A direct approach to Plateau's problem in any codimension

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Abstract: This paper proposes a direct approach to solve the Plateau's problem in codimension higher than one. The problem is formulated as the minimization of the Hausdorff measure among a family of d -rectifiable closed subsets of $\mathbb{R}^n$: following the previous work [13], the existence result is obtained by a compactness principle valid under fairly general assumptions on the class of competitors. Such class is then specified to give meaning to boundary conditions. We also show that the obtained minimizers are regular up to a set of dimension less than $(d-1)$.

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A DIRECT APPROACH TO THE ANISOTROPIC PLATEAU'S PROBLEM

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Abstract. We prove a compactness principle for the anisotropic formulation of the Plateau's problem in codimension one, along the same lines of previous works of the authors [DGM14, DPDRG15]. In particular, we perform a new strategy for proving the rectifiability of the minimal set, avoiding the Preiss' Rectifiability Theorem [Pre87].

1. Introduction

The anisotropic Plateau's problem aims at finding an energy minimizing surface spanning a given boundary when the energy functional is more general than the usual surface area (as in the standard Plateau's problem) and is obtained integrating a general Lagrangian F over the surface. In particular, the integrand depends on the position and the tangent space to the surface.

As in the case of the area integrand, [De 54, FF60, Rei60, Alm68, DS00, HP13, DGM14, DPDRG15], many definitions of boundary conditions (both homological and homotopical), as well as the type of competitors (currents, varifolds, sets) have been considered in the literature. An important existence, regularity and almost uniqueness result in arbitrary dimension and codimension was achieved by Almgren in [Alm68], using refined techniques from geometric measure theory. In more recent times, the Plateau's problem for the area integrand has been investigated in order to give an existence theory which could comprehend several notions of competitors all together [Feu09, HP13, Dav14, DGM14, DPDRG15]. In particular a very elegant notion of boundary for general closed sets has been introduced by Harrison and Pugh in [HP13], which proves the existence and regularity for minimizers of the area functional in codimension 1. The same authors in [HP15] investigated the "inhomogeneous Plateau's problem", where the energy density is isotropic but depends on the space position of the surface, proving existence of a minimizer under a suitable cohomological definition of boundary.

In this paper, we adopt the same strategy as in [DGM14, DPDRG15], namely we prove a general compactness theorem for minimizing sequences in general classes of rectifiable sets. More precisely, we consider the measures naturally associated to any such sequence and we show that, if a sufficiently large class of deformations are admitted, any weak limit is induced by a rectifiable set, thus providing compactness and semicontinuity under very little assumptions. Our result does not address directly the question whether the limiting set belongs to the original class, which is linked to its closure under weak convergence of measures. However, we can easily show that this is the case for the class considered by Harrison and Pugh in [HP13], thus giving a generalization of their existence theorem to any (elliptic) anisotropic functional. While we were completing this paper we learned that analogous results have been obtained at the same time by Harrison and Pugh in [HP16], using different arguments and building upon their previous work [HP15].

One main difficulty in our approach is to prove the rectifiability of the support of the limiting measure. In the paper [DGM14], the key ingredient to obtain such rectifiability is the classical monotonicity formula for the mass ratio of the limiting measure, which allows to apply Preiss' rectifiability theorem for Radon measures [Pre87, De 08]. Such a strategy does not seem feasible for general anisotropic integrands, where the monotonicity of the mass ratio is unlikely to be true, as pointed out in [All74]. Since all the other ingredients of [DGM14] can be easily transported to the anisotropic case, the main goal of this paper is to show how, in codimension one, the rectifiability of the limiting measure follows from the theory of Caccioppoli sets, bypassing the monotonicity formula and the deep result of Preiss. In particular, we are able to prove the results analogous to those of [DGM14] with a strategy which has some similarities with the one used in [Alm68].

In [DPDRG15], a similar theorem for the area functional was proved in any codimension. The most general case of any codimension and anisotropic energies will be addressed in a further paper by the same authors, see [DPDRG16], using however different and more sophisticated PDE techniques.

The structure of this note is the following: in Section 2 we introduce the notation and state the main theorems of the paper. Section 3 is devoted to the proof of the main compactness result. In Section 4 we analyse the applicability of Theorem 2.8 for some specific boundary conditions and investigate the regularity of minimizers.

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2. Notation and main results

The ambient space is the standard euclidean one, $\mathbb{R}^{n+1}$, and $\mathcal{H}^k$ denotes the k-Hausdorff measure; moreover, for every set A, we let $|A| := \mathcal{H}^{n+1}(A)$ be its Lebesgue measure. We will let $U_r(A)$ be the tubular neighborhood of A of radius r. Recall that a set K is said to be n-rectifiable if it can be covered, up to an $\mathcal{H}^n$ negligible set, by countably many C^1 n-dimensional submanifolds, see [Sim83, Chapter 3]; we also denote by $G = G(n+1,n)$ the Grassmannian of unoriented n-dimensional hyperplanes in $\mathbb{R}^{n+1}$. Given an n-rectifiable set K, we denote by $T_K(x)$ the approximate tangent space of K at x, which exists for $\mathcal{H}^n$-almost every point $x \in K$ [Sim83, Chapter 3]. Finally, we let

- $B_{x,r} := \{y \in \mathbb{R}^{n+1} : |x-y| < r\}$;
- $B := B_{0,1}$;
- $Q_r :=]-\frac{r}{2}, \frac{r}{2}[^{n+1}$;
- $R_{2a,2b} := [-a,a]^n \times [-b,b]$;
- $\omega_n := \mathcal{H}^n(B \cap (\mathbb{R}^n \times \{0\}))$ and $\sigma_n := \mathcal{H}^n(\partial B)$.

The anisotropic Lagrangians considered in the rest of the note will be continuous maps

$$F : \mathbb{R}^{n+1} \times G \ni (x,\pi) \mapsto F(x,\pi) \in \mathbb{R}^+,$$

verifying the lower and upper bounds

$$0 < \lambda \le F(x,\pi) \le \Lambda < \infty. \tag{2.1}$$

Given an n-rectifiable set K and an open subset $U \subset \mathbb{R}^{n+1}$, we define:

$$\mathbf{F}(K,U) := \int_{K\cap U} F(x,T_K(x))\, d\mathcal{H}^n(x) \quad \text{and} \quad \mathbf{F}(K) := \mathbf{F}(K,\mathbb{R}^{n+1}). \tag{2.2}$$

It will be also convenient to look at the frozen Lagrangian: for $y \in \mathbb{R}^{n+1}$, we let

$$\mathbf{F}^y(K,U) := \int_{K\cap U} F(y,T_K(x))\, d\mathcal{H}^n(x).$$

Throughout all the paper, $H \subset \mathbb{R}^{n+1}$ will denote a closed subset of $\mathbb{R}^{n+1}$. Assume to have a class $\mathcal{P}(H)$ of relatively closed n-rectifiable subsets K of $\mathbb{R}^{n+1} \setminus H$: one can then formulate the anisotropic Plateau's problem by asking whether the infimum

$$m_0 := \inf \big\{\mathbf{F}(K) : K \in \mathcal{P}(H)\big\} \tag{2.3}$$

is achieved by some set (which is the limit of a minimizing sequence), if it belongs to the chosen class $\mathcal{P}(H)$ and which additional regularity properties it satisfies.

We next outline a set of flexible and rather weak requirements for $\mathcal{P}(H)$.

Definition 2.1 (Cup competitors). Let $K \subset \mathbb{R}^{n+1} \setminus H$ and $B_{x,r} \subset\subset \mathbb{R}^{n+1} \setminus H$. We introduce the following equivalence relation among points of $\overline{B_{x,r}} \setminus K$:

$$y_0 \sim_K y_1 \quad \Longleftrightarrow \quad \exists\, \gamma \in C^0([0,1], \overline{B_{x,r}} \setminus K) : \gamma(0) = y_0,\, \gamma(1) = y_1,\, \gamma(]0,1[) \subset B_{x,r}.$$

We denote by $\Gamma_i(x,r)$ the equivalence classes in $\partial B_{x,r}/\sim_K$; to every $\Gamma_i(x,r)$ we associate

$$\Omega_i(x,r) = \{z \in B_{x,r} \setminus K : \exists\, y \in \Gamma_i(x,r) \text{ such that } z \sim_K y\}.$$

The cup competitor associated to $\Gamma_i(x,r)$ for K in $B_{x,r}$ is

$$\big(K \setminus B_{x,r}\big) \cup \big(\partial B_{x,r} \setminus \Gamma_i(x,r)\big)\,. \tag{2.4}$$

It is easy to see that the associated sets $\Omega_i(x,r)$ are connected components of $B_{x,r} \setminus K$ (possibly not all of them) .

Definition 2.2 (Good class). A family $\mathcal{P}(H)$ of relatively closed subsets $K \subset \mathbb{R}^{n+1} \setminus H$ is called a *good class* if for any $K \in \mathcal{P}(H)$, for every $x \in K$ and for a.e. $r \in (0, \mathrm{dist}(x,H))$ the following holds

$$\inf\big\{\mathbf{F}(J) : J \in \mathcal{P}(H)\,, J \setminus \overline{B_{x,r}} = K \setminus \overline{B_{x,r}}\big\} \le \mathbf{F}(L) \tag{2.5}$$

whenever L is any cup competitor for K in $B_{x,r}$.

Remark 2.3. Observe that the definition of cup competitors is a slight modification of that of [DGM14], where $\Gamma_i(x,r)$ were taken to be connected components of $\partial B_{x,r} \setminus K$: observe however that, for every cup competitor in [DGM14], we can find a cup competitor as above which has at most the same area. Finally good classes in this paper do not assume any kind of comparisons with cones, as it is the case of [DGM14].

The notion of good class is enough to ensure that any weak* limit of a minimizing sequence is a rectifiable measure. In order to obtain additional information on the density of such limit measure, we need however some more comparisons. We recall here the ones introduced in [DPDRG15].

Definition 2.4 (Lipschitz deformations). Let $\mathfrak{D}(x,r)$ be the set of functions $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ for which there exists a C^1 isotopy $\lambda : [0,1] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that

$$\lambda(0,\cdot) = \mathrm{Id}, \quad \lambda(1,\cdot) = \varphi, \quad \lambda(t,h) = h \quad \forall\,(t,h) \in [0,1] \times (\mathbb{R}^{n+1} \setminus B_{x,r}).$$

We finally set $\mathsf{D}(x,r) := \overline{\mathfrak{D}(x,r)}^{w^*-W^{1,\infty}}$, the sequential closure of $\mathfrak{D}(x,r)$ with respect to the uniform convergence with equibounded differentials.

Definition 2.5 (Deformed competitors and deformation class). Let $K \subset \mathbb{R}^{n+1} \setminus H$ be relatively closed and $B_{x,r} \subset\subset \mathbb{R}^{n+1} \setminus H$. A *deformed competitor* for K in $B_{x,r}$ is any set of the form

$$\varphi\,(K) \qquad \text{where} \qquad \varphi \in \mathsf{D}(x,r).$$

A family $\mathcal{P}(H)$ of relatively closed n-rectifiable subsets $K \subset \mathbb{R}^{n+1} \setminus H$ is called a *deformation class* if for every $K \in \mathcal{P}(H)$, for every $x \in K$ and for a.e. $r \in (0, \mathrm{dist}(x,H))$

$$\inf\big\{\mathbf{F}(J) : J \in \mathcal{P}(H)\,, J \setminus \overline{B_{x,r}} = K \setminus \overline{B_{x,r}}\big\} \le \mathbf{F}(L) \tag{2.6}$$

whenever L is any deformed competitor for K in $B_{x,r}$.

Observe that the cup and deformed competitors are not assumed to belong to $\mathcal{P}(H)$, but just to be approximable in energy by elements of $\mathcal{P}(H)$ (conditions (2.5), (2.6)). Moreover we point out that a cup competitor (2.4) is not necessarily a deformed competitor, for homotopical reasons.

We will assume the following ellipticity condition on the energy $\mathbf{F}$, introduced in [Alm68], which is a geometric version of quasiconvexity, cf. [Mor66]:

Definition 2.6 (Elliptic anisotropy, [Alm68, 1.2])**.** The anisotropic Lagrangian F is said to be *elliptic* if there exists $\Gamma \geq 0$ such that, whenever $x \in \mathbb{R}^{n+1}$ and D is an n-disk centered in x and with radius r, then the inequality

$$\mathbf{F}^x(K, B_{x,r}) - \mathbf{F}^x(D, B_{x,r}) \geq \Gamma(\mathcal{H}^n(K \cap B_{x,r}) - \mathcal{H}^n(D))$$

holds for every n-rectifiable set K verifying $K \cap \partial B_{x,r} = \partial D \times \{0\}$ which cannot be deformed into $\partial D \times \{0\}$ via a map $\varphi \in \mathsf{D}(x, r)$.

Remark 2.7. Given an n-rectifiable set K and a deformation $\varphi \in \mathsf{D}(x, r)$, using property (2.1), we deduce the quasiminimality property

$$\mathbf{F}(\varphi(K)) \leq \Lambda \mathcal{H}^n(\varphi(K)) \leq \Lambda(\mathrm{Lip}\,(\varphi))^n \mathcal{H}^n(K) \leq \frac{\Lambda}{\lambda}(\mathrm{Lip}\,(\varphi))^n \mathbf{F}(K). \tag{2.7}$$

Moreover, whenever $V \subset\subset \mathbb{R}^{n+1}$, the following holds

$$\sup_{x,y \in V\ S,T \in G(n,n+1)} |F(x, T) - F(y, S)| \leq \omega_V(|x - y| + \|T - S\|), \tag{2.8}$$

for some modulus of continuity ω_V for F in $V \times G(n, n + 1)$.

We now have all the tools to state our main theorem. A minimizing sequence $\{K_j\} \subset \mathcal{P}(H)$ in Problem (2.3) satisfies the property $\mathbf{F}(K_j) \to m_0$, and throughout the paper we will assume m_0 to be finite. The comparison properties of good classes $\mathcal{P}(H)$ are then enough to obtain a compactness property for the measures naturally associating to $\{K_j\}$, namely the weak converge (up to subsequences) to a rectifiable measure; if in addition $\mathcal{P}(H)$ is a deformation class and F is elliptic, then we conclude the lower semicontinuity of the functional (2.2) if we consider the support of the limit measure as the appropriate limit for the sequence.

Theorem 2.8. *Let $H \subset \mathbb{R}^{n+1}$ be closed and $\mathcal{P}(H)$ be a good class. Let $\{K_j\} \subset \mathcal{P}(H)$ be a minimizing sequence and assume $m_0 < \infty$. Then, up to subsequences, the measures $\mu_j := F(\cdot, T_{K_j}(\cdot))\mathcal{H}^n \llcorner K_j$ converge weakly* in $\mathbb{R}^{n+1} \setminus H$ to a measure $\mu = \theta \mathcal{H}^n \llcorner K$, where $K = \mathrm{spt}\,\mu \setminus H$ is an n-rectifiable set and $\theta \geq \lambda$.*

If in addition $\mathcal{P}(H)$ is a deformation class and F is elliptic, then $\theta(x) = F(x, T_K(x))$ and $\liminf_j \mathbf{F}(K_j) \geq \mathbf{F}(K)$. In particular, if $K \in \mathcal{P}(H)$, then K is a minimum for Problem (2.3).

Since the proof of Theorem 2.8 does not exploit Preiss' rectifiability Theorem, when the Lagrangian is constant (i.e. up to a factor it is the area functional $F \equiv 1$) and we require the energetic inequality in (2.5) to hold for any cup competitors as in (2.4) and for any cone competitor as in [DGM14, Equation 1.2], then the same strategy gives a simpler proof of the conclusions in [DGM14, Theorem 2] and there is no need to introduce the deformed competitors defined in Definition 2.5. In fact, the same strategy can be used for isotropic Lagrangian $F(x, \pi) = f(x)$ with minor adjustments.

One application of Theorem 2.8 yields a generalization of the main result in [HP13] to anisotropic Lagrangians. More precisely, consider the following classes of sets.

Definition 2.9. Let $n \geq 2$ and H be a closed set in $\mathbb{R}^{n+1}$. Let us consider the family

$$\mathcal{C}_H = \left\{\gamma : S^1 \to \mathbb{R}^{n+1} \setminus H : \gamma \text{ is a smooth embedding of } S^1 \text{ into } \mathbb{R}^{n+1}\right\}.$$

We say that $\mathcal{C} \subset \mathcal{C}_H$ is closed by homotopy (with respect to H) if $\mathcal{C}$ contains all elements $\gamma' \in \mathcal{C}_H$ belonging to the same homotopy class $[\gamma] \in \pi_1(\mathbb{R}^{n+1} \setminus H)$ of any $\gamma \in \mathcal{C}$. Given $\mathcal{C} \subset \mathcal{C}_H$ closed by homotopy, we denote by $\mathcal{F}(H, \mathcal{C})$ the family of relatively closed subset K of $\mathbb{R}^{n+1} \setminus H$ such that

$$K \cap \gamma \neq \emptyset \text{ for every } \gamma \in \mathcal{C}\,.$$

Theorem 2.10. *Let $n \geq 2$ and $\mathcal{C}$ be closed by homotopy with respect to H. Let also $\mathcal{P}(H) = \{K \in \mathcal{F}(H, \mathcal{C}) : K \text{ is } n-\text{rectifiable}\}$. Then:*

(a) *$\mathcal{F}(H, \mathcal{C})$ is a good class in the sense of Definition 2.2 and a deformation class in the sense of Definition 2.5.*
(b) *If $\{K_j\} \subset \mathcal{P}(H)$ is a minimizing sequence and K is any set associated to $\{K_j\}$ by Theorem 2.8, then $K \in \mathcal{P}(H)$ and thus K is a minimizer.*
(c) *The set K in (b) is an $(\mathbf{F}, 0, \infty)$-minimal set in $\mathbb{R}^{n+1} \setminus H$ in the sense of Almgren* [Alm76]*.*

As already mentioned, a similar theorem has been obtained independently by Harrison and Pugh in [HP16], building upon a previous paper, [HP15], where the same authors considered the special case of isotropic Lagrangians $F(x, \pi) = f(x)$.

3. Proof of Theorem 2.8

Parts of the proofs follow the isotropic case treated in [DGM14]: we will be brief on these arguments, hoping to convey the main ideas and in order to leave space to the original content.

The proof of Theorem 2.8 goes as follows: we consider the natural measures (μ_j) associated to a minimizing sequence (K_j) and extract a weak limit μ. We first recall that, as a consequence of minimality, μ enjoys density upper and lower bounds on $\mathrm{spt}(\mu)$, leading to the representation $\mu = \theta \mathcal{H}^n \llcorner \mathrm{spt}(\mu)$: this part follows almost verbatim the proof of [DGM14]. Then, via an energy comparison argument, we exclude the presence of purely unrectifiable subsets of $\mathrm{spt}(\mu)$, which is the core novelty of the note. Finally we show that, if the Lagrangian is elliptic and we allow for deformed competitors, then the energy is lower semicontinuous along (K_j).

3.1. Density bound.

In this section we prove the following

Lemma 3.1 (Density bounds)**.** *Suppose that $\mathcal{P}(H)$ is a good class, that $\{K_j\} \subset \mathcal{P}(H)$ is a minimizing sequence for problem* (2.3) *and that*

$$\mu_j = F(\cdot, T_K(\cdot))\mathcal{H}^n \llcorner K_j \overset{*}{\rightharpoonup} \mu$$

in $\mathbb{R}^{n+1} \setminus H$. Then the limit measure μ enjoys density upper and lower bounds:

$$\theta_0\, \omega_n r^n \leq \mu(B_{x,r}) \leq \theta_0^{-1}\, \omega_n r^n\,, \qquad \forall x \in \mathrm{spt}\,\mu\,, \forall r < d_x := \mathrm{dist}(x, H) \tag{3.1}$$

for some positive constant $\theta_0 = \theta_0(n, F) > 0$.

Proof. The density lower bound can be proved as in [DGM14, Theorem 2, Step 1] with the use of cup competitors only, since the energy $\mathbf{F}$ is comparable to the Hausdorff measure by (2.7). The notion of cup competitor in Definition 2.2 slightly differs from the notion in [DGM14, Definition 1], however the key fact is that the latter have larger energy, cf. Remark 2.3. The existence of a density upper bound is trivially true, since we can use a generic sequence $\{\Gamma_j\}$ of cup competitors associated to $\{K_j\}$ in $B_{x,r}$. Indeed, by almost minimality

$$\mu(B_{x,r}) \leq \limsup_j \mu_j(\overline{B_{x,r}}) \leq \limsup_j \mathbf{F}(\partial B_{x,r} \setminus \Gamma_j) \leq \limsup_j \Lambda\, \mathcal{H}^n(\partial B_{x,r} \setminus \Gamma_j) \leq \Lambda\, \sigma_n\, r^n\,. \tag{3.2}$$

□

We remark that, if $\mathcal{P}(H)$ were just a deformation class, instead of a good class, the density lower bound could be proven as in [DPDRG15, Theorem 1.3, Step1]. Moreover, the upper bound could be obtained as in (3.2), but using the slightly different cup competitors defined in [DGM14, Definition 1], which are proven to be deformed competitors in [DGM14, Theorem 7, Step 1].

3.2. **Proof Theorem 2.8: rectifiability.** Up to extracting subsequences, we can assume the existence of a Radon measure μ on $\mathbb{R}^{n+1}\setminus H$ such that

$$\mu_j \overset{*}{\rightharpoonup} \mu\,, \qquad \text{as Radon measures on } \mathbb{R}^{n+1}\setminus H\,. \tag{3.3}$$

We set $K = \operatorname{spt}\mu\setminus H$ and from [Mat95, Theorem 6.9] and Lemma 3.1 we deduce

$$\mu = \theta\mathcal{H}^n \llcorner K, \tag{3.4}$$

with K relatively compact in $\mathbb{R}^{n+1}\setminus H$ and $c_0 \le \theta \le C_0$.

We decompose $K = \mathcal{R}\cup\mathcal{N}$ into a rectifiable $\mathcal{R}$ and a purely unrectifiable $\mathcal{N}$ (see [Sim83, Ch. 3.3]) and assume by contradiction that $\mathcal{H}^n(\mathcal{N}) > 0$. Then, there is $x\in K$ such that

$$\Theta^n(\mathcal{R},x) = \lim_{r\to 0}\frac{\mathcal{H}^n(\mathcal{R}\cap B_{x,r})}{\omega_n r^n} = 0, \qquad \Theta^{*n}(\mathcal{N},x) = a > 0. \tag{3.5}$$

Without loss of generality, we assume that $x = 0$. The overall aim is to show that at 0 the density lower bound of Lemma 3.1 would be false, reaching therefore a contradiction.

For every $\rho > 0$, we let $\Omega_i(\rho)$, with $i\in\mathbb{N}$, be the equivalence classes of $B_r\setminus K$ in the sense of Definition 2.2: by the boundary rectifiability Theorem [Fed69, 4.5.6], we get that $\Omega_i(\rho)$ are sets of finite perimeter and $\sum_i \mathcal{H}^n\llcorner\partial^*\Omega_i(\rho)\cap B_\rho \le 2\mathcal{H}^n\llcorner\mathcal{R}\cap B_\rho$ (as it is customary, given a set of finite perimeter Ω we will use $\partial^*\Omega$ for its reduced boundary).

Since in what follows we will often deal with subsets of the sphere ∂B_ρ, we will use the following notation:

- $\partial_{\partial B_\rho}A$ is the topological boundary of A as subset of ∂B_ρ;
- $\partial^*_{\partial B_\rho}A$ is the reduced boundary of A relative to ∂B_ρ.

Using this notation and the slicing theory for sets of finite perimeter (cf. [AFP00]), we infer

$$\sum_i \mathcal{H}^{n-1}\llcorner\partial^*_{\partial B_t}(\Omega_i(\rho)\cap\partial B_t) \le 2\mathcal{H}^{n-1}\llcorner\mathcal{R}\cap\partial B_t \quad \text{for a.e. } t<\rho. \tag{3.6}$$

Step 1. In this first step we show that, for every $\varepsilon_0 > 0$ and every $r_0 > 0$ small enough, there exists $\rho\in]r_0, 2r_0[$ satisfying

$$\max_i\left\{\mathcal{H}^n(\Gamma_i(\rho))\right\} \ge (\sigma_n - \varepsilon_0)\rho^n. \tag{3.7}$$

Indeed, by (3.5), we consider r_0 so small that $\mathcal{H}^n(\mathcal{R}\cap B_s(x)) \le \varepsilon_0 s^n$ for every $s\le 2r_0$. We first claim the existence of a closed set $R\subset]r_0,2r_0[$ of positive measure such that the following holds $\forall\rho\in R$:

(i)
$$\lim_{\sigma\in R,\sigma\to\rho}\mathcal{H}^{n-1}(\mathcal{R}\cap\partial B_\sigma) = \mathcal{H}^{n-1}(\mathcal{R}\cap\partial B_\rho);$$

(ii) $\mathcal{H}^n(K\cap\partial B_\rho) = 0$;

(iii) $\mathcal{H}^{n-1}(\mathcal{R}\cap\partial B_\rho) \le C\varepsilon_0\rho^{n-1}$.

The existence of a set of positive measure R' such that (iii) holds at any $\rho\in R'$ is an obvious consequence of the coarea formula and of Chebycheff's inequality, provided the universal constant C is larger than $\frac{2^n}{n}$. Moreover, condition (ii) holds at all but countable many radii. Next, since the map $t\mapsto\mathcal{H}^{n-1}(\mathcal{R}\cap\partial B_t)$ is measurable, by Lusin's theorem we can select a closed subset R of R' with positive measure for which (i) holds at every radius.

Fix now a point $\rho \in R$ of density 1 for R: it turns out that ρ satisfies indeed condition (3.7). In order to show that estimate, we first choose $(\rho_k) \subset R$, $\rho_k \uparrow \rho$ such that (3.6) holds for $t = \rho_k$. Observe that, for every sequence of points $x_k \in \partial B_{\rho_k} \cap \Omega_i(\rho)$ converging to some x_∞, we have that $x_\infty \in \overline{\Gamma_i(\rho) \cup (K \cap \partial B_\rho)}$, otherwise there would exist $\tau > 0$ such that $B_\tau(x_\infty) \cap \Omega_i(\rho) = \emptyset$, against the convergence of x_k to x_∞. In particular, rescaling everything at radius ρ, for every $\eta > 0$ there exists $k(\eta)$ such that, for all $k \geq k(\eta)$

$$E_{k,i} := \frac{\rho}{\rho_k} \left(\Omega_i(\rho) \cap \partial B_{\rho_k}\right) \subset U_\eta \left(\Gamma_i(\rho) \cup (K \cap \partial B_\rho)\right) \cap \partial B_\rho =: \Gamma_{i,\eta},$$

where U_η denotes the η-tubular neighborhood.

Observe that $\mathcal{H}^n(\Gamma_{i,\eta}) \downarrow \mathcal{H}^n(\Gamma_i(\rho))$ as $\eta \downarrow 0$, because $\partial_{\partial B_\rho}\Gamma_i(\rho) \subset K \cap \partial B_\rho$ and (ii) holds. On the other hand, for every $\eta > 0$, we can take Λ_η compact subset of $\Gamma_i(\rho)$ with $\mathcal{H}^n(\Gamma_i(\rho) \setminus \Lambda_\eta) < \eta$ and $U_\alpha(\Lambda_\eta) \cap B_\rho \subset \Omega_i(\rho)$ for some small $\alpha(\eta) > 0$. Therefore, for $\rho - \rho_k < \alpha(\eta)$, the following holds

$$E_{k,i} \supset \Lambda_\eta.$$

Since $\Lambda_\eta \subset E_{k,i} \subset \Gamma_{i,\eta}$ for every $k > k(\eta)$, $\Lambda_\eta \subset \Gamma_i(\rho) \subset \Gamma_{i,\eta}$ and $\mathcal{H}^n(\Gamma_{i,\eta} \setminus \Lambda_\eta) \downarrow 0$ as $\eta \downarrow 0$, we easily deduce that $E_{k,i} \to \Gamma_i(\rho)$ in $L^1(\partial B_\rho)$. Moreover, by (3.6)

$$\sum_i \mathcal{H}^{n-1}(\partial^*_{\partial B_{\rho_k}}(\Omega_i(\rho) \cap \partial B_{\rho_k})) \leq 2\mathcal{H}^{n-1}(\mathcal{R} \cap \partial B_{\rho_k}). \tag{3.8}$$

The L^1 convergence shown above, the lower semicontinuity of the perimeter and the definition of $E_{k,i}$ imply that

$$\begin{aligned}\sum_i \mathcal{H}^{n-1}(\partial^*_{\partial B_\rho}\Gamma_i(\rho)) &\leq \liminf_k \sum_i \mathcal{H}^{n-1}(\partial^*_{\partial B_\rho} E_{k,i}) \\ &= \liminf_k \left(\frac{\rho}{\rho_k}\right)^{n-1} \sum_i \mathcal{H}^{n-1}(\partial^*_{\partial B_{\rho_k}}(\Omega_i(\rho) \cap \partial B_{\rho_k})).\end{aligned}$$

Plugging (3.8), conditions (i) and (iii) in the previous equation, we get

$$\sum_i \mathcal{H}^{n-1}(\partial^*_{\partial B_\rho}\Gamma_i(\rho)) \leq 2\liminf_k \mathcal{H}^{n-1}(\mathcal{R} \cap \partial B_{\rho_k}) = 2\mathcal{H}^{n-1}(\mathcal{R} \cap \partial B_\rho) \leq C\varepsilon_0 \rho^{n-1}.$$

Let us denote by $\Gamma_0(\rho)$ the element of largest $\mathcal{H}^n$ measure among the $\Gamma_i(\rho)$: applying the isoperimetric inequality [DGM14, Lemma 9] in ∂B_ρ we get

$$\sigma_n \rho^n - \mathcal{H}^n(\Gamma_0(\rho)) = \sum_{i \geq 1} \mathcal{H}^n(\Gamma_i(\rho)) \leq C\left(\mathcal{H}^{n-1}(\mathcal{R} \cap \partial B_\rho)\right)^{\frac{n}{n-1}} \leq C\varepsilon_0^{\frac{n}{n-1}} \rho^n,$$

namely $\mathcal{H}^n(\Gamma_0(\rho)) \geq (\sigma_n - C\varepsilon_0^{\frac{n}{n-1}})\rho^n$, which proves (3.7).

Step 2. In this second step we let $\Gamma_0^j(\rho)$ be a $\sim_{K_j}$-equivalence class of largest $\mathcal{H}^n$-measure in $\partial B_\rho \setminus K_j$ and we claim that:

$$\liminf_j \mathcal{H}^n(\Gamma_0^j(\rho)) \geq \mathcal{H}^n(\Gamma_0(\rho)), \tag{3.9}$$

(where, consistently, $\Gamma_0(\rho)$ is a $\sim_K$-equivalence class of largest measure in $\partial B_\rho \setminus K$; note that the latter estimate, combined with Step 1, implies, for ε_0 sufficiently small and j sufficiently large, that such equivalence classes of largest $\mathcal{H}^n$ measure are indeed uniquely determined).

Recall that $\Omega_0(\rho)$ is associated to $\Gamma_0(\rho)$ according to Definition 2.2. Let us consider $\delta > 0$ sufficiently small and $\bar{\Gamma} \subset\subset \Gamma_0(\rho)$ verifying

$$\mathcal{H}^n(\bar{\Gamma}) \geq \mathcal{H}^n(\Gamma_0(\rho)) - \delta\,. \tag{3.10}$$

Next, by compactness, we can uniformly separate $\bar\Gamma$ and K, that is we can pick $\eta > 0$ sufficiently small so that

$$V := \bigcup_{s\in[\rho-\eta,\rho]} \frac{s}{\rho}\bar\Gamma = \left\{x \in \overline{B_\rho} \setminus B_{\rho-\eta} : \rho\frac{x}{|x|} \in \bar\Gamma\right\} \subset\subset \Omega_0(\rho). \tag{3.11}$$

Next we choose an open connected subset of $\Omega_0(\rho)$ with smooth boundary, denoted by $\Omega(\rho)$, such that

$$|\Omega_0(\rho) \setminus \Omega(\rho)| < \delta\eta. \tag{3.12}$$

The set $\Omega(\rho)$ can be constructed as follows:

- first one considers $\Lambda \subset\subset \Omega_0(\rho)$ compact with $|\Omega_0(\rho) \setminus \Lambda| < \delta\eta$: this can be achieved for instance looking at a Whitney subdivision of $\Omega_0(\rho)$, taking the union of the cubes with side length bounded from below by a small number;
- Λ can be enlarged to become connected by adding, if needed, a finite number of arcs at positive distance from $\partial\Omega_0(\rho)$;
- we can finally take a C^∞ function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $f|_{\mathbb{R}^{n+1}\setminus\Omega_0(\rho)} = 0$, $f|_\Lambda = 1$ and $0 \le f \le 1$: by the Morse-Sard Theorem, one can choose $t \in]0, 1[$ such that $\{f = t\}$ is a C^∞ submanifold.
- the connected component of $\{f > t\}$ containing Λ satisfies the required assumptions.

Since $\Omega(\rho) \subset\subset \Omega_0(\rho)$, by weak convergence $\mathcal{H}^n(K_j \cap \Omega(\rho)) \to 0$; moreover since $\Omega(\rho)$ is smooth and connected, it satisfies the isoperimetric inequality (see [Mag12, Remark 12.39])

$$|\Omega(\rho) \setminus \Omega^j(\rho)| \le \mathrm{Iso}(\Omega(\rho))\mathcal{H}^n(\Omega(\rho) \cap K_j)^{\frac{n+1}{n}} \to_j 0, \tag{3.13}$$

where $\Omega^j(\rho)$ is the connected component of $\Omega(\rho) \setminus K_j$ of largest volume and $\mathrm{Iso}(\Omega(\rho))$ is the isoperimetric constant of the smooth connected domain $\Omega(\rho)$.

Obviously (3.11) implies $\mathcal{H}^n(K_j \cap V) \to 0$ and, by projecting $K_j \cap V$ on ∂B_ρ via the radial map $\Pi : B_\rho \ni x \mapsto \frac{\rho}{|x|}x \in \partial B_\rho$, we easily get that the set

$$\bar\Gamma^j := \{y \in \bar\Gamma : \Pi^{-1}(y) \cap V \cap K_j = \varnothing\}$$

verifies

$$\begin{aligned}\mathcal{H}^n(\bar\Gamma^j) &= \mathcal{H}^n(\bar\Gamma) - \mathcal{H}^n(\Pi(K_j \cap V))\\ &\ge \mathcal{H}^n(\bar\Gamma) - (\mathrm{Lip}\,\Pi|_{B_\rho\setminus B_{\rho-\eta}})^n\mathcal{H}^n(K_j \cap V) = \mathcal{H}^n(\bar\Gamma) - o_j(1).\end{aligned} \tag{3.14}$$

We deduce from (3.14) that

$$|V \cap \Pi^{-1}(\bar\Gamma^j)| = |V| - |V \setminus \Pi^{-1}(\bar\Gamma^j)| \ge |V| - \eta\mathcal{H}^n(\bar\Gamma \setminus \bar\Gamma^j) \ge |V| - o_j(1). \tag{3.15}$$

The previous inequality, (3.11), (3.12) and (3.13) in turn imply that

$$\begin{aligned}|V \setminus (\Omega^j(\rho) \cap \Pi^{-1}(\bar\Gamma^j))| &\le |V \setminus \Omega^j(\rho)| + |V \setminus \Pi^{-1}(\bar\Gamma^j)|\\ &\overset{(3.11),(3.15)}{\le} |\Omega_0(\rho) \setminus \Omega(\rho)| + |\Omega(\rho) \setminus \Omega^j(\rho)| + o_j(1)\\ &\overset{(3.12),(3.13)}{\le} \eta\delta + o_j(1).\end{aligned} \tag{3.16}$$

If $x \in V \cap \Omega^j(\rho) \cap \Pi^{-1}(\bar\Gamma^j)$, then $x \sim_{K_j} \Pi(x)$ using as a path simply the radial segment $[x, \Pi(x)[$; moreover we can always connect two points belonging to $V \cap \Omega^j(\rho) \cap \Pi^{-1}(\bar\Gamma^j)$ with a path inside $\Omega^j(\rho)$. But, by (3.16), the endpoints $\Pi(x)$ of these segments must cover all but a small fraction G_j of $\bar\Gamma^j$ of measure $o_j(1)$. Indeed we can estimate the complement set

$G_j := \Pi(V \setminus (\Omega^j(\rho) \cap \Pi^{-1}(\bar\Gamma^j)))$ using the coarea formula and the self similarity of the shells:

$$\frac{\rho}{n+1}\left(1-\left(1-\frac{\eta}{\rho}\right)^{n+1}\right)\mathcal{H}^n(G_j) = \int_{\rho-\eta}^{\rho} \mathcal{H}^n\left(\frac{t}{\rho}G_j\right)dt \le |V \setminus (\Omega^j(\rho) \cap \Pi^{-1}(\bar\Gamma^j))| \le \delta\eta + o_j(1),$$

which yields, for η small enough,

$$\mathcal{H}^n(G_j) \le 2\delta + \frac{o_j(1)}{\eta}. \tag{3.17}$$

By concatenating the paths we conclude that $\bar\Gamma^j \setminus G_j$ must be contained in a unique equivalence class $\Gamma^j_i(\rho)$. We remark that for the moment we do not know whether $\Gamma^j_i(\rho)$ is an equivalence class of $\partial B_\rho \setminus K_j$ with largest measure. Summarizing the inequalities achieved so far we conclude

$$\begin{aligned}\mathcal{H}^n(\Gamma^j_0(\rho)) \ge \mathcal{H}^n(\Gamma^j_i(\rho)) &\ge \mathcal{H}^n(\bar\Gamma^j \setminus G_j) \overset{(3.17)}{\ge} \mathcal{H}^n(\bar\Gamma^j) - 2\delta - \frac{o_j(1)}{\eta}\\ &\overset{(3.14)}{\ge} \mathcal{H}^n(\bar\Gamma) - \frac{o_j(1)}{\eta} - 2\delta \overset{(3.10)}{\ge} \mathcal{H}^n(\Gamma_0(\rho)) - \frac{o_j(1)}{\eta} - 3\delta\end{aligned}$$

In particular, letting first $j \uparrow \infty$ and then $\delta \downarrow 0$ we achieve (3.9).

Step 3. We recover a straightforward contradiction since, by the density lower bound (d.l.b.) proven in Lemma 3.1, the good class property (g.c.p.) of $\mathcal{P}(H)$, the lower semicontinuity (l.s.) in the weak convergence (3.3) and the bound (2.1), we get

$$\begin{aligned}c_0\rho^n &\overset{d.l.b.}{\le} \mu(B_{x,\rho}) \overset{l.s.}{\le} \liminf_j \mathbf{F}(K_j, B_{x,\rho})\\ &\overset{g.c.p.}{\le} \liminf_j \mathbf{F}(\partial B_{x,\rho} \setminus \Gamma^j_0(\rho)) \overset{(2.1)}{\le} \Lambda \liminf_j \mathcal{H}^n(\partial B_{x,\rho} \setminus \Gamma^j_0(\rho)).\end{aligned}$$

Plugging in (3.9) and (3.7) (both relative to the complementary sets), we get

$$c_0\rho^n \overset{(3.9)}{\le} \Lambda\mathcal{H}^n(\partial B_{x,\rho} \setminus \Gamma_0(\rho)) \overset{(3.7)}{\le} \Lambda\varepsilon_0\rho^n,$$

which is false for ε_0 small enough. We conclude $\mathcal{H}^n(\mathcal{N}) = 0$, hence the rectifiability of the set K. Finally, the bounds (2.1) are enough to deduce $\theta(x) \ge \lambda$ at every $x \in K$ where the approximate tangent space to K exists: the proof is analogous to Step 3 in the proof of [DGM14].

3.3. **Proof of Theorem 2.8: semicontinuity.** In order to prove the second part of Theorem 2.8, we need the following estimates.

Lemma 3.2. *Let K be the n-rectifiable set obtained in the previous section. At almost every $x \in K$ the following holds: for every $\varepsilon > 0$ there exist $r_0 = r_0(x) \le \frac{1}{\sqrt{n+1}}\operatorname{dist}(x,H)$ such that, for $r \le r_0/2$,*

$$(\lambda\omega_n - \varepsilon)r^n \le \mu(B_{x,r}) \le (\theta(x)\omega_n + \varepsilon)r^n, \qquad (\theta(x) - \varepsilon)r^n < \mu(Q_{x,r}) < (\theta(x) + \varepsilon)r^n, \tag{3.18}$$

$$\sup_{y \in B_{x,r_0},\, S \in G(n,n+1)} |F(y,S) - F(x,S)| \le \varepsilon. \tag{3.19}$$

Moreover, for almost every such r, there exists $j_0(r) \in \mathbb{N}$ such that for every $j \ge j_0$:

$$(\theta(x)\omega_n - \varepsilon)r^n \le \mathbf{F}(K_j, B_{x,r}) \le (\theta(x)\omega_n + \varepsilon)r^n, \tag{3.20}$$

$$(\theta(x) - \varepsilon)r^n \le \mathbf{F}(K_j, Q_{x,r}) \le (\theta(x) + \varepsilon)r^n, \qquad \mathbf{F}(K_j, Q_{x,r} \setminus R_{x,r,\varepsilon r}) < \varepsilon r^n. \tag{3.21}$$

Proof. Fix a point x where $T_K(x)$ exists and θ is approximately continuous: for the sake of simplicity, we can assume $x = 0$ and that, after a rotation, the approximate tangent space at 0 coincides with $T_K = \{x^{n+1} = 0\}$. For almost every $r \leq r_0/2$ we can suppose that $\mu(\partial B_r) = \mu(\partial Q_r) = \mu(\partial R_{r,\varepsilon r}) = 0$; moreover by rectifiability and the density lower bound (3.1), we also know that $B_r \cap K \subset U_{\varepsilon r}(T_K)$ (for the proof see [DGM14, eq 2.11]). The second equation in (3.21) follows than by weak convergence. We also know that, up to further reducing r_0, for $r \leq r_0/2$, (3.4) and [AFP00, Theorem 2.83] imply that

$$(\theta(0)\omega_n - \varepsilon)r^n < \mu(B_r) < (\theta(0)\omega_n + \varepsilon)r^n, \tag{3.22}$$

$$(\theta(0) - \varepsilon)r^n < \mu(Q_r) < (\theta(0) + \varepsilon)r^n. \tag{3.23}$$

Again by weak convergence, we recover (3.20) and the first equation in (3.21). Moreover (3.19) is a consequence of (2.8). Finally (3.18) follows from (3.22), (3.23) and $\theta \geq \lambda$ (proved in the Step 2 of the proof of Theorem 2.8). □

We are now ready to complete the proof of Theorem 2.8, namely to show $\liminf_j \mathbf{F}(K_j) \geq \mathbf{F}(K)$ and $\mu = F(x, T_K(x))\mathcal{H}^n \llcorner K$, when the integrand F is elliptic and the class $\mathcal{P}(H)$ a deformation class (in addition to being a good class).
For the first claim, it will be enough to show $\theta(x) \geq F(x, T_K)$, where $x \in K$ satisfies the properties of Lemma 3.2. Assume w.l.o.g. $x = 0$. Let us fix $\varepsilon < r_0/2$ and choose a radius r such that both r and $(1 - \sqrt{\varepsilon})r$ satisfy properties (3.18)-(3.21): in order to apply the ellipticity of F, we need a map $P \in \mathsf{D}(0, r)$ deforming the set $K_j \cap \partial B_r$ to $T_K \cap \partial B_r$. We can build P as follows: consider a map S satisfying

- $S = Id$ in $\overline{B}_{1-\sqrt{\varepsilon}} \cup (\mathbb{R}^{n+1} \setminus \overline{B}_{1+\sqrt{\varepsilon}})$,
- $S(\partial B \cap U_\varepsilon(T_K)) = \partial B \cap T_K$,
- S stretches $\partial B \setminus U_\varepsilon(T_K)$ onto $\partial B \setminus T_K$.

It is not hard to construct an extension $S \in \mathsf{D}(0, 1)$ fulfilling the previous requirements and such that $S|_{B_{1+\sqrt{\varepsilon}} \setminus B}$ and $S|_{B \setminus B_{1-\sqrt{\varepsilon}}}$ are interpolations between the values of S on the three spheres $S|_{\partial B_{1+\sqrt{\varepsilon}}}$, $S|_{\partial B}$ and $S|_{B \setminus B_{1-\sqrt{\varepsilon}}}$. One can also assume that $\|S - Id\|_\infty + \mathrm{Lip}\,(S - Id) \leq C\sqrt{\varepsilon}$ and then obtain the desired map by rescaling $P(\cdot) = rS(\frac{\cdot}{r})$.

We set $K'_j := P(K_j)$ and since $K_j \cap B_{(1-\sqrt{\varepsilon})r} = K'_j \cap B_{(1-\sqrt{\varepsilon})r}$, using Remark 2.7 and property (3.20), we estimate:

$$\begin{aligned}
\mathbf{F}(K_j, B_r) &\geq \mathbf{F}\left(K'_j, B_{(1-\sqrt{\varepsilon})r}\right) \geq \mathbf{F}\left(K'_j, B_r\right) - \mathbf{F}\left(K'_j, B_r \setminus B_{(1-\sqrt{\varepsilon})r}\right) \\
&\geq \mathbf{F}(K'_j, B_r) - \frac{\Lambda}{\lambda}\mathrm{Lip}\,(P)^n \mathbf{F}\left(K_j, B_r \setminus B_{(1-\sqrt{\varepsilon})r}\right) \\
&= \mathbf{F}(K'_j, B_r) - \frac{\Lambda}{\lambda}\mathrm{Lip}\,(P)^n \left(\mathbf{F}(K_j, B_r) - \mathbf{F}\left(K_j, B_{(1-\sqrt{\varepsilon})r}\right)\right) \\
&\geq \mathbf{F}(K'_j, B_r) - \frac{\Lambda}{\lambda}\mathrm{Lip}\,(P)^n(\theta(0)\omega_n + \varepsilon)r^n + \frac{\Lambda}{\lambda}\mathrm{Lip}\,(P)^n(\theta(0)\omega_n - \varepsilon)(1 - \sqrt{\varepsilon})^n r^n \\
&\geq \mathbf{F}(K'_j, B_r) - C\sqrt{\varepsilon}r^n.
\end{aligned} \tag{3.24}$$

We furthermore observe that $K'_j \cap B_r$ cannot be deformed via any map $Q \in \mathsf{D}(0, r)$ onto $\partial B_r \cap T_K$. Otherwise, being $\mathcal{P}(H)$ a deformation class, there would exist a competitor $J_j \in \mathcal{P}(H)$, εr^n-close in energy to $K''_j := Q(P(K_j))$, with $K''_j \cap B_r = \emptyset$. Since $K''_j \cap (\mathbb{R}^{n+1} \setminus B_{(1+\sqrt{\varepsilon})r}) =$

$K_j \cap (\mathbb{R}^{n+1} \setminus B_{(1+\sqrt{\varepsilon})r})$, using (3.18), (3.22) and equation (2.7), we would get:

$$\begin{aligned}\mathbf{F}(K_j) - \mathbf{F}(J_j) &\geq \mathbf{F}\left(K_j, B_{(1+\sqrt{\varepsilon})r}\right) - \mathbf{F}\left(K_j'', B_{(1+\sqrt{\varepsilon})r}\right) - \varepsilon r^n \\ &\geq \mathbf{F}(K_j, B_r) + \mathbf{F}\left(K_j, B_{(1+\sqrt{\varepsilon})r} \setminus B_r\right) - \mathbf{F}\left(K_j', B_{(1+\sqrt{\varepsilon})r} \setminus B_r\right) - \varepsilon r^n \\ &\geq (\lambda\omega_n - \varepsilon)r^n + (\theta(0)\omega_n + \varepsilon)((1+\sqrt{\varepsilon})^n - 1)\left(1 - \frac{\Lambda}{\lambda}\mathrm{Lip}\,(P)^n\right) r^n - \varepsilon r^n \\ &\geq (\lambda\omega_n - C\sqrt{\varepsilon})r^n > 0,\end{aligned}$$

which contradicts the minimizing property of the sequence $\{K_j\}$ if ε is small enough. Therefore, the ellipticity of F, (3.19), (3.20), (2.7) and (3.1) imply that

$$\mathbf{F}^0(T_K, B_r) \leq \mathbf{F}^0(K_j', B_r) \leq \mathbf{F}(K_j', B_r) + C\varepsilon r^n. \tag{3.25}$$

We can now sum up as follows

$$\begin{aligned}\theta(0)\omega_n r^n &\overset{(3.20)}{\geq} \mathbf{F}(K_j, B_r) - \varepsilon r^n \overset{(3.24)}{\geq} \mathbf{F}(K_j', B_r) - C\sqrt{\varepsilon} r^n \\ &\overset{(3.25)}{\geq} \mathbf{F}^0(T_K, B_r) - C\sqrt{\varepsilon} r^n = F(0, T_K)\omega_n r^n - C\sqrt{\varepsilon} r^n\end{aligned}$$

which easily implies the desired inequality $\theta(0) \geq F(0, T_K)$.

To get the last claim, we prove that $\theta(x) \leq F(x, T_K(x))$ for almost every $x \in K$ (again in the setting of Lemma 3.2). Take w.l.o.g. $x = 0$. Arguing by contradiction, we assume that $\theta(0) = F(0, T_K(0)) + \sigma$ for some $\sigma > 0$ and let $\varepsilon < \min\{\frac{\sigma}{2}, \frac{\lambda\sigma}{4\Lambda}\}$. As a consequence of (3.21), there exist r and $j_0 = j_0(r)$ such that

$$\mathbf{F}(K_j, Q_r) > \left(F(0, T) + \frac{\sigma}{2}\right) r^n, \qquad \mathbf{F}(K_j, Q_r \setminus R_{r,\varepsilon r}) < \frac{\lambda\sigma}{4\Lambda} r^n, \qquad \forall j \geq j_0. \tag{3.26}$$

Consider the map $P \in \mathsf{D}(0, r)$ defined in [DPDRG15, Equation 3.14] which collapses $R_{r(1-\sqrt{\varepsilon}),\varepsilon r}$ onto the tangent plane T_K and satisfies $\|P - Id\|_\infty + \mathrm{Lip}\,(P - Id) \leq C\sqrt{\varepsilon}$. Exploiting the fact that $\mathcal{P}(H)$ is a deformation class and by almost minimality of K_j, we find that

$$\begin{aligned}\mathbf{F}(K_j, Q_r) - o_j(1) \leq{}& \underbrace{\mathbf{F}(P(K_j), P(R_{(1-\sqrt{\varepsilon})r,\varepsilon r}))}_{I_1} + \underbrace{\mathbf{F}(P(K_j), P(R_{r,\varepsilon r} \setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r}))}_{I_2} \\ &+ \underbrace{\mathbf{F}(P(K_j), P(Q_r \setminus R_{r,\varepsilon r}))}_{I_3}.\end{aligned}$$

By the properties of P and (3.19), we get $I_1 \leq (F(0, T_K) + \varepsilon)r^n$, while, by (3.26) and equation (2.7)

$$I_3 \leq \frac{\Lambda}{\lambda}(\mathrm{Lip}\, P)^n\, \mathbf{F}(K_j, Q_r \setminus R_{r,\varepsilon r}) < (1 + C\sqrt{\varepsilon})^n \frac{\sigma}{4} r^n.$$

Since $\mathbf{F}(P(K_j), P(R_{r,\varepsilon r} \setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r})) \leq \frac{\Lambda}{\lambda}(1 + C\sqrt{\varepsilon})^n \mathbf{F}(K_j, R_{r,\varepsilon r} \setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r})$ and $R_{r,\varepsilon r} \setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r} \subset Q_{(1-\sqrt{\varepsilon})r} \setminus Q_r$, by (3.21) we can also bound

$$\begin{aligned}I_2 = \mathbf{F}(P(K_j), P(R_{r,\varepsilon r} \setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r})) &\leq \frac{\Lambda}{\lambda}(1 + C\sqrt{\varepsilon})^n \mathbf{F}(K_j, Q_r \setminus Q_{(1-\sqrt{\varepsilon})r}) \\ &\leq C(1 + C\sqrt{\varepsilon})^n \Big((F(0, T_K) + \sigma + \varepsilon) - (F(0, T_K) + \sigma - \varepsilon)(1 - \sqrt{\varepsilon})^n\Big) r^n \leq C\sqrt{\varepsilon} r^n.\end{aligned}$$

Hence, as $j \to \infty$, by (3.18)

$$\left(F(0, T_K) + \frac{\sigma}{2}\right) r^n \leq (F(0, T_K) + \varepsilon)r^n + C\sqrt{\varepsilon} r^n + (1 + C\sqrt{\varepsilon})^n \frac{\sigma}{4} r^n \;:$$

dividing by r^n and letting $\varepsilon \downarrow 0$ provides the desired contradiction.

We obtain that $\theta(x) \le F(x, T_K(x))$ almost everywhere and, together with the previous step, $\mu = F(x, T_K(x))\mathcal{H}^n \llcorner K$.

4. Proof of Theorem 2.10

Most of the proof of Theorem 2.10 relies on the following elementary geometric remark.

Lemma 4.1. *If $K \in \mathcal{F}(H, \mathcal{C})$, $B_{x,r} \subset\subset \mathbb{R}^{n+1} \setminus H$, and $\gamma \in \mathcal{C}$, then either $\gamma \cap (K \setminus B_{x,r}) \neq \emptyset$, or there exists a connected component σ of $\gamma \cap \overline{B_{x,r}}$ which is homeomorphic to an interval and whose end-points belong to two distinct equivalence classes $\Gamma_i(x,r)$'s of $\partial B_{x,r} \setminus K$ in the sense of Definition 2.2.*

Proof of Lemma 4.1. The proof is analogous to [DGM14, Lemma 10], we briefly sketch the argument to highlight the main differences. Assuming that γ and $\partial B_{x,r}$ intersect transversally (see Step 2 in the proof of [DGM14, Lemma 10] for the reduction to this case), we can find finitely many mutually disjoint closed arcs $I_i \subset S^1$, $I_i = [a_i, b_i]$, such that $\gamma \cap B_{x,r} = \bigcup_i \gamma((a_i, b_i))$ and $\gamma \cap \partial B_{x,r} = \bigcup_i \{\gamma(a_i), \gamma(b_i)\}$. Arguing by contradiction we may assume that for every i there exists an equivalence class $\Gamma_i(x,r)$ of $\partial B_{x,r} \setminus K$ such that $\gamma(a_i), \gamma(b_i) \in \Gamma_i(x,r)$. By connectedness of the associated $\Omega_i(x,r)$ (see the discussion after Definition 2.2) and the definition of $\Gamma_i(x,r)$, for each i we can find a smooth embedding $\tau_i : I_i \to \Omega_i(x,r) \cup \Gamma_i(x,r)$ such that $\tau_i(a_i) = \gamma(a_i)$ and $\tau_i(b_i) = \gamma(b_i)$; moreover since $n \ge 2$, one can easily achieve this by enforcing $\tau_i(I_i) \cap \tau_j(I_j) = \emptyset$. Finally, we define $\bar\gamma$ by setting $\bar\gamma = \gamma$ on $S^1 \setminus \bigcup_i I_i$, and $\bar\gamma = \tau_i$ on I_i. In this way, $[\bar\gamma] = [\gamma]$ in $\pi_1(\mathbb{R}^{n+1} \setminus H)$, with $\bar\gamma \cap K \setminus \overline{B_{x,r}} = \gamma \cap K \setminus \overline{B_{x,r}} = \emptyset$ and $\bar\gamma \cap K \cap \overline{B_{x,r}} = \emptyset$ by construction; that is, $\bar\gamma \cap K = \emptyset$. Since there exists $\widetilde\gamma \in \mathcal{C}_H$ with $[\widetilde\gamma] = [\bar\gamma] = [\gamma]$ in $\pi_1(\mathbb{R}^{n+1} \setminus H)$ which is uniformly close to $\bar\gamma$, we infer $\widetilde\gamma \cap K = \emptyset$, and thus find a contradiction to $K \in \mathcal{F}(H, \mathcal{C})$. □

Proof of Theorem 2.10. (a): We start showing that $\mathcal{F}(H, \mathcal{C})$ is a good class in the sense of Definition 2.2 and moreover a deformation class in the sense of Definition 2.5. To this end, we fix $V \in \mathcal{F}(H, \mathcal{C})$ and $x \in V$, and prove that a.e. $r \in (0, \mathrm{dist}(x, H))$ one has $V' \in \mathcal{F}(H, \mathcal{C})$, where V' is a cup competitor of V in $B_{x,r}$. We thus fix $\gamma \in \mathcal{C}$ and, without loss of generality, we assume that $\gamma \cap (V \setminus B_{x,r}) = \emptyset$. By Lemma 4.1, γ has an arc contained in $\overline{B_{x,r}}$ homeomorphic to $[0,1]$ and whose end-points belong to distinct equivalence classes of $\partial B_{x,r} \setminus V$; we denote by $\sigma : [0,1] \to \overline{B_{x,r}}$ a parametrization of this arc. Since V' must contain all but one $\Gamma_i(x,r)$, either $\sigma(0)$ or $\sigma(1)$ belongs to $\gamma \cap V' \cap \partial B_{x,r}$. This proves that $V' \in \mathcal{F}(H, \mathcal{C})$. $\mathcal{F}(H, \mathcal{C})$ is also a deformation class and the proof is exactly the same as in [DPDRG15, Theorem 1.5].

(b): The second statement of the Theorem, namely that $K \in \mathcal{F}(H, \mathcal{C})$, has exactly the same proof as in [DGM14, Theorem 4(b)]. It follows that all the results of Theorem 2.8 apply.

(c): We observe that K is a $(\mathbf{F}, 0, \infty)$-minimal set, i.e.

$$\mathbf{F}(K) \le \mathbf{F}(\varphi(K))$$

whenever $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a Lipschitz map such that $\varphi = \mathrm{Id}$ on $\mathbb{R}^{n+1} \setminus B_{x,r}$ and $\varphi(B_{x,r}) \subset B_{x,r}$ for some $x \in \mathbb{R}^{n+1} \setminus H$ and $r < \mathrm{dist}(x, H)$.

The above inequality is a consequence of $\varphi(K) \in \mathcal{F}(H, \mathcal{C})$, which can be proved via degree theory as in [DGM14, Theorem 4, Step 3]. □

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