On the Robustness of Consumption-Based Asset Pricing Theory

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Preface

Standard financial economic theory has increasingly been criticized since the advent of the most recent financial crisis. Obviously it was not designed to address some of the most important issues in finance. Many sceptics even believe that there is a need for a radical overhaul, since the theory is built on clearly unrealistic assumptions and struggles to explain various empirical facts.

Do we truly need a completely new theory? What exactly does the existing theory already explain? Which of its features should we keep? One might be led to suspect that the shortcomings of the theory are evident and to focus more on the search for improvements. However, sometimes it is difficult to know right from wrong and tricky to identify the problematic parts of a model. Seemingly problematic assumptions might still be justified and seemingly unproblematic approaches might come with some hidden pitfalls. This thesis contributes to the discussion by presenting three new at first sight surprising results.

The representative agent is an example of an assumption that has been harshly criticized ever since its introduction.[1] It nonetheless seems to be a widely accepted framework to study asset prices and the most prominent models in consumption-based asset pricing build on this assumption.[2] Thus, while many researchers suggest completely abandoning the representative agent, there is still a big community believing in it. Though one has to say that even advocates of the representative agent disagree on

[1] For different points of criticism see Hartley (1996) and Kirman (1992).

[2] See Barro (2006), Bansal and Yaron (2004) and Campbell and Cochrane (1999).

how the objective function should look like: at the center of the debate lies the question whether the representative agent should be perfectly rational or exhibit behavioral traits.

From a theoretical point of view, the representative-agent approach is hard to justify and supportive analytical results require quite restrictive assumptions.[3] Chapter 2 looks at the representative agent from a different angle and takes the discussion to a new theoretical framework called evolutionary finance. Evolutionary finance is a relatively young field of theoretical finance and takes up concepts from evolutionary biology. Financial markets are regarded as an environment where different populations of investment strategies compete for capital. The aim is to gain a better understanding of population dynamics and their impact on financial markets. An evolutionary approach to finance opens up new possibilities for analyzing and interpreting the representative agent.

Chapter 2 employs an evolutionary finance model in the sense of Evstigneev et al. (2016). All agents are assumed to invest according to simple rebalancing rules and to consume the same fraction of wealth. In this simple model, aggregate consumption alone already fails to explain asset prices. Suppose that there are two investors and two assets. Investor 1 invests a higher percentage of wealth into asset A, whereas investor 2 invests a higher percentage of wealth into asset B. Without knowing the wealth distribution between the two investors we would not even know whether the aggregate demand for asset A or B is higher and which asset sells at a higher price.

The search for a representative agent that does not account for the wealth distribution seems pointless in our model. Chapter 2 argues that the definition of a representative agent might still make sense depending on what we intend to explain. There might be no representative agent explaining actual price levels, but one explaining other interesting features of asset prices. In fact, the setup in Chapter 2 allows specifying a representative agent that generates relative prices to which actual relative asset prices tend to in expectation. The respective representative agent maximizes a logarithmic

[3] See for example Rubinstein (1974) and Constantinides (1982).

utility function no matter which investment strategies are present in the market and how much wealth is initially allocated to them.

How come that the relative asset prices tend towards the same relative prices in expectation for any set of rebalancing rules? The intuition is as follows. The influence of an individual on prices depends on an individual's wealth. While investment strategies that drive up the price of overpriced assets suffer from a low dividend yield, strategies investing into underpriced assets benefit from a high dividend yield. The relative wealth and thus price influence of the former strategies therefore decreases. This leads to a slight correction in prices, which further decreases their price influence. All prices therefore tend to move towards a level where the expected returns on all assets corrected for the impact of consumption on wealth are the same. The logarithmic utility function of the representative agent is a consequence of the assumption that investors consume a constant fraction of wealth.

To summarize, Chapter 2 shows how evolutionary finance can provide new tools to reassess and tackle aggregation. The evolutionary approach focuses on identifying and studying the impact of different evolutionary forces. In Chapter 2, one particular market stabilizing force is identified which favors the meaningful formulation of a representative agent. It should serve as a motivation for further research to search for other evolutionary forces and to study their joint effects in order to gain a better understanding of the applicability and functional form of the representative agent.[4]

Chapter 3 refers to another debatable practice in consumption-based asset pricing. Often there are approximations needed to obtain tractable forms in the hope that the error is negligibly small. Though, sometimes even arbitrarily small terms can contain useful information. Chapter 3 provides such an example and discusses the capital asset pricing model (CAPM) as an approximation to the consumption capital asset pricing model (CCAPM) in case of power utility and lognormally distributed dividends. This approximation is quite common because there are no closed-form solutions available for

[4]For example an opposing evolutionary force has been identified by Brock and Hommes (1998). They find that the presence of investors that switch between different investment strategies can lead to chaotic asset price dynamics.

this specification of the CCAPM. Another reason for its widespread use is that a linear pricing rule like the CAPM was important for estimation before the development of the generalized method of moments.[5] Campbell and Viceira (2002) as well as Herings and Kübler (2007) show that the differences between the CAPM and the CCAPM are in fact small which partially justifies the approximation. Chapter 3 takes the view that even though the differences are small, they still deserve some attention: They might shed some light on possible reasons for the appearance of CAPM-anomalies like the low-beta, the value and the small-size premium.[6]

Chapter 3 studies a special case of the CCAPM with power utility and lognormally distributed dividends. An additional distributional assumption, which is motivated by empirical evidence, allows to solve the model in closed form.[7] This makes a deeper analysis of the differences between the CAPM and the CCAPM possible. Chapter 3 shows that using the CAPM as an approximation to the CCAPM leads to a low-beta, a value and a small-size premium. A closer look at the pricing error of the CAPM reveals that these premia increase with the coefficient of relative risk aversion and with consumption volatility. The similarities between cross-sectional puzzles and the equity premium puzzle[8] are striking. Both are predicted by the CCAPM with power utility and lognormal dividends, but the predictions are too small in size. Moreover, the size of both increase with risk aversion and consumption risk.

The insights of Chapter 3 may help in the search for explanations of cross-sectional anomalies. Knowing more about the nature of a problem shapes the way we think about possible solutions. Take for example the equity premium puzzle: The fact that the high equity premium is merely a quantitative puzzle and that its size crucially depends on the coefficient of relative risk aversion and consumption risk has led the research community to focus mainly on explanations based on alternative preferences

[5] See Cochrane (2001).

[6] These three CAPM-anomalies were first discovered by Black et al. (1972), Basu (1977) and Banz (1981).

[7] We assume that dividends are lognormally distributed in the cross-section as empirically documented by Elmiger (2010).

[8] See Mehra and Prescott (1985).

and additional consumption risk.[9] The results in Chapter 3 motivate taking a similar approach to cross-sectional anomalies and searching for a joint explanation to the equity premium puzzle and cross-sectional puzzles.

Chapter 4 takes up the literature on rare disasters. Rare disasters may be an explanation for the high equity premium as well as various other asset pricing puzzles.[10] The term rare disasters refers to large drops in consumption that occur only around once in a hundred years for example due to war. The literature on rare disasters argues that the timespan of currently available consumption data is too short to properly reflect the risk for such events. Thus, consumption risk is actually much higher than suggested by data and justifies the high equity premium.

The effect of rare disasters impressively demonstrates how sensitive this type of model reacts to distributional assumptions. A once-in-a-century event can have a big impact on the model outcome. In fact, incorporating rare disasters not only changes quantitative results, but can even alter qualitative results of the model. Chapter 4 shows such an example and discusses the predictability of stock returns with the labor income to consumption ratio: A change in the risk for a rare disaster can flip the sign of the stock return predictability.[11]

What is the intuition behind this result? Let us start with the case of no or small disaster risk. In this case, there is a negative relation between the labor income to consumption ratio and expected stock returns. The reason is that labor income is relatively stable compared to capital income. When the labor income to consumption ratio is high, there is only little covariation between stock returns and consumption. Therefore the required return on stocks to compensate for risk is low. In contrast, stock returns and the labor income to consumption ratio become positively related when the risk for a rare disaster is high. Labor income in this case becomes more risky, which

[9] For prominent such approaches see for example Barro (2006), Bansal and Yaron (2004) and Campbell and Cochrane (1999).

[10] See for example Rietz (1988), Barro (2006) and Gabaix (2012).

[11] The given example is motivated by the empirical findings of Santos and Veronesi (2006) and Chen and Joslin (2012). They observe a negative relation between stock returns and the labor income to consumption ratio prior to the nineties and a positive relation afterwards.

leads to a diversification effect. If the labor income to consumption ratio is already high, there is less diversification in total income as the labor income to consumption ratio increases. As a consequence, the required return on stocks decreases.

Distributional assumptions can undoubtedly have unexpected and far-reaching consequences for consumption-based asset pricing results. An appropriately accurate estimation of the underlying distributions is therefore crucial. This poses a particular challenge to some currently suggested resolutions for the high equity premium as well as other asset pricing puzzles, since they heavily rely on difficult to verify distributional assumptions.

2

A Heterogeneous-Agent Foundation of the Representative-Agent Approach

The stochastic discount factor is a key concept in the equilibrium asset pricing literature. It allows pricing assets solely by the contribution of their cash flows to aggregate consumption growth. A broad class of asset pricing models based on the stochastic discount factor approach are able to replicate various stylized facts about asset prices.[12] Apparently this approach achieves some empirical successes, even though the theoretical motivation for the full determination of the stochastic discount factor by macroeonomic variables relies on very restrictive assumptions. Either one directly assumes that asset prices are generated by a single representative agent maximizing utility on an aggregate budget set or one assumes that markets are complete and all agents have perfect foresight (see Constantinides (1982)). This chapter provides a theoretical foundation for the use of a representative agent based on a heterogeneous-agent model with not necessarily complete markets and market participants that have no knowledge about the future. We analyze a financial market model in discrete time where agents invest according to simple rebalancing rules. Asset prices therefore depend on the distribution of wealth across individual agents and cannot solely be explained by their cash flows and macroeconomic aggregates in general. However, the direction in which relative asset prices are expected to move in the next period conditional on past information can be determined using a stochastic discount factor generated by a simple representative

[12] For example habit-formation (see Campbell and Cochrane (1999)), long-run risk (see Bansal and Yaron (2004)) and rare disaster models (see Barro (2006)).

agent. We prove analytically that all relative asset prices are expected to move in this direction in the case of two types of agents and two assets. Our simulation results suggest that this property carries over for the majority of assets in the case of more than two types of agents and more than two assets.

The special case with two types of agents and two assets illustrates the intuition behind the results. In the model, all agents have a price impact and asset prices are driven by the relative wealth dynamics. Suppose that one type of agents has a very high demand for one asset compared to the other so that the price of one asset compared to the future stream of dividends is much higher than of the other. The price impact of this type of agents decreases compared to the price impact of the other agents as dividends are paid, because they hold more of the overpriced asset and earn a lower return than the other agents. In a market with several assets and types of agents, there can be cross-effects so that the mechanism identified in the model with two types of agents and two assets does not always work. However, our simulations suggest that this market force is stronger than the cross-effects in case of most assets.

Some closely related arguments for the representative-agent approach have been brought up by the existing literature. An alternative hypothesis based on thoughts by Alchian (1950) and Friedman (1953) is that there exists a dominant investment strategy in terms of profits driving all other strategies out of the market in the long run. It is debatable whether such a dominant strategy actually exists or whether the best long-run performance can be achieved by different investment strategies.[13] Even if there was a dominant strategy, it is unclear how much time it would take until other strategies are driven out of the market. The literature on market selection mainly provides asymptotic results that are difficult to interpret in this respect.[14] Our results suggest that independently of the presence of a dominant strategy the direction in which relative asset prices are expected to move from one time period to the next can be found using a representative agent.

[13] For economies where the market selection hypothesis fails see for example Blume and Easley (2006) and Borovička (2015).

[14] For a survey on market selection and asset pricing see for example Blume and Easley (2009).

Another argument first brought up by Muth (1961) is that non-systematic mistakes in the assessment of future outcomes across individuals cancel on average. Aggregate market behavior can therefore still be determined by the behavior of a single agent fully knowledgeable about the true distribution of future outcomes. However, individual mistakes are unlikely to cancel on average given the systematic biases identified by the behavioural finance literature.[15] We show that even if all individuals use rebalancing rules that over- or underinvest into an asset, relative wealth dynamics still tend to push relative asset prices in expectation closer towards the relative prices of a representative-agent model with perfect foresight.

This chapter analyzes an evolutionary finance model in the style of Evstigneev et al. (2016).[16] The main advantages of this modelling approach is that common restrictive assumptions like utility functions, perfect foresight and market completeness are not imposed. Market participants are directly characterized by their investment rules rather than by utility functions. Investment rules can therefore follow from utility maximization but do not need to. Previous research in this framework focuses on identifying portfolio rules that survive and/or dominate in the long run.[17] Almost sure convergence to a single investment strategy present in the market, which invests proportionally to discounted expected relative dividends, is shown under different assumptions on the initial set of investment strategies and on dividend processes.[18] Compared to these asymptotic studies, our paper focuses on expected relative wealth and price dynamics over one time period and our results hold independently of the presence of a dominant strategy. We find that relative prices are expected to move closer to the relative prices under the discounted-expected-relative-dividend-strategy over one time period even in the absence of this investment strategy.

[15] See Daniel et al. (2002), Barber and Odean (2011) and references therein for common biases of market participants.

[16] More specifically, it is a special case of the setting in Amir et al. (2011).

[17] For related results within the utility-maximizing framework see for example Sandroni (2000), Blume and Easley (2006) and Borovička (2015).

[18] For results on global convergence see Amir et al. (2005) and Evstigneev et al. (2008). For results on local convergence see Evstigneev et al. (2006) and Evstigneev et al. (2011).

The only other study within the evolutionary finance framework in the sense of Evstigneev et al. (2016) that treats the case of no dominant strategy present in the market is the dissertation of Giachini (2015). Chapter 3 addresses short- and medium-term dynamics, whereas Chapter 2 and 4 provide asymptotic analyses.[19] The main difference between the one-period analysis of Chapter 3 and our study is that Giachini (2015) examines the differences in expected next-period relative prices conditional on current dividends given the same relative price, whereas we analyze expected next-period relative prices conditional on current relative prices. The results cannot be compared due to different assumptions on dividends. Neither model is a special case of the other.[20]

The outline of the paper is as follows. Section 2.1 introduces the evolutionary finance model with heterogeneous agents. Section 2.2 states the representative-agent benchmark. Section 2.3 presents analytical results for the case of two strategies and two assets and simulation results for different numbers of strategies and assets.

2.1 The Heterogeneous-Agent Model

The model is set in discrete time $t = 0, 1, \ldots$ There is an arbitrary number of market participants who can invest into $K > 1$ long-lived financial assets. The total number of different investment strategies implemented by market participants is given by $I \geq K$.

2.1.1 Financial assets

The K financial assets are characterized by an exogenously given asset supply and dividend payments. The two stochastic processes $\{V_{t,k}\}_{t=0}^{\infty}$ and $\{D_{t,k}\}_{t=0}^{\infty}$ describe the supply and dividend payments of asset $k \in \{1, \ldots, K\}$ at each point in time. Both stochastic processes are defined over a probability space $(\Omega, \mathscr{F}, \mathscr{P})$, where Ω is the

[19] Chapter 2 discusses conditions for the long-term co-existence of investment strategies and Chapter 4 discusses the wisdom of crowds in prediction markets.

[20] Both our analytical results are derived in the case of two investment strategies and two assets. Giachini (2015) restricts the analysis to two states of the world and treats the case of Arrow securities, whereas we assume that relative dividends are martingales.

sample space, $\mathscr{F}$ is a filtration and $\mathscr{P}$ is the probability measure, and take finite values in $\left(\mathbb{R}_+, \mathscr{B}\right)$. Each asset always remains in positive supply

$$V_{t,k} > 0 \quad \text{for} \quad k = 1, \ldots, K,\ t = 0, 1, \ldots,$$

and at least one asset pays a positive dividend each period,

$$\sum_{k=1}^{K} D_{t,k} > 0 \quad \text{for} \quad t = 0, 1, \ldots$$

Let $D_t \equiv (D_{t,1}, ..., D_{t,K})'$ denote the vector with dividend payments of all assets at time t. The evolutionary finance model is better tractable when stated in relative terms. We therefore define the relative dividends of asset $k \in \{1, \ldots, K\}$ as

$$d_{t,k} \equiv \frac{D_{t,k} V_{t,k}}{\sum_{l=1}^{K} D_{t,l} V_{t,l}}.$$

The price of asset k, $p_{t,k}$, is endogeneously determined. We will show that $\{p_{t,k}\}_{t=0}^{\infty}$ is a positive real-valued stochastic process on $(\Omega, \mathscr{F}, \mathscr{P})$. The vector of asset prices at time t is denoted as $p_t \equiv (p_{t,1}, ..., p_{t,K})'$.

2.1.2 Investment strategies

The I different investment strategies are characterized by the fraction of wealth invested into financial assets and by the proportions invested into the K financial assets. The real-valued adapted process $\{\alpha_t^i\}_{t=0}^{\infty}$ denotes the fraction of wealth that strategy i invests into financial assets at time t and satisfies $0 < \alpha_t^i < 1$. Put differently, the fraction $(1 - \alpha_t^i)$ denotes the consumption rate at time t of investors following strategy i. In addition we exclude the possibility of large asset repurchases (see Assumption 2.1).

Assumption 2.1 *Investment rates are bounded by asset supply growth, i.e.*

$$\max_{i=1,\ldots,I} \alpha_t^i < \min_{k=1,\ldots,K} \frac{V_{t,k}}{V_{t-1,k}} \quad \textit{for} \quad t = 0, 1, \ldots$$

From the total amount of wealth invested into financial assets, strategy i invests a fraction $\lambda^i_{t,k}$ into asset k at time t. We define $\lambda^i_t \equiv (\lambda^i_{t,1}, \ldots, \lambda^i_{t,K})$ to be the vector of investment proportions from strategy i at time t and assume investment proportions to be $\mathscr{F}_t$-measurable. Thus, investors base their decisions on any available present and past information. The investment proportions must sum up to one:

$$\sum_{k=1}^{K} \lambda^i_{t,k} = 1 \quad \text{for} \quad t = 0, 1, \ldots, i = 1, \ldots, I,$$

and there is a positive demand for each asset throughout time:

$$\sum_{i=1}^{I} \lambda^i_{t,k} > 0 \quad \text{for} \quad t = 0, 1, \ldots, k = 1, \ldots, K.$$

In addition, we assume that no short-selling occurs (see Assumption 2.2).

Assumption 2.2 *There are no short-selling investment strategies, i.e.* $\lambda^i_t > 0$ *for all* $i = 1, \ldots, I$ *at each point in time.*

Let us define $x^i_{t,k}$ as the number of assets k that investment strategy i holds in the portfolio at time t. The vector $x^i_t \equiv (x^i_{t,1}, \ldots, x^i_{t,K})'$ denotes the portfolio holdings of strategy i at time t. The total wealth allocated to strategy i at time t is defined as $w^i_t \equiv \langle D_t + p_t, x^i_{t-1} \rangle$, where $\langle ., . \rangle$ denotes the scalar product of the two vectors. For the formulation of the model in relative terms, we define the relative wealth of strategy i at time t as

$$r^i_t \equiv \frac{w^i_t}{\sum_{j=1}^{I} w^j_t}.$$

The wealth of all strategies remains positive at each point in time since the investment rate and prices are positive and no short-selling takes place. Therefore, relative wealth is always well-defined and satisfies $0 < r^i_t < 1$. The vector $r_t \equiv (r^1_t, \ldots, r^I_t)'$ denotes the relative wealth shares of all investors at time t.

2.1.3 Dynamic equilibrium

Suppose that each investment strategy i starts with a positive amount of wealth and is initially endowed with $x^i_{-1} > 0$. All financial assets are traded each period. Prices are endogeneously determined by the market clearing conditions at time $t = 0, 1, \ldots$:

$$p_{t,k} V_{t,k} = \sum_{i=1}^{I} \lambda^i_{t,k} \alpha^i_t \langle D_t + p_t, x^i_{t-1} \rangle, \quad k = 1, ..., K. \tag{2.1}$$

While the supply of each asset, consumption rates, investment proportions and dividends are exogeneously given, asset holdings at time $t = 0, 1 \ldots$ are determined by

$$x^i_{t,k} = \frac{\lambda^i_{t,k} \alpha^i_t \langle D_t + p_t, x^i_{t-1} \rangle}{p_{t,k}}, \qquad i = 1, \ldots, I, k = 1, \ldots, K. \tag{2.2}$$

Note that all exogeneously given variables are $\mathscr{F}_t$-measurable, which implies that the processes $\{p_{t,k}\}_{t=0}^{\infty}$ and $\{w^i_t\}_{t=0}^{\infty}$, where $w^i_t = \langle D_t + p_t, x^i_{t-1} \rangle$, are adapted as well. Furthermore, the appendix shows that the equations in (2.1) uniquely determine asset prices and that prices are positive. The equations in (2.2) are therefore well-defined.

2.1.4 Relative wealth dynamics

The relative wealth dynamics of the evolutionary finance model are more convenient to work with than the random dynamical system defined by the equations (2.1) and (2.2). Before stating them, let us introduce two simplifying assumptions that we impose throughout the remainder of this chapter.

Assumption 2.3 *The supply of all assets remains constant over time, i.e.* $V_{t,k} = V_k$ *for all* $k = 1, \ldots, K$ *and* $t = 0, 1, \ldots$.

Assumption 2.4 *All investors have the same constant investment rate, i.e.* $\alpha^i_t = \alpha$ *for* $i = 1, \ldots, I$ *and* $t = 0, 1, \ldots$.

Now let us state the relative wealth dynamics in this setting. The relative wealth at time $t = 0$ follows directly from the initial endowments and the market clearing condition (2.1). Given $r_0^i > 0$ for all investment strategies i, the relative wealth dynamics are implicitly given by

$$r_t^i = \sum_{k=1}^{K} \big((1-\alpha)d_{t,k} + \alpha\langle \lambda_{t,k}, r_t\rangle\big) \frac{\lambda_{t-1,k}^i r_{t-1}^i}{\langle \lambda_{t-1,k}, r_{t-1}\rangle}, \tag{2.3}$$

where $i = 1, \ldots, I$ and $t = 1, 2, \ldots$. The derivation of the relative wealth dynamics is in the appendix.[21] Since dividends are non-negative and asset holdings as well as asset prices are positive, the relative wealth dynamics are well-defined.

2.2 The Representative-Agent Benchmark

A representative-agent model with a logarithmic utility maximizer will serve us as a benchmark for comparison with the evolutionary finance model. The decision problem of the representative agent is given as

$$\max_{\{x_t\}_{t=0}^{\infty}} \mathbf{E}\left[\sum_{t=0}^{\infty} \delta^t \log\big(c_t\big)\right]$$
$$\text{s.t.} \quad c_t + \sum_{k=1}^{K} p_{t,k} x_{t,k} = \sum_{k=1}^{K} \big(p_{t,k} + D_{t,k}\big)\, x_{t-1,k},$$

where $t = 0, 1, \ldots$ and $x_{-1,k}$ for $k = 1, \ldots, K$ are given. Here δ denotes the time discount factor, c_t denotes consumption at time t, $x_{t,k}$ denotes the number of assets k in the portfolio of the representative agent at time t, $p_{t,k}$ denotes the price of asset k at time t and $D_{t,k}$ denotes the dividend of asset k at time t. In equilibrium all markets clear: $x_{t,k}^* = V_k$ for $k = 1, \ldots, K$ and $c_t^* = \sum_{k=1}^{K} D_{t,k} V_k$ for $t = 0, 1, \ldots$, where V_k denotes the exogenously given supply of asset k.

[21] Amir et al. (2011) derive the relative wealth dynamics in a more general setting with a time-varying investment rate and time-varying asset supply growth.

Asset prices in the representative-agent model are given by

$$p^*_{t,k} = \mathbf{E}_t\left[\sum_{\tau=1}^{\infty} \delta^{\tau} \frac{\sum_{l=1}^{K} D_{t,l} V_l}{\sum_{m=1}^{K} D_{t+\tau,m} V_m} D_{t+\tau,k}\right], \tag{2.4}$$

where $k = 1, \ldots, K$ and $t = 0, 1, \ldots$. The aggregate market capitalization is

$$\sum_{k=1}^{K} p^*_{t,k} V_k = \sum_{k=1}^{K} \mathbf{E}_t\left[\sum_{\tau=1}^{\infty} \delta^{\tau} \frac{\sum_{l=1}^{K} D_{t,l} V_l}{\sum_{m=1}^{K} D_{t+\tau,m} V_m} D_{t+\tau,k}\right] V_k = \frac{\delta}{1-\delta} \sum_{l=1}^{K} D_{t,l} V_l, \tag{2.5}$$

where $t = 0, 1, \ldots$. To better illustrate the connection between this model and the evolutionary finance model, let us also calculate the investment rate and the investment proportions of the representative agent in equilibrium. The investment rate is given by

$$\alpha = \frac{\sum_{k=1}^{K} p^*_{t,k} V_k}{\sum_{l=1}^{K} \left(p^*_{t,l} + D_{t,l}\right) V_l} = \frac{\frac{\delta}{1-\delta} \sum_{k=1}^{K} D_{t,k} V_k}{\left(\frac{\delta}{1-\delta} + 1\right) \sum_{l=1}^{K} D_{t,l} V_l} = \delta. \tag{2.6}$$

Thus, the investment rate equals exactly the discount factor. The fraction of wealth after consumption invested into asset k at time t is

$$\begin{aligned} \lambda^*_{t,k} &= \frac{p^*_{t,k} V_k}{\delta \sum_{l=1}^{K} \left(p^*_{t,l} + D_{t,l}\right) V_l} \\ &= \frac{\mathbf{E}_t\left[\sum_{\tau=1}^{\infty} \delta^{\tau} \frac{\sum_{l=1}^{K} D_{t,l} V_l}{\sum_{m=1}^{K} D_{t+\tau,m} V_m} D_{t+\tau,k}\right] V_k}{\delta \left(\frac{\delta}{1-\delta} + 1\right) \sum_{l=1}^{K} D_{t,l} V_l} \\ &= \mathbf{E}_t\left[\sum_{\tau=1}^{\infty} (1-\delta) \delta^{\tau-1} d_{t+\tau,k}\right]. \end{aligned} \tag{2.7}$$

The representative-agent model with a logarithmic utility maximizer is equivalent to an evolutionary finance model with only one investment strategy present in the market characterized by an investment rate of δ and investment proportions $\{\lambda^*_t\}_{t=0}^{\infty}$ as pointed out by Gerber et al. (2010) and Hens et al. (2011).[22] The following shows that asset

[22] For a comparison of the two models in case of independently and identically distributed dividends see Gerber et al. (2010). Hens et al. (2011) refer to the model specification used in our paper and state relative prices. We contribute to their comparison by comparing investment rates and absolute prices.

prices are indeed the same in both models. The market capitalization of asset k at time t in the respective evolutionary finance model is given by

$$p_{t,k}V_k = \lambda^*_{t,k}\alpha\langle D_t + p_t, x_{t-1}\rangle. \tag{2.8}$$

Since the λ^*-strategy is the only investment strategy present in the market, the wealth of the λ^*-strategy, which is $w_t = \langle D_t + p_t, x_{t-1}\rangle$, equals aggregate wealth. Aggregate wealth equals aggregate dividends plus aggregate market capitalization. The aggregate market capitalization in the evolutionary finance model follows from summing up (2.1) over all $k = 1, \ldots, K$ and noting that $x_{t-1} = V_k$:

$$\sum_{k=1}^{K} p_{t,k}V_k = \frac{\alpha}{1-\alpha}\sum_{k=1}^{K} D_{t,k}V_k,$$

where $t = 0, 1, \ldots$. Thus, aggregate wealth is given by

$$\sum_{k=1}^{K} D_{t,k}V_k + \sum_{k=1}^{K} p_{t,k}V_k = \frac{1}{1-\alpha}\sum_{k=1}^{K} D_{t,k}V_k. \tag{2.9}$$

Inserting aggregate wealth (2.9) and $\lambda^*_{t,k}$ as defined in (2.7) into the expression for the market capitalization (2.8), we obtain

$$p_{t,k}V_k = \lambda^*_{t,k}\frac{\alpha}{1-\alpha}\sum_{l=1}^{K} D_{t,l}V_l = \frac{\alpha}{1-\alpha}\mathbf{E}_t\left[\sum_{\tau=1}^{\infty}(1-\delta)\delta^{\tau-1}\frac{\sum_{l=1}^{K} D_{t,l}V_l}{\sum_{m=1}^{K} D_{t+\tau,m}V_m}D_{t+\tau,k}V_k\right].$$

Dividing by V_k and noting that the investment rate equals the discount factor as shown in (2.6), we obtain

$$p_{t,k} = \mathbf{E}_t\left[\sum_{\tau=1}^{\infty}\delta^{\tau}\frac{\sum_{l=1}^{K} D_{t,l}V_l}{\sum_{m=1}^{K} D_{t+\tau,m}V_m}D_{t+\tau,k}\right].$$

Hence prices are the same as implied by utility maximization (see (2.4)).

For the comparison of the model with a representative logarithmic utility maximizer to the evolutionary finance model with multiple rebalancing rules, we make another simplifying assumption that concerns relative dividends.[23]

Assumption 2.5 *Relative dividends are martingales, i.e. the conditional expectation of relative dividends is* $\mathbf{E}_t[d_{t+1,k}] = d_{t,k}$ *for* $k = 1, \ldots, K$ *and* $t = 0, 1, \ldots$*.*

This assumption takes up the fact that managers tend to be reluctant to change dividend payouts. Lowering dividends could signal bad performance to the market, whereas an increase in dividends would make it more difficult to sustain the higher level of dividend payouts.[24]

Assuming that relative dividends are martingales, the proportions of wealth after consumption that the representative agent invests into asset k at time t becomes

$$\lambda^*_{t,k} = d_{t,k}.$$

When we compare the cross-section of asset prices between the two models, it will be more convenient to move to relative terms. Instead of asset prices, we will focus our discussion on relative market capitalizations:

$$q^*_{t,k} = \frac{p^*_{t,k} V_k}{\sum_{l=1}^{K} p^*_{t,l} V_l} = \mathbf{E}_t \left[\sum_{\tau=1}^{\infty} (1-\delta)\delta^{\tau-1} d_{t+\tau,k} \right] = \lambda^*_{t,k},$$

where $k = 1, \ldots, K$ and $t = 0, 1, \ldots$. In case of relative dividends being martingales, we obtain that relative market capitalizations equal relative dividends

$$q^*_{t,k} = d_{t,k}.$$

[23] Note that we only impose an assumption on relative dividends which is weaker than a respective assumption on absolute dividends.

[24] These arguments were first identified by Lintner (1956) from interviews with managers about their dividend policies. For a more recent survey see Brav et al. (2005).

2.3 Asset Prices in the Evolutionary Finance Model

Asset prices in the evolutionary finance model are in general different from the ones in the representative-agent model with a logarithmic utility maximizer. In fact, the relative market capitalizations depend on the different investment strategies present in the market and the distribution of relative wealth across investment strategies:

$$q_{t,k} = \sum_{i=1}^{I} \lambda_{t,k}^{i} r_{t}^{i},$$

where $k = 1, \ldots, K$ and $t = 0, 1, \ldots$. Asset prices can therefore not be explained by any model that does not take into account information about the cross-section.

Recall that we assume the same consumption and savings behavior across agents.[25] The cross-sectional wealth distribution has therefore no impact on the aggregate market capitalization:[26]

$$\sum_{k=1}^{K} p_{t,k} V_k = \frac{\alpha}{1-\alpha} \sum_{k=1}^{K} D_{t,k} V_k,$$

where $t = 0, 1, \ldots$. We see that the aggregate market capitalization equals discounted expected future aggregate dividends and is the same as in the representative-agent model (2.5) at each point in time.[27]

Does the dependence of prices on cross-sectional information exclude a meaningful specification of a representative agent? We show that this is not generally true. Even though a representative-agent model cannot explain actual price levels, it still provides valuable information about asset prices: It indicates the direction in which relative asset prices tend to move in expectation.

[25] See Assumption 2.4.

[26] The aggregate market capitalization follows from summing up (2.1) over all $k = 1, \ldots, K$ and noting that $\sum_{i=1}^{I} x_{t-1}^{i} = V_k$ for $t = 0, 1, \ldots$.

[27] This follows from the fact that the investment rate of the representative logarithmic utility maximizer equals the discount factor as shown in (2.6).

For the following analysis we assume that relative dividends are martingales and that strategies invest fixed proportions of wealth into assets, i.e. Assumption 2.5 and Assumption 2.6 hold.

Assumption 2.6 *The investment proportions of all strategies are constant over time, i.e.* $\lambda_t^i = \lambda^i$ *for* $i = 1, \dots, I$ *and* $t = 0, 1, \dots$.

Assumption 2.6 means that we focus on the implications of relative wealth dynamics on asset prices and we exclude possible effects from investors revising their investment proportions.

2.3.1 Two investment strategies and two financial assets

First we examine the case of two investment strategies and two financial assets. For this case we prove that the relative market capitalizations of both assets move closer to the respective relative market capitalizations of the representative-agent model from Section 2.2 in expectation. Since we assume investment strategies to invest according to constant proportions of wealth, relative wealth dynamics must be the sole driver of our results.

For a precise statement of the results some more notation is needed. We define the relative wealth levels r_t^{1*} and r_t^{2*} as the ones at which the relative market capitalization of each asset corresponds to the one derived from the decision problem of a representative expected logarithmic utility maximizer. Put differently, r_t^{1*} and r_t^{2*} are the relative wealth levels at which the resulting representative agent in the evolutionary finance model invests according to λ_t^*:

$$q_{t,k} = \sum_{i=1}^{2} \lambda_k^i r_t^{i*} = \lambda_{t,k}^* = d_{t,k} = q_{t,k}^*, \quad \text{where} \quad k = 1, 2.$$

Note that there exist such relative wealth levels if and only if

$$\begin{pmatrix} r_t^{1*} \\ r_t^{2*} \end{pmatrix} = \begin{pmatrix} \lambda_1^1 & \lambda_1^2 \\ \lambda_2^1 & \lambda_2^2 \end{pmatrix}^{-1} \begin{pmatrix} d_{t,1} \\ d_{t,2} \end{pmatrix} = \frac{1}{\lambda_1^1\lambda_2^2 - \lambda_1^2\lambda_2^1} \begin{pmatrix} \lambda_2^2 d_{t,1} - \lambda_1^2 d_{t,2} \\ \lambda_1^1 d_{t,2} - \lambda_2^1 d_{t,1} \end{pmatrix} \tag{2.10}$$

is positive. This is the case if and only if $\lambda_1^1 > d_{t,1} > \lambda_1^2$ or $\lambda_1^1 < d_{t,1} < \lambda_1^2$ or if one of the two strategies invests according to λ_t^*. Suppose that both strategies invest more (less) into the same asset compared to the λ_t^*-strategy. In this case, relative prices can never reach the equilibrium prices of the representative-agent model. The closest relative prices can get is when the strategy closer to the λ_t^*-strategy owns all wealth.

Apparently relative wealth can reach a level where the representative agent invests according to the λ_t^*-strategy if and only if the two strategies do not simultaneously invest more (less) than the λ_t^*-strategy. Moreover, relative wealth shares are expected to move closer to these levels in this case as stated in Theorem 2.1. Theorem 2.1 also says that there is no overshooting in expectation. If the relative wealth share of investor 1 is currently higher (lower) than the relative wealth share that would imply a representative λ_t^*-strategy, it remains higher (lower) in expectation.

Theorem 2.1 *Suppose that $\lambda_1^1 \neq \lambda_1^2$ and $\lambda_1^1 \geq \lambda_{t,1}^* \geq \lambda_1^2$ or $\lambda_1^1 \leq \lambda_{t,1}^* \leq \lambda_1^2$. Then the following two statements hold:*

(A) $\left|\mathbf{E}_t[r_{t+1}^i - r_{t+1}^{i*}]\right| \leq |r_t^i - r_t^{i*}|$ *with equality if and only if* $r_t^i = r_t^{i*}$,

(B) $\operatorname{sgn}\left(\mathbf{E}_t[r_{t+1}^i - r_{t+1}^{i*}]\right) = \operatorname{sgn}\left(r_t^i - r_t^{i*}\right)$,

where $i = 1, 2$.

Considering Theorem 2.1 we expect that the relative market capitalizations also move closer to the ones of the representative-agent model when the two strategies do not simultaneously invest more (less) than the λ_t^*-strategy. In fact, this property holds even in the case of both strategies investing more (less) than the λ_t^*-strategy as stated in Theorem 2.2.

Theorem 2.2 *The following two statements hold:*

(A) $\left|\mathbf{E}_t[q_{t+1,k} - q^*_{t+1,k}]\right| \leq \left|q_{t,k} - q^*_{t,k}\right|$ *with equality if and only if* $q_{t,k} = q^*_{t,k}$*,*

(B) $\operatorname{sgn}\left(\mathbf{E}_t[q_{t+1,k} - q^*_{t+1,k}]\right) = \operatorname{sgn}\left(q_{t,k} - q^*_{t,k}\right)$*,*

where $k = 1, 2$*.*

Theorem 2.2 also says that there is no overshooting of relative market capitalizations in expectation. Note that Theorem 2.1 and Theorem 2.2 make statements on conditional expectations, but not on realized paths. For example overshooting can occasionally take place as illustrated in the following numerical example.

Example (Sample path): Suppose that the dividends of asset 1 and asset 2 at time $t = 0$ are given by $(D_{0,1}, D_{0,2}) = (1, 1)$ and that

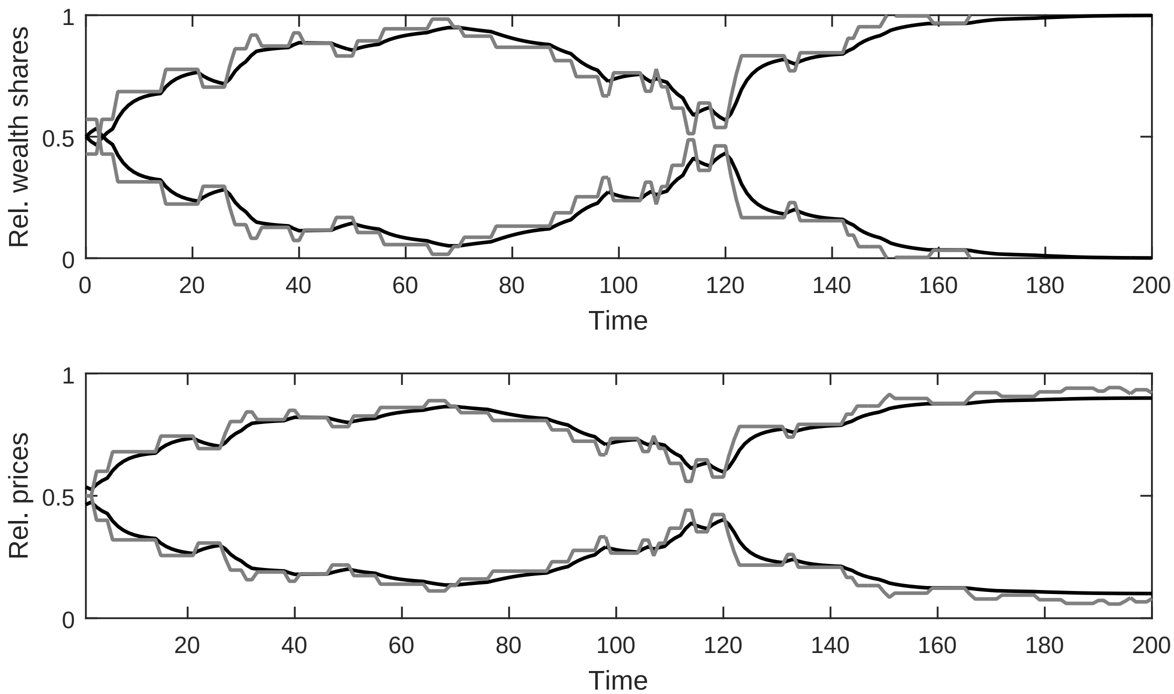
$$(D_{t+1,1}, D_{t+1,2}) = \begin{cases} (D_{t,1} + \Delta_t, D_{t,2} - \Delta_t) & \text{with probability } 0.1, \\ (D_{t,1}, D_{t,2}) & \text{with probability } 0.8, \\ (D_{t,1} - \Delta_t, D_{t,2} + \Delta_t) & \text{with probability } 0.1, \end{cases}$$

where $t = 1, 2, \ldots$ and $\Delta_t = 0.2 \min(D_{t,1}, D_{t,2})$. Furthermore we assume an investment rate of 0.6 and investment proportions are given by $\lambda^1 = (0.2, 0.8)$ and $\lambda^2 = (0.9, 0.1)$. Both investment strategies are initially endowed with the same amount of wealth. The sample path in Figure 2.1 shows that relative wealth shares tend to move towards the relative wealth shares that would imply a representative λ^*-strategy and relative prices tend to move towards the relative prices under the λ^*-strategy. We also see a tendency not to overshoot, even though an overshooting occurs every now and then.

Note that Theorem 2.2 imposes no restrictions on the rebalancing rules. The claims on relative prices hold independently of the presence of a dominant investment strategy as the following two special cases with two identical assets show.[28]

[28] See Figure 2.4 in the appendix for a graphical illustration of both cases.

Figure 2.1: This figure shows a sample path of the relative wealth shares and relative prices (black lines) as well as the respective relative wealth shares that would imply relative prices equivalent to the relative prices in a representative-agent model with a logarithmic utility maximizer (grey lines). The investment rate is 0.6 and the investment proportions are $\lambda^1 = (0.2, 0.8)$ and $\lambda^2 = (0.9, 0.1)$. Dividends increase and decrease by 20% of the smaller current dividend payment with a probability of 0.1, otherwise they stay the same. The dividend of one asset increases when the dividend of the other asset decreases and vice versa.



Example (Dominance): Suppose that there are two identical assets so that relative dividends remain constant over time. There is one strategy investing proportionally to relative dividends and another strategy with a different asset allocation. In this case, the relative-dividend strategy drives the other strategy out of the market over the long run as shown in Evstigneev et al. (2008).

Example (Co-Survival): Suppose that there are two identical assets so that relative dividends remain constant over time. There is one strategy investing as much into one asset as the other strategy invests into the other asset. Due to the symmetry of the model both strategies must survive in the long run.

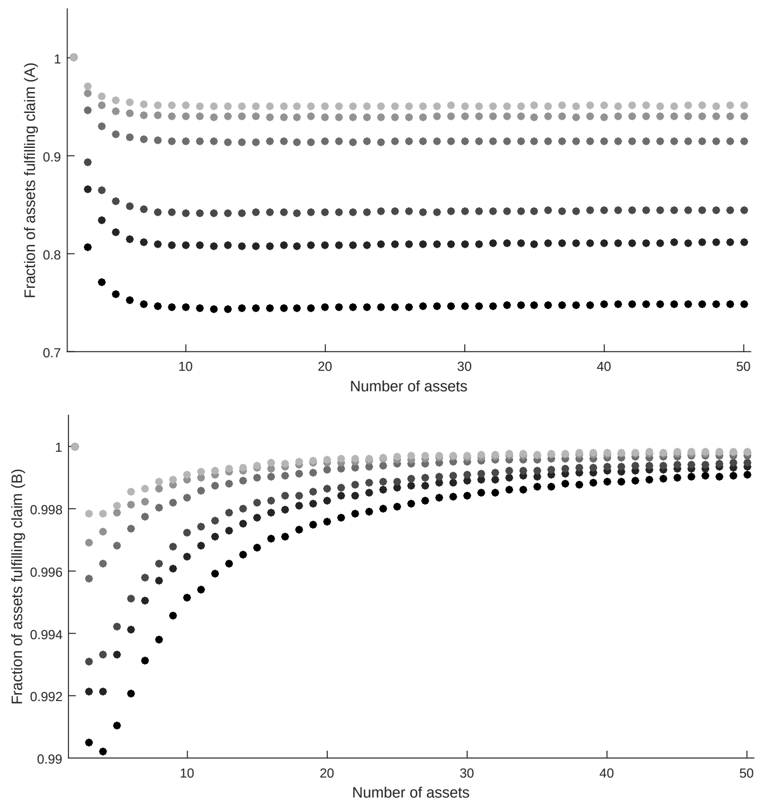
2.3.2 Multiple investment strategies and multiple financial assets

Theorem 2.2 does not generalize to the case of more than two financial assets, though the market capitalizations of most assets still behave as in the two-asset case. The following simulations illustrate the extent to which Theorem 2.2 applies for different numbers of strategies and assets in the market. For each number of strategies and assets, we randomly generate 100000 realized states from different economies. The current realization of dividends, wealth and portfolio holdings as well as the investment rate are drawn from a uniform distribution. Based on these parameters we calculate the current and expected relative market capitalizations of all assets.

How often did expected relative market capitalizations conditional on the current state approach the representative-agent benchmark across all simulated economies, i.e. how often did relation (A) in Theorem 2.2 apply? The upper plot in Figure 2.2 shows the fraction of relative market capitalizations with this property as a function of the number of assets. Each shade of grey stands for a different number of investment strategies compared to the number of assets present in the market. The ratio of the number of strategies to the number of assets from the darkest shade to the lightest is 1, 2, 3, 10, 20, 30. Note that claim (A) always holds in the case of two assets no matter how many strategies are present in the market. We therefore see only one point on the vertical line of the two-asset case. In addition, we observe that claim (A) is not true for a general number of assets. The more assets there are in the market for a fixed number of strategies to number of assets ratio, the lower the fraction of assets that fulfill claim (A) is. However, the curves seem to flatten out. Overall the fraction of assets fulfilling claim (A) is clearly above 70% in all simulated cases. Another interesting observation is that the more investment strategies there are, the more relative market capitalizations approach the representative-agent benchmark in expectation.

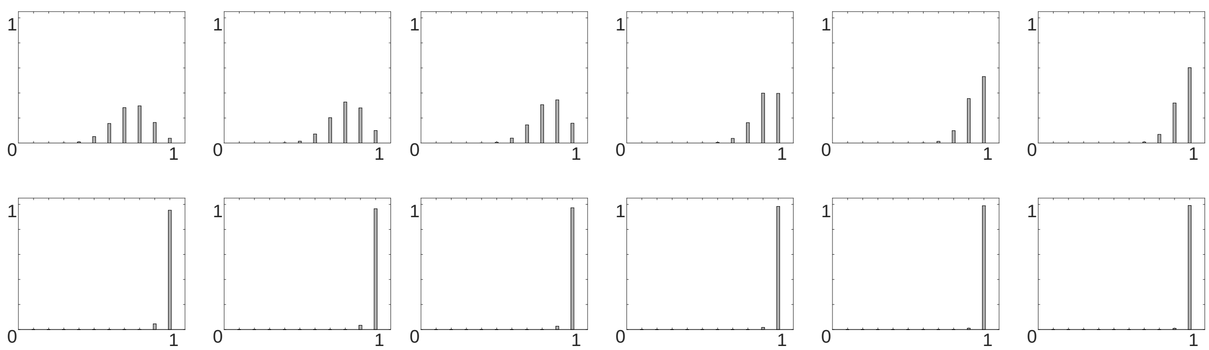
Let us turn to the second claim in Theorem 2.2. The lower plot in Figure 2.2 shows the fraction of relative market capitalizations that did not overshoot the representative-agent benchmark in expectation for different numbers of assets in the economy. The

Figure 2.2: The upper plot shows the fraction of relative market capitalizations that are expected to move closer to the representative-agent benchmark in the next period, i.e. fulfill claim (A) in Theorem 2.2, averaged over 100000 simulated economies. The lower plot shows the fraction of relative market capitalizations that are expected not to overshoot the representative-agent benchmark in in the next period, i.e. fulfill claim (B) in Theorem 2.2, averaged over 10000 simulated economies. The different shades of grey indicate the ratio of the number of strategies to the number of financial assets. The ratio from the darkest shade to the lightest shade: 1, 2, 3, 10, 20, 30.



different shades of grey indicate the same ratios of number of investment strategies to the number of assets as in the upper plot. Claim (B) also holds in the two-asset case, but is not true for a general number of assets. After an initial drop, all curves increase in the number of assets. The fraction of expected overshoots seems to decrease as the number of assets increases for a larger number of assets. Overall we observe that the fraction of expected overshoots is negligibly small in all simulated economies and becomes even smaller as the number of investment strategies increases.

Figure 2.3: The histograms in the upper row show for which fraction of the 100000 simulated economies (y-axis) which fraction of relative market capitalizations are expected to move closer to the representative-agent benchmark in the next period (x-axis), i.e. fulfill claim (A) in Theorem 2.2. The histograms in the row below show for which fraction of the 100000 simulated economies (y-axis) which fraction of relative market capitalizations are expected not to overshoot the representative-agent benchmark in in the next period (x-axis), i.e. fulfill claim (B) in Theorem 2.2. The number of assets is set to 10. The ratio of the of the number of strategies to the number of assets from the left to the right: 1, 2, 3, 10, 20, 30.



The previous results only show averages over all simulated economies. A natural question to ask is whether there are huge differences across economies with respect to the fraction of relative market capitalizations that move closer, but do not overshoot the representative-agent benchmark in expectation. The upper row in Figure 2.3 shows in how many of the generated economies which fraction of market capitalizations fulfills claim (A). The number of assets in the market is held constant at 10 for all histograms, whereas the number of investment strategies varies from the left to the right: 10, 20, 30, 100, 200, 300. We see that there are economies where quite many relative market capitalizations violate claim (A) when there are as many investment strategies as assets in the market. As the number of strategies increases relative to the number of assets, the number of violations decreases and most relative market capitalizations satisfy claim (A) in pretty much all simulated economies. Simulations with different numbers of assets can be found in Figure 2.5 and Figure 2.6 in the appendix.

The lower row in Figure 2.3 shows in how many of the generated economies which fraction of market capitalizations satisfies claim (B). The number of assets in all simulations is 10 and the histograms from the left to the right again correspond to a ratio of the number of strategies to the number of assets of 1, 2, 3, 10, 20, 30. We see that claim (B) is rarely violated in any of the simulated economies. As the number of strategies increases relative to the number of assets, the number of economies exhibiting violations decreases even further.

2.4 Conclusion

The evolutionary approach to finance takes the view that the long-term performance of investment strategies and asset prices are the outcome of the interaction between different evolutionary forces. Our analysis identifies a market force that supports the possible use of a representative agent to study long-term asset prices: The wealth dynamics tend to push relative asset prices towards relative asset prices that follow from the decision problem of a commonly used representative agent. In our setting, the representative agent maximizes a logarithmic utility function and has perfect foresight even though individual agents follow only simple rebalancing rules.

An interesting extension to our work would be to study the effect of wealth dynamics in an environment with other evolutionary forces. Brock and Hommes (1998) for example identify an opposing market force that can lead to barely predictable long-term asset prices. If individual agents keep changing their investment strategies based on their past performance, chaotic asset price dynamics can emerge. For a more thorough assessment of the benefits and limitations of the representative-agent approach, we need to better understand the extent to which different evolutionary forces are present in the market and how they interact.

2.5 Appendix

Existence, uniqueness and positivity of prices:

Suppose that we are at time t. The supply of each financial asset, the consumption rates, investment proportions, dividends and asset holdings of each investment strategy are given. Consider the map from $\mathbb{R}^K_+$ onto itself that maps vector $p = (p_1, \ldots, p_K)'$ to vector $q = (q_1, \ldots, q_K)'$:

$$q_k = V_{t,k}^{-1} \sum_{i=1}^{I} \lambda_{t,k}^i \alpha_t^i \langle D_t + p, x_{t-1}^i \rangle, \quad \text{for} \quad k = 1, \ldots, K.$$

Adding to the proof in Amir et al. (2011) the possibility of heterogeneous consumption rates across investment strategies, we prove that this map is contracting in the norm $\|p\|_V = \sum_{k=1}^K |p_k| V_{t-1,k}$. First let us define

$$\tilde{\alpha} \equiv \max_{i=1,\ldots,I,k=1,\ldots,K} \alpha_t^i V_{t-1,k} V_{t,k}^{-1}.$$

Note that $\tilde{\alpha} < 1$ because of Assumption 2.1.

$$\begin{aligned}
\|q - \tilde{q}\|_V &= \sum_{k=1}^{K} |q_k - \tilde{q}_k| V_{t-1,k} \\
&\leq \sum_{k=1}^{K} \sum_{i=1}^{I} \lambda_{t,k}^i \alpha_t^i V_{t-1,k} V_{t,k}^{-1} |\langle p - \tilde{p}, x_{t-1}^i \rangle| \\
&\leq \tilde{\alpha} \sum_{k=1}^{K} \sum_{i=1}^{I} \lambda_{t,k}^i |\langle p - \tilde{p}, x_{t-1}^i \rangle| \\
&= \tilde{\alpha} \sum_{i=1}^{I} |\langle p - \tilde{p}, x_{t-1}^i \rangle| \\
&\leq \tilde{\alpha} \sum_{i=1}^{I} \sum_{k=1}^{K} |p_k - \tilde{p}_k| x_{t-1,k}^i
\end{aligned}$$

$$
\begin{aligned}
&= \tilde{\alpha} \sum_{k=1}^{K} |p_k - \tilde{p}_k| \sum_{i=1}^{I} x^i_{t-1,k} \\
&= \tilde{\alpha} \sum_{k=1}^{K} |p_k - \tilde{p}_k| V_{t-1,k} \\
&= \tilde{\alpha} \left\| p - \tilde{p} \right\|_V
\end{aligned}
$$

According to the contraction mapping theorem, there exists a unique fixed-point in the metric space $\left(\mathbb{R}_+, \|\cdot\|\right)$. To show that equilibrium prices are positive, we can start the iteration from any point $p^{(0)} \in \mathbb{R}^K_+$. The price of the assets $k = 1, \ldots, K$ after the first iteration step are

$$
p^{(1)}_k = V^{-1}_{t,k} \sum_{i=1}^{I} \lambda^i_{t,k} \alpha^i_t \langle D_t + p^{(0)}, x^i_{t-1} \rangle \geq V^{-1}_{t,k} \sum_{i=1}^{I} \lambda^i_{t,k} \alpha^i_t \langle D_t, x^i_{t-1} \rangle > 0.
$$

This argument applies to all iteration steps, which implies that prices are positive. □

Derivation of the relative wealth dynamics:

Let us start the derivation from the definition of wealth

$$
\begin{aligned}
w^i_t &= \langle D_t + p_t, x^i_{t-1} \rangle \\
&= \sum_{k=1}^{K} \left(D_{t,k} + p_{t,k} \right) x^i_{t-1,k}.
\end{aligned}
\tag{2.11}
$$

Note that asset prices and asset holdings as given in equation (2.1) and (2.2) expressed in terms of relative wealth become

$$
p_{t,k} = \frac{\alpha}{V_k} \langle \lambda_{t,k}, w_t \rangle
$$

and

$$
x^i_{t-1,k} = \frac{\lambda^i_{t-1,k} \alpha w^i_{t-1}}{p_{t-1,k}} = \frac{V_k \lambda^i_{t-1,k} w^i_{t-1}}{\langle \lambda_{t-1,k}, w_{t-1} \rangle}.
$$

Plugging these expressions into the wealth equation (2.11), we obtain

$$w_t^i = \sum_{k=1}^{K} \left(D_{t,k} + \frac{\alpha}{V_k} \langle \lambda_{t,k}, w_t \rangle \right) \frac{V_k \lambda_{t-1,k}^i w_{t-1}^i}{\langle \lambda_{t-1,k}, w_{t-1} \rangle}. \tag{2.12}$$

Aggregate wealth therefore is

$$W_t = \sum_{i=1}^{I} w_t^i = \sum_{k=1}^{K} \left(D_{t,k} V_k + \alpha \langle \lambda_{t,k}, w_t \rangle \right) = \sum_{k=1}^{K} D_{t,k} V_k + \alpha W_t.$$

Solving this equation for aggregate wealth, we obtain

$$W_t = \frac{1}{1-\alpha} \sum_{k=1}^{K} D_{t,k} V_k. \tag{2.13}$$

Dividing the equation for individual wealth (2.12) by aggregate wealth gives us

$$r_t^i = \sum_{k=1}^{K} \left(\frac{D_{t,k} V_k}{W_t} + \alpha \langle \lambda_{t,k}, r_t \rangle \right) \frac{\lambda_{t-1,k}^i r_{t-1}^i}{\langle \lambda_{t-1,k}, r_{t-1} \rangle}.$$

Using the relation between aggregate wealth and aggregate dividends (2.13), we obtain the relative wealth dynamics

$$r_t^i = \sum_{k=1}^{K} \left((1-\alpha) d_{t,k} + \alpha \langle \lambda_{t,k}, r_t \rangle \right) \frac{\lambda_{t-1,k}^i r_{t-1}^i}{\langle \lambda_{t-1,k}, r_{t-1} \rangle}.$$

□

Proof of Theorem 2.1:

First of all, note the following relation that follows from the first equation in (2.10) and from the assumption that relative dividends are martingales:

$$\begin{pmatrix} \mathbf{E}_t \left[r_{t+1}^{1*} \right] \\ \mathbf{E}_t \left[r_{t+1}^{2*} \right] \end{pmatrix} = \begin{pmatrix} \lambda_1^1 & \lambda_1^2 \\ \lambda_2^1 & \lambda_2^2 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{E}_t \left[d_{t+1,1} \right] \\ \mathbf{E}_t \left[d_{t+1,2} \right] \end{pmatrix} = \begin{pmatrix} \lambda_1^1 & \lambda_1^2 \\ \lambda_2^1 & \lambda_2^2 \end{pmatrix}^{-1} \begin{pmatrix} d_{t,1} \\ d_{t,2} \end{pmatrix} = \begin{pmatrix} r_t^{1*} \\ r_t^{2*} \end{pmatrix}.$$

Let us start with the proof of (B). Since $\mathbf{E}_t[r^{i*}_{t+1}] = r^{i*}_t$, $i = 1,2$, we need to show that

1. $r^1_t < r^{1*}_t$ $(r^2_t > r^{2*}_t) \implies \mathbf{E}_t[r^1_{t+1}] < r^{1*}_t$ $(\mathbf{E}_t[r^2_{t+1}] > r^{2*}_t)$,
2. $r^1_t = r^{1*}_t$ $(r^2_t = r^{2*}_t) \implies \mathbf{E}_t[r^1_{t+1}] = r^{1*}_t$ $(\mathbf{E}_t[r^2_{t+1}] = r^{2*}_t)$,
3. $r^1_t > r^{1*}_t$ $(r^2_t < r^{2*}_t) \implies \mathbf{E}_t[r^1_{t+1}] > r^{1*}_t$ $(\mathbf{E}_t[r^2_{t+1}] < r^{2*}_t)$.

For the proof, let us write $\mathbf{E}_t[r^1_{t+1}]$ as a function of r^1_t. Taking the conditional expectation of the relative wealth dynamics (2.3) from time t to $t+1$, using that $r^2_t = 1 - r^1_t$ and $\mathbf{E}_t[r^2_{t+1}] = 1 - \mathbf{E}_t[r^1_{t+1}]$ and solving for $\mathbf{E}_t[r^1_{t+1}]$, we obtain

$$\mathbf{E}_t[r^1_{t+1}] = f_t(r^1_t) \equiv \frac{a(r^1_t) + b_t(r^1_t)}{a(r^1_t) + c(r^1_t)},$$

where

$$\begin{aligned} a(r^1_t) &\equiv \alpha \sum_{k=1}^{K} \lambda^2_k \frac{\lambda^1_k r^1_t}{\langle \lambda_k, r_t \rangle}, \\ b_t(r^1_t) &\equiv (1-\alpha) \sum_{k=1}^{K} \mathbf{E}_t[d_{t+1,k}] \frac{\lambda^1_k r^1_t}{\langle \lambda_k, r_t \rangle}, \\ c(r^1_t) &\equiv 1 - \alpha \sum_{k=1}^{K} \lambda^1_k \frac{\lambda^1_k r^1_t}{\langle \lambda_k, r_t \rangle}. \end{aligned}$$

It suffices to show that $f_t(r^{1*}_t) = r^{1*}_t$ holds and that the map is strictly monotonically increasing. By the definition of r^{1*}_t and using that investment proportions sum up to one, we have

$$a(r^{1*}_t) + b_t(r^{1*}_t) = \left(\alpha \sum_{k=1}^{K} \lambda^2_k \frac{\lambda^1_k}{\langle \lambda_k, r^*_t \rangle} + 1 - \alpha \right) r^{1*}_t,$$

and

$$
\begin{aligned}
a(r_t^{1*}) + c(r_t^{1*}) &= \alpha \sum_{k=1}^{K} \lambda_k^2 \frac{\lambda_k^1}{\langle \lambda_k, r_t^* \rangle} + 1 - \alpha \sum_{k=1}^{K} \frac{\lambda_k^1}{\langle \lambda_k, r_t^* \rangle} \langle \lambda_k, r_t^* \rangle \\
&= \alpha \sum_{k=1}^{K} \lambda_k^2 \frac{\lambda_k^1}{\langle \lambda_k, r_t^* \rangle} + 1 - \alpha.
\end{aligned}
$$

This implies that $f_t(r_t^{1*}) = r_t^{1*}$ holds. The following shows that the expected relative wealth map is strictly monotonically increasing. Note that

$$
\begin{aligned}
a(r_t^1) &= \alpha \sum_{k=1}^{K} \lambda_k^2 \frac{\lambda_k^1 r_t^1}{\langle \lambda_k, r_t \rangle} > 0, \\
b_t(r_t^1) &= (1-\alpha) \sum_{k=1}^{K} \mathbf{E}_t[d_{t+1,k}] \frac{\lambda_k^1 r_t^1}{\langle \lambda_k, r_t \rangle} > 0, \\
c(r_t^1) &= 1 - \alpha \sum_{k=1}^{K} \lambda_k^1 \frac{\lambda_k^1 r_t^1}{\langle \lambda_k, r_t \rangle} > 0.
\end{aligned}
$$

The derivatives of $a(r_t^1)$, $b_t(r_t^1)$ and $c(r_t^1)$ are

$$
a'(r_t^1) = \alpha \sum_{k=1}^{K} \lambda_k^2 \frac{\lambda_k^1}{\left(\lambda_k^1 - \lambda_k^2 + \frac{\lambda_k^2}{r_t^1}\right)^2} \frac{\lambda_k^2}{(r_t^1)^2} > 0,
$$

$$
b_t'(r_t^1) = (1-\alpha) \sum_{k=1}^{K} \mathbf{E}_t[d_{t+1,k}] \frac{\lambda_k^1}{\left(\lambda_k^1 - \lambda_k^2 + \frac{\lambda_k^2}{r_t^1}\right)^2} \frac{\lambda_k^2}{(r_t^1)^2} > 0,
$$

$$
c'(r_t^1) = -\alpha \sum_{k=1}^{K} \lambda_k^1 \frac{\lambda_k^1}{\left(\lambda_k^1 - \lambda_k^2 + \frac{\lambda_k^2}{r_t^1}\right)^2} \frac{\lambda_k^2}{(r_t^1)^2} < 0.
$$

The derivative of the relative wealth map is

$$
f_t'(r_t^1) = \frac{\left(a'(r_t^1) + b_t'(r_t^1)\right)\left(a(r_t^1) + c(r_t^1)\right) - \left(a(r_t^1) + b_t(r_t^1)\right)\left(a'(r_t^1) + c'(r_t^1)\right)}{\left(a(r_t^1) + c(r_t^1)\right)^2}
$$

and since $a(r_t^1), b_t(r_t^1), c(r_t^1) > 0$, $a'(r_t^1), b_t'(r_t^1) > 0$ and $c'(r_t^1) < 0$, we have

$$f_t'(r_t^1) > \frac{a'(r_t^1)\big(c(r_t^1) - b_t(r_t^1)\big)}{\big(a(r_t^1) + c(r_t^1)\big)^2}.$$

Note that $b_t(r_t^1)$ is maximal and $c(r_t^1)$ is minimal for $r_t^1 = 1$. It follows that

$$f_t'(r_t^1) > \frac{a'(r_t^1)\big(c(1) - b_t(1)\big)}{\big(a(r_t^1) + c(r_t^1)\big)^2} = 0,$$

since

$$c(1) - b_t(1) = \left(1 - \alpha \sum_{k=1}^{K} \lambda_k^1\right) - (1-\alpha) \sum_{k=1}^{K} \mathbf{E}_t[d_{t+1,k}] = 0.$$

This concludes the proof of (B). To prove statement (A), we additionally need to show that

1. $r_t^1 < r_t^{1*}$ $(r_t^2 > r_t^{2*}) \implies \mathbf{E}_t[r_{t+1}^1] > r_t^1$ $(\mathbf{E}_t[r_{t+1}^2] < r_t^2)$,
2. $r_t^1 > r_t^{1*}$ $(r_t^2 < r_t^{2*}) \implies \mathbf{E}_t[r_{t+1}^1] < r_t^1$ $(\mathbf{E}_t[r_{t+1}^2] > r_t^2)$.

Let us start with the claim that $r_t^1 < r_t^{1*}$ implies $\mathbf{E}_t[r_{t+1}^1] > r_t^1$. For the proof we show that $f_t'(0) > 1$ and that r_t^{1*} is the unique interior fixed point of $f_t(r_t^1)$. Let us explain how this proves the claim. Suppose that both statements are true. From the definition of the derivative we know that for all $\epsilon > 0$ there is a $\delta > 0$ so that for all r_t^1 with $0 < r_t^1 < \delta$ we have

$$\left| \frac{f_t(r_t^1) - f_t(0)}{r_t^1} - f_t'(0) \right| < \epsilon.$$

Choosing $\epsilon < f_t'(0) - 1$, we see that there exists a $\delta > 0$ so that for all $r_t^1 < \delta$, we know that $\frac{f_t(r_t^1) - f_t(0)}{r_t^1} > 1$. Since $f_t(0) = 0$, this is equivalent to $r_t^1 < f_t(r_t^1) = \mathbf{E}_t[r_{t+1}^1]$. Thus, the claim holds true for r_t^1 close enough to zero. Using a continuity argument, we can show that $\mathbf{E}_t[r_{t+1}^1] > r_t^1$ whenever $r_t^1 < r_t^{1*}$. Suppose that the claim was not true and there was a $r_t^1 < r_t^{1*}$ so that $\mathbf{E}_t[r_{t+1}^1] < r_t^1$. Take another relative wealth share close enough to zero so that $\mathbf{E}_t[r_{t+1}^1] > r_t^1$. Note that $\mathbf{E}_t[r_{t+1}^1] = f_t(r_t^1)$ is a continuous function. Because of the intermediate value theorem, the function $f_t(r_t^1) - r_t^1$ must take the value zero between these two relative wealth share levels. This would mean that there exists another point $r_t^1 < r_t^{1*}$ for which $\mathbf{E}_t[r_{t+1}^1] = r_t^1$, which is a contradiction to r_t^{1*} being the unique fixed point.

Now let us show that $f_t'(0) > 1$. The derivative of $f_t(r_t^1)$ is

$$f_t'(r_t^1) = \frac{\left(a'(r_t^1) + b_t'(r_t^1)\right)\left(a(r_t^1) + c(r_t^1)\right) - \left(a(r_t^1) + b_t(r_t^1)\right)\left(a'(r_t^1) + c'(r_t^1)\right)}{\left(a(r_t^1) + c(r_t^1)\right)^2}.$$

First let us evaluate $a(r_t^1)$, $b_t(r_t^1)$ and $c(r_t^1)$ at the point $r_t^1 = 0$:

$$\begin{aligned} a(0) &= 0, \\ b_t(0) &= 0, \\ c(0) &= 1. \end{aligned}$$

The derivative therefore simplifies to

$$f_t'(0) = a'(0) + b_t'(0).$$

The derivatives of $a(r_t^1)$ and $b_t(r_t^1)$ are

$$\begin{aligned} a'(r_t^1) &= \alpha \sum_{k=1}^{K} \lambda_k^2 \frac{\lambda_k^1\left(\lambda_k^1 r_t^1 + \lambda_k^2(1 - r_t^1)\right) - \lambda_k^1 r_t^1(\lambda_k^1 - \lambda_k^2)}{\left(\lambda_k^1 r_t^1 + \lambda_k^2(1 - r_t^1)\right)^2}. \\ b_t'(r_t^1) &= (1 - \alpha) \sum_{k=1}^{K} \mathbf{E}_t[d_{t+1,k}] \frac{\lambda_k^1\left(\lambda_k^1 r_t^1 + \lambda_k^2(1 - r_t^1)\right) - \lambda_k^1 r_t^1(\lambda_k^1 - \lambda_k^2)}{\left(\lambda_k^1 r_t^1 + \lambda_k^2(1 - r_t^1)\right)^2}. \end{aligned}$$

At $r_t^1 = 0$, we have

$$a'(0) = \alpha \sum_{k=1}^{K} \lambda_k^1 = \alpha,$$

$$b_t'(0) = (1-\alpha)\sum_{k=1}^{K}\mathbf{E}_t[d_{t+1,k}]\frac{\lambda_k^1}{\lambda_k^2}.$$

Since relative dividends are martingales, we have

$$f_t'(0) = \alpha + (1-\alpha)\sum_{k=1}^{K} d_{t,k}\frac{\lambda_k^1}{\lambda_k^2}$$

Note that $f_t'(0) = 1$ if $\lambda^2 = \lambda_t^*$. However, the case that $r_t^1 < r_t^{1*}$ cannot occur if $\lambda^2 = \lambda_t^*$ since $r_t^{1*} = 0$. It remains to show that $\sum_{k=1}^{K} d_{t,k}\frac{\lambda_k^1}{\lambda_k^2} > 1$ for $\lambda^2 \neq \lambda_t^*$ and therefore $f_t'(0) > 1$. We can write the sum as

$$\sum_{k=1}^{K} d_{t,k}\frac{\lambda_k^1}{\lambda_k^2} = d_{t,1}\frac{\lambda_1^1}{\lambda_1^2} + (1-d_{t,1})\frac{1-\lambda_1^1}{1-\lambda_1^2}. \tag{2.14}$$

The derivative with respect to $d_{t,1}$ is

$$\frac{\lambda_1^1}{\lambda_1^2} - \frac{1-\lambda_1^1}{1-\lambda_1^2} = \frac{\lambda_1^1 - \lambda_1^2}{\lambda_1^2(1-\lambda_1^2)}.$$

Thus, the expression (2.14) is increasing in $d_{t,1}$ if $\lambda_1^1 > \lambda_1^2$ and decreasing in $d_{t,1}$ if $\lambda_1^1 < \lambda_1^2$. Suppose that $\lambda_1^1 \geq d_{t,1} > \lambda_1^2$. This means that the expression (2.14) is greater than evaluated at the point $d_{t,1} = \lambda_1^2$ and this means greater than

$$\lambda_1^1 + 1 - \lambda_1^1 = 1.$$

Suppose that $\lambda_1^1 \leq d_{t,1} < \lambda_1^2$. This means that the expression (2.14) is greater than evaluated at $d_{t,1} = \lambda_1^2$ and therefore greater than

$$\lambda_1^1 + 1 - \lambda_1^1 = 1.$$

We conclude that $f_t'(0) > 1$.

Next let us show that r_t^{1*} is the unique interior fixed point of $f_t(r_t^1)$. The relative wealth dynamics (2.3) for an interior fixed point are

$$\begin{aligned} r_t^1 &= \mathbf{E}_t\left[\sum_{k=1}^{K}\big(\alpha\langle\lambda_k, r_{t+1}\rangle + (1-\alpha)d_{t+1,k}\big)\frac{\lambda_k^1 r_t^1}{\langle\lambda_k, r_t\rangle}\right] \\ &= \sum_{k=1}^{K}\big(\alpha\langle\lambda_k, r_t\rangle + (1-\alpha)\mathbf{E}_t\left[d_{t+1,k}\right]\big)\frac{\lambda_k^1 r_t^1}{\langle\lambda_k, r_t\rangle}. \end{aligned}$$

This can be simplified to

$$\begin{aligned} 1 &= \sum_{k=1}^{K}\left(\alpha\lambda_k^i + (1-\alpha)\mathbf{E}_t\left[d_{t+1,k}\right]\frac{\lambda_k^i}{\langle\lambda_k, r_t\rangle}\right) \\ &= \alpha + (1-\alpha)\sum_{k=1}^{K}\mathbf{E}_t\left[d_{t+1,k}\right]\frac{\lambda_k^i}{\langle\lambda_k, r_t\rangle}, \end{aligned}$$

or, equivalently,

$$1 = \sum_{k=1}^{K}\mathbf{E}_t\left[d_{t+1,k}\right]\frac{\lambda_k^i}{\langle\lambda_k, r_t\rangle}.$$

Let us define

$$\zeta_k \equiv \frac{\mathbf{E}_t[d_{t+1,k}]}{\langle\lambda_k, r_t\rangle}.$$

The ζ_k, $k = 1, 2$, are uniquely determined by the system of equations

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1^1 & \lambda_2^1 \\ \lambda_1^2 & \lambda_2^2 \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$$

Since the investment proportions of each agent sum up to one, the solution is $\zeta = (1, 1)'$. This implies that

$$\sum_{i=1}^{2}\lambda_k^i r_t^i = \mathbf{E}_t[d_{t+1,k}] = d_{t,k}, \tag{2.15}$$

but this is the definition of r_t^{1*} and therefore r_t^{1*} is the unique interior fixed point.

To conclude the proof of (A), we need to show that $r_t^1 > r_t^{1*}$ implies $\mathbf{E}_t[r_{t+1}^1] < r_t^1$. Note that the problem is symmetric to the one above. We can replace strategy one by strategy two everywhere and show that $r_t^2 < r_t^{2*}$ implies $\mathbf{E}_t[r_{t+1}^2] > r_t^2$. Since (expected) relative wealth shares sum up to one, we obtain our claim. □

Proof of Theorem 2.2:

First note that

$$\mathbf{E}_t[q_{t+1,k}^*] = \sum_{i=1}^{2} \lambda_k^i \mathbf{E}_t[r_{t+1}^{i*}] = \sum_{i=1}^{2} \lambda_k^i r_t^{i*} = q_{t,k}^*$$

for $k = 1, 2$. To prove (A) and (B) we therefore need to show that

1. $q_{t,k} < q_{t,k}^* \implies q_{t,k} < \mathbf{E}_t[q_{t+1,k}] < q_{t,k}^*$,
2. $q_{t,k} = q_{t,k}^* \implies q_{t,k} = \mathbf{E}_t[q_{t+1,k}] = q_{t,k}^*$,
3. $q_{t,k} > q_{t,k}^* \implies q_{t,k} > \mathbf{E}_t[q_{t+1,k}] > q_{t,k}^*$,

for k=1,2.

Let us consider the case when Theorem 2.1 applies and we have $\lambda_1^1 \geq \lambda_{t,1}^* \geq \lambda_1^2$ or $\lambda_1^1 \leq \lambda_{t,1}^* \leq \lambda_1^2$ and $\lambda_1^1 \neq \lambda_1^2$. We write $q_{t,k}$ as a function of r_t^1:

$$q_{t,k} = g_k(r_t^1) \equiv \lambda_k^1 r_t^1 + \lambda_k^2 (1 - r_t^1).$$

This function is strictly monotonically increasing (decreasing) if $\lambda_k^1 > \lambda_k^2$ ($\lambda_k^1 < \lambda_k^2$). In the proof of Theorem 2.1, we have shown that

1. $r_t^1 < r_t^{1*} \implies r_t^1 < \mathbf{E}_t[r_{t+1}^1] < r_t^{1*}$,
2. $r_t^1 = r_t^{1*} \implies r_t^1 = \mathbf{E}_t[r_{t+1}^1] = r_t^{1*}$,
3. $r_t^1 > r_t^{1*} \implies r_t^1 > \mathbf{E}_t[r_{t+1}^1] > r_t^{1*}$.

Suppose that $\lambda_k^1 > \lambda_k^2$ ($\lambda_k^1 < \lambda_k^2$). Because of the monotonicity property and noting that the function $g_k(\cdot)$ and expectation are interchangeable, we obtain that $r_t^1 < r_t^{1*}$

if and only if $q_{t,k} < q^*_{t,k}$ ($q_{t,k} > q^*_{t,k}$) and that $r^1_t < \mathbf{E}_t[r^1_{t+1}] < r^{1*}_t$ if and only if $q_{t,k} < \mathbf{E}_t[q_{t+1,k}] < q^*_{t,k}$ ($q_{t,k} > \mathbf{E}_t[q_{t+1,k}] > q^*_{t,k}$). Applying the same argument to the other two implications, we see that the claim holds in the case of $\lambda^1_1 \neq \lambda^2_1$ and $\lambda^1_1 \geq \lambda^*_{t,1} \geq \lambda^2_1$ or $\lambda^1_1 \leq \lambda^*_{t,1} \leq \lambda^2_1$.

Let us turn to the case when both investment strategies invest more into one asset than the λ^*-strategy and less into the other asset or vice versa. Without loss of generality we only consider the case $\lambda^1_1 > \lambda^2_1 > \lambda^*_{t,1}$. The other three possible cases directly follow from this case due to symmetry reasons. Suppose that $\lambda^1_1 > \lambda^2_1 > \lambda^*_{t,1}$ holds. Note that the price of asset one cannot get below $q^*_{t,1}$ since

$$q_{t,1} = \sum_{i=1}^{2} \lambda^i_1 r^i_t > \lambda^*_{t,1} = q^*_{t,1}$$

and the opposite holds true for asset two. This argument applies at each time step and for each state, which proves statement (B).

For the proof of statement (A), it remains to show that $q_{t,1} > \mathbf{E}_t[q_{t+1,1}]$. Since the function $g_1(r^1_t)$, which maps the relative wealth of strategy one to the relative price of asset one, is strictly monotonically increasing and interchangeable with the expectation, it suffices to show that $r^1_t > \mathbf{E}_t[r^1_{t+1}]$. Recall from the proof of Theorem 2.1 the function $f_t(r^1_t)$ that maps the relative wealth share at time t to the expected relative wealth share at time $t+1$. To show that $r^1_t > \mathbf{E}_t[r^1_{t+1}]$, we prove that $f'_t(0) < 1$ and that there is no interior fixed point. That there is no interior fixed point follows directly from the derivation of all interior fixed points for the case that $\lambda^1_1 \neq \lambda^2_1$ and $\lambda^1_1 \geq \lambda^*_{t,1} \geq \lambda^2_1$ or $\lambda^1_1 \leq \lambda^*_{t,1} \leq \lambda^2_1$ in the proof of Theorem 2.1. We see that there does not exist a relative wealth share level so that the condition for an interior fixed point (2.15) is fulfilled if $\lambda^1_1 > \lambda^2_1 > \lambda^*_{t,1}$.

From the proof of Theorem 2.1 we also know that

$$f'_t(0) = \alpha + (1-\alpha) \sum_{k=1}^{K} d_{t,k} \frac{\lambda^1_k}{\lambda^2_k}$$

is increasing for larger values of $d_{t,1}$. This implies that the derivative $f_t'(0)$ is bounded from above by

$$\alpha + (1-\alpha)\sum_{k=1}^{K} \lambda_k^2 \frac{\lambda_k^1}{\lambda_k^2} = 1.$$

The remainder of the proof is analogous to the respective part in the proof of Theorem 2.1. Since $f_t'(0) < 1$, we know that there exist a relative wealth share level close enough to zero so that $\mathbf{E}_t[r_{t+1}^1] < r_t^1$. Suppose that there was a larger relative wealth share level with $\mathbf{E}_t[r_{t+1}^1] > r_t^1$. Due to the intermediate value theorem there must be a point in between so that $\mathbf{E}_t[r_{t+1}^1] = r_t^1$, which is a contradiction to the non-existence of an interior fixed point. □

Figure 2.4: This figure shows relative wealth shares and relative prices (black lines) as well as the respective relative wealth shares that would imply relative prices equivalent to the relative prices in a representative-agent model with a logarithmic utility maximizer (grey lines). The two upper plots illustrate a case where one strategy dominates. The investment rate is 0.6 and the investment proportions are $\lambda^1 = (0.01, 0.99)$ and $\lambda^2 = (0.5, 0.5)$. The two lower plots illustrate a case where both strategies survive. The investment rate is 0.8 and the investment proportions are $\lambda^1 = (0.25, 0.75)$ and $\lambda^2 = (0.75, 0.25)$. Dividends are the same for both assets and constant over time.

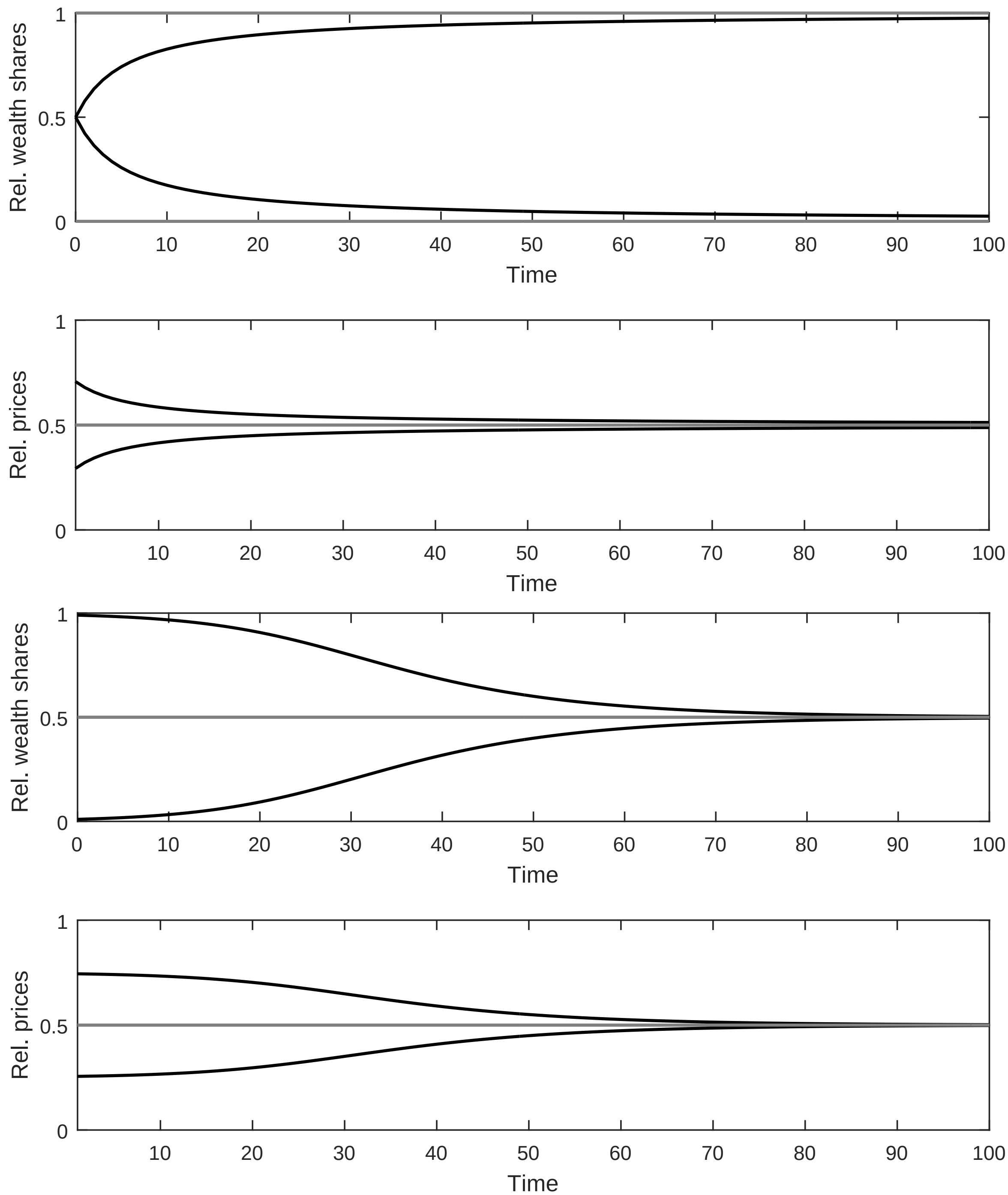


Figure 2.5: The histograms show for which fraction of the 100000 simulated economies (y-axis) which fraction of relative market capitalizations are expected to move closer to the representative-agent benchmark in the next period (x-axis), i.e. fulfill claim (*A*) in Theorem 2.2. The number of assets varies from two to ten (top to bottom row) and the number of investment strategies to assets ratio takes the values 1, 2, 3, 10, 20 and 30 (left to right column).

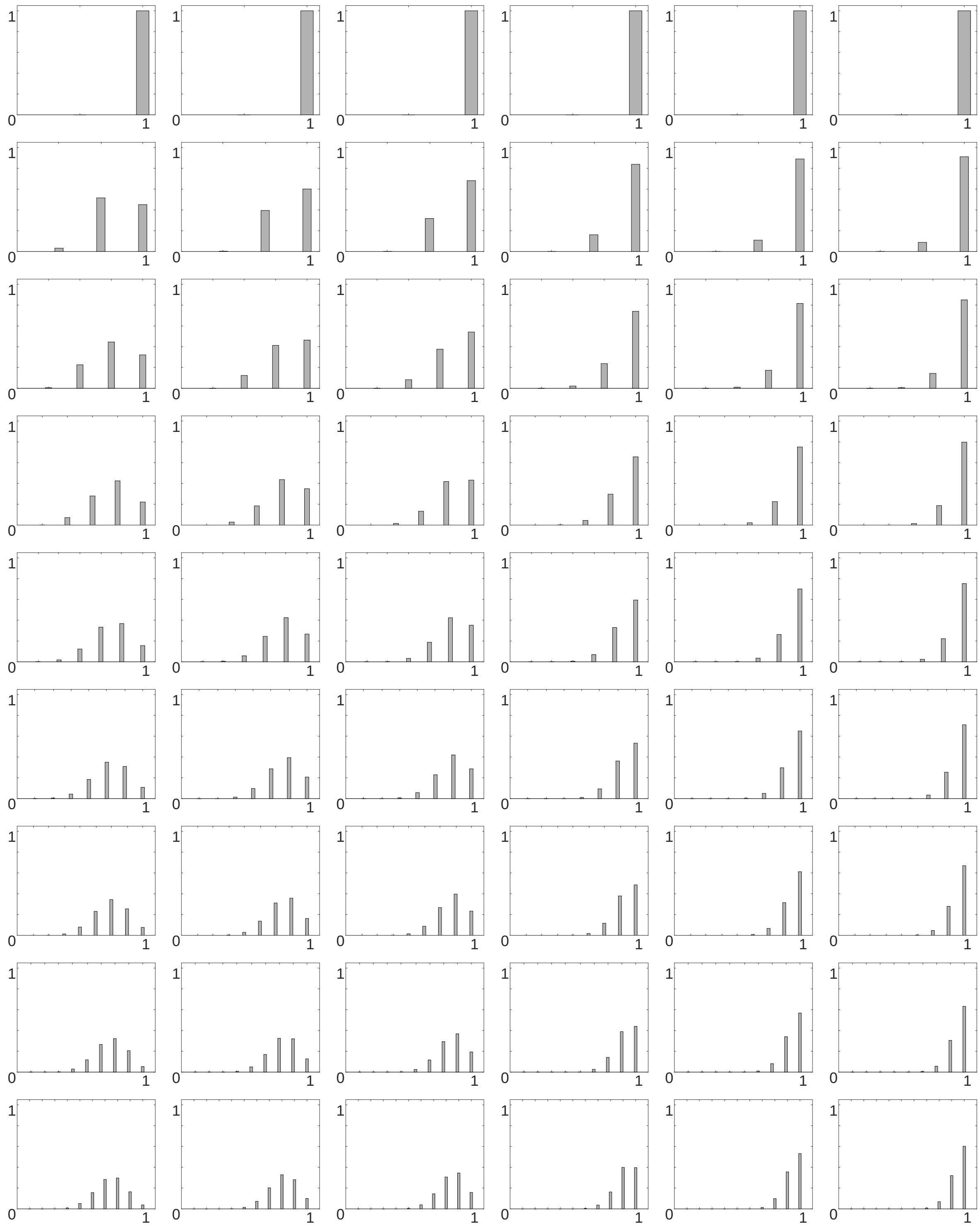
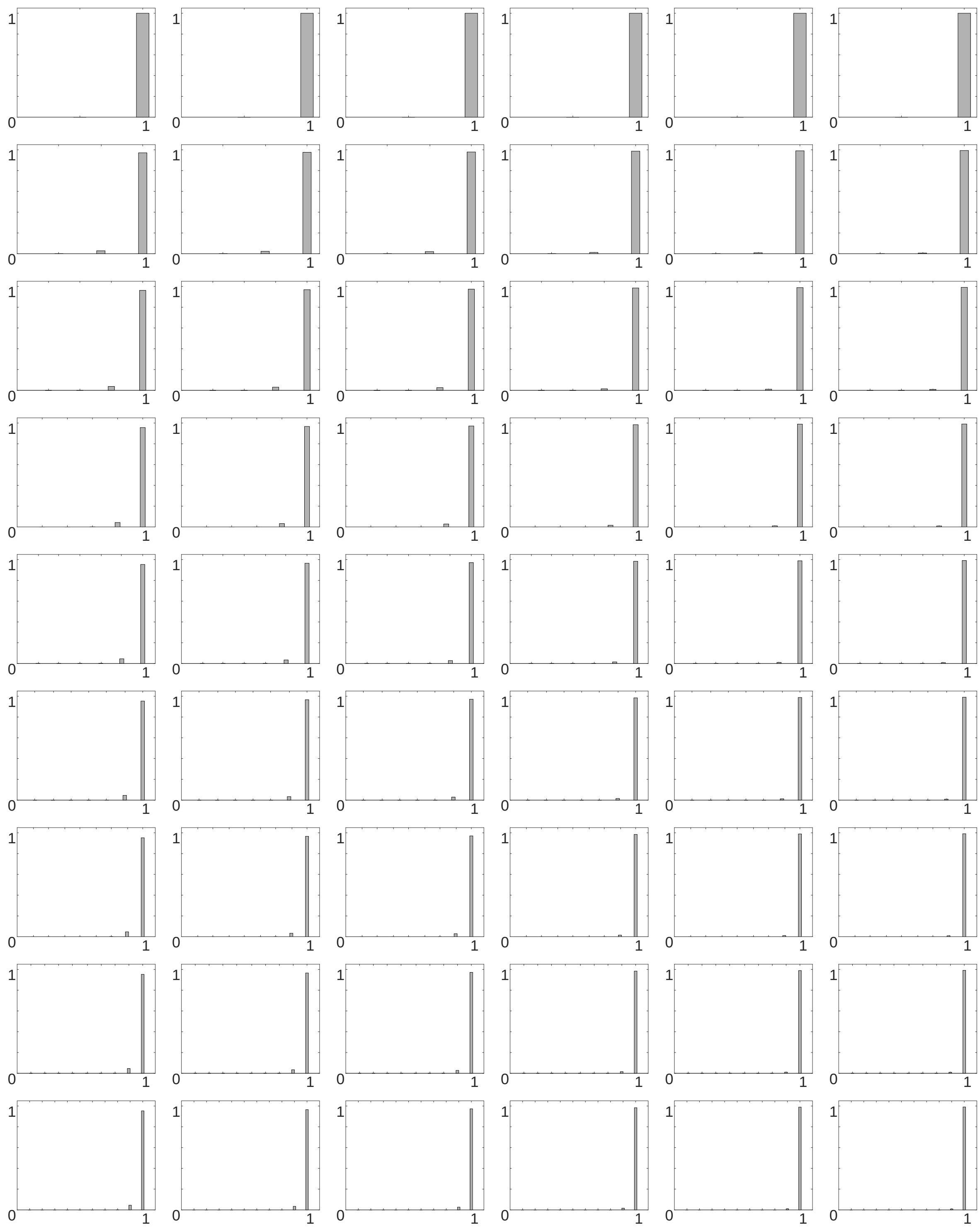


Figure 2.6: The histograms show for which fraction of the 100000 simulated economies (y-axis) which fraction of relative market capitalizations are expected not to overshoot the representative-agent benchmark in in the next period (x-axis), i.e. fulfill claim (B) in Theorem 2.2. The number of assets varies from two to ten (top to bottom row) and the number of investment strategies to assets ratio takes the values 1, 2, 3, 10, 20 and 30 (left to right column).



3

Can the CRRA-Lognormal Framework Explain CAPM-Anomalies in the Cross-Section of Stock Returns?

For cross-sectional asset pricing, the traditional capital asset pricing model (CAPM) of Sharpe (1964), Lintner (1965) and Mossin (1966) is one of the cornerstones of finance. Even though the CAPM is a very elegant pricing model and widely used in practice,[29] its empirical validity is highly debated. A large empirical literature has developed on asset pricing anomalies with respect to the CAPM. This raises the question whether the CAPM is based on reasonable assumptions or whether the assumptions needed to derive the CAPM are too restrictive?[30]

This chapter discusses CAPM-anomalies in a two-period version of the consumption-based capital asset pricing model (CCAPM), developed by Rubinstein (1976), Lucas (1978) and Breeden (1979). The CAPM follows as a special case from the more general CCAPM if, for example, preferences are quadratic or dividends are normally distributed. However, Campbell and Viceira (2002) note that preferences must exhibit constant relative risk aversion (CRRA), because there are no long-term trends in risk premia, even though per capita consumption and wealth considerably increased in the past. Dividends are often assumed to follow a lognormal distribution so that they cannot become negative.

There are plenty of CAPM-anomalies documented in the literature. We restrict our analysis to three stylized facts. First is the fact that the slope of the empirical security market line is smaller than the one in the CAPM, i.e. average returns on stocks with a low beta lie below the security market line of the CAPM whereas average returns on

[29] See for example the survey by Graham and Harvey (2001).
[30] For a list of necessary and sufficient conditions to derive the CAPM see Berk (1997).

stocks with a high beta lie above (see Black et al. (1972) and Frazzini and Pedersen (2014) for more recent results). Second is the value premium puzzle (see Basu (1977)): Average returns on stocks with a high fundamental-to-price ratio (value stocks) are higher than predicted by the CAPM. Third is the size premium puzzle (see Banz (1981)): Stocks with a small market capitalization yield, on average, positive abnormal returns with respect to the CAPM.[31]

The main finding of this chapter is that the CRRA-lognormal framework can at least qualitatively explain the premium on low-beta stocks, the size premium and the value premium. This suggests that some important features of preferences and the dividend distribution to better understand the cross-section of stock returns are not accounted for by the CAPM. However, we also find that the size of the CAPM-anomalies in the CRRA-lognormal model is too small and closely linked to the equity premium puzzle. Our theoretical results therefore support the view that there is a common explanation for the equity premium puzzle and cross-sectional anomalies.

The cross-sectional properties of the CRRA-lognormal framework have attracted relatively little attention in the theoretical literature. One reason is that closed-form solutions for prices and returns in the CRRA-lognormal model with multiple assets have become available only recently: Cochrane et al. (2008) provide closed-form solutions in a continuous-time model with logarithmic utility and two risky assets. Martin (2013) generalizes their results to power utility. The closed-form solutions, however, are rather intractable for further analytical results on CAPM-anomalies in the CRRA-lognormal framework. We therefore take a different route and restrict ourselves to a class of economies within the general CRRA-lognormal framework that yields tractable closed-form solutions. More precisely, we additionally assume that total dividends are lognormally distributed across firms.

A closely related strand of literature is the one on higher-order CAPM. The underlying theory goes back to Kraus and Litzenberger (1983): A monotone increasing strictly

[31]For more recent results on the size and the value premium see for example Fama and French (2012) and references therein.

concave utility with nonincreasing absolute risk aversion, which includes the case of CRRA utility, implies a preference for positive skewness. The CAPM should therefore not only price the covariance with market returns, but also the co-skewness. Dittmar (2002) argues that positive and decreasing marginal utility with decreasing absolute risk aversion and decreasing absolute prudence, which includes the case of CRRA utility, implies not only preference for skewness, but also kurtosis aversion. The CAPM should therefore price the co-kurtosis as well. On one hand, his empirical analysis shows that skewness and kurtosis capture the size and the value premium. On the other hand, the model with CRRA preferences is rejected.

Our study provides theoretical support for a connection between the preference for higher-order moments and CAPM-anomalies. Note that the lognormal distribution is positively skewed and has a positive kurtosis. The preference for skewness and kurtosis aversion should therefore partly offset each other. Our results can be understood as the net pricing impact of all higher-order moments.

Other related studies comparing the CRRA-lognormal model to the CAPM find that the CAPM holds approximately in the CRRA-lognormal framework. In their book, Campbell and Viceira (2002) argue that in discrete time an approximate CAPM holds when time intervals are short. Herings and Kübler (2007) demonstrate computationally that the CAPM holds approximately. This is in line with our quantitative results.

The paper is structured as follows. In Section 3.1, we define the model and report closed-form solutions for price-dividend ratios, expected returns, CAPM betas and CAPM pricing errors in the CRRA-lognormal framework. Section 3.2 discusses the relation between the CAPM pricing error and different CAPM-anomalies. There we show that the CRRA-lognormal framework can qualitatively explain CAPM-anomalies in the cross-section of stock returns. Section 3.3 discusses the size of the generated CAPM-anomalies with respect to different parameter values.

3.1 The Two-Period Economy

In this chapter we analyze a two-period economy. We keep the model as simple as possible in order to get tractable closed-form solutions. Two is the minimum number of periods required to discuss the relation between expected returns and beginning-of-period prices and price-dividend ratios.

The economy consists of a continuum of firms.[32] Firms are characterized by their dividend payments and accordingly indexed by $\rho \in \mathbb{R}$. A firm of type ρ pays a total dividend of D_t^ρ at time $t = 0, 1$. We impose the common assumption that the total dividend of each firm is lognormally distributed. In addition, we assume a lognormal distribution of dividends across firms as specified in Assumption 3.1.[33]

Assumption 3.1 *Total dividends are lognormally distributed across time and across firms. More precisely, a firm of type ρ pays a total dividend*

$$D_t^\rho = e^{\rho y_t} \quad \textit{for } t = 0, 1,$$

where y_0 is a constant, $y_1 \sim \mathcal{N}(\mu, \sigma^2)$ and $\rho \sim \mathcal{N}(\mu_\rho, \sigma_\rho^2)$.

Note that Assumption 3.1 incorporates a common source of risk across assets, but no idiosyncratic risk component. The effect of idiosyncratic risk on our results is discussed in Section 3.2.3.

There is a representative agent who initially owns all firms in the economy. Since there is a continuum of firms, the number of type-ρ firms is described by a density function $f(\rho)$.[34] The value of a type-ρ firm is denoted by q^ρ. The agent chooses

[32] This assumption simplifies notation, but is not necessary. The number of firms can be discretized.

[33] Elmiger (2010) documents that the cross-section of total dividends follows approximately a lognormal distribution except for the tails of the distribution.

[34] The total number of firms is normalized to one.

portfolio holdings θ^ρ to maximize the expected utility derived from consumption c_t at time $t = 0, 1$. The objective function is

$$u(c_0) + \delta \mathbf{E}\left[u(c_1)\right]$$

subject to the budget constraints

$$\begin{aligned} c_0 + \int_{-\infty}^{\infty} q^\rho \theta^\rho d\rho &= \int_{-\infty}^{\infty} (q^\rho + D_0^\rho) f(\rho) d\rho \\ c_1 &= \int_{-\infty}^{\infty} D_1^\rho \theta^\rho d\rho, \end{aligned}$$

The representative agent has CRRA preferences $u(c) = (c)^{1-\gamma}/(1-\gamma)$, where $\gamma > 0$ denotes the coefficient of relative risk aversion. Future expected utility is discounted with the factor $0 < \delta < 1$. In equilibrium the agent owns all firms entirely and consumption equals aggregate dividends: $c_t = D_t^M$ at time $t = 0, 1$. Aggregate dividends at time $t = 0, 1$ are given by

$$D_t^M = \int_{-\infty}^{\infty} D_t^\rho f(\rho) d\rho = \int_{-\infty}^{\infty} e^{\rho y_t} f(\rho) d\rho = e^{\mu_\rho y_t + \frac{1}{2}\sigma_\rho^2 y_t^2},$$

where $f(\rho)$ denotes the density function of the normal distribution $\mathcal{N}(\mu_\rho, \sigma_\rho^2)$. We see that aggregate dividends are, in general, not lognormally distributed. They are only approximately lognormal if the cross-sectional dispersion of dividends, σ_ρ^2, is very small. Throughout the paper, we assume without loss of generality that $\mu_\rho = 1$.[35]

3.1.1 Asset prices and returns

In this section, we report closed-form solutions for price-dividend ratios and expected returns. The derivations of the closed-form solutions are in the appendix.

[35]Results for $\mu_\rho \neq 1$ can be obtained by scaling μ, σ and σ_ρ accordingly.

Proposition 3.1 *The price-dividend ratio of a firm of type ρ is*

$$\frac{q^\rho}{D_0^\rho} = \frac{\delta}{\sqrt{1+\gamma\sigma_\rho^2\sigma^2}} e^{\frac{1}{2\sigma^2}\frac{\left((\rho-\gamma-\gamma\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0)\right)^2}{1+\gamma\sigma_\rho^2\sigma^2}-\frac{1}{2\sigma^2}(\mu-y_0)^2}.$$

Note that the price-dividend ratio depends not only on average expected logarithmic dividend growth $\mu - y_0$, but also on the current average level of logarithmic dividends.

Corollary 3.1 *In the case of a single firm the formula for the price-dividend ratio reduces to*

$$\frac{q}{D_0} = \delta e^{(1-\gamma)(\mu-y_0)+\frac{1}{2}(1-\gamma)^2\sigma^2}.$$

Corollary 3.1 directly follows from Proposition 3.1 by setting $\rho = 1$ and $\sigma_\rho = 0$. Next, we compute expected returns. The return on assets of a type-ρ firm in our two-period model is given by $\frac{D_1^\rho}{q^\rho}$[36] and the return on the market portfolio is given by $\frac{D_1^M}{q^M}$.

Proposition 3.2 *Suppose that $\sigma_\rho^2\sigma^2 < 1$. Then the expected return on an asset of type ρ, the expected return on the market portfolio and the riskless rate are*

$$\mathbf{E}\left[R^\rho\right] = \frac{\sqrt{1+\gamma\sigma_\rho^2\sigma^2}}{\delta} e^{\rho(\mu-y_0)+\frac{1}{2}\rho^2\sigma^2-\frac{1}{2\sigma^2}\frac{\left((\rho-\gamma-\gamma\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0)\right)^2}{1+\gamma\sigma_\rho^2\sigma^2}+\frac{1}{2\sigma^2}(\mu-y_0)^2},$$

$$\mathbf{E}\left[R^M\right] = \frac{\sqrt{1-(1-\gamma)\sigma_\rho^2\sigma^2}}{\delta\sqrt{1-\sigma_\rho^2\sigma^2}} e^{\frac{1}{2\sigma^2}\frac{\left((1+\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0)\right)^2}{1-\sigma_\rho^2\sigma^2}-\frac{1}{2\sigma^2}\frac{\left((1-\gamma)(1+\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0)\right)^2}{1-(1-\gamma)\sigma_\rho^2\sigma^2}},$$

$$R^f = \frac{\sqrt{1+\gamma\sigma_\rho^2\sigma^2}}{\delta} e^{-\frac{1}{2\sigma^2}\frac{(-\gamma(1+\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0))^2}{1+\gamma\sigma_\rho^2\sigma^2}+\frac{1}{2\sigma^2}(\mu-y_0)^2}.$$

The assumption $\sigma_\rho^2\sigma^2 < 1$ is used in the proof and guarantees that the formulas given above are well-defined. We will see later that this condition includes sets of reasonable parameter values.

[36] The return equals total future dividends divided by the current market capitalization since the asset supply is constant over time in our model.

Corollary 3.2 *In the case of a single firm the formula for the expected return on the risky asset and the formula for the riskless asset reduce to*

$$\mathbf{E}[R] = \frac{1}{\delta} e^{\gamma(\mu - y_0) + (\gamma - \frac{1}{2}\gamma^2)\sigma^2},$$
$$R^f = \frac{1}{\delta} e^{\gamma(\mu - y_0) - \frac{1}{2}\gamma^2\sigma^2}.$$

Corollary 3.2 directly follows from Proposition 3.2 by setting $\rho = 1$ and $\sigma_\rho = 0$.

3.1.2 CAPM beta and CAPM pricing error

In the CRRA-lognormal framework under study, the CAPM does not hold. Expected excess returns therefore must equal expected excess returns as predicted by the CAPM plus some error term. Consistent with the notation of the empirical literature on the CAPM we call this systematic deviation from the CAPM alpha. This section provides closed-form solutions for the CAPM beta and the CAPM alpha. A detailed analysis of the qualitative and quantitative properties of the CAPM pricing error follows in Section 3.2 and Section 3.3. Section 3.2 shows that the CAPM pricing error can at least qualitatively explain CAPM-anomalies in the cross-section of stock returns, whereas Section 3.3 discusses the comparative statics.

The CAPM states that the expected excess return on an asset of type ρ is

$$\mathbf{E}[R^\rho] - R^f = \beta^\rho \left(\mathbf{E}\left[R^M\right] - R^f\right), \quad \text{where} \quad \beta^\rho = \frac{\mathrm{Cov}(R^\rho, R^M)}{\mathrm{Var}(R^M)}.$$

Proposition 3.3 *Suppose that $\sigma_\rho^2\sigma^2 < \frac{1}{2}$. Then the CAPM beta of an asset of type ρ is*

$$\beta^\rho = \frac{g(1-\gamma, 0)\left[g(1, \rho\sigma^2)g(0,0) - g(1,0)g(0, \rho\sigma^2)\right]}{g(-\gamma, \rho\sigma^2)\left[g(2,0)g(0,0) - g(1,0)^2\right]},$$

where

$$g(a,b) \equiv \frac{e^{\frac{1}{2\sigma^2}\frac{\left(a(1+\sigma_\rho^2 y_0)\sigma^2+b+(\mu-y_0)\right)^2}{1-a\sigma_\rho^2\sigma^2}}}{\sqrt{1-a\sigma_\rho^2\sigma^2}}.$$

The condition $\sigma_\rho^2\sigma^2 < \frac{1}{2}$ is more restrictive than the condition in Proposition 3.2. However, we will show that it still contains sets of reasonable parameter values. This condition is introduced for the proof of Proposition 3.3 and guarantees that the above formula is well-defined.

In the CCAPM with CRRA preferences we have

$$\mathbf{E}\left[R^\rho\right]-R^f = \beta_{con}^\rho\left(\mathbf{E}\left[R^M\right]-R^f\right), \quad \text{where} \quad \beta_{con}^\rho = \frac{\text{Cov}\left(R^\rho,(R^M)^{-\gamma}\right)}{\text{Cov}\left(R^M,(R^M)^{-\gamma}\right)}.$$

Thus, the CAPM pricing error in the CRRA-lognormal framework is given by

$$\alpha^\rho = \left(\beta_{con}^\rho-\beta^\rho\right)\left(\mathbf{E}[R^M]-R^f\right).$$

Proposition 3.4 *The CAPM pricing error of an asset of type ρ is*

$$\alpha^\rho = \left(\beta_{con}^\rho-\beta^\rho\right)\left(\mathbf{E}[R^M]-R^f\right),$$

where β, $\mathbf{E}[R^M]$, R^f are given in Proposition 3.2 and 3.3, and

$$\beta_{con}^\rho = \frac{h\big((1-\gamma)\eta,1-\gamma\big)\left[h\big(\rho-\gamma\eta,-\gamma\big)h(0,0)-h\big(-\gamma\eta,-\gamma\big)h\big(\rho,0\big)\right]}{h\big(\rho-\gamma\eta,-\gamma\big)\left[h\big((1-\gamma)\eta,1-\gamma\big)h(0,0)-h\big(\eta,1\big)h\big(-\gamma\eta,-\gamma\big)\right]},$$

where

$$h(a,b) \equiv \frac{e^{\frac{1}{2\sigma^2}\frac{\left(a\sigma^2+(\mu-y_0)\right)^2}{1-b\sigma_\rho^2\sigma^2}}}{\sqrt{1-b\sigma_\rho^2\sigma^2}},$$

$$\eta \equiv 1+\sigma_\rho^2 y_0.$$

3.2 CAPM Anomalies

The empirical literature documents plenty of CAPM-anomalies, some of them are related to each other in one way or another. We restrict ourselves to three anomalies that have a long history in the literature: the low-beta premium, the size premium and the value premium.

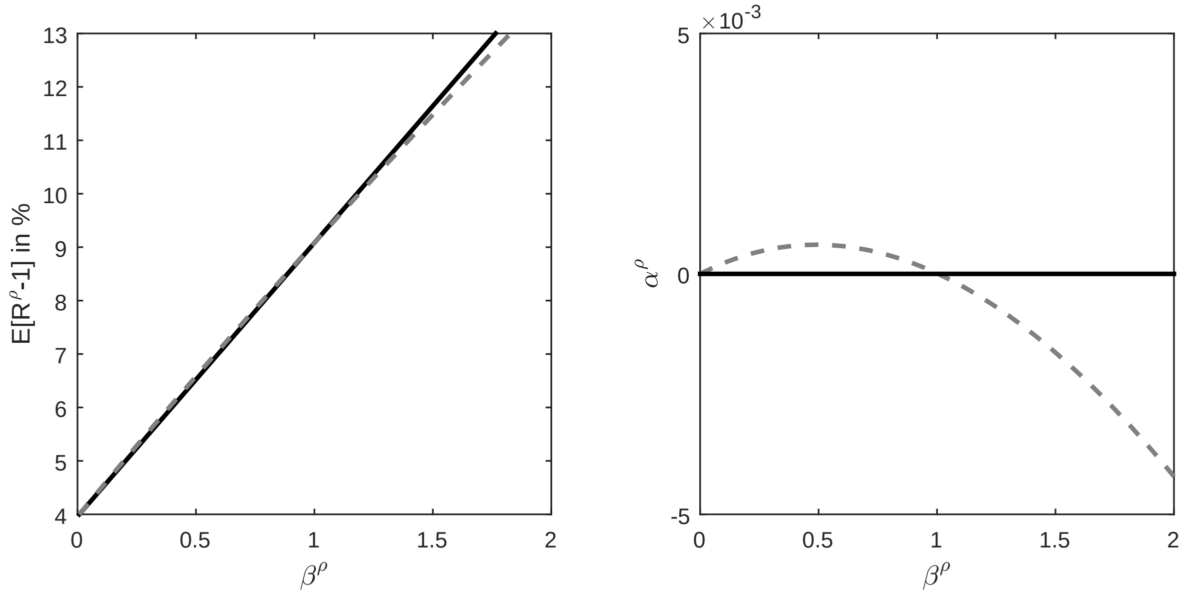
3.2.1 Low-beta premium

Early empirical studies on the validity of the CAPM like Black et al. (1972) or the more recent study by Frazzini and Pedersen (2014) suggest that the security market line in the CAPM is too steep: They find that portfolios with a low beta lie above the security market line, whereas portfolios with a high beta lie below the security market line. Is the CRRA-lognormal model consistent with this stylized fact? We find that the security market line can indeed be steeper in the CRRA-lognormal model than in the CAPM for a very large set of assets.

Let us start with a numerical example to illustrate our point. We set the coefficient of relative risk aversion to one and assume a discount factor of 0.96. The dividend parameters are chosen in a way that the resulting cross-section of total dividends and aggregate dividend growth take reasonable values: $y_0 = 16.70$, $\mu = 16.72$, $\sigma = 0.2$, $\sigma_\rho = 0.076$.[37] The implied expected logarithmic aggregate dividend growth is 2.2% and the standard deviation is 20.1%. The mean dividend payment across assets is \$40 million and the standard deviation is \$80 million. In this numerical example, assets with a small beta yield a positive alpha, whereas assets with a high beta yield a negative alpha (see Figure 3.1). The figure also shows that the CAPM pricing error, α^ρ, is not a monotone function of β^ρ. There are pairs of assets, where the asset with a higher beta yields a higher alpha. The security market line in the CRRA-lognormal model therefore looks more like a curve rather than a straight line. Is this contradictory to the observed

[37] Reasonable means that aggregate dividend growth and the size of dividend payments across assets lie in the range reported by Chen (2009) and DeAngelo et al. (2004).

Figure 3.1: The left plot shows the security market line of the CAPM (black solid line) and of the CCAPM (grey dashed line), whereas the right plot shows the difference between the two security market lines. The parameters are $\delta = 0.96$, $\gamma = 1$, $y_0 = 16.7$, $\mu = 16.72$, $\sigma = 0.2$, $\sigma_\rho = 0.076$. They generate expected logarithmic aggregate dividend growth of 2.2% and standard deviation of 20.1%. The implied cross-section of dividends has a mean of \$40 million and a standard deviation of \$80 million.



almost straight line observed in empirical studies? Note that σ_ρ in our example is quite small. In fact, around 99% of all assets have a beta between 0.7 and 1.1. The security market line of the CRRA-lognormal model is flatter than the security market line of the CAPM everywhere in this range.

In a next step, we will analyze analytically to what extent the above result depends on the chosen parameter values. Can we always find a decreasing relation between α^ρ and β^ρ over a large range of assets? First, let us note that there is not a one-to-one relation between α^ρ and β^ρ as the previous example shows. Therefore α^ρ cannot decrease with respect to β^ρ everywhere. Can we at least find a decreasing relation over the β^ρ-values that occur most often across assets? To address this question, we first discuss the properties of α^ρ and β^ρ as functions of ρ.

The function β^ρ has either one extremum or three extrema depending on the parameter values (see Lemma 3.1). The largest extremum is in either case a minimum, which means that β^ρ is an increasing function for larger values of ρ. This will be important when we answer the question whether α^ρ tends to decrease with β^ρ.

Lemma 3.1 *One of the following holds:*

- *If* $(\gamma+1)e^{-\frac{(\mu\sigma_\rho^2+1)^2}{2(1-\sigma^2\sigma_\rho^2)\sigma_\rho^2}} \geq \gamma(1-\sigma^2\sigma_\rho^2)$ *holds, then* β^ρ *has one local minimum:*

$$\rho_\beta^{min} = -\frac{(\mu\sigma_\rho^2+1)}{\sigma^2\sigma_\rho^2}.$$

- *If* $(\gamma+1)e^{-\frac{(\mu\sigma_\rho^2+1)^2}{2(1-\sigma^2\sigma_\rho^2)\sigma_\rho^2}} < \gamma(1-\sigma^2\sigma_\rho^2)$ *holds, then* β^ρ *has one local maximum and two local minima:*

$$\rho_\beta^{max} = -\frac{(\mu\sigma_\rho^2+1)}{\sigma^2\sigma_\rho^2},$$

$$\rho_\beta^{min,1} = \frac{\sqrt{2(\sigma^2\sigma_\rho^2-1)\ln\left(-\frac{(\gamma+1)}{\gamma(\sigma^2\sigma_\rho^2-1)}\right)\sigma_\rho^2+(\mu\sigma_\rho^2+1)^2}-\mu\sigma_\rho^2-1}{\sigma^2\sigma_\rho^2},$$

$$\rho_\beta^{min,2} = \frac{-\sqrt{2(\sigma^2\sigma_\rho^2-1)\ln\left(-\frac{(\gamma+1)}{\gamma(\sigma^2\sigma_\rho^2-1)}\right)\sigma_\rho^2+(\mu\sigma_\rho^2+1)^2}-\mu\sigma_\rho^2-1}{\sigma^2\sigma_\rho^2}.$$

Lemma 3.1 is straightforward, but lengthy to prove. We use a symbolic calculator to compute the zeros of the first derivative and to determine the signs of the second derivative.[38]

Recall that the CAPM pricing error is given by $\alpha^\rho = (\beta_{con}^\rho - \beta^\rho)\big(\mathbf{E}[R^M]-R^f\big)$. Since the expected market excess return does not depend on ρ, we focus our analysis on $(\beta_{con}^\rho - \beta^\rho)$. The function $(\beta_{con}^\rho - \beta^\rho)$ has either one extremum or three extrema depending on the choice of parameters (see Lemma 3.2). Note that the largest extremum is in either case a maximum. $(\beta_{con}^\rho - \beta^\rho)$ therefore is a decreasing function for values larger than the respective maximum.

[38] Intermediate results are available on request.

Lemma 3.2 *Define*

$$
\begin{aligned}
C \equiv & \sqrt{\gamma\sigma^2\sigma_\rho^2+1}\sqrt{1-\sigma^2\sigma_\rho^2}\,e^{\frac{(\mu\sigma_\rho^2+1)^2}{2((\gamma-1)\sigma^2\sigma_\rho^2+1)\sigma_\rho^2}+\frac{\gamma(\mu\sigma_\rho^2+1)^2}{2(\gamma\sigma^2\sigma_\rho^2+1)\sigma_\rho^2}+\frac{(\mu\sigma_\rho^2+1)^2}{2(\sigma^2\sigma_\rho^2-1)\sigma_\rho^2}} \\
& -\sqrt{(\gamma-1)\sigma^2\sigma_\rho^2+1}\,e^{\frac{\gamma(\mu\sigma_\rho^2+1)^2}{2((\gamma-1)\sigma^2\sigma_\rho^2+1)\sigma_\rho^2}}, \\
D \equiv & \sqrt{(\gamma-1)\sigma^2\sigma_\rho^2+1}(\sigma^2\sigma_\rho^2-1)^2 e^{\frac{\gamma(\mu\sigma_\rho^2+1)^2}{2((\gamma-1)\sigma^2\sigma_\rho^2+1)\sigma_\rho^2}+\frac{(\mu\sigma_\rho^2+1)^2}{2(\sigma^2\sigma_\rho^2-1)\sigma_\rho^2}} \\
& -\sqrt{\gamma\sigma^2\sigma_\rho^2+1}\left(1-\sigma^2\sigma_\rho^2\right)^{\frac{3}{2}}\sqrt{1-2\sigma^2\sigma_\rho^2}\,e^{\frac{(\mu\sigma_\rho^2+1)^2}{2((\gamma-1)\sigma^2\sigma_\rho^2+1)\sigma_\rho^2}+\frac{\gamma(\mu\sigma_\rho^2+1)^2}{2(\gamma\sigma^2\sigma_\rho^2+1)\sigma_\rho^2}+\frac{(\mu\sigma_\rho^2+1)^2}{(2\sigma^2\sigma_\rho^2-1)\sigma_\rho^2}}.
\end{aligned}
$$

Then one of the following holds:

- *If* $(\gamma+1)\sqrt{1-2\sigma^2\sigma_\rho^2}\,Ce^{-\frac{(\mu\sigma_\rho^2+1)^2}{(1-\sigma^2\sigma_\rho^2)\sigma_\rho^2}}+\gamma D \leq 0$ *holds, then* $(\beta_{con}^\rho-\beta^\rho)$ *has one local maximum:*

$$
\rho_{(\beta_{con}-\beta)}^{max} = -\frac{(\mu\sigma_\rho^2+1)}{\sigma^2\sigma_\rho^2}.
$$

- *If* $(\gamma+1)\sqrt{1-2\sigma^2\sigma_\rho^2}\,Ce^{-\frac{(\mu\sigma_\rho^2+1)^2}{(1-\sigma^2\sigma_\rho^2)\sigma_\rho^2}}+\gamma D > 0$ *holds, then* $(\beta_{con}^\rho-\beta^\rho)$ *has one local minimum and two local maxima:*

$$
\begin{aligned}
\rho_{(\beta_{con}-\beta)}^{min} &= -\frac{(\mu\sigma_\rho^2+1)}{\sigma^2\sigma_\rho^2}, \\
\rho_{(\beta_{con}-\beta)}^{max,1} &= \frac{\sqrt{2(\sigma^2\sigma_\rho^2-1)\left((2\sigma^2\sigma_\rho^2-1)\ln\left(-\frac{(\gamma+1)\sqrt{1-2\sigma^2\sigma_\rho^2}\,C}{\gamma D}\right)\sigma_\rho^2+(\mu\sigma_\rho^2+1)^2\right)}-(\mu\sigma_\rho^2+1)\sqrt{2\sigma^2\sigma_\rho^2-1}}{\sigma^2\sigma_\rho^2\sqrt{2\sigma^2\sigma_\rho^2-1}}, \\
\rho_{(\beta_{con}-\beta)}^{max,2} &= \frac{-\sqrt{2(\sigma^2\sigma_\rho^2-1)\left((2\sigma^2\sigma_\rho^2-1)\ln\left(-\frac{(\gamma+1)\sqrt{1-2\sigma^2\sigma_\rho^2}\,C}{\gamma D}\right)\sigma_\rho^2+(\mu\sigma_\rho^2+1)^2\right)}-(\mu\sigma_\rho^2+1)\sqrt{2\sigma^2\sigma_\rho^2-1}}{\sigma^2\sigma_\rho^2\sqrt{2\sigma^2\sigma_\rho^2-1}}.
\end{aligned}
$$

Suppose that the expected market risk premium is positive and α^ρ exhibits the same properties as $(\beta_{con}^\rho-\beta^\rho)$. Combining the findings in Lemma 3.1 and Lemma 3.2, we conclude that α^ρ is decreasing in β^ρ for values of ρ larger than the largest extremum of both α^ρ and β^ρ. We can therefore determine a number of assets for which the CAPM pricing error is a decreasing function of the CAPM beta (see Proposition 3.5).

Proposition 3.5 *Suppose that the expected market risk premium is positive.* α^{ρ} *then is decreasing in* β^{ρ} *for at least a fraction* $\left(1-\Phi\left(\frac{\rho^*_{SML}-\mu_\rho}{\sigma_\rho}\right)\right)$ *of the firms, where* ρ^*_{SML} *denotes the largest extremum of* $(\beta^{\rho}_{con}-\beta^{\rho})$ *and* β^{ρ} *and* $\Phi(\cdot)$ *denotes the complementary cumulative distribution function of the standard normal distribution.*

Note that the fraction of firms $\left(1-\Phi\left(\frac{\rho^*_{SML}-1}{\sigma_\rho}\right)\right)$ corresponds to the number of firms, since we normalized the total number of firms to one. Proposition 3.5 follows immediately from Lemma 3.1 and 3.2 and from the assumption that ρ is normally distributed with mean μ_ρ and standard deviation σ_ρ. Table 3.1 shows that the lower bound to the range of firms where the security market line in the CCAPM is flatter than the security market line of the CAPM is quite large for different parameter values.

3.2.2 Size and value premium

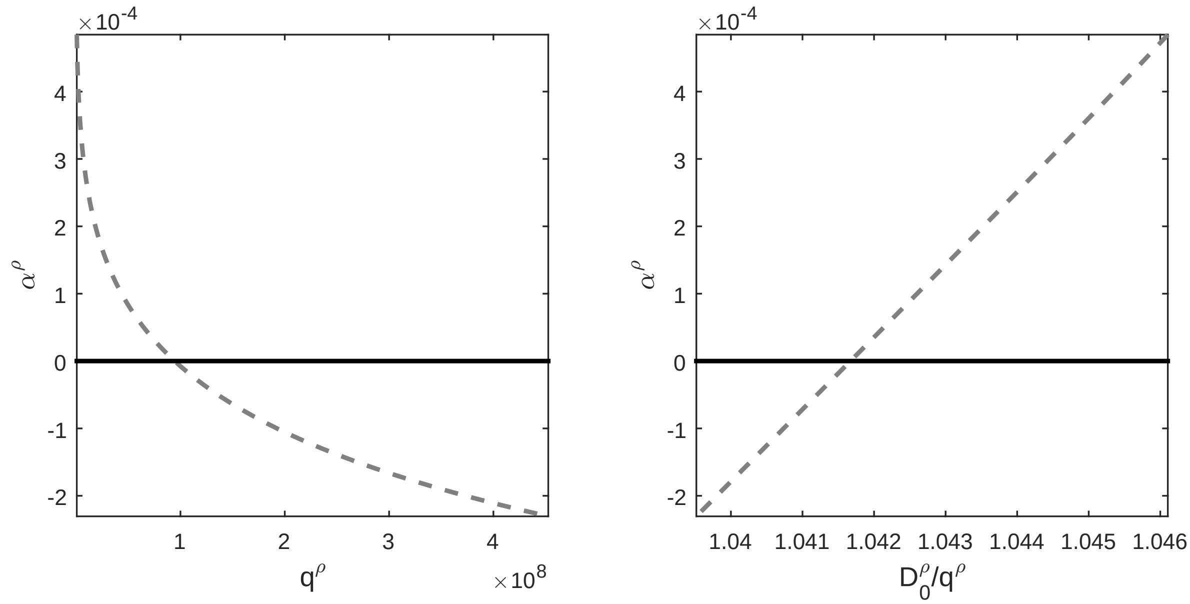
The size and the value premium go back to the empirical studies of Banz (1981) and Basu (1977). Banz (1981) shows that stocks with small market capitalizations have higher average returns after adjustment for market risk than stocks with large market capitalizations. In other words, small stocks have a higher alpha than large stocks. Basu (1977) finds that stocks with low price-earnings ratios (value stocks) have higher average returns after adjustment for market risk than stocks with high price-earnings ratios (growth stocks). Thus, value stocks have a higher alpha than growth stocks. Empirical studies on other price-to-fundamental and fundamental-to-price ratios yield similar results (see the survey by Fama and French (2004) and references therein). Our analysis will focus on the dividend yield.

Let us first illustrate the size and the value premium using the numerical example from the previous section. Figure 3.2 shows the CAPM pricing error in relation to the market capitalization and dividend yield for 99% of all firms. We see that there is a decreasing relation between the CAPM pricing error and the market capitalization, whereas the CAPM pricing error increases with the dividend yield.

Table 3.1: This table shows the standard normal cumulative distribution function for different parameter values. The complementary cumulative distribution function is a lower bound to the fraction of firms, where the security market line in the CCAPM is flatter than the security market line in the CAPM.

δ	γ	μ	σ	y_0	σ_ρ	$\mathbf{E}\left[\ln\left(\frac{D_1^M}{D_0^M}\right)\right]$	$\sigma\left(\ln\left(\frac{D_1^M}{D_0^M}\right)\right)$	$\overline{D_0^\rho}$	$\overline{\lvert D_0^\rho - \overline{D_0^\rho}\rvert}$	ρ^*_{SML}	$\Phi\left(\frac{\rho^*_{SML}-1}{\sigma_\rho}\right)$
0.96	1	16.72	0.2	16.7	0.076	2.2%	20.1 %	$40 mn	$80 mn	0.5577	3.0e-9
0.96	1	16.72	0.2	16.7	0.1	2.4%	20.3 %	$70 mn	$300 mn	0.5963	2.7e-5
0.96	1	16.72	0.2	16.7	0.12	2.5%	20.6 %	$100 mn	$1 bn	0.6364	0.0012
0.96	1	16.72	0.2	16.7	0.14	2.7%	21.1 %	$300 mn	$4 bn	0.6839	0.0120
0.96	1	16.72	0.2	16.7	0.16	2.9%	21.8 %	$600 mn	$23 bn	0.7389	0.0513
0.96	1	16.72	0.2	16.7	0.18	3.2%	22.8 %	$2 bn	$200 bn	0.8015	0.1351
0.96	1	16.72	0.2	16.7	0.2	3.4%	24.1 %	$5 bn	$1 tn	0.8721	0.2612
0.96	1	16.72	0.4	16.7	0.076	2.2%	40.2 %	$40 mn	$80 mn	0.5782	1.4e-8
0.96	1	16.72	0.6	16.7	0.076	2.3%	60.3 %	$40 mn	$80 mn	0.6121	1.7e-7
0.96	1	16.72	0.8	16.7	0.076	2.4%	80.4 %	$40 mn	$80 mn	0.6584	3.4e-6
0.96	1	16.8	0.2	16.7	0.076	11.0%	20.1 %	$40 mn	$80 mn	0.5580	3.0e-9
0.96	1	17	0.2	16.7	0.076	32.9%	20.1 %	$40 mn	$80 mn	0.5586	3.2e-9
0.96	3	16.72	0.2	16.7	0.076	2.2%	20.1 %	$40 mn	$80 mn	0.5665	5.9e-9
0.96	5	16.72	0.2	16.7	0.076	2.2%	20.1 %	$40 mn	$80 mn	0.5753	1.1e-8
0.96	1	16.71	0.1	16.7	0.076	1.1%	10.1 %	$40 mn	$80 mn	0.5526	2.0e-9
0.96	1	16.71	0.2	16.7	0.076	1.1%	20.1 %	$40 mn	$80 mn	0.5577	2.9e-9

Figure 3.2: The left plot illustrates the small size premium and shows the CAPM pricing error as a function of market capitalization. The right plot illustrates the value premium and shows the CAPM pricing error as a function of the dividend yield. The parameters are $\delta = 0.96$, $\gamma = 1$, $y_0 = 16.7$, $\mu = 16.72$, $\sigma = 0.2$, $\sigma_\rho = 0.076$. They generate expected logarithmic aggregate dividend growth of 2.2% and standard deviation of 20.1%. The implied cross-section of dividends has a mean of $40 million and a standard deviation of $80 million.



The question is whether the size and the value premium generally arise in this model or depend on our choice of parameters? Since one can show that there is no one-to-one relationship between the CAPM pricing error and the market capitalization or dividend yield, we first analyze the market capitalization and dividend yield as functions of ρ. We then combine the results with Lemma 3.2 to understand the relation between the CAPM pricing error and the market capitalization or dividend yield.

Lemma 3.3 *Market capitalization q^ρ has one local minimum:*

$$\rho_q^{min} = \frac{\gamma\sigma^2 - \mu}{\sigma^2}.$$

Market capitalization q^ρ as a function of ρ has one local minimum and the largest extremum of $(\beta^\rho_{con} - \beta^\rho)$ is a maximum (see Lemma 3.3 and Lemma 3.2). This implies that there exists a threshold ρ_q^*, above which market capitalization is increasing and the difference in betas $(\beta^{\rho}_{con} - \beta^{\rho})$ is decreasing. In this range of ρ-values, $(\beta^{\rho}_{con} - \beta^{\rho})$ is decreasing in q^{ρ}. If the expected market risk premium is positive, the CAPM pricing error $\alpha^{\rho} = \left(\beta^{\rho}_{con} - \beta^{\rho}\right)\left(\mathbf{E}[R^M] - R^f\right)$ is decreasing with an increasing market capitalization q^{ρ}. Thus, we can determine the number of assets for which the CAPM pricing error is a decreasing function of the market capitalization (see Proposition 3.6). Note that we do not include a table with values of $\left(1 - \Phi\left(\frac{\rho^*_q - \mu_\rho}{\sigma_\rho}\right)\right)$ for different parameter specifications, since the values happen to coincide with the ones in Table 3.1: $\rho^*_q = \rho^*_{SML}$.

Proposition 3.6 *Suppose that the expected market risk premium is positive. The CAPM pricing error α^{ρ} then is decreasing in the market capitalization q^{ρ} for at least a fraction $\left(1 - \Phi\left(\frac{\rho^*_q - \mu_\rho}{\sigma_\rho}\right)\right)$ of all firms, where ρ^*_q denotes the largest extremum of $(\beta^{\rho}_{con} - \beta^{\rho})$ and q^{ρ} and $\Phi(\cdot)$ denotes the complementary cumulative distribution function of the standard normal distribution.*

In order to analyze the value premium, we proceed analogously to the analysis of the low-beta premium and size premium. The dividend yield as a function of ρ has one local maximum (see Lemma 3.4). This means that there is a threshold value above which the dividend yield decreases with increasing values of ρ.

Lemma 3.4 *The dividend yield D^{ρ}_0/q^{ρ} has one local maximum:*

$$\rho^{max}_{D/q} = \frac{\gamma\sigma^2(\sigma^2_\rho y_0 + 1) + y_0 - \mu}{\sigma^2}.$$

The fact that the dividend yield D^{ρ}_0/q^{ρ} as well as the difference in betas $(\beta^{\rho}_{con} - \beta^{\rho})$ decrease above a certain threshold (see Lemma 3.2 and 3.4) implies that $(\beta^{\rho}_{con} - \beta^{\rho})$ increases with increasing values of D^{ρ}_0/q^{ρ}. Therefore if the expected market risk premium is positive, the CAPM pricing error $\alpha^{\rho} = \left(\beta^{\rho}_{con} - \beta^{\rho}\right)\left(\mathbf{E}[R^M] - R^f\right)$ increases with the dividend yield. We can therefore determine the number of assets for which the CAPM pricing error is an increasing function of the dividend yield (see Proposition 3.7).

Proposition 3.7 *Suppose that the expected market risk premium is positive. The CAPM pricing error α^{ρ} then is increasing in the dividend yield D^{ρ}_0/q^{ρ} for at least a fraction*

$\left(1-\Phi\left(\frac{\rho^*_{D/q}-\mu_\rho}{\sigma_\rho}\right)\right)$ *of all firms, where* $\rho^*_{D/q}$ *denotes the largest extremum of* $(\beta^\rho_{con}-\beta^\rho)$ *and* D^ρ_0/q^ρ *and* $\Phi(\cdot)$ *denotes the complementary cumulative distribution function of the standard normal distribution.*

Table 3.2 shows the lower bound to the fraction of firms, where firms with a high dividend-price ratio pay a higher return adjusted for market risk than firms with a low dividend-price ratio. We see that the choice of parameters has a larger influence on the range of firms, where value firms pay a premium. For some parameter values, less than half the firms belong to this range.

3.2.3 Impact of idiosyncratic risk

This section discusses how firm-specific risk affects part of our results. For notational convenience we extend the model to firm-type-specific sources of risk instead of firm-specific sources of risk. However, all arguments carry over to the case of firm-specific risk. While all our results on low-beta stocks remain the same, idiosyncratic risk affects price-related variables and therefore our results on the size and the value premium.

In this section dividends are given by

$$D^\rho_1 = e^{\rho y_1+\epsilon_\rho},$$

where ϵ_ρ denotes a firm-type-specific source of risk. Market capitalizations are

$$q^\rho = \mathbf{E}\left[\delta\left(\frac{c_0}{c_1}\right)^\gamma D^\rho_1\right] = \mathbf{E}\left[\delta\left(\frac{c_0}{c_1}\right)^\gamma e^{\rho y_1+\epsilon_\rho}\right] = \mathbf{E}\left[\delta\left(\frac{c_0}{c_1}\right)^\gamma e^{\rho y_1}\right]\mathbf{E}\left[e^{\epsilon_\rho}\right].$$

We see that market capitalizations depend on the expectation of the idiosyncratic risk factor. On the other hand, idiosyncratic risk has no impact on the CAPM beta and the pricing error. The CAPM beta in the model with idiosyncratic risk is given by

$$\beta^\rho = \frac{\mathrm{Cov}\left(R^\rho, R^M\right)}{\mathrm{Var}\left(R^M\right)} = \frac{\mathrm{Cov}\left(\frac{e^{\rho y_1+\epsilon_\rho}}{q^\rho}, R^M\right)}{\mathrm{Var}\left(R^M\right)} = \frac{\mathrm{Cov}\left(e^{\rho y_1}, R^M\right)\frac{\mathbf{E}[e^{\epsilon_\rho}]}{q^\rho}}{\mathrm{Var}\left(R^M\right)}.$$

Table 3.2: This table shows the standard normal cumulative distribution function for different parameter values. The complementary cumulative distribution function is a lower bound to the fraction of firms, where firms with a higher dividend-price ratio pay higher returns adjusted for market risk than firms with a lower dividend-price ratio.

δ	γ	μ	σ	y_0	σ_ρ	$\mathbf{E}\left[\ln\left(\frac{D_1^M}{D_0^M}\right)\right]$	$\sigma\left(\ln\left(\frac{D_1^M}{D_0^M}\right)\right)$	$\overline{D_0^\rho}$	$\overline{\lvert D_0^\rho - \overline{D_0^\rho}\rvert}$	$\rho^*_{D/q}$	$\Phi\left(\frac{\rho^*_{D/q}-1}{\sigma_\rho}\right)$
0.96	1	16.72	0.2	16.7	0.076	2.2%	20.1 %	\$40 mn	\$80 mn	0.5965	5.5e-8
0.96	1	16.72	0.2	16.7	0.1	2.4%	20.3 %	\$70 mn	\$300 mn	0.6670	4.4e-4
0.96	1	16.72	0.2	16.7	0.12	2.5%	20.6 %	\$100 mn	\$1 bn	0.7405	0.0053
0.96	1	16.72	0.2	16.7	0.14	2.7%	21.1 %	\$300 mn	\$4 bn	0.8273	0.1087
0.96	1	16.72	0.2	16.7	0.16	2.9%	21.8 %	\$600 mn	\$23 bn	0.9275	0.3253
0.96	1	16.72	0.2	16.7	0.18	3.2%	22.8 %	\$2 bn	\$200 bn	1.0411	0.5903
0.96	1	16.72	0.2	16.7	0.2	3.4%	24.1 %	\$5 bn	\$1 tn	1.1680	0.7995
0.96	1	16.72	0.4	16.7	0.076	2.2%	40.2 %	\$40 mn	\$80 mn	0.9715	0.3538
0.96	1	16.72	0.6	16.7	0.076	2.3%	60.3 %	\$40 mn	\$80 mn	1.0409	0.7048
0.96	1	16.72	0.8	16.7	0.076	2.4%	80.4 %	\$40 mn	\$80 mn	1.0652	0.8045
0.96	1	16.8	0.2	16.7	0.076	11.0%	20.1 %	\$40 mn	\$80 mn	0.5580	3.0e-9
0.96	1	17	0.2	16.7	0.076	32.9%	20.1 %	\$40 mn	\$80 mn	0.5586	3.2e-9
0.96	3	16.72	0.2	16.7	0.076	2.2%	20.1 %	\$40 mn	\$80 mn	2.7894	1
0.96	5	16.72	0.2	16.7	0.076	2.2%	20.1 %	\$40 mn	\$80 mn	4.9823	1
0.96	1	16.71	0.1	16.7	0.076	1.1%	10.1 %	\$40 mn	\$80 mn	0.5526	2.0e-9
0.96	1	16.71	0.2	16.7	0.076	1.1%	20.1 %	\$40 mn	\$80 mn	0.8465	0.0217

Inserting the above expression for market capitalizations, we obtain

$$\beta^\rho = \frac{\mathrm{Cov}\left(e^{\rho y_1}, R^M\right) \mathbf{E}\left[\delta \left(\frac{c_0}{c_1}\right)^\gamma e^{\rho y_1}\right]^{-1}}{\mathrm{Var}\left(R^M\right)}.$$

Hence β^ρ does not depend on the idiosyncratic risk component ϵ_ρ. For computing the CAPM pricing error, we proceed analogously. Recall that the pricing error is given by

$$\alpha^\rho = \left(\frac{\mathrm{Cov}\left((R^M)^{-\gamma}, R^\rho\right)}{\mathrm{Cov}\left((R^M)^{-\gamma}, R^M\right)} - \frac{\mathrm{Cov}(R^\rho, R^M)}{\mathrm{Var}(R^M)}\right)\left(\mathbf{E}[R^M] - R^f\right).$$

The only two terms that could possibly depend on idiosyncratic risk are $\mathrm{Cov}\left((R^M)^{-\gamma}, R^\rho\right)$ and $\mathrm{Cov}(R^\rho, R^M)$. The second term does not depend on idiosyncratic risk as shown above and

$$\mathrm{Cov}\left((R^M)^{-\gamma}, R^\rho\right) = \mathrm{Cov}\left((R^M)^{-\gamma}, \frac{e^{\rho y_1 + \epsilon_\rho}}{q^\rho}\right) = \mathrm{Cov}\left((R^M)^{-\gamma}, e^{\rho y_1}\right)\frac{\mathbf{E}[e^{\epsilon_\rho}]}{q^\rho}.$$

Inserting the above expression for market capitalizations, we have

$$\mathrm{Cov}\left((R^M)^{-\gamma}, R^\rho\right) = \mathrm{Cov}\left((R^M)^{-\gamma}, e^{\rho y_1}\right)\mathbf{E}\left[\delta\left(\frac{c_0}{c_1}\right)^\gamma e^{\rho y_1}\right]^{-1}.$$

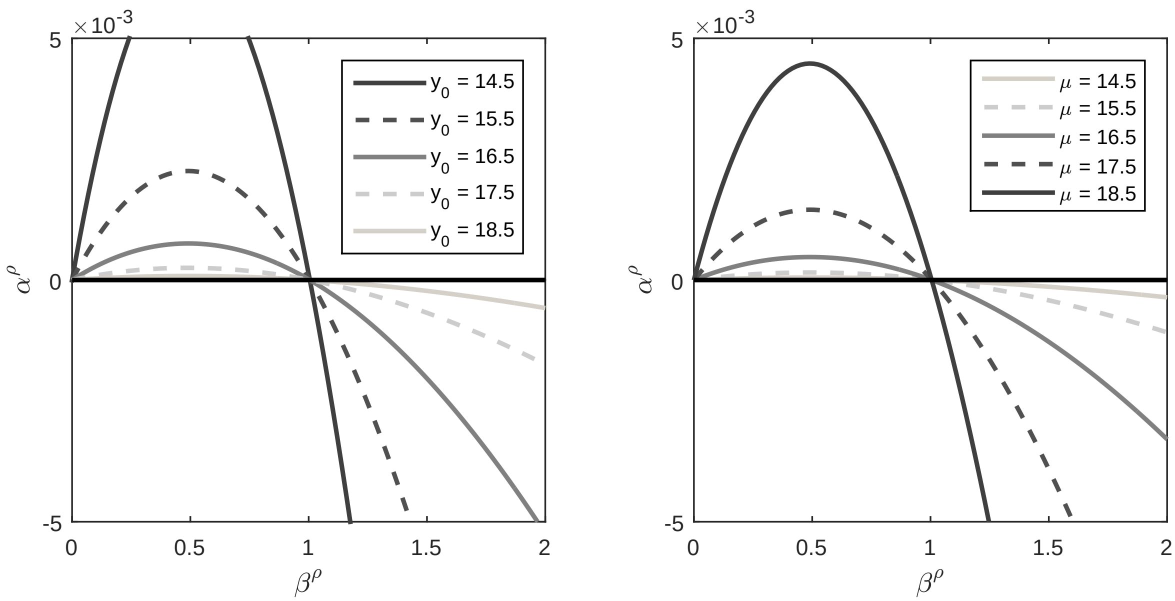
Idiosyncratic risk therefore affects neither the CAPM pricing error nor the CAPM beta. Since the CAPM beta as well as the pricing error do not depend on idiosyncratic risk, we conclude that our previous results in a model without idiosyncratic risk still hold in the presence of idiosyncratic risk.

Apparently idiosyncratic risk has an impact on market capitalizations, but not on the CAPM pricing error. Therefore we cannot say anything about the relation between α^ρ and q^ρ or D_0^ρ/q^ρ respectively without knowing the distribution of idiosyncratic risk across firms. Since we are not aware of any empirical results on the distribution of idiosyncratic risk across firms, we do not attempt to model it.

3.3 Comparative Statics

The CRRA-lognormal framework can at least qualitatively explain different CAPM-anomalies in the cross-section of stock-returns over a large range of firms. The previous results focused mostly on how large this range of firms is depending on different parameter values. We did not discuss the size of the CAPM pricing error. The following analysis shows that the CAPM pricing error is far too small with an order of thousandths for reasonable parameter values. In a sense we should expect it to be too small: the CAPM pricing error $\alpha^{\rho} = \left(\beta^{\rho}_{con} - \beta^{\rho}\right)\left(\mathbf{E}[R^M] - R^f\right)$ is proportional to the equity premium, which is generally known to be too small in the CRRA-lognormal framework. In the following, we discuss graphically how the CAPM pricing error relates to different parameter specifications.[39]

Figure 3.3: The left plot shows the CAPM pricing error as a function of the CAPM beta for different values of y_0, whereas the right plot shows the CAPM pricing error as a function of the CAPM beta for different values of μ. The parameters are $\delta = 0.96$, $\gamma = 1$, $y_0 = 16.7$, $\mu = 16.72$, $\sigma = 0.2$, $\sigma_{\rho} = 0.076$ unless specified differently in the legend of the plot.



[39] The memory of our symbolic calculator does not suffice for an analytical discussion.

3.3.1 Dividend distribution parameters

The two parameters y_0 and μ determine expected logarithmic dividend growth:

$$\mathbf{E}\left[\ln(D_1^\rho/D_0^\rho)\right] = \rho(\mu - y_0).$$

The CAPM pricing error increases with increasing expected logarithmic dividend growth (see Figure 3.3). This follows from the observation that the CAPM pricing error increases with μ and decreases with y_0.

The parameter σ determines the volatility of logarithmic dividend growth $\ln(D_1^\rho/D_0^\rho)$ which equals $\rho\sigma$. The CAPM pricing error increases with increasing values of σ (see Figure 3.4). This brings out the relation between the cross-sectional anomalies and the equity premium puzzle even more. The CRRA-lognormal framework struggles to explain the magnitude of cross-sectional anomalies as well as the high equity premium partly due to the low volatility of consumption growth, which equals aggregate dividend growth in our model.

The cross-sectional parameter σ_ρ controls for the average deviation of logarithmic dividend payments $\ln(D_t^\rho)$ from the mean at time $t = 0, 1$ which equals $\sigma_\rho y_t$. A rise in σ_ρ increases the magnitude of cross-sectional anomalies (see Figure 3.4).

3.3.2 Preference parameters

The impact of the time discount factor δ on the pricing error is easily determined. Note that the discount factor affects neither the CAPM beta nor the CCAPM beta. It only changes the endogenously determined equity premium. An increase in the discount factor decreases the equity premium and thus the CAPM pricing error (see Figure 3.5).

The coefficient of relative risk aversion γ increases the magnitude of cross-sectional anomalies (see Figure 3.5). This again highlights the connection between the equity premium puzzle and the cross-sectional puzzles. Both are low partly due to the low coefficient of relative risk aversion that investors have.

Figure 3.4: The left plot shows the CAPM pricing error as a function of the CAPM beta for different values of σ, whereas the right plot shows the CAPM pricing error as a function of the CAPM beta for different values of σ_ρ. The parameters are $\delta = 0.96$, $\gamma = 1$, $y_0 = 16.7$, $\mu = 16.72$, $\sigma = 0.2$, $\sigma_\rho = 0.076$ unless specified differently in the legend of the plot.

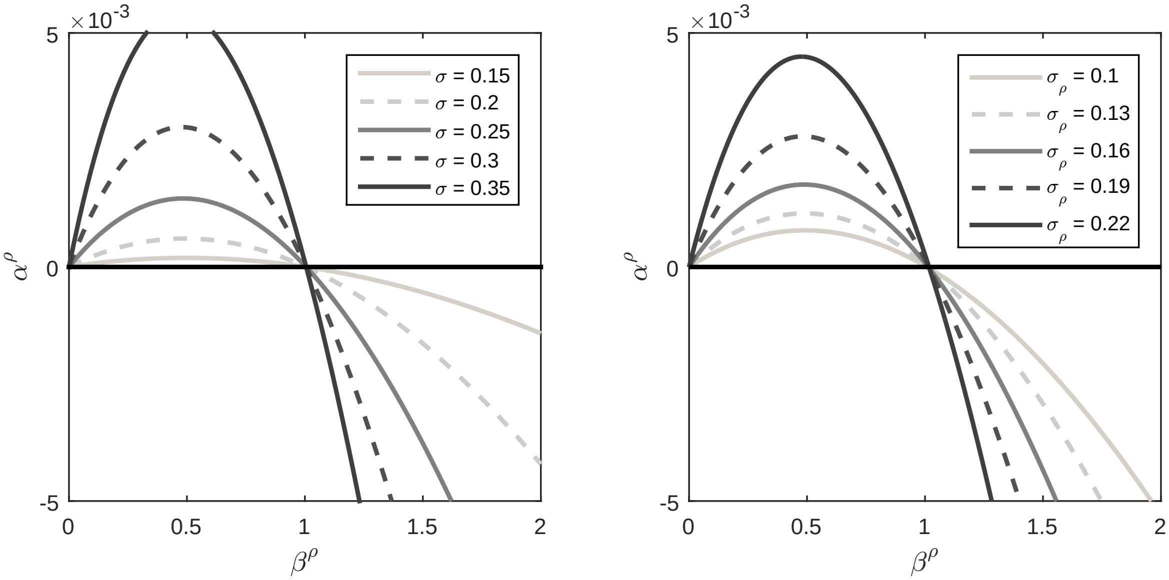
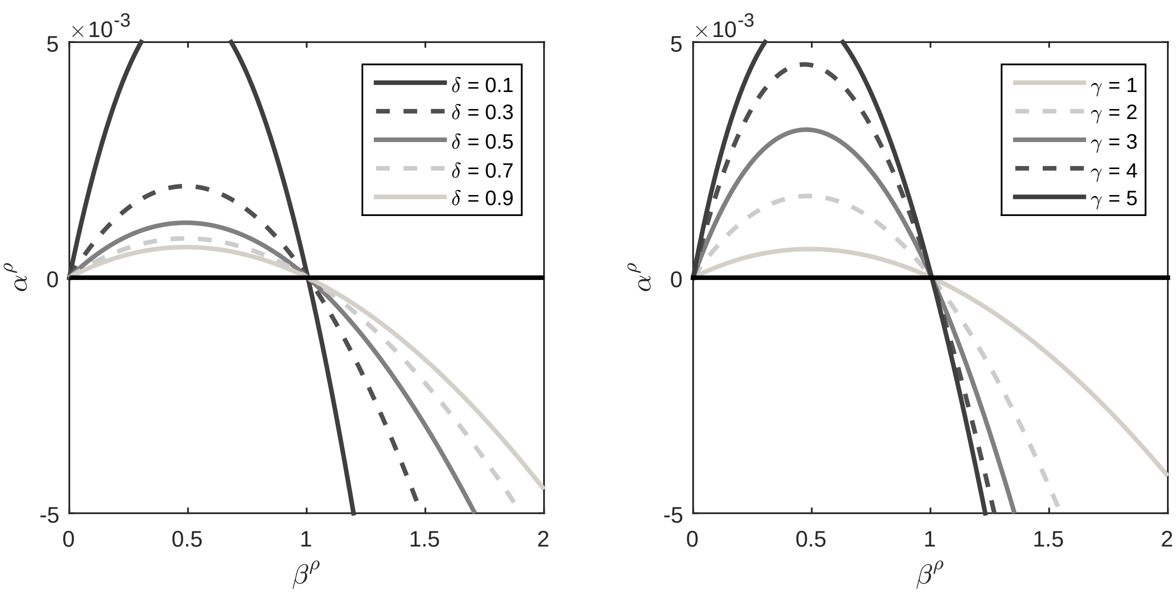


Figure 3.5: The left plot shows the CAPM pricing error as a function of the CAPM beta for different values of γ, whereas the right plot shows the CAPM pricing error as a function of the CAPM beta for different values of δ.The parameters are $\delta = 0.96$, $\gamma = 1$, $y_0 = 16.7$, $\mu = 16.72$, $\sigma = 0.2$, $\sigma_\rho = 0.076$ unless specified differently in the legend of the plot.



3.4 Conclusion

The presented example economy shows that the CCAPM with CRRA preferences and lognormal returns can qualitatively explain CAPM-anomalies in the cross-section of stock returns. The model implies a security market line that is flatter than the one in the CAPM over the range of most assets. Most low-beta assets therefore pay higher returns adjusted for market risk as measured by the CAPM than high-beta assets. Furthermore, the smaller slope of the security market line leads to a size premium and a value premium.

Another interesting finding is that the CAPM pricing error might not capture price-related anomalies in the presence of idiosyncratic risk. The problem is that prices can have an idiosyncratic risk component, whereas the CAPM pricing error cannot. Suppose that the CAPM pricing error truly is a result of neglected higher-order moments and an investor tries to perform better than the CAPM. Investment strategies based on price-related financial variables can then be outperformed by strategies based on the CAPM beta.

The slope of the security market line and thus the size of the CAPM-anomalies strongly depend on the dividend distribution and preference parameters. For reasonable parameter values, the security market line is still too steep. In a sense the cross-sectional puzzles are very similar to the equity premium puzzle. We find for example that the security market line becomes flatter for increasing risk aversion and increasing consumption volatility. A higher level of risk aversion and consumption volatility are also known to increase the equity premium. Possible explanations of the high equity premium in the literature might therefore also explain the size of CAPM-anomalies.

Our discussion of CAPM-anomalies in the cross-section of stock returns is based on a simple two-period economy with a specific correlation structure of dividends. An interesting extension would be to introduce multiple time-periods so that for example momentum could be discussed. Of course it would also be interesting to generalize the results to a broader class of correlation structures.

3.5 Appendix

Derivation of the Expectation:

In the proofs we often encounter expectations of the form $\mathbf{E}\left[e^{a\Delta y+b(\Delta y)^2}\right]$ with different coefficients a and b and $\Delta y \sim \mathcal{N}(\mu - y_0, \sigma^2)$. Here we show the common derivation of this expectation using the completion of the square method.

$$\begin{aligned}
\mathbf{E}\left[e^{a\Delta y+b(\Delta y)^2}\right] &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{a\Delta y+b(\Delta y)^2} e^{-\frac{((\Delta y-(\mu-y_0))^2}{2\sigma^2}} d\Delta y \\
&= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{1}{\sigma^2}-2b\right)\left((\Delta y)^2-2\frac{a\sigma^2+(\mu-y_0)}{1-2b\sigma^2}\Delta y+\frac{(\mu-y_0)^2}{1-2b\sigma^2}\right)} d\Delta y \\
&= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{1}{\sigma^2}-2b\right)\left(\frac{(\mu-y_0)^2}{1-2b\sigma^2}-\left(\frac{a\sigma^2+(\mu-y_0)}{1-2b\sigma^2}\right)^2\right)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{1}{\sigma^2}-2b\right)\left(\Delta y-\frac{a\sigma^2+(\mu-y_0)}{1-2b\sigma^2}\right)^2} d\Delta y.
\end{aligned}$$

The integrand is proportional to a normal probability density function with mean $\frac{a\sigma^2+(\mu-y_0)}{1-2b\sigma^2}$ and standard deviation $\sqrt{\frac{\sigma^2}{1-2b\sigma^2}}$. The expectation therefore is equal to

$$\begin{aligned}
\mathbf{E}\left[e^{a\Delta y+b(\Delta y)^2}\right] &= \frac{1}{\sqrt{1-2b\sigma^2}} e^{-\frac{1}{2}\left(\frac{1}{\sigma^2}-2b\right)\left(\frac{(\mu-y_0)^2}{1-2b\sigma^2}-\left(\frac{a\sigma^2+(\mu-y_0)}{1-2b\sigma^2}\right)^2\right)} \\
&= \frac{1}{\sqrt{1-2b\sigma^2}} e^{\frac{1}{2\sigma^2}\frac{\left(a\sigma^2+(\mu-y_0)\right)^2}{1-2b\sigma^2}-\frac{1}{2\sigma^2}(\mu-y_0)^2}.
\end{aligned}$$

□

Proof of Proposition 3.1:

The price-dividend ratio of an asset of type ρ is

$$\begin{aligned}
\frac{q^\rho}{D_0^\rho} &= \mathbf{E}\left[\delta\left(\frac{c_0}{c_1}\right)^\gamma \frac{D_1^\rho}{D_0^\rho}\right] \\
&= \delta\mathbf{E}\left[e^{-\gamma(y_1-y_0)-\frac{\gamma}{2}\sigma_\rho^2(y_1^2-y_0^2)} e^{\rho(y_1-y_0)}\right].
\end{aligned}$$

To shorten notation let us define $\Delta y \equiv y_1 - y_0$ and $\Delta y \sim \mathcal{N}\left(\mu - y_0, \sigma^2\right)$. The price-dividend ratio then is

$$\begin{aligned}\frac{q^\rho}{D_0^\rho} &= \delta \mathbf{E}\left[e^{-\gamma\Delta y - \frac{\gamma}{2}\sigma_\rho^2 \Delta y (y_1+y_0)} e^{\rho \Delta y}\right] \\ &= \delta \mathbf{E}\left[e^{-\gamma\Delta y - \frac{\gamma}{2}\sigma_\rho^2 \left((\Delta y)^2 + 2y_0\Delta y\right)} e^{\rho \Delta y}\right] \\ &= \delta \mathbf{E}\left[e^{(\rho-\gamma-\gamma\sigma_\rho^2 y_0)\Delta y - \frac{\gamma}{2}\sigma_\rho^2 (\Delta y)^2}\right].\end{aligned}$$

Using the formula for the expectation we obtain

$$\frac{q^\rho}{D_0^\rho} = \frac{\delta}{\sqrt{1+\gamma\sigma_\rho^2\sigma^2}} e^{\frac{1}{2\sigma^2}\frac{\left((\rho-\gamma-\gamma\sigma_\rho^2 y_0)\sigma^2 + (\mu - y_0)\right)^2}{1+\gamma\sigma_\rho^2\sigma^2} - \frac{1}{2\sigma^2}(\mu-y_0)^2}.$$

□

Proof of Proposition 3.2:

The expected return of an asset of type ρ is

$$\begin{aligned}\mathbf{E}\left[R^\rho\right] &= \mathbf{E}\left[\frac{D_1^\rho}{q^\rho}\right] = \frac{\mathbf{E}\left[e^{\rho\Delta y}\right] D_0^\rho}{q^\rho} \\ &= \frac{\sqrt{1+\gamma\sigma_\rho^2\sigma^2}}{\delta} e^{\rho(\mu-y_0) + \frac{1}{2}\rho^2\sigma^2 - \frac{1}{2\sigma^2}\frac{\left((\rho-\gamma-\gamma\sigma_\rho^2 y_0)\sigma^2 + (\mu - y_0)\right)^2}{1+\gamma\sigma_\rho^2\sigma^2} + \frac{1}{2\sigma^2}(\mu-y_0)^2}.\end{aligned}$$

The price-dividend of the market portfolio is

$$\begin{aligned}\frac{q^M}{D_0^M} &= \mathbf{E}\left[\delta\left(\frac{c_0}{c_1}\right)^\gamma \frac{D_1^M}{D_0^M}\right] \\ &= \delta \mathbf{E}\left[e^{(1-\gamma)(y_1-y_0) + \frac{1-\gamma}{2}\sigma_\rho^2 (y_1^2 - y_0^2)}\right] \\ &= \delta \mathbf{E}\left[e^{(1-\gamma)\Delta y + \frac{1-\gamma}{2}\sigma_\rho^2 \Delta y (y_1+y_0)}\right] \\ &= \delta \mathbf{E}\left[e^{(1-\gamma)\Delta y + \frac{1-\gamma}{2}\sigma_\rho^2 \left((\Delta y)^2 + 2y_0\Delta y\right)}\right] \\ &= \delta \mathbf{E}\left[e^{(1-\gamma)(1+\sigma_\rho^2 y_0)\Delta y + \frac{1-\gamma}{2}\sigma_\rho^2 (\Delta y)^2}\right].\end{aligned}$$

Using the formula for the expectation we obtain

$$\frac{q^M}{D_0^M} = \frac{\delta}{\sqrt{1-(1-\gamma)\sigma_\rho^2\sigma^2}} e^{\frac{1}{2\sigma^2}\frac{\left((1-\gamma)(1+\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0)\right)^2}{1-(1-\gamma)\sigma_\rho^2\sigma^2}-\frac{1}{2\sigma^2}(\mu-y_0)^2}.$$

Therefore the expected return on the market portfolio is

$$\begin{aligned}\mathbf{E}\left[R^M\right] &= \mathbf{E}\left[\frac{D_1^M}{q^M}\right] = \frac{\mathbf{E}\left[e^{\Delta y+\frac{1}{2}\sigma_\rho^2(y_1^2-y_0^2)}\right]D_0^M}{q^M} = \frac{\mathbf{E}\left[e^{(1+\sigma_\rho^2 y_0)\Delta y+\frac{1}{2}\sigma_\rho^2(\Delta y)^2}\right]D_0^M}{q^M} \\ &= \frac{\sqrt{1-(1-\gamma)\sigma_\rho^2\sigma^2}}{\delta\sqrt{1-\sigma_\rho^2\sigma^2}} e^{\frac{1}{2\sigma^2}\frac{\left((1+\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0)\right)^2}{1-\sigma_\rho^2\sigma^2}-\frac{1}{2\sigma^2}\frac{\left((1-\gamma)(1+\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0)\right)^2}{1-(1-\gamma)\sigma_\rho^2\sigma^2}}.\end{aligned}$$

The riskless rate is

$$\begin{aligned}R^f &= \frac{1}{\mathbf{E}\left[\delta\left(\frac{c_0}{c_1}\right)^\gamma\right]} \\ &= \frac{1}{\delta\mathbf{E}\left[e^{-\gamma(y_1-y_0)-\frac{\gamma}{2}\sigma_\rho^2(y_1^2-y_0^2)}\right]} \\ &= \frac{1}{\delta\mathbf{E}\left[e^{-\gamma\Delta y-\frac{\gamma}{2}\sigma_\rho^2\Delta y(y_1+y_0)}\right]} \\ &= \frac{1}{\delta\mathbf{E}\left[e^{-\gamma\Delta y-\frac{\gamma}{2}\sigma_\rho^2\left((\Delta y)^2+2y_0\Delta y\right)}\right]} \\ &= \frac{1}{\delta\mathbf{E}\left[e^{-\gamma(1+\sigma_\rho^2 y_0)\Delta y-\frac{\gamma}{2}\sigma_\rho^2(\Delta y)^2}\right]}.\end{aligned}$$

Using the formula for the expectation we obtain

$$R^f = \frac{\sqrt{1+\gamma\sigma_\rho^2\sigma^2}}{\delta} e^{-\frac{1}{2\sigma^2}\frac{(-\gamma(1+\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0))^2}{1+\gamma\sigma_\rho^2\sigma^2}+\frac{1}{2\sigma^2}(\mu-y_0)^2}.$$

□

Proof of Proposition 3.3:

The CAPM beta β^ρ is

$$
\begin{aligned}
\beta^\rho &= \frac{\mathrm{Cov}\left(R^\rho, R^M\right)}{\mathrm{Var}\left(R^M\right)} = \frac{q^M}{q^\rho}\frac{\mathrm{Cov}\left(D_1^\rho, D_1^M\right)}{\mathrm{Var}\left(D_1^M\right)} \\
&= \frac{q^M}{q^\rho}\frac{\mathbf{E}\left[e^{(1+\rho)y_1+\frac{1}{2}\sigma_\rho^2 y_1^2}\right]-\mathbf{E}\left[e^{\rho y_1}\right]\mathbf{E}\left[e^{y_1+\frac{1}{2}\sigma_\rho^2 y_1^2}\right]}{\mathbf{E}\left[e^{2y_1+\sigma_\rho^2 y_1^2}\right]-\mathbf{E}\left[e^{y_1+\frac{1}{2}\sigma_\rho^2 y_1^2}\right]^2} \\
&= \frac{q^M}{q^\rho}e^{(\rho-1)y_0-\frac{1}{2}\sigma_\rho^2 y_0^2}\frac{\mathbf{E}\left[e^{(1+\rho+\sigma_\rho^2 y_0)\Delta y+\frac{1}{2}\sigma_\rho^2(\Delta y)^2}\right]-\mathbf{E}\left[e^{\rho\Delta y}\right]\mathbf{E}\left[e^{(1+\sigma_\rho^2 y_0)\Delta y+\frac{1}{2}\sigma_\rho^2(\Delta y)^2}\right]}{\mathbf{E}\left[e^{2(1+\sigma_\rho^2 y_0)\Delta y+\sigma_\rho^2(\Delta y)^2}\right]-\mathbf{E}\left[e^{(1+\sigma_\rho^2 y_0)\Delta y+\frac{1}{2}\sigma_\rho^2(\Delta y)^2}\right]^2}.
\end{aligned}
$$

Using the formula for the expectations and multiplying the numerator and denominator by $e^{\frac{1}{2\sigma^2}(\mu-y_0)^2}$ we obtain

$$
\beta^\rho = \frac{q^M}{q^\rho}e^{(\rho-1)y_0-\frac{1}{2}\sigma_\rho^2 y_0^2}\frac{\frac{e^{\frac{1}{2\sigma^2}\frac{\left((1+\rho+\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0)\right)^2}{1-\sigma_\rho^2\sigma^2}}}{\sqrt{1-\sigma_\rho^2\sigma^2}}-\frac{e^{\rho(\mu-y_0)+\frac{1}{2}\rho^2\sigma^2+\frac{1}{2\sigma^2}\frac{\left((1+\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0)\right)^2}{1-\sigma_\rho^2\sigma^2}}}{\sqrt{1-\sigma_\rho^2\sigma^2}}}{\frac{e^{\frac{1}{2\sigma^2}\frac{\left(2(1+\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0)\right)^2}{1-2\sigma_\rho^2\sigma^2}}}{\sqrt{1-2\sigma_\rho^2\sigma^2}}-\frac{e^{\frac{1}{\sigma^2}\frac{\left((1+\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0)\right)^2}{1-\sigma_\rho^2\sigma^2}-\frac{1}{2\sigma^2}(\mu-y_0)^2}}{1-\sigma_\rho^2\sigma^2}}.
$$

Using the formula for the price-dividend ratio we obtain

$$
\beta^\rho = \frac{\frac{e^{\frac{1}{2\sigma^2}\frac{\left((1-\gamma)(1+\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0)\right)^2}{1-(1-\gamma)\sigma_\rho^2\sigma^2}}}{\sqrt{1-(1-\gamma)\sigma_\rho^2\sigma^2}}\left(\frac{e^{\frac{1}{2\sigma^2}\frac{\left((1+\rho+\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0)\right)^2}{1-\sigma_\rho^2\sigma^2}}}{\sqrt{1-\sigma_\rho^2\sigma^2}}-\frac{e^{\rho(\mu-y_0)+\frac{1}{2}\rho^2\sigma^2+\frac{1}{2\sigma^2}\frac{\left((1+\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0)\right)^2}{1-\sigma_\rho^2\sigma^2}}}{\sqrt{1-\sigma_\rho^2\sigma^2}}\right)}{\frac{e^{\frac{1}{2\sigma^2}\frac{\left((\rho-\gamma-\gamma\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0)\right)^2}{1+\gamma\sigma_\rho^2\sigma^2}}}{\sqrt{1+\gamma\sigma_\rho^2\sigma^2}}\left(\frac{e^{\frac{1}{2\sigma^2}\frac{\left(2(1+\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0)\right)^2}{1-2\sigma_\rho^2\sigma^2}}}{\sqrt{1-2\sigma_\rho^2\sigma^2}}-\frac{e^{\frac{1}{\sigma^2}\frac{\left((1+\sigma_\rho^2 y_0)\sigma^2+(\mu-y_0)\right)^2}{1-\sigma_\rho^2\sigma^2}-\frac{1}{2\sigma^2}(\mu-y_0)^2}}{1-\sigma_\rho^2\sigma^2}\right)}.
$$

□

Proof of Proposition 3.4:

The CCAPM beta β^{ρ}_{con} is

$$\begin{aligned}
\beta^{\rho}_{con} &= \frac{\mathrm{Cov}\left(R^{\rho},(R^{M})^{-\gamma}\right)}{\mathrm{Cov}\left(R^{M},(R^{M})^{-\gamma}\right)} = \frac{q^{M}}{q^{\rho}}\frac{\mathrm{Cov}\left(D_1^{\rho},\left(D_1^{M}\right)^{-\gamma}\right)}{\mathrm{Cov}\left(D_1^{M},\left(D_1^{M}\right)^{-\gamma}\right)} \\
&= \frac{q^{M}}{q^{\rho}}\frac{\mathbf{E}\left[e^{(\rho-\gamma)y_1-\frac{\gamma}{2}\sigma_{\rho}^2 y_1^2}\right]-\mathbf{E}\left[e^{\rho y_1}\right]\mathbf{E}\left[e^{-\gamma y_1-\frac{\gamma}{2}\sigma_{\rho}^2 y_1^2}\right]}{\mathbf{E}\left[e^{(1-\gamma)y_1+\frac{1-\gamma}{2}\sigma_{\rho}^2 y_1^2}\right]-\mathbf{E}\left[e^{y_1+\frac{1}{2}\sigma_{\rho}^2 y_1^2}\right]\mathbf{E}\left[e^{-\gamma y_1-\frac{\gamma}{2}\sigma_{\rho}^2 y_1^2}\right]} \\
&= \frac{q^{M}}{q^{\rho}}\frac{e^{(\rho-1)y_0-\frac{1}{2}\sigma_{\rho}^2 y_0^2}\left(\mathbf{E}\left[e^{(\rho-\gamma\eta)\Delta y-\frac{\gamma}{2}\sigma_{\rho}^2(\Delta y)^2}\right]-\mathbf{E}\left[e^{\rho\Delta y}\right]\mathbf{E}\left[e^{-\gamma\eta\Delta y-\frac{\gamma}{2}\sigma_{\rho}^2(\Delta y)^2}\right]\right)}{\mathbf{E}\left[e^{(1-\gamma)\eta\Delta y+\frac{1-\gamma}{2}\sigma_{\rho}^2(\Delta y)^2}\right]-\mathbf{E}\left[e^{\eta\Delta y+\frac{1}{2}\sigma_{\rho}^2(\Delta y)^2}\right]\mathbf{E}\left[e^{-\gamma\eta\Delta y-\frac{\gamma}{2}\sigma_{\rho}^2(\Delta y)^2}\right]},
\end{aligned}$$

where η is defined as $\eta \equiv 1+\sigma_{\rho}^2 y_0$. Using the formula for the expectations and multiplying the numerator and denominator by $e^{\frac{1}{2\sigma^2}(\mu-y_0)^2}$ we obtain

$$\beta^{\rho}_{con} = \frac{q^{M}}{q^{\rho}}\frac{e^{(\rho-1)y_0-\frac{1}{2}\sigma_{\rho}^2 y_0^2}\left(\frac{e^{\frac{1}{2\sigma^2}\frac{\left((\rho-\gamma\eta)\sigma^2+(\mu-y_0)\right)^2}{1+\gamma\sigma_{\rho}^2\sigma^2}}}{\sqrt{1+\gamma\sigma_{\rho}^2\sigma^2}}-\frac{e^{\rho(\mu-y_0)+\frac{1}{2}\rho^2\sigma^2+\frac{1}{2\sigma^2}\frac{\left(-\gamma\eta\sigma^2+(\mu-y_0)\right)^2}{1+\gamma\sigma_{\rho}^2\sigma^2}}}{\sqrt{1+\gamma\sigma_{\rho}^2\sigma^2}}\right)}{\frac{e^{\frac{1}{2\sigma^2}\frac{\left((1-\gamma)\eta\sigma^2+(\mu-y_0)\right)^2}{1-(1-\gamma)\sigma_{\rho}^2\sigma^2}}}{\sqrt{1-(1-\gamma)\sigma_{\rho}^2\sigma^2}}-\frac{e^{\frac{1}{2\sigma^2}\frac{\left(\eta\sigma^2+(\mu-y_0)\right)^2}{1-\sigma_{\rho}^2\sigma^2}+\frac{1}{2\sigma^2}\frac{\left(-\gamma\eta\sigma^2+(\mu-y_0)\right)^2}{1+\gamma\sigma_{\rho}^2\sigma^2}-\frac{1}{2\sigma^2}(\mu-y_0)^2}}{\sqrt{1-\sigma_{\rho}^2\sigma^2}\sqrt{1+\gamma\sigma_{\rho}^2\sigma^2}}}.$$

Using the formula for the price-dividend ratio we obtain

$$\beta^{\rho}_{con} = \frac{\frac{e^{\frac{1}{2\sigma^2}\frac{\left((1-\gamma)\eta\sigma^2+(\mu-y_0)\right)^2}{1-(1-\gamma)\sigma_{\rho}^2\sigma^2}}}{\sqrt{1-(1-\gamma)\sigma_{\rho}^2\sigma^2}}\left(\frac{e^{\frac{1}{2\sigma^2}\frac{\left((\rho-\gamma\eta)\sigma^2+(\mu-y_0)\right)^2}{1+\gamma\sigma_{\rho}^2\sigma^2}+\frac{1}{2\sigma^2}(\mu-y_0)^2}}{\sqrt{1+\gamma\sigma_{\rho}^2\sigma^2}}-\frac{e^{\frac{1}{2\sigma^2}\frac{\left(-\gamma\eta\sigma^2+(\mu-y_0)\right)^2}{1+\gamma\sigma_{\rho}^2\sigma^2}+\frac{1}{2\sigma^2}\left(\rho\sigma^2+(\mu-y_0)\right)^2}}{\sqrt{1+\gamma\sigma_{\rho}^2\sigma^2}}\right)}{\frac{e^{\frac{1}{2\sigma^2}\frac{\left((\rho-\gamma\eta)\sigma^2+(\mu-y_0)\right)^2}{1+\gamma\sigma_{\rho}^2\sigma^2}}}{\sqrt{1+\gamma\sigma_{\rho}^2\sigma^2}}\left(\frac{e^{\frac{1}{2\sigma^2}\frac{\left((1-\gamma)\eta\sigma^2+(\mu-y_0)\right)^2}{1-(1-\gamma)\sigma_{\rho}^2\sigma^2}+\frac{1}{2\sigma^2}(\mu-y_0)^2}}{\sqrt{1-(1-\gamma)\sigma_{\rho}^2\sigma^2}}-\frac{e^{\frac{1}{2\sigma^2}\frac{\left(\eta\sigma^2+(\mu-y_0)\right)^2}{1-\sigma_{\rho}^2\sigma^2}+\frac{1}{2\sigma^2}\frac{\left(-\gamma\eta\sigma^2+(\mu-y_0)\right)^2}{1+\gamma\sigma_{\rho}^2\sigma^2}}}{\sqrt{1-\sigma_{\rho}^2\sigma^2}\sqrt{1+\gamma\sigma_{\rho}^2\sigma^2}}\right)}.$$

□

4

Asset Pricing with Labor Income and Rare Disasters

The article by Mehra and Prescott (1985) has entailed a multiplicity of attempts to explain the equity premium puzzle ever since its publication. A major obstacle in the attempt to explain observed equity premia and riskless rates with Lucas (1978)-type models is the low variability of consumption growth. In the presence of labor income, the representative agents' consumption equals labor income plus aggregate dividends. While aggregate dividend growth exhibits substantial variability, labor income growth varies very little. Since labor income constitutes the lion's share of total consumption, it is probably the main reason why the volatility of consumption growth is so low.

A natural approach to explain the equity premium puzzle is to question measured consumption growth variability. For example Weil (1992) and Constantinides and Duffie (1996) introduce heterogeneous agents and argue that the labor income risk of the representative agent does not reflect the much higher labor income risk faced by an individual consumer.

Another possible approach is to compensate for the low volatility of consumption by tackling the skewness of the distribution. Rietz (1988), Barro (2006) and Martin (2013) demonstrate that including low-probability economic disasters increases predicted equity premia and decreases the riskless rate despite the low variability of consumption growth. Disasters leave the first- and second-order moments of consumption growth merely unchanged, but have a big impact on the skewness of the distribution. We show that the impact of introducing rare disasters on equity premia and riskless rates heavily depends on the labor income to consumption ratio. Furthermore, we separate and analyze the channels through which disasters raise equity premia and lower riskless rates.

Rare disasters affect the model in three ways. First, the decline in mean consumption growth and increase in volatility during times of disaster are hard to gauge. Since disasters occur very infrequently, shorter sample periods are likely to lead to a slight overestimation of mean consumption growth and underestimation of the volatility. Second, including low-probability events changes the distributional assumption on dividends and labor income. Third, major economic shocks affect both labor income and dividends simultaneously and introduce some correlation.

The impact of miscalibrated volatility has been previously studied by Weil (1992), Constantinides and Duffie (1996), Gollier and Schlesinger (2002), and others. They introduce additional risk to labor income and asset payoffs that arise from individual consumers' risk assessment. These kinds of risk are unobservable from aggregate data. A market analyst would therefore underestimate the volatility of labor income and dividends. Rare disasters imply labor income and dividend risk that a market analyst would most probably underestimate as well. They therefore offer an alternative explanation for miscalibrated risk.

Rietz (1988), Barro (2006) and Martin (2013) study the effect of changing the distributional assumption on consumption growth. Our paper is most closely related to Martin (2013). Martin (2013) studies the impact of economic disasters in a market with multiple assets. While Martin (2013) models logarithmic dividend growth of single assets by a normal distribution combined with a compound Poisson distribution, we introduce labor income and model logarithmic aggregate dividend growth and labor income growth by a normal distribution combined with a compound Poisson distribution.

The main difficulty in such models as discussed by Martin (2013) and references therein is the combination of consumption equaling the sum of lognormally distributed random variables and the structure of the stochastic discount factor implied by the assumption of constant relative risk aversion (CRRA). Closed-form solutions for asset prices are difficult to obtain in such a setting and have only been derived for special cases. That is why we take a numerical approach as suggested by Martin (2013).

Our simulations show that accounting for rare disasters alters the predictions of the model. Santos and Veronesi (2006) derive closed-form solutions in a market with labor income and multiple assets and find that a high share of labor income to consumption predicts low future aggregate returns consistent with data from 1948-2001. Chen and Joslin (2012) show that the predictability changes in the data over the period from 1990-2010. A high share of labor income to consumption then predicts high future aggregate returns. Chen and Joslin (2012) generalize the solution procedure of Martin (2013) and allow for time-varying dividend and labor income risk. They find that the predictability discovered by Santos and Veronesi (2006) almost vanishes, when the volatilities and the correlation between labor income and dividends are low. We show that including rare disasters can explain a change in predictability.

Unlike Martin (2013), Santos and Veronesi (2006) and Chen and Joslin (2012), our model is placed in discrete time. Dividends are usually paid in lump sums and not in rates. Therefore we prefer the discrete-time framework. Another possibility would be to model cumulative dividends as a jump process in continuous time. According to Aase (2008), there is some confusion about the continuous-time analogue of the pricing formula in the literature. He presents a continous-time pricing formula that can be viewed as a limiting case of the usual discrete-time formulation. Martin (2013) and Santos and Veronesi (2006) use another continuous-time pricing formula, since they model directly the dividend process and not cumulative dividends. We show that their formulation can be viewed as a limiting case of the discrete-time framework as well.

The paper is organized as follows. In Section 4.1, the economy is defined and a convergence proof to the continuous-time formulation is given. In Section 4.2, we solve the model for prices, returns and interest rates. Section 4.3 describes the simulation setup and how the parameters are calibrated to data. In Section 4.4, we discuss the simulation results and analyze the different channels through which rare disasters increase the equity premium and lower the riskless rate and how they affect stock return predictability. Section 4.5 concludes. In order to improve readability, we moved all the mathematical proofs to the appendix.

4.1 The Economy

First, we present the model in a discrete-time setting. Our specification of the exchange economy is the discrete-time counterpart of the continuous-time model used by Martin (2013). We will show that the continuous-time model can be viewed as a limiting case of our model when the time interval tends to zero.

4.1.1 The discrete-time economy

We consider a Lucas (1978)-type one-good exchange economy where time is discrete. Consumption is financed by labor income, the dividends of $N-1$ risky assets indexed by $i = 2, \dots, N$ and discount bonds of all maturities. Labor income at time t is denoted by D_t^1. The risky asset i pays a random dividend D_t^i and has ex-dividend price P_t^i at time $t = 0, 1, \dots$ To ensure that dividends remain positive, we assume that $D_t^i = e^{y_t^i}$. $(y_t^i)_{t=0,1,\dots}$ is a discrete-time stochastic process with initial value $\bar{y}_0^i$. Logarithmic labor income growth and dividend growth $y_t^i - y_{t-1}^i$ are independent and identically distributed over time, according to the sum of a normal and a compound Poisson distribution. More precisely, let $\boldsymbol{y}_t \equiv (y_t^1, \dots, y_t^N)'$ and

$$\boldsymbol{y}_t - \boldsymbol{y}_{t-1} = \boldsymbol{M}_t + \sum_{k=1}^{K_t} \boldsymbol{J}_t^k,$$

where $\boldsymbol{M}_t$ follows an N-variate normal distribution $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, K_t follows a Poisson distribution Pois(λ) and $\boldsymbol{J}_t^k$ are N-dimensional independent and identically distributed random variables. We will later use the compound Poisson distribution to model the impact of economic disasters on labor income growth and dividend growth. Bonds pay one unit of consumption good at maturity. Let B_t^T denote the price of a bond at time t with time to maturity T.

There is an infinitely-lived representative agent who maximizes a time-separable utility function for lifetime consumption

$$\mathbf{E}_t\left[\sum_{s=0}^{\infty}\frac{1}{(1+\beta)^s}u(C_{t+s})\right]$$

subject to the budget constraint

$$C_t+\sum_{i=2}^{N}P_t^i\theta_t^i+\sum_{T=1}^{\infty}\psi_t^T B_t^T \;=D_t^1+\sum_{i=2}^{N}(P_t^i+D_t^i)\theta_{t-1}^i+\sum_{T=1}^{\infty}\psi_{t-1}^T B_t^{T-1},$$

where β denotes the time discount rate, C_t denotes consumption at time t and $u(.)$ is a CRRA utility function with coefficient of relative risk aversion γ. $(\theta_t^i)_{i=2,\ldots,N}$ denote the agent's equity holdings and $(\psi_t^T)_{T=1,2,\ldots}$ denote the agent's bond holdings in period t. $\mathbf{E}_t[.]$ denotes the conditional expectation given all information up to time t. The information up to time t includes the labor income history and the securities' dividend and price histories. The consumption good is not storable, so that the representative agent consumes the aggregate amount of dividends and labor income each period $C_t=\sum_{i=1}^{N}D_t^i$.

The equilibrium price of asset i at time t follows from the Euler equations and is given by

$$P_t^i \;=\; \mathbf{E}_t\left[\sum_{s=1}^{\infty}\frac{1}{(1+\beta)^s}\left(\frac{C_{t+s}}{C_t}\right)^{-\gamma}D_{t+s}^i\right]. \qquad (4.1)$$

Before we further elaborate the discrete-time model, let us study the relation to the continuous-time counterpart. We will demonstrate that the limiting pricing formula as the time interval tends to zero is equal to the pricing formula in the continuous-time model used by Martin (2013) and Santos and Veronesi (2006).

4.1.2 The continuous-time limit

Let us first define a sequence of discrete-time models with decreasing time intervals $\Delta t = \frac{1}{n}$. The representative agent's lifetime utility is given by

$$\mathbf{E}_t \left[\lim_{T\to\infty} \sum_{s=0}^{T-1} \sum_{k=0}^{n-1} \left(e^{-\rho\left(s+\frac{k}{n}\right)} u(C_{t+s+\frac{k}{n}}) \frac{1}{n} \right) + e^{-\rho T} u(C_{t+T}) \frac{1}{n} \right],$$

where ρ denotes a time discount parameter with

$$e^{-\rho} = \frac{1}{1+\beta}.$$

The representative agent faces the budget constraint

$$C_t \Delta t + \sum_{i=2}^{N} P_t^i \theta_t^i = D_t^1 \Delta t + \sum_{i=2}^{N} (P_t^i + D_t^i \Delta t) \theta_{t-\Delta t}^i$$

in each period for all possible realizations of dividend streams. Since discount bonds do not enter the Euler equations of risky asset holdings and do not affect asset prices, we exclude them in this section to shorten notation.

We assume that logarithmic labor income and dividend growth are independent and identically distributed according to

$$\boldsymbol{y}_t - \boldsymbol{y}_{t-\Delta t} = \boldsymbol{M}_t + \sum_{k=1}^{K_t} \boldsymbol{J}_t^k,$$

where $\boldsymbol{M}_t$ follows an N-variate normal distribution $\mathcal{N}(\boldsymbol{\mu}\Delta t, \boldsymbol{\Sigma}\Delta t)$, K_t is distributed according to a Poisson distribution $\text{Pois}(\lambda\Delta t)$ and $\boldsymbol{J}_t^k$ are N-dimensional independent and identically distributed random variables.

Furthermore, we impose the transversality condition

$$\lim_{T\to\infty} \mathbf{E}_t \left[\left(\frac{C_{t+T}}{C_t} \right)^{-\gamma} P_{t+T}^i \right] = 0 \tag{4.2}$$

to rule out any rational bubbles.

Proposition 4.1 *The price of asset i at time t in the continuous-time limit as $\Delta t \to 0$ is*

$$P_t^i = \mathbf{E}_t \left[\int_t^\infty e^{-\rho s} \left(\frac{C_{t+s}}{C_t} \right)^{-\gamma} D_{t+s}^i ds \right],$$

assuming that the transversality condition (4.2) *holds.*

We note that the $\Delta t = 1$ describes the discrete-time case in Section 4.1.1. Hence, Proposition 4.1 states that our discrete-time model is consistent with the continuous-time model used by Martin (2013).

4.2 Prices, Returns and Interest Rates

In this section, we solve the model for prices, expected returns and interest rates following the solution method of Martin (2013) and Chen and Joslin (2012). The main problem is that we know too little about the distribution of consumption. In the standard model without disasters, consumption is the sum of lognormal random variables. The distribution of the sum of lognormal random variables is hitherto unknown. Even though we know some properties of the resulting distribution, they do not suffice to solve the model. The properties of the expectation operator also cannot solve the problem because of the structure of the pricing formula under CRRA preferences.

Even though we have not managed to entirely solve for prices, expected returns and interest rates, the solution techniques of Martin (2013) and Chen and Joslin (2012) yield at least a partial solution to the problem. First, we will exchange the sum and expectation in the pricing formula (4.1) and write the expression in the expectation in terms of a function from which we know the Fourier transform. Replacing the function by its inverse Fourier transform, we get an expression in terms of the product of powers of dividends. Then we can apply the expectation operator, since the moment-generating function of logarithmic dividends is known.

We will state all our results in terms of the cumulant-generating function following Martin (2013). For our specification of the logarithmic dividend growth distribution, the cumulant-generating function is

$$c(\boldsymbol{\theta}) \equiv \log\left(\mathbf{E}\left[e^{\boldsymbol{\theta}'(y_t-y_{t-1})}\right]\right) = \boldsymbol{\theta}'\boldsymbol{\mu}+\frac{1}{2}\boldsymbol{\theta}'\boldsymbol{\Sigma}\boldsymbol{\theta}+\lambda\left(\mathbf{E}[e^{\boldsymbol{\theta}'\boldsymbol{J}}]-1\right),$$

where $\boldsymbol{J}$ can be any of the $(\boldsymbol{J}_t^k)_{k=1,2,\ldots}$ since they are identically distributed. The cumulant-generating function is the same over each time period, since logarithmic dividend growth is identically distributed as well.

The resulting formulas for asset prices, expected returns and interest rates depend on

$$s_t^i \equiv \frac{D_t^i}{D_t^1+\cdots+D_t^N}$$

for $i=1,\ldots,N$. s_t^i denotes the relative share of labor income to consumption for $i=1$ and the relative share of dividends to consumption for $i=2,\ldots,N$. For notational convenience we will work with the logarithmic dividends relative to labor income instead

$$u_t^i \equiv \log\left(\frac{s_t^i}{s_t^1}\right) = y_t^i - y_t^1.$$

Let us introduce some more notation in order to get simpler expressions for the pricing and return formulas. We define the $(N-1)$-dimensional vector $\boldsymbol{u}_t \equiv (u_t^2,\ldots,u_t^N)'$, the N-dimensional vectors $\boldsymbol{u}_t^+ \equiv (u_t^1,\ldots,u_t^N)'$ and $\boldsymbol{\gamma} \equiv (\gamma,\ldots,\gamma)'$ and the $(N-1)\times N$ matrix

$$U \equiv \begin{pmatrix} -1 & 1 & 0 & \ldots & 0 \\ -1 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -1 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_N)$ be a multiindex and

$$\begin{aligned} D_t^\alpha &= (D_t^1)^{\alpha_1} \ldots (D_t^N)^{\alpha_N} \\ P_t^\alpha &= (P_t^1)^{\alpha_1} \ldots (D_t^N)^{\alpha_N}. \end{aligned}$$

Thus, the prices and dividends of asset i can be described by an α with a one at the ith entry and zeros elsewhere.

Proposition 4.2 *The price-dividend ratio of an asset that pays the dividend stream D_t^α is*

$$\frac{P_t^\alpha}{D_t^\alpha} = e^{-\frac{\gamma' u_t^+}{N}} \left(e^{u_t^1} + \cdots + e^{u_t^N}\right)^\gamma \int\limits_{\mathbb{R}^{N-1}} \mathscr{F}_\gamma^N(\boldsymbol{z}) e^{i\boldsymbol{z}' u} \frac{e^{c\left(\boldsymbol{\alpha} - \frac{\gamma}{N} + i\boldsymbol{U}'\boldsymbol{z}\right)}}{1 + \beta - e^{c\left(\boldsymbol{\alpha} - \frac{\gamma}{N} + i\boldsymbol{U}'\boldsymbol{z}\right)}} \boldsymbol{dz},$$

where

$$\mathscr{F}_\gamma^N(\boldsymbol{z}) \equiv \frac{\Gamma(\frac{\gamma}{N} + iz_1 + \cdots + iz_{N-1})}{(2\pi)^{N-1}\Gamma(\gamma)} \prod_{k=1}^{N-1} \Gamma\left(\frac{\gamma}{N} - iz_k\right).$$

The proof of Proposition 4.2 shows that prices are only finite if the condition

$$\left| \frac{1}{1+\beta} e^{c(\boldsymbol{\alpha} - \frac{\gamma}{N} + i\boldsymbol{Q}'\boldsymbol{z})} \right| < 1 \tag{4.3}$$

holds. From now on, we assume that the finiteness condition (4.3) holds for all assets.

We introduce the same notation for asset returns as for prices and dividends

$$R_{t+1}^\alpha = (R_{t+1}^1)^{\alpha_1} \ldots (R_{t+1}^N)^{\alpha_N}, \quad \text{where} \quad R_{t+1}^i \equiv \frac{P_{t+1}^i - P_t^i + D_{t+1}^i}{P_t^i}.$$

Proposition 4.3 then gives us an expression for expected returns in terms of relative dividends.

Proposition 4.3 *The expected return of an asset that pays the dividend stream D_t^α is*

$$\mathbf{E}_t[R_{t+1}^\alpha] = \left(\sum_{|m|=\gamma} \binom{\gamma}{m} e^{(m-\frac{\gamma}{N})'\boldsymbol{u}_t^+} \Phi_m^\alpha + e^{c(\alpha)} \right) \frac{D_t^\alpha}{P_t^\alpha},$$

where

$$\Phi_m^\alpha \equiv \int\limits_{\mathbb{R}^{N-1}} \frac{\mathscr{F}_\gamma^N(\boldsymbol{z}) e^{i\boldsymbol{z}'\boldsymbol{u}_t} e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\boldsymbol{U}'\boldsymbol{z})}(e^{c(\alpha-\frac{\gamma}{N}+m+i\boldsymbol{U}'\boldsymbol{z})}-1)}{1+\beta-e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\boldsymbol{U}'\boldsymbol{z})}} \boldsymbol{dz}.$$

Next, let us consider a zero-coupon bond B_t^T at time t that pays one unit of consumption good T periods later. Let $\mathscr{Y}_t(T)$ be the yield to maturity $t+T$ at time t, which is defined by

$$B_t^T = \frac{1}{(1+\mathscr{Y}_t(T))^{(T-t)}}.$$

Proposition 4.4 gives us expressions for the bond yield and the risk-free rate R_{t+1}^f from time t to $t+1$ in terms of relative dividends.

Proposition 4.4 *The yield of a zero-coupon bond with time to maturity T is*

$$\mathscr{Y}_t(T) = \frac{(1+\beta)}{\left(e^{-\frac{\gamma' u_t^+}{N}} (e^{u_t^1} + \cdots + e^{u_t^N})^\gamma \int\limits_{\mathbb{R}^{N-1}} \mathscr{F}_\gamma^N(\boldsymbol{z}) e^{i\boldsymbol{z}'\boldsymbol{u}_t} e^{c(-\frac{\gamma}{N}+i\boldsymbol{U}'\boldsymbol{z})T} \boldsymbol{dz} \right)^{\frac{1}{T}}} - 1.$$

The risk-free rate is

$$R_{t+1}^f = \frac{(1+\beta)}{e^{-\frac{\gamma' u_t^+}{N}} (e^{u_t^1} + \cdots + e^{u_t^N})^\gamma \int\limits_{\mathbb{R}^{N-1}} \mathscr{F}_\gamma^N(\boldsymbol{z}) e^{i\boldsymbol{z}'\boldsymbol{u}_t} e^{c(-\frac{\gamma}{N}+i\boldsymbol{U}'\boldsymbol{z})} \boldsymbol{dz}} - 1.$$

We did not manage to solve the integrals in Proposition 4.2, 4.3 and 4.4 so far. All the simulations in the following sections rely on numerical integration.

4.3 Simulation Setup

Since the multiple integrals in Proposition 4.2, 4.3 and 4.4 are computationally intensive, we restrict ourselves to aggregate dividends instead of modeling the dividend process of single assets[40]. From now on, let s_t^w denote the share of labor income on consumption. Let $\boldsymbol{\mu} = (\mu_w, \mu_d)'$ and

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_w^2 & 0 \\ 0 & \sigma_d^2 \end{pmatrix}.$$

Further, let $(\boldsymbol{J}_t^k)_{k=1,2,...}$ be normally distributed random variables with mean $\boldsymbol{\mu}^J = (\mu_J, \mu_J)'$ and covariance matrix

$$\boldsymbol{\Sigma}^J = \begin{pmatrix} \sigma_J^2 & 0 \\ 0 & \sigma_J^2 \end{pmatrix}.$$

In the following, we will restate the formulas for prices, expected excess returns and interest rates in terms of the labor income to consumption ratio and the aggregate dividend to consumption ratio.

4.3.1 Prices and returns with labor income and aggregate dividends

In the context of labor income and aggregate dividends, Proposition 4.2 gives us the aggregate market capitalization to aggregate dividend ratio. The formula reduces to

$$\frac{P_t}{D_t} = \left(2\cosh\left(\frac{u_t}{2}\right)\right)^{\gamma} \int_{-\infty}^{\infty} \mathscr{F}_{\gamma}(z) e^{izu_t} \frac{e^{c(-\frac{\gamma}{2}-iz, 1-\frac{\gamma}{2}+iz)}}{1+\beta - e^{c(-\frac{\gamma}{2}-iz, 1-\frac{\gamma}{2}+iz)}} dz,$$

[40]Note that the results do not necessarily carry over to the case where the dividends of single assets are modeled by the same process as aggregate dividends in our simulation setup. The distribution of aggregate dividends may then follow a very different process.

where

$$u_t = \ln\left(\frac{s_t^w}{1-s_t^w}\right).$$

Proposition 4.3 then gives us the expected returns on the aggregate stock market. The return formula can be simplified to

$$\mathbf{E}_t[R_{t+1}] = \frac{\sum_{m=0}^{\gamma}\binom{\gamma}{m}e^{-mu_t}\int_{-\infty}^{\infty} h(z)(e^{c(-\frac{\gamma}{2}+m-iz,1+\frac{\gamma}{2}-m+iz)}-1)dz}{\sum_{m=0}^{\gamma}\binom{\gamma}{m}e^{-mu_t}\int_{-\infty}^{\infty} h(z)dz} + e^{c(0,1)}\frac{D_t}{P_t},$$

where

$$h(z) \equiv \mathscr{F}_\gamma(z)\frac{e^{izu_t}e^{c(-\frac{\gamma}{2}-iz,1-\frac{\gamma}{2}+iz)}}{1+\beta-e^{c(-\frac{\gamma}{2}-iz,1-\frac{\gamma}{2}+iz)}}.$$

Proposition 4.4 gives us the yield on a zero-coupon bond with time to maturity T

$$\mathscr{Y}_t(T) = \frac{(1+\beta)}{\left((2\cosh(\frac{u_t}{2}))^\gamma \int_{-\infty}^{\infty}\mathscr{F}_\gamma(z)e^{izu_t}e^{c(-\frac{\gamma}{2}-iz,-\frac{\gamma}{2}+iz)T}dz\right)^{\frac{1}{T}}} - 1.$$

The long rate is

$$\lim_{T\to\infty}\mathscr{Y}_t(T) = \max_{\theta\in[-\frac{\gamma}{2},\frac{\gamma}{2}]}(1+\beta)e^{-c(-\frac{\gamma}{2}+\theta,-\frac{\gamma}{2}-\theta)} - 1.$$

Comparing the formulas for the yield of a zero-coupon bond in discrete time and continuous time, the long rate follows directly from the results in continuous time derived by Martin (2013). Note that the long rate is independent of the relative shares of labor income and aggregate dividends to consumption.

4.3.2 Calibration to data

The data on aggregate dividends and labor income are obtained from the Bureau of Economic Analysis. We follow the approach of Lettau and Ludvigson (2001) to define labor income and measure labor income as compensation of employees plus personal current transfer receipts less contributions for government social insurance and taxes. Taxes are defined as personal current taxes adjusted by the ratio of wage and salary disbursements to the sum of wage and salary disbursements, proprietors' income, rental income and personal income receipts on assets.

Dividends are defined as personal dividend income less taxes. Taxes are calculated analogously as for labor income. Both time series are quarterly and our sample period includes the years $1948 - 2007$. The labor income and aggregate dividend series are deflated by the implicit PCE deflator with base year 2005 and adjusted for population growth.

In our model, consumption equals labor income plus aggregate dividends. It differs from the more common measure of consumption that uses nondurable goods plus services. However, the time series of our measure of consumption and the time series of nondurable goods and services move very closely. The annualized mean consumption growth rate implied by our definition using labor income and dividends is 2.4% and is about the same as the 2.3% mean growth rate of nondurable goods plus services. The standard deviation of 2.0% is higher than the 1.0% we would get with the definition of nondurable goods plus services.

Prices, expected returns and interest rates depend considerably on the relative share of labor income to consumption. Figure 4.1 plots the relative share of labor income to consumption over time. We see that consumption is financed primarily by labor income. The relative share of labor income varies between 91% and 97%.

The disaster parameters are chosen as in Martin (2013). The frequency of low-probability events λ is set to 0.017 and complies with the frequency estimated in the empirical analysis of Barro (2006). A major decline in consumption arises around once

Figure 4.1: This plot shows the labor income to consumption ratio over time. The data is quarterly and includes the sample period from 1948 to 2007.

every sixty years. The mean and standard deviation of the disaster size are -0.38 and 0.25. They approximately match the disaster size distribution estimated by Barro (2006). The size distribution parameters should be interpreted with caution, since Barro (2006) estimates declines in output growth rates rather than consumption growth rates. However, he selects for events that were accompanied by a decline in consumption growth to better estimate the decline in consumption growth rates.

The remaining parameters to specify are the time preference rate β and the coefficient of relative risk aversion γ. We choose values for β that match a long rate of 7% for given values of γ as in Martin (2013). For γ, we choose possible positive integer values that satisfy the finiteness condition (4.2) and for which the implied β is positive.

4.4 Simulation Results

In this section, we present and analyze the simulation results of the price-dividend ratio[41], expected excess returns and riskless rates. The plots show the results for all possible values of the labor income to consumption ratio s^w. However, our discussion of the results will mainly focus on values of s^w equal to 0.9 and above. Figure 4.1 in Section 4.3.2 shows that the labor income to consumption ratio never went below 0.9 over the period 1948-2007.

4.4.1 Main results

First, let us compare the model with rare disasters to the standard model with lognormal labor income and lognormal aggregate dividends. The only reasonable values for γ in the standard model are one and two. For higher coefficients of relative risk aversion, we would need negative rates of time preference β to match the targeted long rate. Including disasters, γ can take values up to 10 and still imply positive rates of time preference. We only present simulation results for values up to 6. When γ equals 6, the model already generates expected equity premia that are too high and negative riskless rates. Table 4.1 summarizes our parameter specification.

The simulation results of the price-dividend ratio, expected excess returns and riskless rates obtained from the model with and without disasters are plotted in Figure 4.2. The plots show that the price-dividend ratio increases in the labor income to consumption ratio s^w for large values of s^w in both models.

When dividends account for a small portion of consumption, fluctuations in the dividend stream have a smaller impact on the agent's consumption path. The agent demands less compensation for holding equity, which drives prices up. This relation is consistent with the results from the model of Santos and Veronesi (2006). Including

[41] We will use the terms market capitalization to aggregate dividends ratio and price-dividend ratio interchangeably, since the aggregate stock market is modeled by a single asset in our model.

Table 4.1: This table shows the parameters used for simulation. The mean μ_l and standard deviation σ_l of the labor income growth rate and the mean μ_d and standard deviation σ_d of the aggregate dividend growth rate are estimated using quarterly data from 1948-2007. The frequency of rare disasters λ and the mean μ_J and standard deviation σ_J of the disaster size distribution are chosen to match the empirical findings of Barro (2006). All values are annualized. $\beta^{Ln}_{\gamma=i}$ denotes the time preference rates chosen to match a long rate of 7% in the standard model without disasters when the coefficient of relative risk aversion equals i. $\beta^{Dis}_{\gamma=i}$ denotes the time preference rates chosen to match a long rate of 7% in the model with disasters when the coefficient of relative risk aversion equals i.

μ_l	σ_l	μ_d	σ_d	μ_J	σ_J	λ	
0.0232	0.0199	0.0343	0.0667	-0.3800	0.2500	0.0170	
$\beta^{Ln}_{\gamma=1}$	$\beta^{Ln}_{\gamma=2}$	$\beta^{Dis}_{\gamma=1}$	$\beta^{Dis}_{\gamma=2}$	$\beta^{Dis}_{\gamma=3}$	$\beta^{Dis}_{\gamma=4}$	$\beta^{Dis}_{\gamma=5}$	$\beta^{Dis}_{\gamma=6}$
0.0356	0.0080	0.0442	0.0321	0.0327	0.0466	0.0904	0.1924

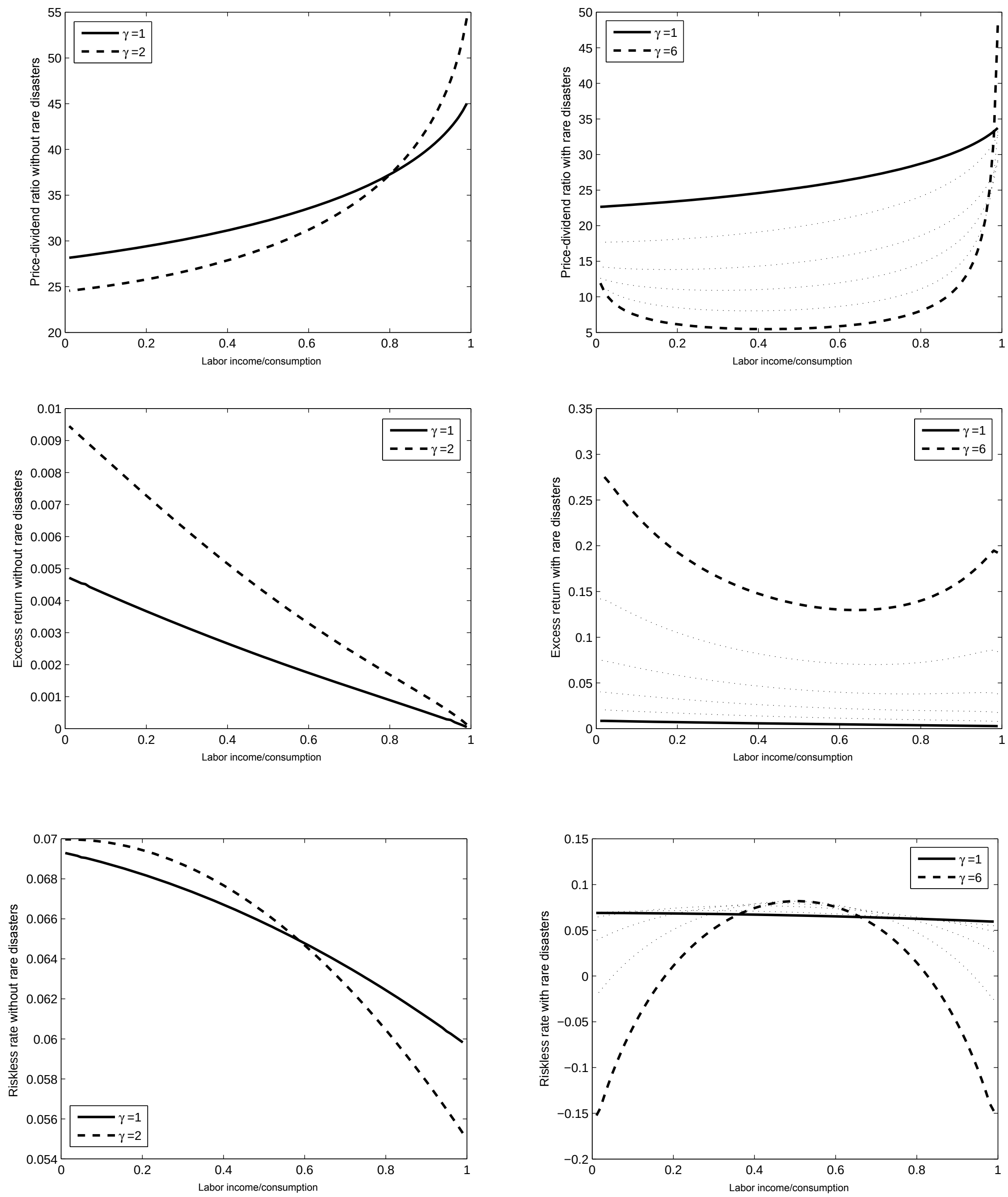
rare disasters lowers the price-dividend ratio, since it increases the risk of holding stocks. The stock-price sensitivity to s^w increases considerably for larger values of s^w.

Expected excess returns in the standard model are basically zero. The problem is that the possible choices of values for γ are very limited. If we account for rare disasters, much higher values for γ still match the targeted long rate, support finite prices and imply a positive time discount rate. Expected equity premia in the model with disasters increase rapidly as γ increases. Values of γ between 4 and 5 suffice to generate equity premia in the range of the usually reported historical averages around 6% to 7%.

The expected excess return plots are also interesting in connection with return predictability. Santos and Veronesi (2006) find that a high share of labor income to consumption predicts low future excess returns. Our findings in the simulations with the standard model point in the same direction. The curve is downward sloping, even though only slightly.

Expected excess returns in the model with rare disasters show a very different picture. For higher values of relative risk aversion γ, the simulations suggest predictability in the opposite direction. Expected excess returns are increasing in the labor income to consumption ratio s^w for high values of s^w. Thus, a change in disaster risk might explain

Figure 4.2: The plots on the left show the simulated price-dividend ratio, expected excess returns and riskless rates plotted against the labor income share to consumption for different coefficients of relative risk aversion γ in the standard model without disasters. The plots on the right show the same in the model with economic disasters.



a change in the predictability of excess returns with the labor income to consumption ratio as documented by Chen and Joslin (2012).

The riskless rate in the standard model is very high and lies above 5%. The problem is again the restriction on possible choices of γ. In the model with rare disasters, γ can take again higher values. For γ between 4 and 5, we can generate riskless rates in the usually reported range of historical averages around 1% to 2%.

4.4.2 Miscalibration of parameters

Large economic disasters are very rare and occur around once every sixty years on average. Shorter data samples may not contain any major disaster. For example our data set excludes the Great Depression and the two world wars. According to the empirical study of Barro (2006), the only episodes of declines in real GDP by more than 15% in the twentieth century occurred during the Great Depression and the aftermath of the Second World War. Our sample therefore does not allow to assess the impact of rare disasters on the mean and variance of logarithmic labor income growth and dividend growth. Since the mean growth rate is lower and the volatility is higher during times of disaster, we probably overestimate the mean growth rate and underestimate the volatility of the labor income growth rate and aggregate dividend growth rate.

Even if we would take a longer sample, we would probably underestimate the volatility. The longer sample would contain probably one or two disasters, which do not suffice to estimate the volatility during times of disaster. Thus, including disasters in our model changes our calibration of the mean and variance of logarithmic dividend growth and labor income growth. It decreases the mean and increases the variance. For our disaster calibration, the resulting mean and standard deviation of logarithmic labor income growth and dividend growth are reported in Table 4.2. While the mean labor income growth rate declines by less than 1%, the standard deviation increases by more than 4%.

Table 4.2: This table shows the simulation parameters adjusted for rare disasters. The mean μ_l and standard deviation σ_l of the labor income growth rate and the mean μ_d and standard deviation σ_d of the aggregate dividend growth rate are adjusted for the occurence of rare disasters and compared to the unadjusted counterparts.

	μ_l	σ_l	μ_d	σ_d
With disasters	0.0167	0.0626	0.0278	0.0893
Without disasters	0.0232	0.0199	0.0343	0.0667

The simulation results are given in Figure 4.3. Previous articles like Weil (1992), Constantinides and Duffie (1996) and Gollier and Schlesinger (2002) have studied the impact of miscalibrated variance of labor income and dividends before. They find that correcting for the underestimation of the variance can increase the equity premium. In our setting, adjusting parameters for the overestimated mean and underestimated variance leads to higher equity premia as well. However, the increase in expected excess returns does not suffice to explain observed historical averages.

The model with adjusted parameters allows to attain lower values of the riskless rate for larger values of γ that are not supported by the model with miscalibrated parameters. The generated riskless rates still do not match historical averages.

4.4.3 Misspecification of the distribution

Modeling disastrous low-probability events requires a change in the distributional assumption on logarithmic labor income growth and dividend growth. In the following, we study the impact of replacing the normal distribution by the sum of a normal and a compound Poisson distribution on the price-dividend ratio, expected excess returns and interest rates. The resulting distribution is negatively skewed, which probably is the main cause for changes in the results. We assume that disasters strike labor income and aggregate dividends independently in order to analyze the effect from the change in distribution separately from correlation effects.

Figure 4.4 compares the price-dividend ratio, expected excess returns and interest rates in the model with disasters to the standard model with lognormal dividends and

Figure 4.3: The plots on the left show the simulated price-dividend ratio, expected excess returns and riskless rates plotted against the labor income share to consumption when parameters are calibrated to data. The plots on the right show the same when parameters are adjusted for the occurrence of rare disasters.

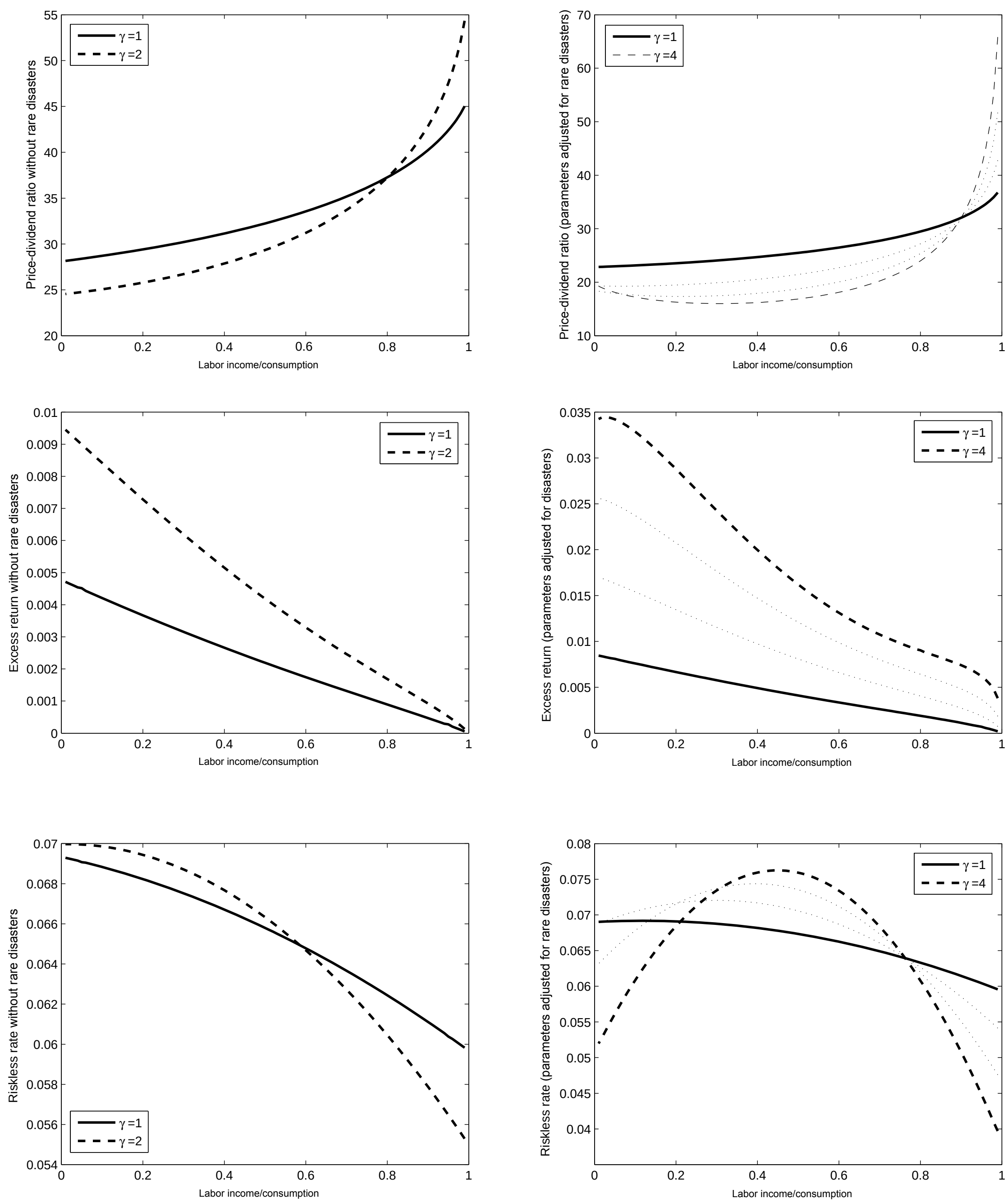
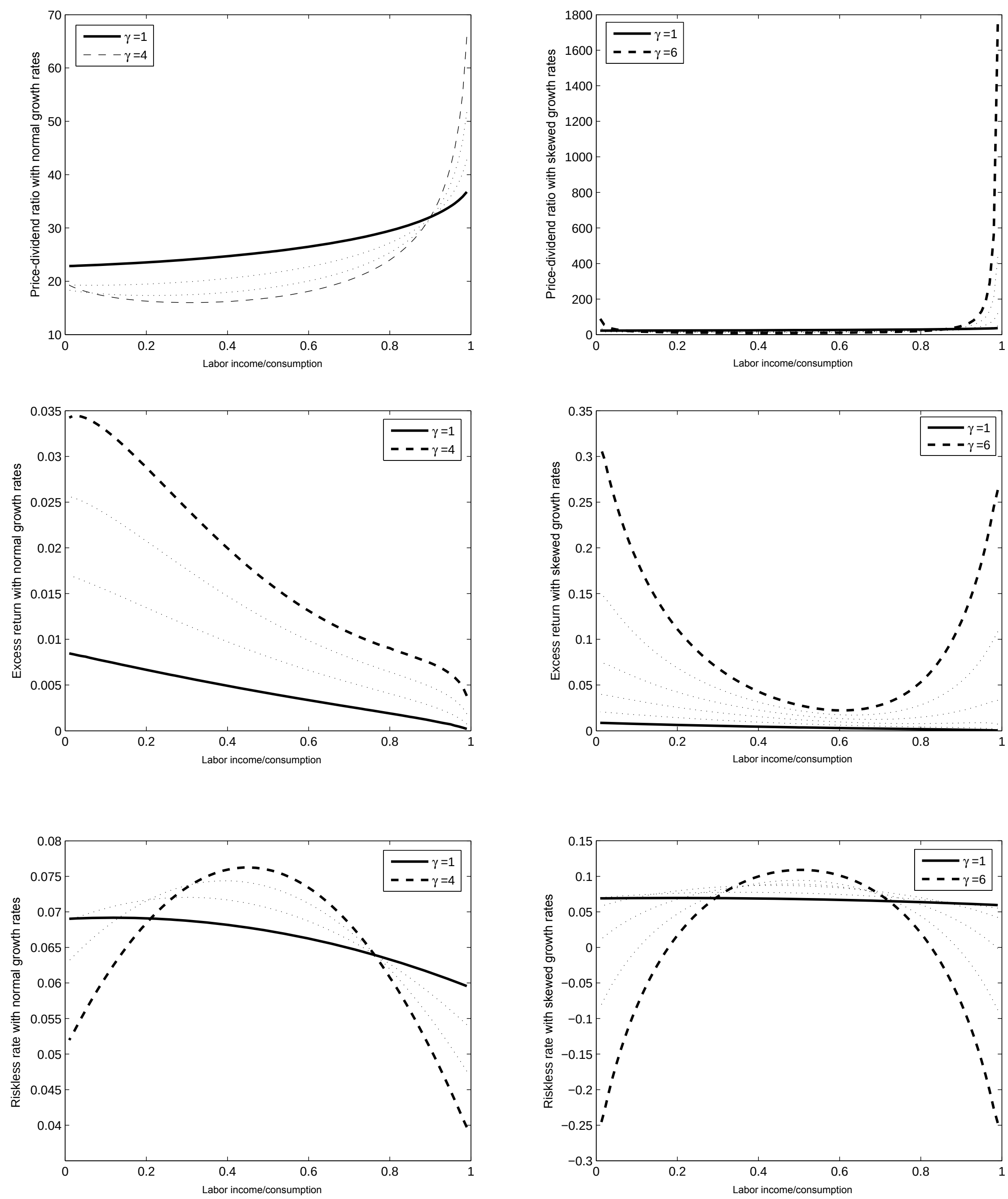


Figure 4.4: The plots on the left show the simulated price-dividend ratio, expected excess returns and riskless rates plotted against the labor income share to consumption with normal growth rates. The plots on the right show the same with growth rates distributed according to the sum of a normal and a compound Poisson distribution.



labor income. The growth rate parameters in the model with lognormal growth are the adjusted ones reported in Table 4.2. We note that we only plot curves for values of γ up to 6, even though values higher than 6 are admissible in the model with disasters. Higher values of γ lead to results outside the range of interest.

The plots show that prices become increasingly sensitive to changes in the labor income to consumption ratio s^w for larger values of s^w as we increase the coefficient of relative risk aversion γ. The expected excess return curve changes completely. For large values of s^w, expected excess returns become increasing in s^w. A misspecification of the labor income and dividend distribution may therefore lead to different conclusions about the predictive power of the labor income to consumption ratio. The modified labor income and dividend distributions allow to match substantially higher equity premia by only slightly increasing γ.

Riskless rates decrease substantially for higher values of γ and s^w. If the coefficient of relative risk aversion is larger or equal to 5, riskless rates become even significantly negative for large values of s^w.

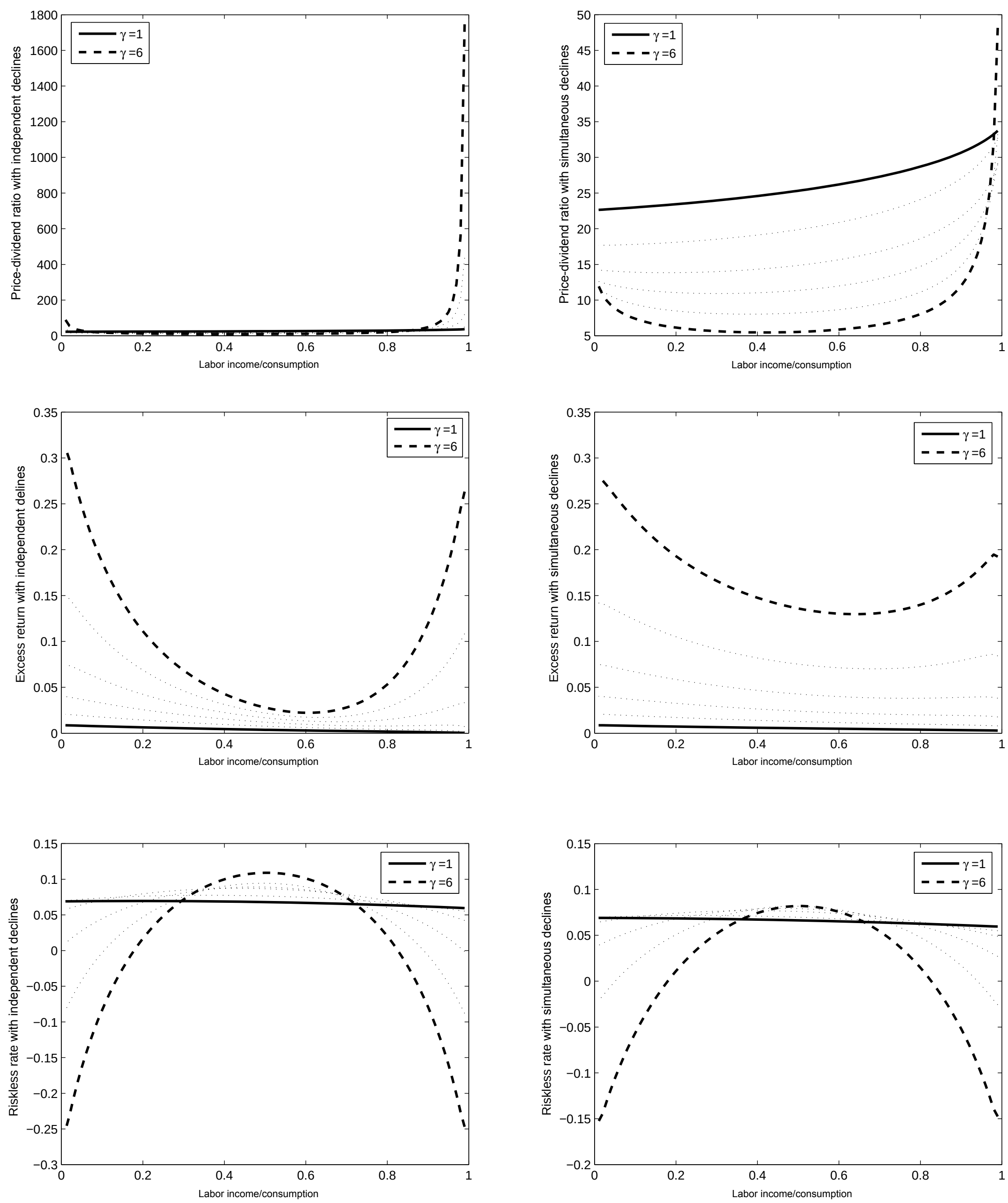
4.4.4 Correlation effects

Disasters usually hit labor income and aggregate dividends at around the same time. That is why we assume that substantial declines occur simultaneously in labor income and dividends. The following analysis compares the difference in results between independent declines and simultaneous declines. Allowing the correlation between labor income growth and dividend growth to be positive increases the risk of consumption.

Figure 4.5 shows the simulation results for the price-dividend ratio, expected excess returns and riskless rates for the model with disasters striking labor income and aggregate dividends independently and simultaneously. Prices increase more slowly for larger values of the labor income share to consumption s^w when disasters are correlated.

Expected excess returns become less sensitive to changes in s^w when disasters in labor income and aggregate dividends occur at the same time. They increase the

Figure 4.5: The plots on the left show the simulated price-dividend ratio, expected excess returns and riskless rates plotted against the labor income share to consumption when disasters strike labor income and dividends independently. The plots on the right show the same when disasters occur simultaneously.



most compared to the uncorrelated case when the labor income share to consumption and the dividend share to consumption are more or less balanced and there is more diversification. Introducing positive correlation between labor income and dividends lessens the reduction in risk due to diversification and thus leads to higher equity premia.

The riskless rate decreases more slowly for larger values of the labor income to consumption ratio s^w when we introduce correlation. However, the impact of correlation on the riskless rate is less pronounced.

4.5 Conclusion

We have studied the impact of including rare economic disasters into a Lucas (1978)-type economy with labor income. Labor income and aggregate dividends are assumed to be lognormally distributed if we neglect the existence of low-probability disastrous events. Our simulations support the findings of Rietz (1988), Barro (2006) and Martin (2013) that rare economic disasters can explain the high equity premium and the low riskless rate in a model with labor income.

We analyze three channels through which the inclusion of rare disasters affect equity premia and riskless rates. First, we study the impact of a miscalibration of distribution parameters. Even though including rare disasters markedly raises labor income volatility, the model with adjusted parameters fails to generate sufficiently high equity premia and low riskless rates. Second, we analyze the impact of changing the distributional assumption on labor income and aggregate dividends. Modifying the distribution and imposing negative skewness allows for much higher equity premia and substantially lower riskless rates. Third, we examine correlation effects. Since disasters strike labor income and aggregate dividends more or less simultaneously, they introduce correlation between the two variables in our model. The impact on equity premia and the riskless rate is rather small in the range of the labor income to consumption ratio observed in data. We conclude that it is crucial to specify the growth rate distributions correctly.

Our results show that modeling rare disasters also yields new predictions about the predictability of excess returns. In contrast to previous models without disasters, a higher labor income to consumption ratio predicts higher excess returns in our model.

An interesting extension to our analysis would be to reassess our disaster calibration. Currently, disasters are assumed to equally affect labor income and dividends. Our disaster parameters are calibrated to the empirical findings about output declines of Barro (2006). If declines in output have a different impact on labor income and dividends, our model might be able to explain the higher equity premia and lower riskless rates during recessions.

4.6 Appendix

Proof of Proposition 4.1:

From the Euler equation of the representative agent's optimization problem, we obtain

$$P_t^i = e^{-\rho\Delta t} E_t \left[\left(\frac{C_{t+\Delta t}}{C_t} \right)^{-\gamma} (P_{t+\Delta t}^i + D_{t+\Delta t}^i \Delta t) \right]. \tag{4.4}$$

By forward iteration of equation (4.4), we get

$$\begin{aligned} P_t^i &= \sum_{s=0}^{T-1} \sum_{k=1}^{n} e^{-\rho(s+\frac{k}{n})} \mathbf{E}_t \left[\left(\frac{C_{t+s+\frac{k}{n}}}{C_t} \right)^{-\gamma} D_{t+s+\frac{k}{n}} \frac{1}{n} \right] \\ &\quad + e^{-\rho T} \mathbf{E}_t \left[\left(\frac{C_{t+T}}{C_t} \right)^{-\gamma} P_{t+T}^i \right]. \end{aligned}$$

Taking the limits $T \to \infty$ and $n \to \infty$ and using the transversality condition (4.2), the Riemann sum converges to the respective Riemann integral and we obtain the pricing formula (4.3) in Proposition 4.1. □

Proof of Proposition 4.2:

$$\begin{aligned} P_t^\alpha &= \mathbf{E}_t \left[\sum_{s=1}^{\infty} \frac{1}{(1+\beta)^s} \left(\frac{C_{t+s}}{C_t} \right)^{-\gamma} D_{t+s}^\alpha \right] \\ &= (C_t)^\gamma \sum_{s=1}^{\infty} \frac{1}{(1+\beta)^s} \mathbf{E}_t \left[\frac{D_{t+s}^\alpha}{(C_{t+s})^\gamma} \right] \\ &= (D_t^1 + \cdots + D_t^N)^\gamma \sum_{s=1}^{\infty} \frac{1}{(1+\beta)^s} \mathbf{E}_t \left[\frac{D_{t+s}^\alpha}{(D_{t+s}^1 + \cdots + D_{t+s}^N)^\gamma} \right] \\ &= (e^{y_t^1} + \cdots + e^{y_t^N})^\gamma \sum_{s=1}^{\infty} \frac{1}{(1+\beta)^s} \underbrace{\mathbf{E}_t \left[\frac{e^{\boldsymbol{\alpha}' \mathbf{y}_{t+s}}}{(e^{y_{t+s}^1} + \cdots + e^{y_{t+s}^N})^\gamma} \right]}_{(I)} \end{aligned} \tag{4.5}$$

Let us define the logarithmic dividend growth of asset i between time t and s as $\tilde{y}^i_{t,s} \equiv y^i_{t+s} - y^i_t$ and the vector of changes in the dividend growth rate of all assets $\tilde{\boldsymbol{y}}_{t,s} \equiv (\tilde{y}^1_{t,s}, \ldots, \tilde{y}^N_{t,s})'$. Then

$$
\begin{aligned}
(I) &= e^{\boldsymbol{\alpha}' \boldsymbol{y}_t} \mathbf{E}_t \left[\frac{e^{\boldsymbol{\alpha}' \tilde{\boldsymbol{y}}_{t,s}}}{(e^{y^1_t + \tilde{y}^1_{t,s}} + \cdots + e^{y^N_t + \tilde{y}^N_{t,s}})^\gamma} \right] \\
&= D^{\alpha}_t \mathbf{E}_t \left[\frac{e^{\boldsymbol{\alpha}' \tilde{\boldsymbol{y}}_{t,s}}}{(e^{y^1_t + \tilde{y}^1_{t,s}} + \cdots + e^{y^N_t + \tilde{y}^N_{t,s}})^\gamma} \right]. \qquad (4.6)
\end{aligned}
$$

In a next step, we will rewrite the expression in the expectation in terms of a function from which the Fourier transform is known (see Martin, 2013, pg. 54). Define the $(N-1) \times N$ matrix $\boldsymbol{Q}$ and the N-dimensional vectors $\boldsymbol{q}_i$ and $\boldsymbol{\gamma}$ as

$$
\boldsymbol{q}'_1 \equiv (N-1, -1, \ldots, -1), \qquad (4.7)
$$

$$
\boldsymbol{Q} \equiv \begin{pmatrix} \boldsymbol{q}'_2 \\ \vdots \\ \boldsymbol{q}'_N \end{pmatrix} \equiv \begin{pmatrix} -1 & N-1 & -1 & \ldots & -1 \\ -1 & -1 & N-1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ -1 & -1 & \ldots & -1 & N-1 \end{pmatrix} \qquad (4.8)
$$

and

$$
\boldsymbol{\gamma}' \equiv (\gamma, \ldots, \gamma). \qquad (4.9)
$$

Using (4.7), (4.8) and (4.9), we can write (4.6) as

$$
\begin{aligned}
(I) &= D^{\alpha}_t \mathbf{E}_t \left[\frac{e^{\boldsymbol{\alpha}' \tilde{\boldsymbol{y}}_{t,s}}}{e^{\frac{\boldsymbol{\gamma}'(\boldsymbol{y}_t + \tilde{\boldsymbol{y}}_{t,s})}{N}} (e^{\frac{\boldsymbol{q}'_1(\boldsymbol{y}_t + \tilde{\boldsymbol{y}}_{t,s})}{N}} + \cdots + e^{\frac{\boldsymbol{q}'_N(\boldsymbol{y}_t + \tilde{\boldsymbol{y}}_{t,s})}{N}})^\gamma} \right] \\
&= D^{\alpha}_t e^{-\frac{\boldsymbol{\gamma}' \boldsymbol{y}_t}{N}} \mathbf{E}_t \left[e^{(\boldsymbol{\alpha} - \frac{\boldsymbol{\gamma}}{N})' \tilde{\boldsymbol{y}}_{t,s}} \underbrace{\frac{1}{(e^{\frac{\boldsymbol{q}'_1(\boldsymbol{y}_t + \tilde{\boldsymbol{y}}_{t,s})}{N}} + \cdots + e^{\frac{\boldsymbol{q}'_N(\boldsymbol{y}_t + \tilde{\boldsymbol{y}}_{t,s})}{N}})^\gamma}}_{(II)} \right] \qquad (4.10)
\end{aligned}
$$

The Fourier transform of (II) is known (see Martin, 2013, pg. 55). Using the Fourier inversion theorem, we can write (4.10) as

$$(I) \;=\; D_t^{\alpha} e^{-\frac{\gamma' y_t}{N}} \mathbf{E}_t \left[e^{(\boldsymbol{\alpha}-\frac{\gamma}{N})'\tilde{\mathbf{y}}_{t,s}} \int\limits_{\mathbb{R}^{N-1}} \mathscr{G}_{\gamma}^{N}(\boldsymbol{z}) e^{i\boldsymbol{z}'\mathbf{Q}(\mathbf{y}_t+\tilde{\mathbf{y}}_{t,s})} \boldsymbol{dz} \right], \tag{4.11}$$

where

$$\mathscr{G}_{\gamma}^{N} \equiv N^{N-2} \frac{\Gamma(\frac{\gamma}{N}+iz_1+\cdots+iz_{N-1})}{(2\pi)^{N-1}\Gamma(\gamma)} \prod_{k=1}^{N-1} \Gamma(\frac{\gamma}{N}+iz_1+\cdots+iz_{N-1}-Niz_k).$$

Applying Fubini's theorem and exchanging the order of integration, expression (4.11) becomes

$$\begin{aligned}(I) \;&=\; D_t^{\alpha} e^{-\frac{\gamma' y_t}{N}} \int\limits_{\mathbb{R}^{N-1}} \mathscr{G}_{\gamma}^{N}(\boldsymbol{z}) e^{i\boldsymbol{z}'\mathbf{Q}\mathbf{y}_t} \mathbf{E}_t \left[e^{(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\mathbf{Q}'\boldsymbol{z})'\tilde{\mathbf{y}}_{t,s}} \right] \boldsymbol{dz} \\ &=\; D_t^{\alpha} e^{-\frac{\gamma' y_t}{N}} \int\limits_{\mathbb{R}^{N-1}} \mathscr{G}_{\gamma}^{N}(\boldsymbol{z}) e^{i\boldsymbol{z}'\mathbf{Q}\mathbf{y}_t} e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\mathbf{Q}'\boldsymbol{z})s} \boldsymbol{dz}. \end{aligned} \tag{4.12}$$

Inserting (4.12) in (4.5), we get

$$\begin{aligned}\frac{P_t^{\alpha}}{D_t^{\alpha}} \;&=\; (e^{y_t^1}+\cdots+e^{y_t^N})^{\gamma} \sum_{s=1}^{\infty} \frac{e^{-\frac{\gamma' y_t}{N}}}{(1+\beta)^s} \int\limits_{\mathbb{R}^{N-1}} \mathscr{G}_{\gamma}^{N}(\boldsymbol{z}) e^{i\boldsymbol{z}'\mathbf{Q}\mathbf{y}_t} e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\mathbf{Q}'\boldsymbol{z})s} \boldsymbol{dz} \\ &=\; (e^{y_t^1}+\cdots+e^{y_t^N})^{\gamma} e^{-\frac{\gamma' y_t}{N}} \int\limits_{\mathbb{R}^{N-1}} \mathscr{G}_{\gamma}^{N}(\boldsymbol{z}) e^{i\boldsymbol{z}'\mathbf{Q}\mathbf{y}_t} \sum_{s=1}^{\infty} \left(\frac{e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\mathbf{Q}'\boldsymbol{z})}}{1+\beta} \right)^s \boldsymbol{dz} \\ &=\; (e^{y_t^1}+\cdots+e^{y_t^N})^{\gamma} e^{-\frac{\gamma' y_t}{N}} \int\limits_{\mathbb{R}^{N-1}} \mathscr{G}_{\gamma}^{N}(\boldsymbol{z}) e^{i\boldsymbol{z}'\mathbf{Q}\mathbf{y}_t} \frac{\frac{e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\mathbf{Q}'\boldsymbol{z})}}{1+\beta}}{1-\frac{e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\mathbf{Q}'\boldsymbol{z})}}{1+\beta}} \boldsymbol{dz} \\ &=\; (e^{\frac{q_1' y_t}{N}}+\cdots+e^{\frac{q_N' y_t}{N}})^{\gamma} \int\limits_{\mathbb{R}^{N-1}} \mathscr{G}_{\gamma}^{N}(\boldsymbol{z}) \frac{e^{i\boldsymbol{z}'\mathbf{Q}\mathbf{y}_t} e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\mathbf{Q}'\boldsymbol{z})}}{1+\beta-e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\mathbf{Q}'\boldsymbol{z})}} \boldsymbol{dz} \end{aligned} \tag{4.13}$$

Note that for the geometric series to converge, we need the following restriction

$$\left|\frac{1}{1+\beta}e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\boldsymbol{Q}'\boldsymbol{z})}\right| < 1. \tag{4.14}$$

We will refer to the inequality (4.14) as the finiteness condition. If it is not satisfied, the geometric series diverges and the asset's price is infinite.

The price-dividend ratio (4.13) depends on the N realizations of the logarithmic dividends $y_t^1, \ldots, y_t^N$. We can rewrite (4.13) in terms of the $N-1$ logarithmic dividends relative to the dividends of the first asset $u_t^i \equiv y_t^i - y_t^1$ instead of $\boldsymbol{y}_t$. Let $\boldsymbol{u}_t = (u_t^2, \ldots, u_t^N)'$ and $\boldsymbol{u}_t^+ = (0, u_t^2, \ldots, u_t^N)'$. Let us replace the factor in front of the integral in (4.13)

$$e^{-\frac{\boldsymbol{\gamma}'\boldsymbol{u}_t^+}{N}}(e^{u_t^1} + \cdots + e^{u_t^N})^{\gamma} \int\limits_{\mathbb{R}^{N-1}} \mathcal{G}_\gamma^N(\boldsymbol{z}) e^{i\boldsymbol{z}'\boldsymbol{Q}\boldsymbol{y}_t} \frac{e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\boldsymbol{Q}'\boldsymbol{z})}}{1+\beta-e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\boldsymbol{Q}'\boldsymbol{z})}} \boldsymbol{d}\boldsymbol{z}. \tag{4.15}$$

To write the integral in (4.15) in terms of $\boldsymbol{u}_t$, we need a change of variable. Let $\hat{\boldsymbol{z}} \equiv \boldsymbol{B}\boldsymbol{z}$, where

$$\boldsymbol{B} \equiv \begin{pmatrix} N-1 & -1 & \ldots & -1 \\ -1 & N-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \ldots & -1 & N-1 \end{pmatrix}.$$

The inverse of $\boldsymbol{B}$ is

$$\boldsymbol{B}^{-1} = \frac{1}{N}\begin{pmatrix} 2 & 1 & \ldots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \ldots & 1 & 2 \end{pmatrix}.$$

The determinant of $\boldsymbol{B}^{-1}$ can be determined using the matrix determinant lemma and is equal to $\frac{1}{N^{N-2}}$ (see Martin, 2013, p. 55). With the change of variable, (4.15) becomes

$$e^{-\frac{\gamma' u_t^+}{N}}(e^{u_t^1}+\cdots+e^{u_t^N})^\gamma \int\limits_{\mathbb{R}^{N-1}} \mathscr{G}_\gamma^N(\boldsymbol{B}^{-1}\hat{\boldsymbol{z}}) \frac{e^{i(\boldsymbol{B}^{-1}\hat{\boldsymbol{z}})'\boldsymbol{Q}\boldsymbol{y}_t} e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\boldsymbol{Q}'\boldsymbol{B}^{-1}\hat{\boldsymbol{z}})}}{1+\beta-e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\boldsymbol{Q}'\boldsymbol{B}^{-1}\hat{\boldsymbol{z}})}} \frac{\boldsymbol{d}\hat{\boldsymbol{z}}}{N^{N-2}}$$

Using $\hat{z}_k = Nz_k - z_1 - \cdots - z_{N-1}$ and $\hat{z}_1 + \cdots + \hat{z}_{N-1} = z_1 + \cdots + z_{N-1}$, we get

$$\mathscr{G}_\gamma^N = \frac{N^{N-2}}{(2\pi)^{N-1}} \frac{\Gamma(\frac{\gamma}{N}+i\hat{z}_1+\cdots+i\hat{z}_{N-1})}{\Gamma(\gamma)} \prod_{k=1}^{N-1} \Gamma(\frac{\gamma}{N}-i\hat{z}_k).$$

Let us define $\boldsymbol{U} \equiv \boldsymbol{B}^{-1}\boldsymbol{Q}$. It follows that $\boldsymbol{u}_t = \boldsymbol{U}\boldsymbol{y}_t$. Hence, the price-dividend ratio becomes

$$\frac{P_t^\alpha}{D_t^\alpha} = e^{-\frac{\gamma' u_t^+}{N}}(e^{u_t^1}+\cdots+e^{u_t^N})^\gamma \int\limits_{\mathbb{R}^{N-1}} \mathscr{F}_\gamma^N(\hat{\boldsymbol{z}}) e^{i\hat{\boldsymbol{z}}'\boldsymbol{u}_t} \frac{e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\boldsymbol{U}'\hat{\boldsymbol{z}})}}{1+\beta-e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\boldsymbol{U}'\hat{\boldsymbol{z}})}} \boldsymbol{d}\hat{\boldsymbol{z}},$$

where

$$\mathscr{F}_\gamma^N(\hat{\boldsymbol{z}}) \equiv \frac{\mathscr{G}_\gamma^N(\boldsymbol{B}^{-1}\hat{\boldsymbol{z}})}{N^{N-2}}.$$

□

Proof of Proposition 4.3:

Let us derive the expected return based on our results in 4.6.

$$\begin{aligned} \mathbf{E}_t[R_{t+1}^\alpha] &= \mathbf{E}_t\left[\frac{P_{t+1}^\alpha - P_t^\alpha + D_{t+1}^\alpha}{P_t^\alpha}\right] \\ &= \underbrace{\mathbf{E}_t\left[\frac{P_{t+1}^\alpha - P_t^\alpha}{P_t^\alpha}\right]}_{(I)} + \underbrace{\mathbf{E}_t\left[\frac{D_{t+1}^\alpha}{P_t^\alpha}\right]}_{(II)} \end{aligned}$$

We start by calculating the expected capital gain (I). First recall the price formula from (4.13).

$$
\begin{aligned}
P_t^\alpha &= (e^{y_t^1}+\cdots+e^{y_t^N})^\gamma e^{-\frac{\gamma' y_t}{N}} \int_{\mathbb{R}^{N-1}} e^{i\boldsymbol{z}'\boldsymbol{Q}\boldsymbol{y}_t} \underbrace{\frac{\mathcal{G}_\gamma^N(\boldsymbol{z}) e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\boldsymbol{Q}'\boldsymbol{z})}}{1+\beta-e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\boldsymbol{Q}'\boldsymbol{z})}}}_{\equiv h^\alpha(\boldsymbol{z})} D_t^\alpha \boldsymbol{dz} \\
&= (e^{y_t^1}+\cdots+e^{y_t^N})^\gamma \int_{\mathbb{R}^{N-1}} e^{(\alpha-\frac{\gamma}{N}+i\boldsymbol{Q}'\boldsymbol{z})'\boldsymbol{y}_t} h^\alpha(\boldsymbol{z}) \boldsymbol{dz} \qquad (4.16)
\end{aligned}
$$

By applying the multinomial formula to (4.16), we get

$$
P_t^\alpha = \sum_{|m|=\gamma} \binom{\gamma}{m} \int_{\mathbb{R}^{N-1}} e^{(\alpha-\frac{\gamma}{N}+m+i\boldsymbol{Q}'\boldsymbol{z})'\boldsymbol{y}_t} h^\alpha(\boldsymbol{z}) \boldsymbol{dz},
$$

where m is a multiindex. The expected prices of the next time step are

$$
\begin{aligned}
\mathbf{E}_t[P_{t+1}^\alpha] &= \sum_{|m|=\gamma} \binom{\gamma}{m} \int_{\mathbb{R}^{N-1}} \mathbf{E}_t[e^{(\alpha-\frac{\gamma}{N}+m+i\boldsymbol{Q}'\boldsymbol{z})'\boldsymbol{y}_{t+1}}] h^\alpha(\boldsymbol{z}) \boldsymbol{dz} \\
&= \sum_{|m|=\gamma} \binom{\gamma}{m} \int_{\mathbb{R}^{N-1}} \mathbf{E}_t[e^{(\alpha-\frac{\gamma}{N}+m+i\boldsymbol{Q}'\boldsymbol{z})'(\boldsymbol{y}_t+\tilde{\boldsymbol{y}}_{t,t+1})}] h^\alpha(\boldsymbol{z}) \boldsymbol{dz} \\
&= \sum_{|m|=\gamma} \binom{\gamma}{m} \int_{\mathbb{R}^{N-1}} e^{(\alpha-\frac{\gamma}{N}+m+i\boldsymbol{Q}'\boldsymbol{z})'\boldsymbol{y}_t} e^{c(\alpha-\frac{\gamma}{N}+m+i\boldsymbol{Q}'\boldsymbol{z})} h^\alpha(\boldsymbol{z}) \boldsymbol{dz}
\end{aligned}
$$

Hence, the expected capital gains are

$$
\begin{aligned}
\mathbf{E}_t\left[\frac{P_{t+1}^\alpha - P_t^\alpha}{P_t^\alpha}\right] &= \frac{\mathbf{E}_t[P_{t+1}^\alpha] - P_t^\alpha}{P_t^\alpha} \\
&= \Phi^\alpha \frac{D_t^\alpha}{P_t^\alpha}, \qquad (4.17)
\end{aligned}
$$

where

$$\Phi^\alpha \equiv \sum_{|m|=\gamma} \binom{\gamma}{m} \int_{\mathbb{R}^{N-1}} e^{(-\frac{\gamma}{N}+m+i\boldsymbol{Q}'\boldsymbol{z})'\boldsymbol{y}_t}(e^{c(\alpha-\frac{\gamma}{N}+m+i\boldsymbol{Q}'\boldsymbol{z})}-1)h^\alpha(\boldsymbol{z})\boldsymbol{d}\boldsymbol{z}.$$

Now let us derive the expected dividend gain (II)

$$\begin{aligned} \mathbf{E}_t\left[\frac{D^\alpha_{t+1}}{P^\alpha_t}\right] &= \frac{\mathbf{E}_t[D^\alpha_1]}{P^\alpha_t} \\ &= e^{c(\alpha)}\frac{D^\alpha_t}{P^\alpha_t}. \end{aligned} \tag{4.18}$$

From (4.17) and (4.18) the expected return follows

$$\mathbf{E}_t[R^\alpha_{t+1}] = (\Phi^\alpha + e^{c(\alpha)})\frac{D^\alpha_t}{P^\alpha_t}.$$

The expected return depends on the N realizations of the logarithmic dividends $y^1_t, \ldots, y^N_t$. We can apply the same change of variable as on the price-dividend ratio in 4.6 to obtain an expression depending only on the $N-1$ logarithmic dividends relative to the dividends of the first asset.

$$\mathbf{E}_t[R^\alpha_{t+1}] = (\Phi^\alpha + e^{c(\alpha)})\frac{D^\alpha_t}{P^\alpha_t},$$

where

$$\Phi^\alpha = \sum_{|m|=\gamma} \binom{\gamma}{m} e^{(m-\frac{\gamma}{N})'\boldsymbol{u}^+_t} \int_{\mathbb{R}^{N-1}} \mathscr{F}^N_\gamma(\hat{\boldsymbol{z}}) e^{i\hat{\boldsymbol{z}}'\boldsymbol{u}_t} \frac{e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\boldsymbol{U}'\hat{\boldsymbol{z}})}(e^{c(\alpha-\frac{\gamma}{N}+m+i\boldsymbol{U}'\hat{\boldsymbol{z}})}-1)}{1+\beta-e^{c(\boldsymbol{\alpha}-\frac{\gamma}{N}+i\boldsymbol{U}'\hat{\boldsymbol{z}})}}\boldsymbol{d}\hat{\boldsymbol{z}}$$

□

Proof of Proposition 4.4:

Let us derive the yield of a zero-coupon bond with maturity T.

$$
\begin{aligned}
B_t^{(T)} &= \mathbf{E}_t\left[\frac{1}{(1+\beta)^T}\left(\frac{C_{t+T}}{C_t}\right)^{-\gamma}\right] \\
&= \frac{(C_t)^\gamma}{(1+\beta)^T}\mathbf{E}_t\left[\frac{1}{(C_{t+T})^\gamma}\right] \\
&= \frac{(D_t^1+\cdots+D_t^N)^\gamma}{(1+\beta)^T}\mathbf{E}_t\left[\frac{1}{(D_{t+T}^1+\cdots+D_{t+T}^N)^\gamma}\right] \\
&= \frac{(e^{y_t^1}+\cdots+e^{y_t^N})^\gamma}{(1+\beta)^T}\mathbf{E}_t\left[\frac{1}{(e^{y_{t+T}^1}+\cdots+e^{y_{t+T}^N})^\gamma}\right]
\end{aligned}
\tag{4.19}
$$

The expectation in (4.19) is equal to the expectation (I) in (4.5) of 4.6 with $\alpha = \mathbf{0}$ and $s = T$. Inserting (4.12), we get

$$
B_t^{(T)} = \frac{1}{(1+\beta)^T}(e^{y_t^1}+\cdots+e^{y_t^N})^\gamma e^{-\frac{\boldsymbol{\gamma}'\boldsymbol{y}_t}{N}} \int\limits_{\mathbb{R}^{N-1}} \mathscr{G}_\gamma^N(\boldsymbol{z})e^{i\boldsymbol{z}'\mathbf{Q}\boldsymbol{y}_t}e^{c(-\frac{\boldsymbol{\gamma}}{N}+i\mathbf{Q}'\boldsymbol{z})T}\boldsymbol{dz}.
$$

Applying the same change of variable as in 4.6 and 4.6, we obtain

$$
B_t^{(T)} = \frac{1}{(1+\beta)^T}e^{-\frac{\boldsymbol{\gamma}'\boldsymbol{u}_t^+}{N}}(e^{u_t^1}+\cdots+e^{u_t^N})^\gamma \int\limits_{\mathbb{R}^{N-1}} \mathscr{F}_\gamma^N(\hat{\boldsymbol{z}})e^{i\hat{\boldsymbol{z}}'\boldsymbol{u}_t}e^{c(-\frac{\boldsymbol{\gamma}}{N}+i\boldsymbol{U}'\hat{\boldsymbol{z}})T}\boldsymbol{d}\hat{\boldsymbol{z}}.
$$

Now we can determine $\mathscr{Y}_t(T)$ from

$$
\frac{1}{(1+\mathscr{Y}_t(T))^T} = \frac{e^{-\frac{\boldsymbol{\gamma}'\boldsymbol{u}_t^+}{N}}(e^{u_t^1}+\cdots+e^{u_t^N})^\gamma}{(1+\beta)^T} \int\limits_{\mathbb{R}^{N-1}} \mathscr{F}_\gamma^N(\hat{\boldsymbol{z}})e^{i\hat{\boldsymbol{z}}'\boldsymbol{u}_t}e^{c(-\frac{\boldsymbol{\gamma}}{N}+i\boldsymbol{U}'\hat{\boldsymbol{z}})T}\boldsymbol{d}\hat{\boldsymbol{z}}.
$$

This is equivalent to

$$
\mathscr{Y}_t(T) = \frac{(1+\beta)}{\left(e^{-\frac{\boldsymbol{\gamma}'\boldsymbol{u}_t^+}{N}}(e^{u_t^1}+\cdots+e^{u_t^N})^\gamma \int\limits_{\mathbb{R}^{N-1}} \mathscr{F}_\gamma^N(\hat{\boldsymbol{z}})e^{i\hat{\boldsymbol{z}}'\boldsymbol{u}_t}e^{c(-\frac{\boldsymbol{\gamma}}{N}+i\boldsymbol{U}'\hat{\boldsymbol{z}})T}\boldsymbol{d}\hat{\boldsymbol{z}}\right)^{\frac{1}{T}}} - 1.
$$

The risk-free rate is accordingly

$$\begin{aligned} R^f_{t+1} &= \mathscr{Y}_t(1) \\ &= \frac{(1+\beta)}{e^{-\frac{\gamma' u_t^+}{N}}(e^{u_t^1}+\cdots+e^{u_t^N})^\gamma \int\limits_{\mathbb{R}^{N-1}} \mathscr{F}_\gamma^N(\hat{\boldsymbol{z}}) e^{i\hat{\boldsymbol{z}}'\boldsymbol{u}_t} e^{c(-\frac{\gamma}{N}+i\boldsymbol{U}'\hat{\boldsymbol{z}})} \boldsymbol{d}\hat{\boldsymbol{z}}} - 1. \end{aligned}$$

□

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