin and Vial, 1978). However, we will use the terminology of conflict because it is more suggestive and also because it allows to make an important distinction (homogeneous vs. heterogeneous valuations) that is absent from the original theory but crucially needed below.[22]

A network game G will be called a *conflict network* if the bilateral game G_{ij} is a conflict for each pair $(i, j) \in E$. We will say that a conflict network has *pairwise homogeneous valuations* if valuations of two players coincide in any pairwise conflict, i.e., if $v_{ij} = v_{ji}$ for any two players i and j with $(i, j) \in E$. An example is Franke and Öztürk's (2015) transfer network where the net valuations of winning a conflict are assumed to be identical across players. If valuations are not pairwise homogeneous, we will (somewhat loosely) say that the conflict network exhibits heterogeneous valuations.

Proposition 1 can be extended to these additional classes of games as follows.

[22] This distinction is similar in spirit to the difference between exact and weighted potential games.

Proposition 2. *Let G be either (i) a conflict network with pairwise homogeneous valuations, or (ii) an acyclic conflict network with heterogeneous valuations. Then, G has the continuous-time fictitious-play property.*

Proof. *(i)* Starting from the conflict network G, we construct another network game $\widetilde{G}$ on the same graph and with identical strategy sets by letting payoffs in the bilateral game $\widetilde{G}_{ij}$ be given by

$$\widetilde{u}_{ij}(s_{ij}, s_{ji}) = u_{ij}(s_{ij}, s_{ji}) - \frac{v_{ij}}{2} + c_{ji}(s_{ji}). \tag{20}$$

Then, clearly, each player i's payoffs in $\widetilde{G}$ satisfy

$$\widetilde{u}_i(x_i, x_{N(i)}) \equiv \sum_{j \in N(i)} \widetilde{u}_{ij}(\sigma^j(x_i), \sigma^i(x_j)) = u_i(x_i, x_{N(i)}) - \sum_{j \in N(i)} \left\{ \frac{v_{ij}}{2} - c_{ji}(\sigma^i(x_j)) \right\}, \tag{21}$$

which shows that $\widetilde{G}$ is best-response equivalent in mixed strategies to G.[23] Moreover, for any pair $(i, j) \in E$, using (17-19) and $v_{ij} = v_{ji}$, we have

$$\begin{aligned}
&\widetilde{u}_{ij}(s_{ij}, s_{ji}) + \widetilde{u}_{ji}(s_{ji}, s_{ij}) \\
&= u_{ij}(s_{ij}, s_{ji}) - \frac{v_{ij}}{2} + c_{ji}(s_{ji}) + u_{ji}(s_{ji}, s_{ij}) - \frac{v_{ji}}{2} + c_{ij}(s_{ij}) && (22)\\
&= p_{ij}(s_{ij}, s_{ji})v_{ij} - c_{ij}(s_{ij}) - \frac{v_{ij}}{2} + c_{ji}(s_{ji}) + p_{ji}(s_{ji}, s_{ij})v_{ji} - c_{ji}(s_{ji}) - \frac{v_{ji}}{2} + c_{ij}(s_{ij}) && (23)\\
&= 0, && (24)
\end{aligned}$$

Thus, $\widetilde{G}$ is a network of two-person zero-sum games and, hence, a zero-sum network. By Proposition 1, CTFP converges in $\widetilde{G}$. Using the best-response equivalence in mixed strategies, both the set of continuous-time fictitious plays and the set of Nash equilibria are the same for G and $\widetilde{G}$. Hence, CTFP converges also in G.

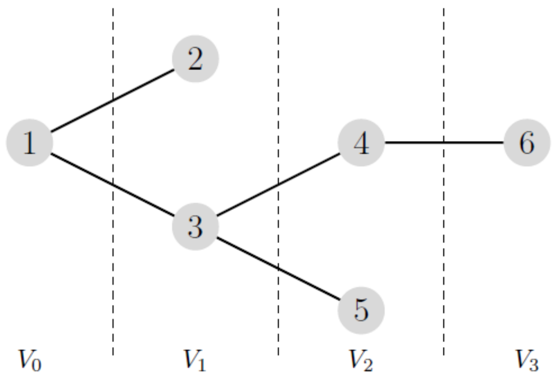

Figure 3. Partitioning the set of nodes in an acyclic network.

[23] Two n-player games G and G' with identical strategy spaces $X_1, ..., X_n$ are called *best-response equivalent in mixed strategies* (Monderer and Shapley, 1996b) if for any $i = 1, .., n$, and any $\mu_{-i} \in \Delta_1 \times ... \times \Delta_{i-1} \times \Delta_{i+1} \times ... \times \Delta_n$ (or any $\mu_{-i} \in \Delta_{-i}$), $\arg\max_{x_i \in X_i} u_i(x_i, \mu_{-i}) = \arg\max_{x_i \in X_i} u_i'(x_i, \mu_{-i})$.

(ii) Without loss of generality, the network may be assumed to be connected. The set of players can then be partitioned into a finite number of subsets $V_0, V_1, ..., V_K$, where $i \in V_k$ for $k \in \{0, ..., K\}$ if and only if the graph-theoretic distance between players 1 and i equals k. See Figure 3 for illustration. Thus,

$$V_0 = \{1\}, \tag{25}$$

$$V_1 = N(1), \text{ and} \tag{26}$$

$$V_{k+1} = \{\bigcup_{i \in V_k} N(i)\} \backslash V_{k-1} \qquad (k = 1, ..., K-1). \tag{27}$$

We will describe an iterative construction that transforms the conflict network with bilateral payoff function u_{ij} and *heterogeneous* valuations v_{ij} into a conflict network with bilateral payoff functions $\widehat{u}_{ij}$ and *homogeneous* valuations $\widehat{v}_{ij}$. For this, we initialize the iteration by letting $\widehat{u}_{1j} = u_{1j}$ and $\widehat{v}_{1j} = v_{1j}$ for any neighbor $j \in N(1)$. We start now with $k = 1$ and consider some player $i \in V_k$. Since the network is acyclic, there is precisely one player $l \in V_{k-1}$ such that $i \in N(l)$. We rescale player i's payoff function u_{ij} in his relationship with any neighbor $j \in N(i)$ by letting

$$\widehat{u}_{ij}(s_{ij}, s_{ji}) = \frac{\widehat{v}_{li}}{v_{il}} \cdot u_{ij}(s_{jl}, s_{lj}), \tag{28}$$

so that player i's valuation becomes

$$\widehat{v}_{ij} = \frac{\widehat{v}_{li}}{v_{il}} \cdot v_{ij}. \tag{29}$$

Since the factor $(\widehat{v}_{li}/v_{il})$ does not depend on the neighbor $j \in N(i)$, such rescaling does not affect player j's multilateral best-response correspondence. Moreover, the resulting bilateral game $\widehat{G}_{ij}$ between players i and j (with payoff function $\widehat{u}_{ij}$ for player i, and payoff function $\widehat{u}_{ji}$ for player j) is a conflict with homogeneous valuations, since $\widehat{v}_{ji} = \widehat{v}_{ij}$ by equation (29). Once this is accomplished for any $i \in V_k$, the index k is incremented, and the rescaling procedure repeated. After the iteration has reached $k = K$, we end up with a conflict network $\widehat{G}$ with pairwise homogeneous valuations that is best-response equivalent in mixed strategies to G. Convergence of CTFP follows, therefore, from part (i). □

The intuition is as follows. If valuations are homogeneous, then each bilateral game is essentially a constant-sum game with costly strategies.[24] Suppose that the cost of a player in any bilateral conflict

[24] This useful interpretation is borrowed from Ben-Sasson et al. (2007).

is not lost, but reaches the other player as a subsidy. Then, as the size of the subsidy does not depend on the player's choice of strategy, his best-response correspondence remains unaffected. However, if the subsidy is implemented in any bilateral conflict, the network game becomes constant-sum, and we are done. If valuations are heterogeneous, and the underlying network structure is acyclic, then the conflict network can be transformed into a conflict network with pairwise homogeneous valuations by a simple iteration that starts at player 1 and works its way through the tree, where in each step, valuations and cost functions of a new set of players are rescaled so as to render the backward-looking conflict homogeneous.

However, rescaling does not work in general conflict networks. Indeed, as the following example illustrates, fictitious play need not converge in cyclic conflict networks with heterogeneous valuations.

Example 2. (Network of conflicts) Suppose there are three players $i = 1, 2, 3$, where $X_1 = X_2 = X_3 = \{\text{L,H}\}$. The network structure is a triangle. In the network game G^2, each bilateral game is a conflict with heterogeneous valuations. Specifically, one assumes

$$v_{i,i+1} = 6,\ v_{i,i-1} = 3, \tag{30}$$

$$c_{i,i+1}(\text{L}) = c_{i,i-1}(\text{L}) = 0,\ c_{i,i+1}(\text{H}) = c_{i,i-1}(\text{H}) = 1 \tag{31}$$

$$p_{i,i+1}(\text{H},\text{L}) = \tfrac{3}{4},\ p_{i,i+1}(\text{L},\text{H}) = \tfrac{1}{3},\ p_{i,i+1}(\text{L},\text{L}) = p_{i,i+1}(\text{H},\text{H}) = \tfrac{1}{2}, \tag{32}$$

where $i + 1$ refers to player 1 if $i = 3$, and similarly, $i - 1$ refers to player 3 if $i = 1$. Denote by $r_i = \text{pr}\{x_i = \text{H}\}$ the probability that player i uses strategy H. There is a unique Nash equilibrium $(r_1^*, r_2^*, r_3^*) = (0, 0, 0)$.[25] Moreover, fictitious play need not converge to equilibrium, but may follow a triangle-shaped path that runs in a round-robin fashion through the points

$$... \to p_1 = (\tfrac{2}{7}, \tfrac{4}{7}, \tfrac{1}{7}) \to p_2 = (\tfrac{1}{7}, \tfrac{2}{7}, \tfrac{4}{7}) \to p_3 = (\tfrac{4}{7}, \tfrac{1}{7}, \tfrac{2}{7}) \to ..., \tag{33}$$

as illustrated in Figure 4. Thus, the dynamic conflict does not settle down, which shows that the assumption of homogeneous valuations cannot be dropped unless the network is acyclic.

[25] So all players would choose the effort level L.

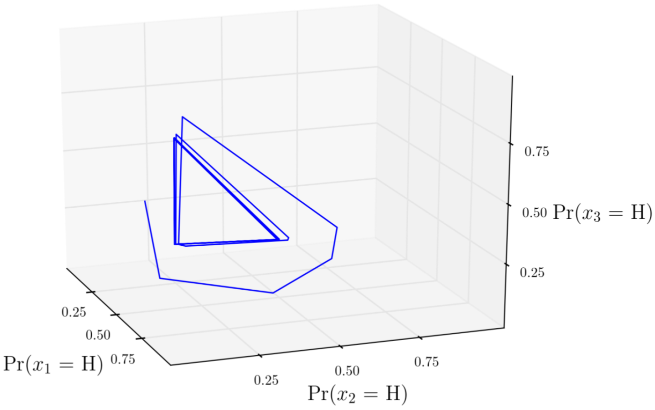

Figure 4. Fictitious play need not converge in a network of conflicts.

4.2 Potential games

We introduce notions of increasing flexibility first for bilateral games and then for network games. A bilateral game G_{ij} is said to possess *identical payoff functions* if $u_{ij}(s_{ij}, s_{ji}) = u_{ji}(s_{ji}, s_{ij})$ for all $s_{ij} \in S_{ij}$ and all $s_{ji} \in S_{ji}$.[26] A bilateral game G_{ij} is an *exact potential game* (Monderer and Shapley, 1996a) if there exists a potential function $P_{ij} : S_{ij} \times S_{ji} \to \mathbb{R}$ such that

$$u_{ij}(s_{ij}, s_{ji}) - u_{ij}(\widehat{s}_{ij}, s_{ji}) = P_{ij}(s_{ij}, s_{ji}) - P_{ij}(\widehat{s}_{ij}, s_{ji}), \tag{34}$$

$$u_{ji}(s_{ji}, s_{ij}) - u_{ji}(\widehat{s}_{ji}, s_{ij}) = P_{ij}(s_{ij}, s_{ji}) - P_{ij}(s_{ij}, \widehat{s}_{ji}), \tag{35}$$

for all $s_{ij}, \widehat{s}_{ij} \in S_{ij}$ and all $s_{ji}, \widehat{s}_{ji} \in S_{ji}$. Next, a bilateral game G_{ij} is a *weighted potential game* (Monderer and Shapley, 1996a) if there exists a potential function $P_{ij} : S_{ij} \times S_{ji} \to \mathbb{R}$ as well as weights $w_{ij} > 0$ and $w_{ji} > 0$ such that

$$u_{ij}(s_{ij}, s_{ji}) - u_{ij}(\widehat{s}_{ij}, s_{ji}) = w_{ij} \left\{ P_{ij}(s_{ij}, s_{ji}) - P_{ij}(\widehat{s}_{ij}, s_{ji}) \right\}, \tag{36}$$

$$u_{ji}(s_{ji}, s_{ij}) - u_{ji}(\widehat{s}_{ji}, s_{ij}) = w_{ji} \left\{ P_{ij}(s_{ij}, s_{ji}) - P_{ij}(s_{ij}, \widehat{s}_{ji}) \right\}, \tag{37}$$

for all $s_{ij}, \widehat{s}_{ij} \in S_{ij}$ and all $s_{ji}, \widehat{s}_{ji} \in S_{ji}$.[27] A network game G will be said to be an *exact potential network* if all bilateral games G_{ij} are exact potential games. Finally, a network game G will be referred to as a *weighted potential network* if all bilateral games G_{ij} are weighted potential games.

[26] A game *with identical interests* (Monderer and Shapley, 1996b) is a game that is best-response equivalent in mixed strategies to a game with identical payoff functions. Thus, games with identical interests relate to games with identical payoff functions in the same way as strategically zero-sum games relate to zero-sum games.

[27] The following observation shows that the notation need not lead to confusion: Let G_{ij} be a weighted potential game with potential P_{ij} and weight w_{ij} for player i and weight w_{ji} for player j. Define a potential $P_{ji} : S_{ji} \times S_{ij} \to \mathbb{R}$ by $P_{ji}(s_{ji}, s_{ij}) = P_{ij}(s_{ij}, s_{ji})$. Then, the bilateral game G_{ji} (i.e., the game G_{ij} with the roles of players i and j exchanged) is a weighted potential game with potential P_{ji} and weight w_{ji} for player j and weight w_{ij} for player i.

Cai and Daskalakis (2011) have shown that if all bilateral games in a network have identical payoff functions then the network game allows an exact potential (the welfare function). The following result applies their reasoning, but also offers some extensions in so far that bilateral games may merely be exact potential games or even weighted potential games. We also use a slightly different proof. Specifically, we construct the potential of the network game as the sum over all potentials rather than as the sum over all payoff functions. Moreover, in the case of weighted potential games, we employ a similar induction argument as in the proof of Proposition 2(ii).

Proposition 3. *Let G be either (i) an exact potential network, or (ii) a weighted potential network on an acyclic graph. Then, G has the continuous-time fictitious-play property.*

Proof. *(i)* Suppose that the network game G is an exact potential network. Then, since any bilateral game G_{ij} is an exact potential game, there exist a potential function $P_{ij} : S_{ij} \times S_{ji} \to \mathbb{R}$ such that

$$u_{ij}(s_{ij}, s_{ji}) - u_{ij}(\widehat{s}_{ij}, s_{ji}) = P_{ij}(s_{ij}, s_{ji}) - P_{ij}(\widehat{s}_{ij}, s_{ji}), \tag{38}$$

$$u_{ji}(s_{ji}, s_{ij}) - u_{ji}(\widehat{s}_{ji}, s_{ij}) = P_{ij}(s_{ij}, s_{ji}) - P_{ij}(s_{ij}, \widehat{s}_{ji}), \tag{39}$$

for any $s_{ij}, \widehat{s}_{ij} \in S_{ij}$ and any $s_{ji}, \widehat{s}_{ji} \in S_{ji}$. In particular, by exchanging the roles of players i and j in equation (39) and comparing with (38), we obtain

$$P_{ij}(s_{ij}, s_{ji}) - P_{ij}(\widehat{s}_{ij}, s_{ji}) = P_{ji}(s_{ji}, s_{ij}) - P_{ji}(s_{ji}, \widehat{s}_{ij}). \tag{40}$$

Consider now the aggregate potential $\mathcal{P} : X \to \mathbb{R}$ defined through

$$\mathcal{P}(x) = \frac{1}{2} \sum_{(i,j)\in E} P_{ij}(s_{ij}, s_{ji}), \tag{41}$$

where $s_{ij} = \sigma^j(x_i) \in S_{ij}$ denotes player i's bilateral strategy vis-a-vis player j under the multilateral strategy x_i and, analogously, $s_{ji} = \sigma^i(x_j) \in S_{ji}$ denotes player j's bilateral strategy vis-a-vis player i under the profile of multilateral strategies $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$. It is claimed that $\mathcal{P}$ is an exact potential for G. Indeed, for any $x_i \in X_i$, $\widehat{x}_i \in X_i$, and $x_{-i} \in X_{-i}$, writing $\widehat{s}_{ij} = \sigma^j(\widehat{x}_i) \in S_{ij}$, it is straightforward to check that

$$
\begin{aligned}
& u_i(x_i, x_{-i}) - u_i(\widehat{x}_i, x_{-i}) \\
&= \sum_{j \in N(i)} \{u_{ij}(s_{ij}, s_{ji}) - u_{ij}(\widehat{s}_{ij}, s_{ji})\} && (42)\\
&= \sum_{j \in N(i)} \{P_{ij}(s_{ij}, s_{ji}) - P_{ij}(\widehat{s}_{ij}, s_{ji})\} && (43)\\
&= \frac{1}{2}\left\{ \sum_{(i,j)\in E} \{P_{ij}(s_{ij}, s_{ji}) - P_{ij}(\widehat{s}_{ij}, s_{ji})\} + \sum_{(j,i)\in E} \{P_{ji}(s_{ji}, s_{ij}) - P_{ji}(s_{ji}, \widehat{s}_{ij})\} \right\} && (44)\\
&= \frac{1}{2}\left\{ \sum_{(i,j)\in E} P_{ij}(s_{ij}, s_{ji}) + \sum_{(j,i)\in E} P_{ji}(s_{ji}, s_{ij}) \right\} - \frac{1}{2}\left\{ \sum_{(i,j)\in E} P_{ij}(\widehat{s}_{ij}, s_{ji}) + \sum_{(j,i)\in E} P_{ji}(s_{ji}, \widehat{s}_{ij}) \right\} && (45)\\
&= \mathcal{P}(x_i, x_{-i}) - \mathcal{P}(\widehat{x}_i, x_{-i}). && (46)
\end{aligned}
$$

Hence, $\mathcal{P}$ is indeed an exact potential for the n-player game G. Therefore, from Harris (1998), CTFP converges.

(ii) From the weighted potential network, an exact potential network is constructed as follows. First, players are assigned to subsets $V_0, V_1, ..., V_K$ according to their distance k from player 1, as in the proof of Proposition 2. Then, we initiate an iteration by letting $\widetilde{u}_{1j} = u_{1j}$ for all $j \in N(1)$. We begin with $k = 1$. Consider any player $j \in V_k$, and recall that there is precisely one player $i \in V_{k-1}$ such that $j \in N(i)$.[28] By assumption, the bilateral game G_{ij} is a weighted potential game. Therefore, there exists a potential function $P_{ij} : S_{ij} \times S_{ji} \to \mathbb{R}$, as well as weights $w_{ij} > 0$ and $w_{ji} > 0$ such that

$$u_{ij}(s_{ij}, s_{ji}) - u_{ij}(\widehat{s}_{ij}, s_{ji}) = w_{ij}\{P_{ij}(s_{ij}, s_{ji}) - P_{ij}(\widehat{s}_{ij}, s_{ji})\}, \tag{47}$$

$$u_{ji}(s_{ji}, s_{ij}) - u_{ji}(\widehat{s}_{ji}, s_{ij}) = w_{ji}\{P_{ij}(s_{ij}, s_{ji}) - P_{ij}(s_{ij}, \widehat{s}_{ji})\}, \tag{48}$$

for any $s_{ij}, \widehat{s}_{ij} \in S_{ij}$ and any $s_{ji}, \widehat{s}_{ji} \in S_{ji}$. Without loss of generality, $w_{ij} = 1$. Given this normalization, we rescale all bilateral payoff functions of player j by letting $\widetilde{u}_{jl} = u_{jl}/w_{ji}$, for any $l \in N(j)$. Then, the bilateral game $\widetilde{G}_{ij}$ with payoff functions $\widetilde{u}_{ij} = u_{ij}$ for player i and $\widetilde{u}_{ji} = u_{ji}/w_{ji}$ for player j is easily seen to allow the exact potential P_{ij}. Moreover, the linear aggregation of payoffs implies that $\widetilde{u}_j = u_j/w_{ji}$, so that player j's best-response correspondence is not affected. In particular, any bilateral game G_{jl} with $l \in N(j)\backslash\{i\}$ remains a weighted potential game. Once all players

[28] Note that this notation differs from the one used in the proof of Proposition 2(ii).

$j \in V_k$ have been dealt with, the running index k is incremented. Clearly, when the iteration ends at $k = K$, the bilateral games $\widetilde{G}_{ij}$ form an exact potential network that is best-response equivalent in mixed strategies to the weighted potential network we started from. Hence, invoking part (i), CTFP converges to equilibrium. □

The intuition of the first part is simple. Because bilateral games possess exact potentials, the aggregate potential reflects incentives precisely as the network game.[29] The intuition of the second part is very similar to Proposition 2(ii).

Sela (1999, Prop. 12) has shown that a star-shaped network of generic weighted potential two-by-two games is, when reduced to a two-player game, best-response equivalent in mixed strategies to a game with identical interests. This implies the fictitious-play property. Proposition 3 shows that the "one-against-all" assumption, the assumption on the number of strategies, and the genericity of payoffs may be dropped without weakening the conclusion.

The following example shows that a general network consisting of weighted potential games need not have the fictitious-play property.

Example 3. (Network of weighted potential games) Consider the following game G^3 between three players $i = 1, 2, 3$, each of them having two compound strategies, i.e., $X_1 = X_2 = X_3 = \{\text{H,T}\}$. Bilateral payoffs are specified in Figure 5.

	x_2 = H	x_2 = T
x_1 = H	1, 5	0, 0
x_1 = T	0, 0	1, 5

	x_3 = H	x_3 = T
x_2 = H	1, 5	0, 0
x_2 = T	0, 0	1, 5

	x_1 = H	x_1 = T
x_3 = H	0, 0	1, 5
x_3 = T	1, 5	0, 0

Figure 5. A network of weighted potential games.

Thus, each player is involved in two coordination games, where for player i, coordination with player $i-1$, is more valuable than coordination with player $i+1$.[30] Moreover, there is a twist in the coordination between players 1 and 3. The game G^3 allows a unique Nash equilibrium in which each

[29] Extending the proof, one can easily convince oneself that arbitrary networks (so-called hypergraphs) of multiplayer exact potential games possess an exact potential.

[30] We use the same notational conventions as in the previous example.

player randomizes with equal probability over his two alternatives. However, CTFP goes through the hexagonal cycle (the entries correspond to the respective probability of playing strategy H)

$$\begin{aligned} \ldots \to p_1 = (a, b, c) \to p_2 = (1-c, a, b) \to p_3 = (1-b, 1-c, a) \to \\ \to p_4 = (1-a, 1-b, 1-c) \to p_5 = (c, 1-a, 1-b) \to p_6 = (b, c, 1-a) \to \ldots, \end{aligned} \tag{49}$$

where $(a, b, c) = (\frac{4}{9}, \frac{8}{9}, \frac{7}{9})$. A numerical fictitious-play path approaching the cycle is illustrated in Figure 6.

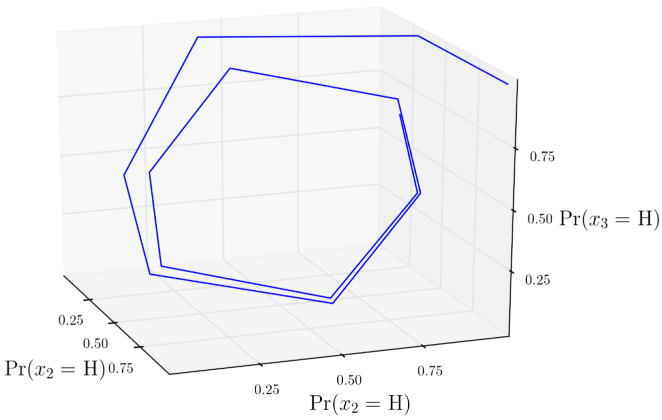

Figure 6. A network of weighted potential games need not possess the CTFP property.

4.3 Two-by-two games

Finally, we consider two-by-two games, i.e., two-player games in which each player has just two strategies.[31] It will be recalled (e.g., Krishna and Sjöström, 1997, Prop. 3) that any two-by-two game without weakly dominated strategies is best-response equivalent in mixed strategies to either a zero-sum game or to a game with identical payoff functions. Miyasawa's theorem is therefore customarily presented as a corollary of the corresponding results for zero-sum and identical interest games. Similarly, our network generalization of Miyasawa's theorem will be derived as an implication of our respective results for conflict and potential networks.

A network game G will be called a *network of strategically similar two-by-two games* if (i) X_i has precisely two elements, for any $i \in \{1, ..., n\}$, and (ii) either all bilateral games G_{ij} are conflicts with pairwise homogeneous valuations, or all bilateral games G_{ij} allow an exact potential.

[31] Miyasawa (1961) has shown that every two-by-two game has the fictitious-play property. That result was later seen to depend on a particular tie-breaking rule (Monderer and Sela, 1996). With generic payoffs, however, the theorem holds. For further discussion of this important special case, see Metrick and Polak (1994) and Sela (1999).

Proposition 4. *Let G be a network of strategically similar two-by-two games. Then, G has the continuous-time fictitious-play property.*

Proof. Immediate from Propositions 2 and 3. □

Proposition 4 extends Miyasawa's theorem to arbitrary network structures. As seen above in Propositions 2 and 3, the assumptions on the bilateral games can be further relaxed when the underlying network structure is acyclic. This implies, in particular, a related result by Sela (1999, Cor. 13) for star-shaped networks. However, Example 2 above shows that it is not possible to generalize Proposition 4 to arbitrary networks of strategically zero-sum games. Neither is it possible, in view of Example 3, to extend the result to arbitrary networks of weighted potential games.[32]

5. Discrete-time fictitious play

While the continuous-time variant of fictitious play considered above is analytically more convenient, there are reasons to be interested also in the discrete-time variant. For instance, the first major result in the literature by Robinson (1951) concerned the discrete-time process in two-person zero-sum games. It has often been suggested that the two processes should behave similarly. This is also intuitive because the incremental changes in the discrete time process become smaller and smaller over time. Harris (1998) has developed a very useful approach that, indeed, allows transferring results for the continuous-time case to the discrete-time case. Below, we will exploit his arguments to extend some of our conclusions to the case of discrete-time fictitious play.[33]

In contrast to the analysis so far, time progresses now in stages. In each stage $t \in \mathbb{N} = \{1, 2, 3, ...\}$, each player $i \in \{1, ..., n\}$ chooses a pure multilateral strategy $x_i(t) \in X_i$. The initial strategy profile $x(0) \in X$ is considered given. Then, the empirical frequencies of pure-strategy choices made before stage t are reflected in the *(discrete independent) average*

$$\alpha^d(t) \equiv \alpha^d(t, x(.)) = \left(\frac{1}{t} \sum_{t'=0}^{t-1} x_1(t'), ..., \frac{1}{t} \sum_{t'=0}^{t-1} x_n(t') \right) \qquad (t = 1, 2, 3, ...). \tag{50}$$

[32] Similarly, it does not seem possible to obtain a general convergence result in mixed networks, i.e., networks where some edge games are zero-sum, while others reflect identical interests.

[33] The idea is it to make a simple change in the time scale, such that the differential inclusion defining the continuous-time process becomes *autonomous* and, in fact, equivalent to the best-response population dynamics, as in Gilboa and Matsui (1991) and Matsui (1992). Thereby, it is feasible to exploit the near-convergence of a sufficiently delayed discrete-time process. As discussed in Hofbauer and Sorin (2006), that method of proof extends to arbitrary finite n-player games in which fictitious play converges uniformly across initial conditions. As we have seen in Corollary 1, the class of zero-sum networks satisfies this condition.

For a mixed strategy profile $\mu \in \Delta(X_1) \times ... \times \Delta(X_n)$, we denote by $\mathrm{BR}(\mu) = \mathrm{MBR}(\mu) \cap X$ the set of best-response profiles to μ that consist of pure strategies only.

Definition 3 (DTFP). *A discrete-time fictitious play in the network game G is a sequence of multilateral pure-strategy profiles $x(.) = \{x(t)\}_{t=1}^{\infty}$ such that $x(t) \in \mathrm{BR}(\alpha^d(t))$ for all $t \in \mathbb{N}$.*

For a given DTFP $x(.) = \{x(t)\}_{t=1}^{\infty}$, the corresponding sequence of averages, defined through $\mu^d(t) = \alpha^d(t, x(.))$, is a sequence in $\Delta(X_1) \times ... \times \Delta(X_n)$. We denote by $\mathcal{A}^d(x(.))$ the set of all accumulation points of $\{\mu^d(t)\}_{t=1}^{\infty}$.

Definition 4. *A sequence of multilateral pure-strategy profiles $x(.) = \{x(t)\}_{t=1}^{\infty}$ is said to converge to equilibrium if any accumulation point $\mu^* \in \mathcal{A}^d(x(.))$ of the corresponding process of discrete-time averages $\{\mu^d(t)\}_{t=1}^{\infty}$ is a Nash equilibrium in G.*

The following result generalizes Robinson's (1951) theorem to n-player zero-sum networks.

Proposition 5. *Let G be an arbitrary zero-sum network. Then any discrete-time fictitious play in G converges to equilibrium.*

Proof. Take any DTFP $x(.) = \{x(t)\}_{t=1}^{\infty}$ in G, and consider the associated process of discrete averages defined through $\mu^d(t) = \alpha^d(t, x(.))$. Then, for any $i = 1, ..., n$,

$$\mu_i^d(t) = \frac{1}{t} \sum_{t'=0}^{t-1} x_i(t') \qquad (t = 1, 2, 3, ...). \tag{51}$$

By simple algebraic manipulation,

$$\mu_i^d(t+1) = \frac{1}{t+1} \sum_{t'=0}^{t} x_i(t') = \frac{1}{t+1} x_i(t) + \frac{1}{t+1} \sum_{t'=0}^{t-1} x_i(t') = \frac{1}{t+1} x_i(t) + \frac{t}{t+1} \mu_i^d(t). \tag{52}$$

Hence, recalling that $x_i(t) \in \mathrm{BR}_i(\mu_{N(i)}^d(t)) \subseteq \mathrm{MBR}_i(\mu_{N(i)}^d(t))$, the process of averages satisfies

$$\mu_i^d(t+1) \in \frac{1}{t+1} \mathrm{MBR}_i(\mu_{N(i)}^d(t)) + \frac{t}{t+1} \mu_i^d(t) \qquad (i \in \{1, ..., n\},\ t = 1, 2, 3, ...). \tag{53}$$

By Corollary 1, there exists τ^* such that any solution of the autonomous differential inclusion

$$\frac{\partial \mu}{\partial \ln \tau} \in_{\text{a.e.}} \mathrm{MBR}(\mu) - \mu \tag{54}$$

starting anywhere at $\tau = 1$ is ε-close to the set of Nash equilibria of G from time τ^* onwards. Clearly, the correspondence MBR is u.s.c., compact-, and convex-valued. Applying now Hofbauer and Sorin

(2006, Prop. 7), it follows that there exists t^* such that $\mu^d(t)$ is ε-close to the set of Nash equilibria of G from time t^* onwards. Hence, any accumulation point of the sequence $\{\mu^d(t)\}_{t=1}^{\infty}$ is a Nash equilibrium, which proves the proposition. □

Proposition 5 extends Sela's (1999, Prop. 7) result for star-shaped networks of zero-sum games in three ways. First, the restriction regarding the network structure is dropped. Second, Proposition 5 imposes the zero-sum assumption only on the network, rather than on each of the bilateral games. Finally, no assumptions regarding tie-breaking are used here. In sum, it is feasible to address additional applications such as security games (Cai et al., 2016) and, as has been seen, conflict networks.

As in the case of two-player zero-sum games, the transformation of the DTFP process into a continuous-time process that is "close" to a CTFP process comes at a cost, which is the slower rate of convergence. More specifically, the discrete-time process is known to "overshoot," which makes it generally hard to nail down the rate of convergence of the discrete process. For two-player zero-sum games, Robinson's proof allows to derive an upper bound for the rate of convergence (Shapiro, 1958). The resulting estimate is of the order $O(t^{-1/(\nu_1+\nu_2-2)})$, where ν_i is the number of strategies for player $i = 1, 2$. If the game is symmetric, and consequently both players have the same number of strategies $\nu_1 = \nu_2 \equiv \nu$, the upper bound may be sharpened to $O(t^{-1/(\nu-1)})$. Given the lack of a direct extension of Robinson's proof to zero-sum networks, however, that upper bound is not easily generalized to zero-sum networks. Improving on Shapiro's upper bound, Karlin's strong conjecture says (or more precisely, said) that DTFP converges at rate $O(t^{-1/2})$ in two-person zero-sum games, regardless of the number of strategies. Daskalakis and Pan (2014) have recently disproved that conjecture, showing that the rate of convergence in an asymmetric two-player zero-sum game in which both players have the same number of strategies ν may be as low as $O(t^{-1/\nu})$. That lower bound holds, obviously, also for zero-sum networks.[34]

The so-called "Rosy Theorem" of Monderer et al. (1997) says that, in DTFP, a player's expected payoffs at any given stage are weakly higher than his average payoff experience. Using Robinson's (1951) theorem on the convergence of DTFP in two-player zero-sum games, it can be concluded that every DTFP in a two-person zero-sum game is *belief-affirming*, which means here that a player's average payoff experience converges against his value of the game. It is tempting to suggest that

[34] For related work, see Gjerstad (1996), Conitzer (2009), and Brandt et al. (2013).

an extension to zero-sum network games is feasible. However, Nash equilibrium payoffs are not necessarily unique in zero-sum networks (cf. Cai et al., 2016). Thus, even though the Rosy Theorem holds for zero-sum networks, the average payoff experience of an individual player may vary as much as his equilibrium payoff. Hence, Monderer et al. (1997, Th. B), which states an equality between a player's long-run payoff experience and the value of the game in two-player zero-sum games, cannot be easily generalized to zero-sum networks.

Cai and Daskalakis (2011) have shown that, if every node in a separable multiplayer game plays a *no-regret* sequence of pure strategies in discrete time, then the resulting frequency distributions of pure strategies ultimately form an ε-equilibrium. No-regret is a property that is less stringent than being belief-affirming. Specifically, under no-regret learning, a player's expected payoffs are assumed to be asymptotically weakly below his average payoff experience, while in a belief-affirming process like fictitious play in two-person zero-sum games, a player's expected payoffs are asymptotically even equal to his average payoff experience. In other words, by combining the conclusions of Robinson's theorem and Monderer et al. (1997, Th. B), DTFP in two-player zero-sum games is recognized as a no-regret learning algorithm. There is, consequently, no simple way to deduce Proposition 5 from the existing literature.

The following should now be immediate.

Corollary 2. *Let G be a network game that satisfies the assumptions of any of the Propositions 1 through 4. Then any DTFP process in G converges to equilibrium.*

Proof. For zero-sum networks, the claim follows from Proposition 5. Since conflict networks with pairwise homogeneous valuations, and also acyclic conflict networks with heterogeneous valuations are best-response equivalent in mixed strategies to zero-sum networks, the claim holds also for these classes of network games. Regarding exact bilateral potential games, it was shown above that the corresponding network game allows an exact potential. Hence, the claim follows in this case from Monderer and Shapley (1996a, 1996b). Finally, to deal with weighted potential networks on acyclic graphs, it suffices to recall the best-response equivalence in mixed strategies to an exact potential network. □

6. The case of joint beliefs

Above, it was assumed that each player i's beliefs about his opponents' play are independent across opponents. However, in a network with more than two players, a player might observe correlations between the behavior of two or more of his neighbors.[35] We will therefore explore in this section the implications of assuming that players take account of such correlations when deciding about their best responses.

Let $\widetilde{\mu}_{N(i)} \in \Delta(X_{N(i)})$ be player i's *joint belief* over strategies chosen by his neighbors, where the tilde indicates that correlation is feasible. Player i's expected payoff from player $x_i \in X_i$ is written as $u_i(x_i, \widetilde{\mu}_{N(i)}) = E[u_i(x_i, x_{N(i)})]$, where the expectation is taken with respect to $\widetilde{\mu}_{N(i)}$. Player i's mixed best-response correspondence MBR_i extends as usual to joint beliefs, $\widetilde{\mu}_{N(i)} \in \Delta(X_{N(i)})$, in the sense that $\text{MBR}_i(\widetilde{\mu}_{N(i)})$ is the set of mixed strategies $\mu_i^* \in \Delta(X_i)$ such that $u_i(\mu_i^*, \widetilde{\mu}_{N(i)}) = \max_{\mu_i \in \Delta(X_i)} u_i(\mu_i, \widetilde{\mu}_{N(i)})$. Similarly, the mixed best-response correspondence MBR of G extends to arbitrary probability distributions $\widetilde{\mu} \in \Delta(X)$ by assigning the Cartesian product $\text{MBR}(\widetilde{\mu}) = \bigtimes_{i=1}^{n} \text{MBR}_i(\widetilde{\mu}_{N(i)})$, where $\widetilde{\mu}_{N(i)}$ denotes then the marginal of $\widetilde{\mu}$ on $X_{N(i)}$. Let $m : [0, \infty) \to \Delta(X_1) \times ... \times \Delta(X_n)$ be a measurable path specifying each player i's mixed strategy at any time τ. Then the *continuous-time joint average* $\widetilde{\alpha}$ is defined as

$$\widetilde{\alpha}(\tau) \equiv \widetilde{\alpha}(\tau, m) = \frac{1}{\tau} \int_0^{\tau} m(\tau')d\tau' \qquad (\tau \geq 1), \tag{55}$$

where the integral is now taken in $\Delta(X)$. Similarly, if $\{x(t)\}_{t=0}^{\infty}$ is a sequence in $\bigtimes_{i=1}^{n} \Delta(X_i)$, we may define the *discrete-time joint average* $\widetilde{\alpha}^d$ as

$$\widetilde{\alpha}^d(t) \equiv \widetilde{\alpha}^d(t, x(.)) = \frac{1}{t-1} \sum_{t'=0}^{t-1} x(t') \qquad (t = 1, 2, 3, ...), \tag{56}$$

where the sum is again taken in $\Delta(X)$. The following definition of joint fictitious play should contain no surprises.

Definition 5. ($\widetilde{\textbf{CTFP}}$, $\widetilde{\textbf{DTFP}}$) *A continuous-time fictitious play with joint beliefs is a measurable mapping $m : [0, \infty) \to \bigtimes_{i=1}^{n} \Delta(X_i)$ such that $m(\tau) \in \text{MBR}(\widetilde{\alpha}(\tau))$ for any $\tau \geq 1$.*[36] *Similarly, a*

[35] Indeed, correlation of the limit profile is a well-documented possibility (Young, 1993; Fudenberg and Kreps, 1993; Jordan, 1993; Fudenberg and Levine, 1998).

[36] Adapting the proof of Harris (1998), existence of a $\widetilde{\text{CTFP}}$ can be verified for any n-player game. For network games, however, existence follows more easily from the proof of Proposition 6 below.

discrete-time fictitious play with joint beliefs is a sequence $\{x(t)\}_{t=0}^{\infty}$ *in* $\bigtimes_{i=1}^{n}\Delta(X_i)$ *such that* $x(t) \in \mathrm{BR}(\widetilde{\alpha}^d(t))$ *for any* $t \in \mathbb{N}$.

The definition of convergence is adapted as follows. Denote by $\widetilde{\mathcal{A}}(m)$ and $\widetilde{\mathcal{A}}^d(x(.))$, respectively, the set of all accumulation points of $\widetilde{\alpha}(.)$ and $\widetilde{\alpha}^d(.)$. We will say that $\widetilde{\mu}^* \in \Delta(X)$ is an *observational equilibrium* when, for each player $i = 1, ..., n$, player i's marginal distribution $\widetilde{\mu}_i^* \in \Delta(X_i)$ is a mixed best response to the marginal $\widetilde{\mu}_{N(i)}^* \in \Delta(X_{N(i)})$, i.e., when $u_i(x_i, \widetilde{\mu}_{N(i)}^*) \leq u_i(\widetilde{\mu}_i^*, \widetilde{\mu}_{N(i)}^*)$ for any player $i = 1, ..., n$ and for any strategy $x_i \in X_i$.[37] We will further say that a path m, or a sequence $x(.)$, *converges observationally to Nash* if any $\widetilde{\mu}^* \in \widetilde{\mathcal{A}}(m)$, or $\widetilde{\mu}^* \in \widetilde{\mathcal{A}}^d(x(.))$, is an observational equilibrium.

In the proof of the following result, we generalizes an insight of Sela (1999, Lemma 1) to arbitrary networks, and apply it to the present situation.

Proposition 6. *Let* G *be an arbitrary network game satisfying assumptions of any of the Propositions 1 through 4. Then any* $\widetilde{CTFP}$*, and likewise any* $\widetilde{DTFP}$*, converges observationally to Nash.*

Proof. Because interactions are bilateral, and linear expectations ignore correlations, expected payoffs satisfy

$$u_i(\mu_i, \widetilde{\mu}_{N(i)}) = \sum_{j \in N(i)} u_{ij}(\mu_i, \mu_j), \tag{57}$$

for any player $i = 1, ..., n$. Therefore, if $\widetilde{\mu} \in \Delta(X)$ is a probability distribution over pure strategy profiles in G, and if $\mu \in \Delta(X_1) \times ... \times \Delta(X_n)$ is the corresponding profile composed of marginal distributions, then $\mathrm{MBR}(\widetilde{\mu}) = \mathrm{MBR}(\mu)$. The claim follows. □

Intuitively, correlation is irrelevant for the best-response correspondence in a network game G because all interactions are bilateral. As a consequence, a path m is a CTFP if and only if it is a $\widetilde{\mathrm{CTFP}}$, and a sequence $x(.)$ is a DTFP if and only if it is a $\widetilde{\mathrm{DTFP}}$. Therefore, in the considered class of network games, fictitious play with joint beliefs converges to a potentially correlated profile in which each player's marginal distribution is a best response to the marginal distribution, formed either independently or jointly, of his neighbor's mixed strategies.

[37] For example, it follows from the analysis of Cai et al. (2016) that any coarse correlated equilibrium in a zero-sum network is an observational equilibrium.

7. Concluding remarks

In this paper, we have identified new and large classes of network games with the fictitious-play property. Using entirely elementary arguments, we have derived simple and natural conditions on either bilateral payoffs or on the network structure sufficient to guarantee convergence of the naive learning procedure even when a player's decisions across bilateral games are interdependent. We have also constructed several examples of multiplayer games that show that these conditions cannot be easily relaxed.

Applications of our results are manifold and include security games, conflict networks, and decentralized wireless channel selection, for instance. Our findings confirm the intuition that equilibrium behavior in important types of social interaction can be reached without assuming strong forms of economic rationality.

Our results on the discrete-time variant of fictitious play entail, in particular, a substantial extension of Robinson's (1951) classic result. This might serve as a basis for further analysis and simulation exercises. Our derivation also provides an additional illustration of the intimate relationship between the continuous-time and the discrete-time processes that has been suggested in many studies.

We did not address all open issues on fictitious play in network games. For instance, we did not examine stochastic fictitious play. It should be noted, however, that the convergence of stochastic fictitious play follows directly from our results for networks of exact potential games and, similarly, for acyclic networks of weighted potential games. Moreover, we conjecture that the techniques developed by Hofbauer and Sandholm (2002) apply also to the zero-sum networks and conflict networks discussed in the present paper.

Last but not least, there is some recent work that digs deeply into the differential topology and projective geometry of fictitious-play paths in two-person zero-sum games (van Strien, 2011; Berger, 2012). Exploring the potentially interesting implications of such approaches for zero-sum network games remains, however, beyond the scope of the present study.[38]

[38] Further, one might want to seek conditions that ensure that the results of the present paper continue to hold if all bilateral games are dominance solvable (Milgrom and Roberts, 1991) or exhibit strategic complementarities and diminishing returns (Krishna, 1992; Berger, 2008). However, as discussed in Sela (1999), this last route seems less promising because interdependencies between choices in bilateral games undermine such structural properties of the bilateral games.

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Appendix (not for publication): Details on Examples 2 and 3

Details on Example 2. Payoffs in the bilateral conflict $G_{i,i+1}$, where $i \in \{1,2,3\}$, are shown in the left panel of Figure 7.

	x_{i+1} = H	x_{i+1} = L
x_i = H	2, .5	3.5, .75
x_i = L	2, 1	3, 1.5

	x_3 = H			x_3 = L	
	x_2 = H	x_2 = L		x_2 = H	x_2 = L
x_1 = H	2.5, 2.5, 2.5	4, 2.75, 3	x_1 = H	3, 4, 2.75	4.5, 3.75, 3.5
x_1 = L	2.75, 3, 4	3.75, 3.5, 4.5	x_1 = L	3.5, 4.5, 3.75	4.5, 4.5, 4.5

Figure 7. Bilateral payoffs (left panel), and the normal form representation of G^2 (right panel).

Lemma A.1. The best-response correspondence in G^2 may be described as follows: Player $i \in \{1,2,3\}$ optimally chooses H if $r_{i-1} - 2r_{i+1} \geq 0$; player i optimally chooses L if $r_{i-1} - 2r_{i+1} \leq 0$.

Proof. Looking at bilateral payoffs, one notes that

$$u_i(\mathrm{H}, r_{i+1}, r_{i-1}) - u_i(\mathrm{L}, r_{i+1}, r_{i-1})$$

$$= u_{i,i+1}(\mathrm{H}, r_{i+1}) - u_{i,i+1}(\mathrm{L}, r_{i+1}) + u_{i,i-1}(\mathrm{H}, r_{i-1}) - u_{i,i-1}(\mathrm{L}, r_{i-1}) \tag{58}$$

$$= r_{i+1} \cdot 0 + (1 - r_{i+1}) \cdot \tfrac{1}{2} + r_{i-1} \cdot (-\tfrac{1}{4}) + (1 - r_{i-1}) \cdot (-\tfrac{1}{2}) \tag{59}$$

$$= \tfrac{1}{4}(r_{i-1} - 2r_{i+1}). \tag{60}$$

The claim follows. □

Lemma A.2. The game G^2 has a unique Nash equilibrium, that is given by $(r_1^*, r_2^*, r_3^*) = (0,0,0)$.

Proof. One easily checks that G^2 has the payoffs shown in the right panel of Figure 7. Clearly, (L,L,L) is a pure-strategy Nash equilibrium in G^2. Next, we show that the equilibrium is unique. Suppose first that there is a completely mixed-strategy equilibrium (r_1, r_2, r_3). Then, it follows from Lemma A.1 that $r_3 - 2r_2 = 0$, $r_1 - 2r_3 = 0$, and $r_2 - 2r_1 = 0$. But the sole solution of this system is $(r_1, r_2, r_3) = (0,0,0)$. Hence, G^2 does not allow a completely mixed equilibrium. Moreover, if another equilibrium exists, at least one player must choose a pure strategy. Without loss of generality, suppose that this is player 3. Assume first that $r_3 = 1$, so that player 3 chooses H. Then the iterated elimination of strictly dominated strategies implies that player 2 chooses L and that player 1 chooses H, but then player 3 would want to deviate to $x_3 = \text{L}$. Conversely, assume that player 3 chooses L. In this case, if $r_1 > 0$, then player 2 chooses H and consequently $r_1 = 0$, which is impossible. If, however, player 1 chooses L with probability one, then player 3 would want to deviate unless also player 2 chooses L with probability one. But that only brings us back to $(r_1, r_2, r_3) = (0, 0, 0)$. This proves uniqueness, and hence, the lemma. □

Next, recall the coordinates of the points at which the process changes its direction:

$$... \to p_1 = (\tfrac{2}{7}, \tfrac{4}{7}, \tfrac{1}{7}) \to p_2 = (\tfrac{1}{7}, \tfrac{2}{7}, \tfrac{4}{7}) \to p_3 = (\tfrac{4}{7}, \tfrac{1}{7}, \tfrac{2}{7}) \to ..., \tag{61}$$

The following lemma shows that this process is indeed a stable cycle.

Lemma A.3. (i) At p_i, with $i \in \{1, 2, 3\}$, player i optimally chooses L, while players $i-1$ and $i+1$ are both indifferent. (ii) There is a $\lambda > 1$ such that

$$\lambda(p_1 - (0, 1, 0)) = p_3 - (0, 1, 0), \tag{62}$$

$$\lambda(p_2 - (0, 0, 1)) = p_1 - (0, 0, 1), \tag{63}$$

$$\lambda(p_3 - (1, 0, 0)) = p_2 - (1, 0, 0). \tag{64}$$

Proof. (i) From Lemma A.1, player 1 optimally chooses L if $r_3 - 2r_2 \leq 0$. But, at $p_1 = (\frac{2}{7}, \frac{4}{7}, \frac{1}{7})$, we even have $r_3 - 2r_2 = -1 < 0$. Moreover, players 2 and 3 are indifferent at p_1 because $r_1 - 2r_3 = 0$ and $r_2 - 2r_1 = 0$. The analysis of the points p_2 and p_3 follows by symmetry. This proves the first claim. *(ii)* Note that with $\lambda = 2 > 1$,

$$\lambda(p_1 - (0, 0, 1)) = 2 \cdot (\tfrac{2}{7}, \tfrac{1}{7}, -\tfrac{3}{7}) = (\tfrac{4}{7}, \tfrac{2}{7}, -\tfrac{6}{7}) = p_3 - (0, 0, 1). \tag{65}$$

The other two equations follow, again, by symmetry. This proves the second claim, and hence, the lemma. □

Details on Example 3. The payoff matrix of G^3 looks as follows:

	$x_3 = H$			$x_3 = L$	
	$x_2 = H$	$x_2 = L$		$x_2 = H$	$x_2 = L$
$x_1 = H$	1, 6, 5	0, 0, 0	$x_1 = H$	6, 5, 1	5, 1, 6
$x_1 = L$	5, 1, 6	6, 5, 1	$x_1 = L$	0, 0, 0	1, 6, 5

Figure 8. The game G^3.

Lemma A.4. The best-response correspondence in G^3 is given as follows: Player 1 optimally chooses H if $5r_3 - r_2 \leq 2$; he optimally chooses T if $5r_3 - r_2 \geq 2$. Player 2 optimally chooses H if $5r_1 + r_3 \geq 3$; he optimally chooses T if $5r_1 + r_3 \leq 3$. Player 3 optimally chooses H if $5r_2 - r_1 \geq 2$; he optimally chooses T if $5r_2 - r_1 \leq 2$.

Proof. As for player 1, one notes that

$$u_1(\mathrm{H}, r_2, r_3) - u_1(\mathrm{T}, r_2, r_3)$$

$$= r_2 r_3 \cdot (-4) + (1 - r_2) r_3 \cdot (-6) + r_2(1 - r_3) \cdot 6 + (1 - r_2)(1 - r_3) \cdot 4 \tag{66}$$

$$= 4 + 2r_2 - 10r_3. \tag{67}$$

This proves the claim regarding player 1. Similarly,

$$u_2(r_1, \mathrm{H}, r_3) - u_2(r_1, \mathrm{T}, r_3)$$

$$= r_1 r_3 \cdot 6 + (1 - r_1) r_3 \cdot (-4) + r_1(1 - r_3) \cdot 4 + (1 - r_1)(1 - r_3) \cdot (-6) \tag{68}$$

$$= -6 + 10r_1 + 2r_3, \tag{69}$$

proving the claim regarding player 2. Finally,

$$u_3(r_1, r_2, \mathrm{H}) - u_3(r_1, r_2, \mathrm{T})$$

$$= r_1 r_2 \cdot 4 + (1 - r_1) r_2 \cdot 6 + r_1(1 - r_2) \cdot (-6) + (1 - r_1)(1 - r_2) \cdot (-4) \tag{70}$$

$$= 4 - 2r_1 + 10r_2. \tag{71}$$

This proves the final claim, and hence, the lemma. □

Lemma A.5. The game G^3 has a unique Nash equilibrium, that is given by $(r_1^*, r_2^*, r_3^*) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

Proof. Suppose first that player 1 chooses H with probability one. Then, by iterated elimination of strictly dominated strategies, player 2 chooses H, and so does player 3. But then, player 1 would deviate to T. Suppose next that player 1 chooses T with probability one. Then, by the iterated elimination of strictly dominated strategies, player 2 chooses T, and so does player 3. But then player 1 would deviate to H. Thus, there is no equilibrium in which player 1 plays a pure strategy. Thus, $r_1 \in (0, 1)$, and by Lemma 3.A, $5r_3 - r_2 = 2$. Clearly, this precludes $r_3 = 0$ and $r_3 = 1$, so that also player 3 must randomize. By Lemma 3.A, $5r_2 - r_1 = 2$. But this excludes $r_2 = 0$ and $r_2 = 1$. Hence, any equilibrium is necessarily completely mixed. There is a unique completely mixed equilibrium, because the system of indifference relationships $-r_2 + 5r_3 = 2$, $5r_1 + r_3 = 3$, and $-r_1 + 5r_2 = 2$ has the unique solution $(r_1, r_2, r_3) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. □

Recall the cycle

$$\begin{aligned} ... \to p_1 = (a, b, c) \to p_2 = (1-c, a, b) \to p_3 = (1-b, 1-c, a) \to \\ \to p_4 = (1-a, 1-b, 1-c) \to p_5 = (c, 1-a, 1-b) \to p_6 = (b, c, 1-a) \to ..., \end{aligned} \tag{72}$$

where $(a, b, c) = (\frac{4}{9}, \frac{8}{9}, \frac{7}{9})$. The following two lemmas show that the cyclic process is indeed a continuous-time fictitious play.

Lemma A.6. (i) At p_1, players 1 optimally chooses T, player 2 is indifferent, and player 3 optimally chooses H. (ii) At p_2, players 1 and 2 optimally choose T, whereas player 3 is indifferent. (iii) At p_3, player 1 is indifferent, while players 2 and 3 optimally choose T. (iv) At p_4, player 1 optimally chooses H, player 2 is indifferent, and player 3 optimally chooses T. (v) At p_5, players 1 and 2 optimally choose H, whereas player 3 is indifferent. (vi) At p_6, player 1 is indifferent, while players 2 and 3 optimally choose H.

Proof. (i) By Lemma A.4, player 1 optimally chooses T if $5r_3 - r_2 \geq 2$. But, at $p_1 = (a, b, c) = (\frac{4}{9}, \frac{8}{9}, \frac{7}{9})$, we even have $5r_3 - r_2 = 3 > 1$. Next, player 2 is indifferent if $5r_1 + r_3 = 3$. But this is true at p_1. Finally, player 3 optimally chooses H if $5r_2 - r_1 \geq 2$. But, at p_1, we even have $5r_2 - r_1 = 4 > 1$. *(ii)* Player 1 optimally chooses T if $5r_3 - r_2 \geq 2$. But, at $p_2 = (1 - c, a, b) = (\frac{2}{9}, \frac{4}{9}, \frac{8}{9})$, we even

have $5r_3 - r_2 = 4 > 1$. Next, player 2 optimally choses T if $5r_1 + r_3 \leq 3$. But, at p_2, we even have $5r_1 + r_3 = 2 < 3$. Finally, player 3 is indifferent if $5r_2 - r_1 = 2$. And at p_2, we indeed have $5r_2 - r_1 = 2$.
(iii) Player 1 is indifferent if $5r_3 - r_2 = 2$. And indeed, at $p_3 = (1-b, 1-c, a) = (\frac{1}{9}, \frac{2}{9}, \frac{4}{9})$, we have $5r_3 - r_2 = 2$. Player 2 optimally choses T if $5r_1 + r_3 \leq 3$. But, at p_3, we even have $5r_1 + r_3 = 1 < 3$. Finally, player 3 optimally chooses T if $5r_2 - r_1 \leq 2$. But, at p_3, we even have $5r_2 - r_1 = 1 < 2$.
(iv) Player 1 optimally chooses H if $5r_3 - r_2 \leq 2$. But, at $p_4 = (1-a, 1-b, 1-c) = (\frac{5}{9}, \frac{1}{9}, \frac{2}{9})$, we even have $5r_3 - r_2 = 1 < 2$. Next, player 2 is indifferent if $5r_1 + r_3 = 3$. But this is true at p_4. Finally, player 3 optimally chooses T if $5r_2 - r_1 \leq 2$. But, at p_4, we even have $5r_2 - r_1 = 0 < 1$.
(ii) Player 1 optimally chooses H if $5r_3 - r_2 \leq 2$. But, at $p_5 = (c, 1-a, 1-b) = (\frac{7}{9}, \frac{5}{9}, \frac{1}{9})$, we even have $5r_3 - r_2 = 0 < 2$. Next, player 2 optimally choses H if $5r_1 + r_3 \geq 3$. But, at p_5, we even have $5r_1 + r_3 = 4 > 3$. Finally, player 3 is indifferent if $5r_2 - r_1 = 2$. And at p_5, we indeed have this.
(iii) Player 1 is indifferent if $5r_3 - r_2 = 2$. And indeed, at $p_6 = (b, c, 1-a) = (\frac{8}{9}, \frac{7}{9}, \frac{5}{9})$, we have this. Player 2 optimally choses H if $5r_1 + r_3 \geq 3$. But, at p_6, we even have $5r_1 + r_3 = 5 > 3$. Finally, player 3 optimally chooses H if $5r_2 - r_1 \geq 2$. But, at p_6, we even have $5r_2 - r_1 = 3 > 2$. This proves the last claim and therefore the lemma. □

Lemma A.7. There is a $\lambda > 1$ such that

$$\lambda(p_2 - (0,0,1)) = p_1 - (0,0,1) \tag{73}$$

$$\lambda p_3 = p_2 \tag{74}$$

$$\lambda(p_4 - (1,0,0)) = p_3 - (1,0,0) \tag{75}$$

$$\lambda(p_5 - (1,1,0)) = p_4 - (1,1,0) \tag{76}$$

$$\lambda(p_6 - (1,1,1)) = p_5 - (1,1,1) \tag{77}$$

$$\lambda(p_1 - (0,1,1)) = p_6 - (0,1,1). \tag{78}$$

Proof. One can easily verify that equations (73-78) are satisfied for $\lambda = 2$. □