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Monotone Equilibria in Signalling Games

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Abstract

We study the monotonicity of sender's equilibrium strategy with respect to her type in signalling games. We use counterexamples to show that when the sender's payoff is non-separable, the Spence-Mirrlees condition cannot rule out equilibria in which the sender uses non-monotone strategies. These equilibria can survive standard refinements as incentives are strict and the sender plays every action with positive probability. We provide sufficient conditions under which the sender's strategy is monotone in every Nash equilibrium. Our conditions require the sender's payoff to have strictly increasing differences between the state and the action profile and monotone with respect to each player's action. We also identify and fully characterize a novel property on the sender's payoff that we call *increasing absolute differences over distributions*, under which every pair of distributions over the receiver's actions can be ranked endogenously. Our sufficient conditions fit into a number of applications, including advertising, warranty provision, education and job assignment, etc.

Keywords: signalling game, monotone equilibrium, Spence-Mirrlees condition, monotone-supermodular payoff, quasi-concavity preserving, increasing absolute differences over distributions

JEL Classification: C72, D82

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1 Introduction

Starting from the seminal contribution of Spence (1973), signalling games have become powerful tools to study strategic interactions under asymmetric information. In a typical signalling model, an informed *sender*, who has private information about the payoff environment (or her *type*), takes an action which can influence the behavior of an uninformed *receiver*. This game theoretic model helps researchers to understand phenomena such as education, limit pricing, the peacock's tail, etc.

In virtually all applications of signalling games, players' payoffs satisfy the well-known Spence-Mirrlees condition: The sender's actions and types are ranked, such that a higher type has a comparative advantage in taking higher actions comparing with a lower type. For example, it is less costly for a talented worker to receive more education (Spence 1973), and is more profitable for an efficient firm to cut prices (Milgrom and Roberts 1982). Under this regularity condition, an intuitive prediction is that the sender's action should be non-decreasing in her type, or the game's outcome is *monotone*.

In this paper, we assess the robustness of the above monotonicity prediction. We focus on signalling games in which the set of types and actions are complete lattices (Topkis 1998) and the sender's payoff exhibits strictly increasing differences between the state and her own action. This generalizes the well-known Spence-Mirrlees condition by allowing for discrete and multi-dimensional state and action spaces. An equilibrium is monotone if a higher type sender never plays a strictly lower action than a lower type.

We examine the monotonicity of all (Bayes) Nash equilibria in these signalling games. The motivations for this question is twofold. First, as pointed out by Fudenberg, Kreps and Levine (1988) as well as Weinstein and Yildiz (2013), refinements in extensive form games are sensitive to the modeling details, so it is important to evaluate the robustness of the monotonicity prediction against equilibrium selection. Second, monotone equilibria have desirable properties, making them straightforward to interpret, tractable to analyze and easy to compute (Athey 2001). Therefore, a result establishing the monotonicity of all equilibria can facilitate the characterization of the set of equilibrium strategies and outcomes.[1]

We start from displaying a counterexample which shows that the Spence-Mirrlees condition cannot guarantee the monotonicity of the sender's strategy in all equilibria. These non-monotone equilibria exist even when both players' payoffs are strictly supermodular functions with respect to the sender's type and

[1] To address the concern that there is a plethora of equilibria in signalling games, we adopt the following "double standard": For counterexamples, we will require stronger solution concepts such as sequential equilibrium (Kreps and Wilson 1982), and equilibria that can survive standard refinements in Kohlberg and Mertens (1986), Cho and Kreps (1987), Banks and Sobel (1987), etc. In contrast, when presenting positive results, we will focus on weaker solution concepts such as Nash equilibrium, rationalizability, etc.

the action profile. Furthermore, they can survive standard refinements as both players have strict incentives and the sender plays every action with positive probability.

Comparing with Spence (1973), non-monotone equilibria exist due to the *non-separability* of the sender's payoff. Especially, the sender's returns from the receiver's action depend on her type. As a result, a high type sender may have an incentive to play a low action if doing so can induce the receiver to play a high action. The receiver has an incentive to do so if he believes that the state is high when the sender's action is low, which becomes self-fulfilling due to the sender's non-monotone strategy.

We then proceed to provide sufficient conditions under which all Nash equilibria are monotone despite payoffs being non-separable. At the heart of our analysis is a *monotone-supermodular* condition, which requires, in addition to the Spence-Mirrlees condition, the sender's payoff being strictly decreasing in her own action, strictly increasing in the receiver's action, and having increasing differences between her type and the receiver's action. This fits into a number of applications, including the education signalling game in which an employer assigns workers job positions after observing their years of education, the warranty provision game in which a firm chooses the length of warranty and the amount of refund before a consumer chooses the quantity to purchase, etc.

Our first result (Theorem 1) shows that every equilibrium is monotone when the sender's payoff is monotone-supermodular and the receiver's action choice is binary. This fits into the warranty provision game when the consumer has unit demand. Intuitively, thanks to the binary action assumption, every pair of distributions over the receiver's action can be ranked according to first-order stochastic dominance (FOSD). Since playing a higher action is more costly for the sender, she only has an incentive to do so when it induces a more favorable response from the receiver. This implies that the ranking over the sender's equilibrium actions must coincide with the ranking over the receiver's actions that they induce. Since a high type sender has a stronger preference towards higher action profiles, she will never play a strictly lower action than a low type in equilibrium.

However, in games where the receiver has three or more actions, non-monotone equilibria can arise despite the sender's payoff being monotone-supermodular, as it is no longer the case that every pair of the receiver's mixed actions can be ranked according to FOSD. Consequently, there can exist two equilibrium *action profiles* which cannot be ranked albeit the corresponding actions for the sender can be ranked.

We introduce two sets of sufficient conditions to address this issue. First, we show in Theorem 2 that every equilibrium is monotone if the sender's payoff is monotone-supermodular, the ranking over the receiver's action set is complete, and the receiver's payoff satisfies a *quasiconcavity-preserving property*

(QPP). QPP requires the receiver's payoff to be strictly quasi-concave in his own action under every belief about the state. A sufficient condition for QPP is that the receiver's payoff being strictly concave in his own action, which fits into the warranty provision game when the consumer faces decreasing marginal returns to quantity.[2] QPP implies that the receiver has at most two pure best replies in every circumstance, which must be adjacent elements in his action set. As a result, every pair of his mixed *best replies* can be ranked according to FOSD.

Second, we identify a novel condition on the sender's payoff under which every pair of the receiver's mixed actions can be ranked endogenously. We call this property *increasing absolute differences over distributions* (IADD). Theorem 3 shows that if the sender's payoff is monotone-supermodular and satisfies IADD, then every Nash equilibrium is monotone.[3] Unlike Theorem 2, Theorem 3 (as well as Theorem 1) makes no reference to the receiver's payoff and incentives, so the conclusion extends to richer environments such as the sender is signalling to a population of heterogeneous receivers, the receiver having private information about his payoff, etc. We also establish a representation theorem that fully characterizes IADD (Proposition 1), which facilitates its verification in applications.

Literature Review. Starting from Spence (1973), the monotonicity of outcomes has become a natural prediction in various applications of signalling games to labor economics, industrial organization, corporate finance and biology. However, to the best of our knowledge, the question that when it is without loss of generality to focus on monotone equilibria has not been systematically analyzed. Our results provide sufficient conditions under which all equilibria are monotone. Our sufficient conditions highlight the economic forces behind equilibrium monotonicity and our counterexamples illustrate how monotonicity can fail once they are relaxed. Furthermore, these conditions are easy to verify given the functional forms of players' payoffs, which are useful for future applied works.

This paper also contributes to the literature on supermodular incomplete information games. Most of the papers in this literature focus on simultaneous move games and establish the existence of monotone pure strategy Nash equilibrium (e.g. Athey 2001; McAdams 2003; Van Zandt and Vives 2007; Reny 2011).[4] In contrast, we analyze the monotonicity of equilibria in *sequential* move games where players'

[2] In Appendix B, we relate QPP to a strict version of the *signed-ratio monotonicity condition* in Quah and Strulovici (2012) and provide a full characterization.

[3] A similar result is also obtained in Kartik, Lee and Rappoport (2017). We will elaborate this more in section 5 and show that our result neither nests nor is nested by theirs.

[4] Several papers establish the monotonicity of all equilibria in *simultaneous* move supermodular games, for example, Morris and Shin (1998) and the vast literature on global games, McAdams (2006) on uniform price auctions, etc.

payoff functions are supermodular, with one-shot signalling games a natural starting point.[5]

Mensch (2016) studies dynamic incomplete information games with strategic complementarities and establishes the existence of monotone perfect Bayesian equilibrium. Complementary to his work, we focus on one-shot signalling games while emphasizing the *robustness* of equilibrium monotonicity. Our results are also applicable to the study of robust predictions in repeated signalling games where the sender's payoff is monotone-supermodular. For example, our Theorems 1 and 3 are intermediate steps towards establishing the commitment payoff bound and the consistency of the sender's equilibrium behavior in infinitely repeated reputation games with interdependent valuations (Pei 2016).

2 The Model

Consider the following two-player signalling game. Player 1 (or sender, she) privately observes the realization of a payoff relevant state $\theta \in \Theta$ (call it her *type*) and then chooses an action $a_1 \in A_1$. Player 2 (or receiver, he) has a prior belief $\pi \in \Delta(\Theta)$ about θ. He chooses $a_2 \in A_2$ after observing a_1. Player i's payoff is $u_i(\theta, a_1, a_2)$ with $i \in \{1,2\}$. Both players are expected utility maximizers. Abusing notation, let $u_i(\theta, a_1, \alpha_2) \equiv \int_{a_2} u_i(\theta, a_1, a_2) d\alpha_2$ with $\alpha_2 \in \Delta(A_2)$.

Throughout the paper, we assume that Θ, A_1 and A_2 are finite lattices and π has full support.[6] We will comment on cases in which Θ, A_1 and A_2 are infinite after stating our main results. We use $\succ$ and $\succsim$ to denote strict and weak orders on lattice sets. For two lattices X and Y, a mapping $f: X \times Y \to \mathbb{R}$ exhibits *increasing differences* if for every $x, x' \in X$ and $y, y' \in Y$ with $x \succ x'$ and $y \succ y'$:

$$f(x,y) - f(x',y) \geq f(x,y') - f(x',y'), \tag{2.1}$$

and it exhibits *strictly increasing differences* if the above inequality is strict (Topkis 1998). We introduce a regularity condition on the sender's payoff, which generalizes the Spence-Mirrlees condition to discrete lattice sets:

[5] Complementarities in dynamic games are explored by Echenique (2004a, 2004b), who explains how dynamic incentives can weaken the implications of complementarity and supermodularity. Despite we have established the monotonicity of all Nash equilibria in a signalling game context, a signalling game with supermodular payoff functions is not necessarily supermodular in its normal form and, therefore, the other attractive properties of simultaneous move supermodular games, such as the existence of extremal equilibria, monotone comparative statics, the tâtonnement algorithm to compute the set of equilibria, etc. cannot be applied to our setting.

[6] A set X is a lattice if there exists a partial order $\succeq$ such that for every $x, x' \in X$, $x \vee x', x \wedge x' \in X$, where $x \vee x'$ is the smallest element above both x and x', $x \wedge x'$ is the largest element below both x and x'.

Definition 1 (Generalized Spence-Mirrlees Condition). *u_1 satisfies the generalized Spence-Mirrlees condition if it exhibits strictly increasing differences in (θ, a_1).*

Intuitively, this generalized Spence-Mirrlees condition requires that a higher type sender has a comparative advantage in playing higher actions comparing with a lower type. This is satisfied in most applications of signalling theory, including the education game in which receiving education is less costly for a more talented worker (Spence 1973), the beer-quiche game in which drinking beer is more pleasant for the strong sender (Cho and Kreps 1987), the warranty provision game in which providing lengthier warranty is less costly for a high quality firm (Gal-Or 1989), etc.

Strategies & Equilibrium. The sender's strategy is $\sigma_1 : \Theta \to \Delta(A_1)$ and the receiver's strategy is $\sigma_2 : A_1 \to \Delta(A_2)$. Let $\sigma_1^\theta \in \Delta(A_1)$ be the (possibly mixed) action played by type θ, which gives $\sigma_1 = \left(\sigma_1^\theta\right)_{\theta\in\Theta}$.

The solution concept is Nash equilibrium (henceforth equilibrium), which consists of a strategy profile $\sigma \equiv (\sigma_1, \sigma_2)$ such that σ_i best responds to σ_{-i} for every $i \in \{1,2\}$. Since the game is finite, an equilibrium exists. Next, we introduce the definitions of *monotone strategy* and *monotone equilibrium*:

Definition 2 (Monotone Strategy & Monotone Equilibrium). *σ_1 is a monotone strategy if for every $\theta \succ \theta'$, there exist no $a_1 \in supp(\sigma_1^\theta)$ and $a_1' \in supp(\sigma_1^{\theta'})$ such that $a_1 \prec a_1'$. An equilibrium (σ_1, σ_2) is monotone if σ_1 is a monotone strategy.*

In words, a strategy is monotone if a low type sender never plays a strictly higher action than a high type. When the order on A_1 is complete (or A_1 is one-dimensional), the monotonicity of σ_1 implies that

$$\min_{a_1}\left\{\text{supp}(\sigma_1^\theta)\right\} \succsim \max_{a_1}\left\{\text{supp}(\sigma_1^{\theta'})\right\} \text{ for every } \theta \succ \theta'. \tag{2.2}$$

That is to say, if type θ' plays a_1 with positive probability, then every type higher than θ' must be playing actions that are higher or equal to a_1 with probability 1. In particular, (2.2) implies that supp(σ_1^θ) dominates supp$(\sigma_1^{\theta'})$ in strong set order for every $\theta \succ \theta'$. Nevertheless, it is worth to note that all our counterexamples violate the weaker notion of monotonicity in terms of strong set order, making them more convincing. In contrast, all the positive results apply under our more demanding monotonicity requirement, which actually strengthens their implications.

We are interested in examining the monotonicity of *all* Nash equilibria in signalling games, and in particular, games in which the sender's payoff satisfies the generalized Spence-Mirrlees condition. The choice of a weak solution concept makes our positive results presented in section 4 robust against equilibrium selection. However, due to the plethora of equilibria in signalling games, one might argue that we should restrict attention to a subset of equilibria that can survive standard refinements instead of all of them. To address this concern, whenever presenting counterexamples we will adopt the more stringent solution concept of sequential equilibrium (Kreps and Wilson 1982). To make it even more convincing, we will also require that the equilibria in the counterexamples can survive the refinements proposed in Kohlberg and Mertens (1986), Cho and Kreps (1987), Banks and Sobel (1987), etc.

3 Counterexample: Existence of Non-Monotone Equilibria

In this section, we present a counterexample which shows that the generalized Spence-Mirrlees condition cannot guarantee the monotonicity of the sender's equilibrium strategy even in $2 \times 2 \times 2$ games.

Example 1. Consider the following $2 \times 2 \times 2$ game:

$\theta = \theta_1$	h	l
H	$2,2$	$0,0$
L	$1,1$	$0,0$

$\theta = \theta_0$	h	l
H	$-1,-1/2$	$1,0$
L	$0,-1$	$5/2,1/4$

We leave the receiver's prior belief unspecified as it plays no role. One can check that according to the orders $\theta_1 \succ \theta_0$, $H \succ L$ and $h \succ l$, the generalized Spence-Mirrlees condition is satisfied. In fact, these payoffs even satisfy a stronger notion of complementarity, that is, both u_1 and u_2 are strictly supermodular functions of the triple (θ, a_1, a_2).[7] Intuitively, there are complementarities between players' actions as well as between the state and the action profile.

However, we can find the following non-monotone equilibrium: The sender plays L if her type is θ_1 and plays H if her type is θ_0. The receiver, who could perfectly learn the state from the sender's action, plays l after observing H and plays h after observing L. Clearly, the sender's strategy is non-monotone. Nevertheless, the strategy profile is an equilibrium as no player has any incentive to deviate.

In fact, since players have strict incentives and there are no off-path beliefs, the above strategy profile and its induced belief system also form a sequential equilibrium (Kreps and Wilson 1982). Moreover, it

[7] Let X be a lattice. A function $f: X \to \mathbb{R}$ is *strictly supermodular* if $f(x \vee x') + f(x \wedge x') \geq f(x) + f(x')$ for every $x, x' \in X$, and the inequality is strict if $\{x, x'\} \neq \{x \vee x', x \wedge x'\}$.

cannot be refined away using the selection criteria proposed in Kohlberg and Mertens (1986), Cho and Kreps (1987), Banks and Sobel (1987). Since players' incentives are strict, this equilibrium is also robust against perturbations on the game's payoff matrices.

We argue that this non-monotone equilibrium is driven by three features of the game: sequential moves, non-separable payoff (of the sender) and interdependent values. Since players move *sequentially*, every a_1 induces a distribution over a_2. As a result, the sender is effectively choosing a distribution over *action profiles* instead of just her own action. Because u_1 is *non-separable* with respect to θ and a_2, her preferences over the receiver's actions also vary with the state. Therefore, her state contingent action choice depends not only on her comparative advantage in a_1 but also on her preferences over a_2. Since the receiver's best response to a_1 depends on his belief about the state (i.e. values are *interdependent*), choosing h after observing L and choosing l after observing H can be rationalized despite there are complementarities between players' actions. In our non-monotone equilibrium, the receiver believes that the state is θ_1 after observing L and the state is θ_0 after observing H, which provides the sender an incentive to use non-monotone strategies and makes the receiver's belief self-fulfilling.

While sequential moves and interdependent values are standard in signalling games, non-separability of the sender's payoff distinguishes our model from the classic education signalling game (Spence 1973) and the beer-quiche game (Cho and Kreps 1987). In these examples, the sender's valuations of money and fighting do not depend on her type. Nevertheless, non-separable payoffs arise naturally in many other economic applications. For example, consider a firm (receiver) offering a worker (sender) a job after observing her education. The worker's preferences over jobs depend on her type (for example, her taste and talent) no matter whether the jobs are horizontally differentiated (Roy 1951) or vertically differentiated (Waldman 1984, Gibbons and Waldman 1999).[8] Non-separability also occurs in various applications of signalling games in industrial organization, some of which will be discussed in details in section 5.

We close this section by observing that the existence of non-monotone equilibrium in our couterexample does not contradict the well-known results in Van Zandt and Vives (2007) on Bayesian supermodular games. This is because once we maintain the pre-specified orders over players' strategies, a signalling game with supermodular payoffs as Example 1 is *not* necessarily supermodular in its normal form.[9]

[8] When jobs are horizontally differentiated, different types of workers prefer different kinds of jobs, as in Roy (1951)'s hunting-fishing example. When jobs are vertically differentiated, the worker's gain from a job position depends on her talent due to the piece-rate incentive schemes, the prospects of promotion, etc.

[9] Echenique (2004a) shows that every game that has at least two pure Nash equilibria is supermodular if the analyst can flexibly choose the order over players' strategies. However, the results that are established under an arbitrary order cannot imply the monotonicity of the sender's action with respect to her type.

4 Sufficient Conditions for Monotone Equilibrium

In this section, we introduce sufficient conditions that guarantee the monotonicity of all equilibria. At the heart of our analysis is the following *monotone-supermodular* condition on the sender's payoff:

Definition 3 (Monotone-Supermodular Condition)**.** *The sender's payoff is monotone-supermodular if (i) u_1 is strictly decreasing in a_1 and strictly increasing in a_2, and (ii) u_1 exhibits strictly increasing differences in (θ, a_1) and increasing differences in (θ, a_2).*

Comparing with the more demanding requirement that both u_1 and u_2 are strictly supermodular functions of (θ, a_1, a_2), our monotone-supermodular condition does not require any complementarities within players' actions, nor does it impose any restrictions on the receiver's payoff function. Nevertheless, it incurs two important requirements in addition to the generalized Spence-Mirrlees condition. First, the sender's payoff exhibits increasing differences between the state and the receiver's action. This includes the separable payoff (i.e., there exist $f : A_1 \times A_2 \to \mathbb{R}$ and $c : \Theta \times A_1 \to \mathbb{R}$ such that $u_1(\theta, a_1, a_2) = f(a_1, a_2) + c(\theta, a_1)$) as a special case. It also fits into many other applications where payoffs are non-separable by nature. For example, in warranty provision games, the firm's per unit profit (sales price minus the expected refund paid to the consumers) increases with its product quality, and therefore, its total profit exhibits increasing differences between its quality and the quantity sold. In education signalling games in which a firm assigns workers to various job positions after observing their years of education, more talented workers receive higher benefits from higher level jobs due to the piece-rate incentive schemes and better prospects of promotion.

Second, the sender's payoff is monotone with respect to her own action and the receiver's action in appropriate directions.[10] This is natural in many applications. For example, it is costly for the firm to provide lengthier warranties and higher refunds, but it can benefit when consumers increase their purchasing quantities. Similarly, workers face higher opportunity costs to receive more education but they can benefit from more decent job positions.

In the next two subsections (4.1 and 4.2), we state results that establish the monotonicity of equilibria in signalling games where the sender's payoff is monotone-supermodular. The role of the monotone-supermodular condition in our results will be discussed in subsection 4.3. We will elaborate in section

[10]Our results also hold when the sender's payoff is increasing in her own action and decreasing in the receiver's action. However, non-monotone equilibria can exist when the sender's payoff is strictly increasing (or strictly decreasing) in both players' actions (see Example 3 in subsection 4.3).

5 how the assumptions of our results fit into the classic applications of signalling games in industrial organization and labor economics, and outline their implications in these settings.

4.1 Binary Action Games

In this subsection, we study games in which the receiver's action choice is binary, i.e. $|A_2| = 2$. This fits into the warranty provision game when the consumer has unit demand, i.e. $a_2 \in \{0,1\}$. Our first result below states that for these games, monotone-supermodularity alone is sufficient to guarantee the monotoncity of all equilibria.

Theorem 1. *If $|A_2| = 2$ and the sender's payoff is monotone-supermodular, then every equilibrium is monotone.*

PROOF. Let $A_2 \equiv \{\overline{a}_2, \underline{a}_2\}$ with $\overline{a}_2 \succ \underline{a}_2$. Suppose towards a contradiction that in some equilibrium σ, there exist $\theta \succ \theta'$ and $a_1 \succ a_1'$ such that $\sigma_1^{\theta}(a_1') > 0$ and $\sigma_1^{\theta'}(a_1) > 0$. Let $\alpha_2 \equiv \sigma_2(a_1)$ and $\alpha_2' \equiv \sigma_2(a_1')$ be the mixed actions played by the receiver after observing a_1 and a_1', respectively. Since type θ prefers (a_1', α_2') to (a_1, α_2) and type θ' prefers (a_1, α_2) to (a_1', α_2'), we have:

$$u_1(\theta, a_1', \alpha_2') \geq u_1(\theta, a_1, \alpha_2) \tag{4.1}$$

and

$$u_1(\theta', a_1, \alpha_2) \geq u_1(\theta', a_1', \alpha_2'). \tag{4.2}$$

These together imply that:

$$u_1(\theta, a_1', \alpha_2') - u_1(\theta, a_1, \alpha_2) \geq 0 \geq u_1(\theta', a_1', \alpha_2') - u_1(\theta', a_1, \alpha_2). \tag{4.3}$$

Because u_1 is strictly decreasing in a_1, (4.2) also implies that $u_1(\theta', a_1', \alpha_2) > u_1(\theta', a_1, \alpha_2) \geq u_1(\theta', a_1', \alpha_2')$. This further implies that α_2 must attach a higher probability to $\overline{a}_2$ comparing to α_2', as the sender's payoff is strictly increasing in a_2. Therefore, we have $\theta \succ \theta'$, $a_1 \succ a_1'$ and α_2 dominates α_2' in the sense of first-order stochastic dominance (FOSD). Since u_1 has strictly increasing differences in (θ, a_1) and increasing differences in (θ, a_2), we have:

$$u_1(\theta, a_1', \alpha_2') - u_1(\theta, a_1, \alpha_2) < u_1(\theta', a_1', \alpha_2') - u_1(\theta', a_1, \alpha_2), \tag{4.4}$$

which contradicts (4.3). $\square$

The intuition of Theorem 1 is as follows. When $|A_2| = 2$, every pair of distributions over the receiver's action can be ranked according to FOSD. Since playing a higher action is more costly for the sender, she only has an incentive to do so when it induces a more favorable response from the receiver. This implies that the ranking over the sender's equilibrium actions must coincide with the ranking over the receiver's (possibly mixed) actions that they induce. Because a high type sender has a stronger preference towards higher action profiles, she will never play a strictly lower action than a low type.

Since the above proof makes no reference to the receiver's incentives, our monotonicity property also applies to every ex ante rationalizable strategy (Bernheim 1984; Pearce 1984). In fact, only monotone strategies can survive the first round of elimination. The irrelevance of the receiver's incentives also makes it clear that our result immediately extends to cases where the receiver has private information about his preferences, the sender is signalling to a population of receivers with heterogeneous preferences, etc.

Provided that an equilibrium exists, Theorem 1 can also be generalized to signalling games with infinite A_1 and Θ with two cautions. First, when the type space is infinite, Nash equilibrium needs to be defined at the interim stage after the sender observes her type. This is to ensure that the sender will play a best reply at every state. Second, when A_1 is infinite, some of the actions in the support of a sender's strategy can be suboptimal. Hence, monotonicity condition in Definition 2 can fail in some equilibria. Nevertheless, we show in Appendix C that the sender's equilibrium strategy must be *almost surely monotone* in the following sense: For every $\theta \succ \theta'$ and $a_1 \in \text{supp}(\sigma_1^\theta)$, the probability that type θ' plays an action strictly higher than a_1 equals to zero.

Theorem 1 also has the following implication on *repeated* signalling games where the sender's payoff is monotone-supermodular. Suppose that in a Nash equilibrium, always playing the highest action is a best reply for a low type sender, then a high type must be playing the highest action with probability 1 at every on-path history. As shown in Pei (2016), this is an intermediate step towards establishing the commitment payoff bound and the uniqueness of equilibrium behavior in reputation games.[11]

[11] However, the monotonicity of equilibria in the normal form representation of a repeated game does not imply the monotonicity of the sender's behavior strategy at every on-path history. Therefore, it cannot guarantee that the receiver will always positively update his belief about the sender's type after observing the highest on-path action.

4.2 Games with $|A_2| \geq 3$

In this subsection, we generalize our findings in binary action games to ones in which the receiver has more than two actions. To illustrate the difficulties, we first present a counterexample showing that the sender's payoff being monotone-supermodular is no longer sufficient to guarantee the monotonicity of all equilibria.

Example 2. Consider the following $2 \times 2 \times 3$ game:

$\theta = \theta_1$	h	m	l
H	$2-\varepsilon, 1$	$1-2\varepsilon, 0$	$-3\varepsilon, -2$
L	$2, 0$	$1, 1$	$0, 0$

$\theta = \theta_0$	h	m	l
H	$2-3\varepsilon, 0$	$1-3\varepsilon, 0$	$4\varepsilon, 0$
L	$2-\varepsilon, 0$	$1, 2$	$8\varepsilon, 3$

Suppose $\varepsilon \in (0, 1/8)$ and apply the rankings $\theta_1 \succ \theta_0$, $H \succ L$ and $h \succ m \succ l$. One can verify that the sender's payoff is monotone-supermodular. However, consider the following strategy profile: The sender plays L if $\theta = \theta_1$, and plays H if $\theta = \theta_0$. The receiver plays m after observing L, and plays h and l with equal probabilities after observing H. One can check that the sender's strategy is non-monotone although this strategy profile and its induced belief constitute a sequential equilibrium.[12]

Example 2 highlights the following issue: When $|A_2| \geq 3$, the *distributions* over the receiver's actions cannot be completely ranked via FOSD. In particular, u_1 being monotone-supermodular does not imply the following:

- For every $\alpha_2, \alpha_2' \in \Delta(A_2)$, if α_2 is preferred to α_2' for some $\theta \in \Theta$ when she plays $a_1 \in A_1$, then the sender's expected payoff difference between α_2 and α_2' is increasing in her type conditional on a_1.

We proceed along two directions to address this challenge, leading to two monotonicity results. First, we introduce a property on the receiver's payoff under which every pair of his (pure or mixed) *best replies* can be ranked via FOSD. This together with the monotone-supermodular condition on the sender's payoff imply the monotonicity of all equilibria (Theorem 2). Second, we identify a class of utility functions u_1 which can endogenously generate a complete order on $\Delta(A_2)$. When the sender's payoff function belongs to this class and is monotone-supermodular, every equilibrium is monotone irrespective of the receiver's payoff (Theorem 3).

[12] This counterexample is not driven by the receiver's non-generic payoff. Even when the receiver has strict preferences over A_2 conditional on $(\theta, a_1) = (\theta_0, H)$, there still exists a non-monotone partial pooling equilibrium in which type θ_1 mixes between H and L, and type θ_0 always plays H.

4.2.1 Quasiconcavity-Preserving Property

For this part, we assume that $A_2 \equiv \{a_2^1, ..., a_2^n\}$ is completely ordered with $a_2^1 \prec a_2^2 \prec ... \prec a_2^n$.[13] For every $(\theta, a_1) \in \Theta \times A_1$ and $i \in I \equiv \{1, 2, ..., n-1\}$, let

$$\gamma_\theta^{a_1}(i) \equiv u_2(\theta, a_1, a_2^i) - u_2(\theta, a_1, a_2^{i+1}) \tag{4.5}$$

be the receiver's payoff gain by decreasing his action locally, and let

$$\Gamma_{\tilde{\pi}}^{a_1}(i) \equiv \int \gamma_\theta^{a_1}(i) d\tilde{\pi} \tag{4.6}$$

be his expected payoff gain under belief $\tilde{\pi} \in \Delta(\Theta)$. We now recall the definition of *strict single-crossing* function in Milgrom and Shannon (1994):

Definition 4. *Function $\gamma : I \to \mathbb{R}$ satisfies strict single-crossing property (SSCP) if for every $i \in I$, $\gamma(i) \geq 0$ implies that $\gamma(j) > 0$ for every $j \in I$ with $j > i$.*

If $\gamma_\theta^{a_1}(\cdot)$ satisfies SSCP for every $(\theta, a_1) \in \Theta \times A_1$, then $u_2(\theta, a_1, \cdot)$ is strictly quasi-concave in a_2. In that case, the receiver has at most two pure best replies to every (θ, a_1), which must be adjacent elements in A_2. This further implies that every pair of his (pure or mixed) best replies to a degenerate distribution on $\Theta \times A_1$ can be ranked according to FOSD. However, some of the sender's actions may induce non-degenerate beliefs in some equilibria, so the issue of aggregating single-crossing property arises. This motivates us to introduce the following *quasiconcavity-preserving property* (QPP) on the receiver's payoff.

Definition 5 (Quasiconcavity-Preserving Property)**.** *The receiver's payoff satisfies QPP if $\Gamma_{\tilde{\pi}}^{a_1} : I \to \mathbb{R}$ satisfies SSCP for every $(\tilde{\pi}, a_1) \in \Delta(\Theta) \times A_1$.*[14]

A sufficient condition for QPP is $\gamma_\theta^{a_1}(\cdot)$ being strictly increasing for every $(\theta, a_1) \in \Theta \times A_1$. This fits into the warranty provision game when the consumer faces decreasing marginal returns with respect to quantity. Intuitively, u_2 is *strictly concave* in a_2 when $\gamma_\theta^{a_1}(\cdot)$ is strictly increasing. The latter implies QPP as strict concavity is preserved under positive linear aggregation. Nevertheless, strict concavity is by no

[13] Our monotonicity result in this subsection (Theorem 2) can also be extended to settings where A_2 is a multi-dimensional convex set.

[14] A more general version of the QPP property when A_2 is any subset of $\mathbb{R}$ is introduced and characterized by Choi and Smith (2017). In the case where A_2 is finite, our condition is equivalent to a strict version of theirs.

means necessary for QPP. In Appendix B, we provide a full characterization of QPP by relating it to a strict version of the signed-ratio monotonicity condition introduced in Quah and Strulovici (2012).

Under QPP, the receiver's (pure or mixed) best replies to every $(\tilde{\pi}, a_1) \in \Delta(\Theta) \times A_1$ can be ranked according to FOSD. This leads to our second result:

Theorem 2. *If (i) the order on A_2 is complete, (ii) the sender's payoff is monotone-supermodular, and (iii) the receiver's payoff satisfies QPP, then every equilibrium is monotone.*

The proof follows along the same line as that of Theorem 1, which we omit to avoid repetition. Note that despite the extra condition on the receiver's payoff function, Theorem 2 only requires him to play a best reply against some $\tilde{\pi} \in \Delta(\Theta)$ after every $a_1 \in A_1$ on the equilibrium path. Therefore, the above monotonicity result does not depend on the receiver's belief updating process and applies to all outcomes under weaker solution concepts such as $S^\infty W$ (Dekel and Fudenberg 1990) and iterative conditional dominance (Shimoji and Watson 1998), which are variants of rationalizability that can rule out the receiver's suboptimal plays at off-path information sets.[15] Moreover, when applying the elimination procedure for $S^\infty W$, all non-monotone strategies will be deleted after one round elimination of weakly dominated strategy followed by another round elimination of strictly dominated strategy. When applying iterative conditional dominance, all surviving strategies are monotone after two rounds of elimination.

4.2.2 Increasing Absolute Differences over Distributions

In what follows, we take an alternative approach by introducing a condition on the sender's payoff that can guarantee the monotonicity of equilibria irrespective of the receiver's payoff. Unlike the previous subsection, we allow the order on A_2 to be incomplete. As illustrated in Example 2, the main obstacle against equilibrium monotonicity is the lack of a complete order on $\Delta(A_2)$. We introduce the following *increasing absolute differences over distributions* condition (IADD) on the sender's payoff under which a complete order on $\Delta(A_2)$ can be constructed endogenously.

[15] $S^\infty W$ is the solution concept when applying one round elimination of weakly dominated strategies followed by iterative elimination of strictly dominated strategies. Dekel and Fudenberg (1990) show that it characterizes the set of rationalizable strategies when players entertain small amount of uncertainty about their opponents' payoffs. Shimoji and Watson (1998) show that iterative conditional dominance generalizes rationalizability in normal form games to extensive form games.

Definition 6 (Increasing Absolute Differences over Distributions)**.** *The sender's payoff satisfies IADD if for every $a_1 \in A_1$ and every $\alpha_2, \alpha_2' \in \Delta(A_2)$, we have $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_1, \alpha_2')$ being either increasing in θ and non-negative for all $\theta \in \Theta$, or decreasing in θ and non-positive for all $\theta \in \Theta$.*

To make sense of the terminology, note that IADD implies that the absolute value of the difference $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_2, \alpha_2')$ is increasing in θ.[16] Intuitively, if u_1 satisfies IADD, then for every $a_1 \in A_1$, there exists a complete ordinal preference on $\Delta(A_2)$ (denoted by $\succsim_{a_1}$) that is shared among all types of senders. In addition, this ordinal ranking coincides with the one based on the intensity of preferences. In other words, if $\alpha_2 \succsim_{a_1} \alpha_2'$, then the difference in the sender's payoffs between (a_1, α_2) and (a_1, α_2') must be increasing in θ. This leads to our last monotonicity result:

Theorem 3. *If u_1 is monotone-supermodular and satisfies IADD, then every equilibrium is monotone.*

PROOF. Suppose towards a contradiction that in some equilibrium (σ_1, σ_2), there exist $\theta \succ \theta'$ and $a_1 \succ a_1'$ such that $\sigma_1^{\theta}(a_1') > 0$ and $\sigma_1^{\theta'}(a_1) > 0$. Let $\alpha_2 \equiv \sigma_2(a_1)$, $\alpha_2' \equiv \sigma_2(a_1')$ with $\alpha_2, \alpha_2' \in \Delta(A_2)$. Since type θ prefers (a_1', α_2') to (a_1, α_2), and type θ' prefers (a_1, α_2) to (a_1', α_2'), we have:

$$u_1(\theta, a_1', \alpha_2') \geq u_1(\theta, a_1, \alpha_2) \tag{4.7}$$

and

$$u_1(\theta', a_1, \alpha_2) \geq u_1(\theta', a_1', \alpha_2'). \tag{4.8}$$

Since u_1 is strictly decreasing in a_1, we have $u_1(\theta', a_1', \alpha_2) > u_1(\theta', a_1, \alpha_2)$. Inequality (4.8) then implies that $u_1(\theta', a_1', \alpha_2) > u_1(\theta', a_1', \alpha_2')$. Applying (4.7) and (4.8) we have:

$$u_1(\theta, a_1', \alpha_2') - u_1(\theta, a_1, \alpha_2) \geq u_1(\theta', a_1', \alpha_2') - u_1(\theta', a_1, \alpha_2). \tag{4.9}$$

Meanwhile, note that

$$u_1(\cdot, a_1', \alpha_2') - u_1(\cdot, a_1, \alpha_2) = u_1(\cdot, a_1', \alpha_2') - u_1(\cdot, a_1', \alpha_2) + u_1(\cdot, a_1', \alpha_2) - u_1(\cdot, a_1, \alpha_2).$$

[16] IADD is also necessary for $\left|u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_2, \alpha_2')\right|$ to be increasing in θ when Θ is a continuum and u_1 is a continuous function of θ.

Since u_1 exhibits strictly increasing differences between θ and a_1, we have:

$$u_1(\theta, a_1', \alpha_2) - u_1(\theta, a_1, \alpha_2) < u_1(\theta', a_1', \alpha_2) - u_1(\theta', a_1, \alpha_2). \tag{4.10}$$

In addition, IADD and $u_1(\theta', a_1', \alpha_2) - u_1(\theta', a_1', \alpha_2') > 0$ imply that:

$$u_1(\theta, a_1', \alpha_2') - u_1(\theta, a_1', \alpha_2) \leq u_1(\theta', a_1', \alpha_2') - u_1(\theta', a_1', \alpha_2). \tag{4.11}$$

Summing up (4.10) and (4.11), we obtain a contradiction against (4.9). Therefore, every equilibrium must be monotone. □

As a remark, since the order on $\Delta(A_2)$ can be constructed endogenously under IADD, our result does not rely on the pre-specified order on A_2, nor does it rely on the monotone-supermodularity condition on u_1 with respect to a_2. In fact, it is clear from the above proof that once u_1 satisfies IADD, all equilibria are monotone if u_1 is strictly decreasing in a_1 and exhibits strictly increasing differences in (θ, a_1).

Furthermore, since the proof of Theorem 3 makes no reference to the receiver's rationality and incentives, it possesses the same robustness properties as Theorem 1. That is, all ex ante rationalizable strategies of the sender must also be monotone. This monotonicity result continues to hold when the receiver has private information about his payoff, when the sender is signalling to a population of receivers with heterogeneous preferences, etc. In addition, Theorem 3 immediately extends to cases where A_2 is infinite as the cardinality of A_2 plays no role in the above proof. Finally, extensions of Theorem 3 to cases where Θ and A_1 are infinite are subject to the same cautions mentioned in subsection 4.1 and Appendix C, with the main issues being the existence of equilibrium as well as the sender playing suboptimal actions.

In order to apply Theorem 3, it is necessary to verify whether IADD is satisfied. To facilitate this process, we fully characterize the functional form restrictions of IADD in the following proposition:

Proposition 1. *u_1 satisfies IADD if and only if there exist functions $f : A_1 \times A_2 \to \mathbb{R}$, $v : \Theta \times A_1 \to \mathbb{R}$ and $c : \Theta \times A_1 \to \mathbb{R}$ with $v(\theta, a_1)$ increasing in θ and $\min_{\theta \in \Theta} v(\theta, a_1) \geq 0$ for every $a_1 \in A_1$ such that:*

$$u_1(\theta, a_1, a_2) = f(a_1, a_2) v(\theta, a_1) + c(\theta, a_1). \tag{4.12}$$

PROOF. See Appendix A. □

Remark on IADD. IADD enables us to construct endogenous orders on $\Delta(A_2)$. Nevertheless, as shown in Kartik et al. (2017) and Kushnir and Liu (2017), there are other conditions on players' payoff functions under which we could obtain a complete order over distributions. These include the *single-crossing expectational differences* (SCED) and the *monotone expectational differences* (MED) in Kartik et al.(2017), and the *increasing differences over distributions* (IDD) in Kushnir and Liu (2017).

When applying to the same probability space, IADD is more demanding than MED and SCED. This is because, for example, IADD on $\Delta(A_2)$ requires that (i) $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_2, \alpha_2')$ does not change sign when we vary θ and (ii) its absolute value is increasing in θ. These together imply that the expected difference $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_2, \alpha_2')$ is monotone in θ.[17]

However, as shown in the next subsection, neither MED, SCED nor IDD on $\Delta(A_2)$ is sufficient for our purpose. It is also important to note that IADD on $\Delta(A_2)$ neither implies nor is implied by MED or SCED on $\Delta(A_1 \times A_2)$. We will further elaborate on this in the context of education signalling (section 5.2), which helps clarify the novel implications of Theorem 3 compared to a related result in Kartik et al. (2017).

4.3 Discussion

In this subsection, we argue that the monotone-supermodular condition on the sender's payoff plays an indispensable role in our analysis. In particular, we will show that neither the monotoncitiy nor the supermodularity part of the condition can be replaced by other appealing alternatives.

4.3.1 Alternative Monotonicity Conditions

One may conjecture that the existence of non-monotone equilibria (e.g. the one in Example 1) is driven by the state dependence of the sender's *ordinal* preferences over a_2, or whether we could modify the monotonicity assumption on the sender's payoff by letting it to be strictly increasing (or strictly decreasing) in both a_1 and a_2. However, the following counterexample suggests that these conjectures are not true.

[17] In our signalling game context, IDD on $\Delta(A_2)$ would require that for every $a_1 \in A_1$ and every $\alpha_2, \alpha_2' \in \Delta(A_2)$, the expected payoff differences $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_1, \alpha_2')$ is either strictly increasing, strictly decreasing, or constant in θ. In contrast, IADD only implies that these differences are either increasing or decreasing. Thus, in general Kushnir and Liu (2017)'s IDD is only implied by the strict version of our IADD (i.e. for every $a_1 \in A_1$ and $\alpha_2, \alpha_2' \in \Delta(A_2)$, $|u_2(\theta, a_1, \alpha_2) - u_2(\theta, a_1, \alpha_2')|$ is either constant of strictly increasing in θ).

Example 3. Consider the following $2 \times 2 \times 2$ game:

$\theta = \theta_1$	h	l
H	$2,2$	$0,0$
L	$1,1$	$-1/2,0$

$\theta = \theta_0$	h	l
H	$1/4,-1/2$	$1/8,0$
L	$0,-1$	$-1/16,1/4$

One can verify that according to the order $\theta_1 \succ \theta_0$, $H \succ L$ and $h \succ l$, the sender's payoff satisfies the generalized Spence-Mirrlees condition. Moreover, as in Example 1, both u_1 and u_2 are supermodular functions of the triple (θ, a_1, a_2). Different from Example 1, the sender's ordinal preferences over a_1 and a_2 are state independent. In particular, the sender's payoff is strictly increasing in both a_1 and a_2. However, her cardinal preferences over the receiver's actions still depend on the state. As a result, there exists a non-monotone equilibrium in which type θ_1 plays L, type θ_0 plays H, and the receiver plays h after observing L and plays l after observing H. One can also construct similar counterexamples in which the sender's payoff exhibits strictly increasing differences in (θ, a_1), increasing differences in (θ, a_2) but is strictly decreasing in both a_1 and a_2.

4.3.2 Single-Crossing Differences vs. Increasing Differences

In this part, we show that our strictly increasing difference condition on u_1 cannot be replaced with the *strict single-crossing difference property* in Milgrom and Shannon (1994), which is well-known in the monotone comparative statics literature.

Definition 7. *The sender's payoff has strict single-crossing differences (SSCD) if for every $\theta \succ \theta'$ and every $(a_1, a_2) \succ (a_1', a_2')$, $u_1(\theta', a_1, a_2) - u_1(\theta', a_1', a_2') \geq 0$ implies that $u_1(\theta, a_1, a_2) - u_1(\theta, a_1', a_2') > 0$.*

By definition, SSCD is a weaker property than strictly increasing differences. The following example shows that SSCD is not sufficient for guaranteeing the monotonicity of equilibria in signalling games, even when the sender's payoff satisfies our monotonicity condition.

Example 4. Consider the following $2 \times 2 \times 2$ game:

$\theta = \theta_1$	h	l
H	$1,2$	$-3,0$
L	$3,1$	$-1,0$

$\theta = \theta_0$	h	l
H	$1,0$	$-2,0$
L	$2,-1$	$-1,0$

Consider the orders $\theta_1 \succ \theta_0$, $H \succ L$ and $h \succ l$. One can check that, first, u_1 is strictly increasing in a_2 and is strictly decreasing in a_1. Second, u_1 has SSCD although it fails to have increasing differences. Let $\alpha_2 \equiv \frac{2}{3}h + \frac{1}{3}l$ and $\alpha_2' \equiv \frac{1}{3}h + \frac{2}{3}l$ be two mixed actions of the receiver, we have:

$$u_1(\theta_0, h, \alpha_2) - u_1(\theta_0, l, \alpha_2') = 0 > -\frac{2}{3} = u_1(\theta_1, h, \alpha_2) - u_1(\theta_1, l, \alpha_2'). \tag{4.13}$$

When the receiver's prior belief attaches probability $1/3$ to state θ_1, one can proceed to construct the following non-monotone equilibrium: Type θ_1 plays L for sure, type θ_0 plays H and L each with probability $1/2$, the receiver plays α_2 after observing H and α_2' after observing L.

In the above example, the receiver's best reply to the sender's action is mixed. SSCD only requires that $u_1(\theta, a_1, a_2) - u_1(\theta, a_1', a_2')$ has the strict single-crossing property for every pair of *pure* action profiles that can be ranked. This does not imply that $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_1', \alpha_2')$ is also strict single-crossing for every $(a_1, \alpha_2), (a_1', \alpha_2') \in A_1 \times \Delta(A_2)$ with $a_1 \succ a_1'$ and α_2 FOSDs α_2'.[18] This leaves open the possibility of (4.13), which leads to the existence of non-monotone equilibria.

5 Applications

In this section, we revisit two classic applications of signalling games in industrial organization and labor economics. We apply our sufficient conditions to establish the monotonicity of all equilibria in these games and discuss the relationships between our results and the existing ones in the literature.

5.1 Advertising and Warranty Provision

Consider a firm (sender) selling products to a consumer (receiver). Let $\theta \in \Theta \subset \mathbb{R}$ be the product's quality, which is the firm's private information. For simplicity, we assume that the per unit sales price is exogenous, which is normalized to 1. Every product sold has a positive probability of breakdown, which depends on its quality. The firm chooses a 3-dimensional action: $a_1 \equiv (a_1^{ad}, a_1^{len}, a_1^{re}) \in A_1 \subset \mathbb{R}_+ \times \mathbb{R}_+ \times [0,1]$, where a_1^{ad} is the intensity of advertising, a_1^{len} is the length of warranty, and a_1^{re} is the (per unit) refund the firm commits to pay if the product breaks down during the length of the warranty. The consumer chooses how many units to purchase after observing a_1, which is denoted by $a_2 \in A_2 \subset \mathbb{N}$.

[18] In fact, since $|A_1| = |A_2| = 2$ in this example, the payoff function u_1 also has SCED on both $\Delta(A_1)$ and $\Delta(A_2)$ (Kartik et al. 2017). However, it does not have SCED on the larger space $\Delta(A_1 \times A_2)$.

The sender's payoff being monotone-supermodular requires that (i) u_1 is strictly decreasing in the triple $(a_1^{ad}, a_1^{len}, a_1^{re})$ and is strictly increasing in a_2, and (ii) u_1 has strictly increasing differences in (θ, a_1^{ad}), (θ, a_1^{len}) and (θ, a_1^{re}), and increasing differences in (θ, a_2). The monotone part is straightforward, as advertising, providing lengthier warranty and more refund are all costly for the firm. Keeping other factors fixed, the firm's profit is higher when the consumer purchases larger quantities.

Next, we justify the supermodular part. First, there are complementarities between θ and a_1^{ad} when the cost of promoting a good product is lower than the cost of promoting a bad one. This can be driven by regulation policies, reputation concerns, umbrella branding (Wernerfelt 1988), etc. Second, there are complementarities between θ and a_1^{re} when higher quality product has lower probability of breakdown, so therefore, committing to a higher per unit refund is less costly. Similarly, the firm's per unit profit (defined as sales price minus expected refund payment) is strictly increasing in the product's quality, and therefore, u_1 has strictly increasing differences in (θ, a_2). Finally, there are complementarities between θ and a_1^{len} when breakdown arrives according to a time homogeneous Poisson process with intensity strictly decreasing in the product's quality (Gal-Or 1989).

As the firm's payoff is monotone-supermodular, it will use a monotone strategy in every equilibrium when the consumer has unit demand (Theorem 1), when the consumer faces decreasing marginal returns to quantities (Theorem 2), or when its payoff takes the following functional form (Theorem 3):

$$u_1(\theta, a_1, a_2) = \Big(1 - \underbrace{g(\theta, a_1^{len})}_{\text{prob. of breakdown within } a_1^{len} \text{ periods}} \overbrace{a_1^{re}}^{\text{per unit refund}} + f(\theta)\Big) a_2 - \underbrace{c(\theta, a_1^{ad})}_{\text{cost of advertising}}, \tag{5.1}$$

where

(i) $g : \Theta \times \mathbb{R}_+ \to [0,1]$ is strictly decreasing in θ, strictly increasing in a_1^{len} and exhibits strictly decreasing differences in (θ, a_1^{len}),

(ii) $f : \Theta \to \mathbb{R}_+$ is strictly increasing, which captures the firm's benefit from initial sales beyond the sales price in a reduced form,[19] and

(iii) $c : \Theta \times \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing in a_1^{ad} and exhibits strictly decreasing differences in (θ, a_1^{ad}).

[19] This is relevant when the product is a newly introduced experience good (Nelson 1974; Milgrom and Roberts 1986). Nevertheless, the absence of $f(\theta)$ will not affect the applicability of our monotonicity result.

5.2 Education Signalling with Vertically Differentiate Jobs

Consider the following variant of the Spence (1973) education signalling model. Let $\theta \in \Theta$ be the talent of the worker, $a_1 \in A_1$ be the education the worker receives, and $a_2 \in A_2$ be the job offered by the employer after he observes a_1.

The sender's payoff being monotone-supermodular implies that (i) u_1 is strictly decreasing in a_1 and strictly increasing in a_2, (ii) u_1 exhibits strictly increasing differences between θ and (a_1, a_2). The monotonicity assumption requires that receiving education is costly, and the jobs are *vertically* differentiated so that every worker prefers a higher level job.[20] For the supermodularity assumption, first, u_1 exhibits strictly increasing differences in (θ, a_1) when receiving education is less costly for more talented workers (Spence 1973). Second, u_1 exhibits strictly increasing differences in (θ, a_2) when the returns from a higher level job (relative to a lower level one) increases with the worker's talent, which is a well-established fact in the personnel economics literature.[21]

When the worker's payoff is monotone-supermodular, more talented workers receive more education in every equilibrium when there are only two jobs (Theorem 1). If the employer's payoff function is strictly concave in a_2, then Theorem 2 guarantees the monotonicity of all equilibria even when there are three or more jobs.[22] Alternatively, suppose the worker's payoff function takes the following form:

$$u_1(\theta, a_1, a_2) = \underbrace{f(\theta)g(a_2)}_{\text{worker's return from job assignment}} - \underbrace{c(\theta, a_1)}_{\text{cost of education}}, \tag{5.2}$$

where both $f : \Theta \to \mathbb{R}_+$ and $g : A_2 \to \mathbb{R}$ are strictly increasing, and $c : \Theta \times A_1 \to \mathbb{R}$ is strictly increasing in a_1 and has strictly decreasing differences in (θ, a_1). According to Theorem 3, every equilibrium is monotone regardless of how the employer evaluates various matches between jobs, talent and education.

Kartik et al. (2017) study a similar application, with $a_2 \in \mathbb{R}_+$ being the wage offered by the firm. They show that if the worker's payoff function has SCED over $\Delta(A_1 \times A_2)$, then every equilibrium of this game is monotone. According to their characterization result, u_1 has SCED over $\Delta(A_1 \times A_2)$ if and only if it

[20] If jobs are instead horizontally differentiated (Roy 1951), then the resulting payoff structure resembles Example 1, in which we have shown that non-monotone equilibria can exist.

[21] See for example, Waldman (1984) and Gibbons and Waldman (1999). Different types of workers receive different returns to a higher level job can be due to, for example, that the talent of a worker affects her prospects of promotion, the expected compensation she receives under piece-rate incentive schemes, etc.

[22] Concavity of u_2 is satisfied if for every $(\theta, a_1) \in \Theta \times A_1$, there exists an ideal job assignment $a_2^*(\theta, a_1)$ that maximizes $u_2(\theta, a_1, \cdot)$, and the employer incurs a quadratic loss when there is a mismatch between talent and jobs.

takes the following functional form:

$$u_1(\theta, a_1, a_2) = g_1(a_1, a_2) f_1(\theta) + g_2(a_1, a_2) f_2(\theta) + h(\theta), \tag{5.3}$$

where both f_1 and f_2 are single-crossing functions that satisfy a ratio-ordered condition.[23] Their results provide insights on cheap talk games and education signalling games when the receiver's payoffs are unknown to the sender. However, their results are not applicable to education signalling games where the worker's payoff is given by (5.2) and $c(\theta, a_1)$ cannot be written as the product of two functions $c_1(\theta)$ and $c_2(a_1)$.[24] Our Theorem 3 accommodates these cases and implies the monotonicity of all equilibria.

6 Conclusion

This paper makes two contributions to the signalling game literature. First, we show that equilibrium monotonicity does not follow trivially from the well-known Spence-Mirrlees condition nor is it implied by the complementarities and supermodularity of players' payoff functions. Our counterexamples are robust against equilibrium refinements and highlight the problems that arise when the sender's returns from the receiver's action depend on her type. Second, we provide sufficient conditions under which all Nash equilibria are monotone. These conditions are easy to verify and fit into a number of applications, including advertising, warranty provision, education and job assignment, etc. In these scenarios, our results imply that it is without loss of generality to focus on monotone equilibria.

[23] According to Kartik et al. (2017), for two single-crossing functions $f_1, f_2 : \Theta \to \mathbb{R}$, f_1 ratio dominates f_2 if (i) $\forall \theta \succeq \theta'$, $f_1(\theta') f_2(\theta) \leq f_1(\theta) f_2(\theta')$, and (ii) $\forall \theta \succeq \hat{\theta} \succeq \theta'$, $f_1(\theta') f_2(\theta) = f_1(\theta) f_2(\theta')$ if and only if $f_1(\theta') f_2(\hat{\theta}) = f_1(\hat{\theta}) f_2(\theta')$ and $f_1(\hat{\theta}) f_2(\theta) = f_1(\theta) f_2(\hat{\theta})$. Functions f_1 and f_2 are ratio-ordered if either f_1 ratio dominates f_2 or f_2 ratio dominates f_1.

[24] The cost function is not multiplicative separable in applications when $c(\theta, a_1) = k(\theta) a_1 + t(a_1)$, with $t(a_1)$ being the a fixed cost, interpreted as the cost of tuition, and $k(\theta) a_1$ being a variable cost which depends on the worker's talent.

Appendices

A Proof of Proposition 1

We establish Proposition 1 by proving a series of equivalence statements.

Lemma A1. *u_1 has IADD if and only if for every $a_1 \in A_1$ and every $\alpha_2, \alpha_2' \in \Delta(A_2)$, we have*

$$u_1(\tilde{\theta}, a_1, \alpha_2) > u_1(\tilde{\theta}, a_1, \alpha_2') \text{ for some } \tilde{\theta} \in \Theta \Longrightarrow u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_1, \alpha_2') \text{ is increasing in } \theta. \quad \text{(A.1)}$$

PROOF. The only-if part of the lemma is straightforward. Let us focus on the if part. To show that (A.1) implies IADD, it suffices to show that if $u_1(\tilde{\theta}, a_1, \alpha_2) > u_1(\tilde{\theta}, a_1, \alpha_2')$ for some $\tilde{\theta}$, then $u_1(\theta, a_1, \alpha_2) \geq u_1(\theta, a_1, \alpha_2')$ for all $\theta \in \Theta$.

Suppose towards a contradiction that there exist $a_1 \in A_1, \alpha_2, \alpha_2' \in \Delta(A_2)$, and $\tilde{\theta}, \hat{\theta} \in \Theta$, such that $u_1(\tilde{\theta}, a_1, \alpha_2) > u_1(\tilde{\theta}, a_1, \alpha_2')$ and $u_1(\hat{\theta}, a_1, \alpha_2) < u_1(\hat{\theta}, a_1, \alpha_2')$. Then, condition (A.1) implies that we have both $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_1, \alpha_2')$ and $u_1(\theta, a_1, \alpha_2') - u_1(\theta, a_1, \alpha_2)$ being increasing in θ. Hence, $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_1, \alpha_2')$ must be constant for every θ, which leads to a contradiction. □

Next, notice that an immediate implication of u_1 satisfying IADD is that for every $a_1 \in A_1$ and $\alpha_2, \alpha_2' \in \Delta(A_2)$, the expected payoff difference $u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_1, \alpha_2')$ is monotone in θ. The following lemma fully characterizes this necessary condition of IADD.

Lemma A2. *$u_1(\theta, a_1, \alpha_2) - u_1(\theta, a_1, \alpha_2')$ is monotone in θ for every $(a_1, \alpha_2, \alpha_2') \in A_1 \times \Delta(A_2) \times \Delta(A_2)$ if and only if the sender's payoff has the following representation:*

$$u_1(\theta, a_1, a_2) = f(a_1, a_2) v(\theta, a_1) + c(\theta, a_1) + g(a_1, a_2), \quad \text{(A.2)}$$

where $v : \Theta \times A_1 \to \mathbb{R}$ is an increasing function of θ.

The proof of Lemma A2 is omitted as it immediately follows from the characterization results in Kartik et al. (2017) and Kushnir and Liu (2017). Therefore, it is without loss of generality to assume u_1 taking the functional form in (A.2), which we will do for the rest of the proof.

We now proceed to characterize condition (A.1). To do this, let us first introduce some useful notation.

Let $A_2 \equiv \{a_2^1, .., a_2^n\}$ with $n \geq 2$. For every $a_1 \in A_1$, let

$$\underline{v}^{a_1} \equiv \min_{\theta \in \Theta} v(\theta, a_1) \in \mathbb{R},$$

and

$$f^{a_1} \equiv (f(a_1, a_2^1), ..., f(a_1, a_2^n)),\ g^{a_1} \equiv (g(a_1, a_2^1), ..., g(a_1, a_2^n)) \in \mathbb{R}^n.$$

Finally, let $\Gamma \equiv \{\gamma \in \mathbb{R}^n | \mathbf{1} \cdot \gamma = 0\}$, where $\mathbf{1} \equiv (1, 1, ..., 1) \in \mathbb{R}^n$ and '$\cdot$' denotes the inner product of two vectors. We establish the following result.

Lemma A3. *Suppose that u_1 has representation (A.2). Then, u_1 satisfies* (A.1) *if and only if*

$$\forall (a_1, \gamma) \in A_1 \times \Gamma,\ (\underline{v}^{a_1} f^{a_1} + g^{a_1}) \cdot \gamma > 0 \Longrightarrow f^{a_1} \cdot \gamma \geq 0. \tag{A.3}$$

PROOF. (*If statement*) First, note that given the representation (A.2), condition (A.1) is equivalent to the requirement that for every $(a_1, \gamma) \in A_1 \times \Gamma$ and every $v \geq \underline{v}^{a_1}$, $(v f^{a_1} + g^{a_1}) \cdot \gamma > 0 \Rightarrow f^{a_1} \cdot \gamma \geq 0$. Suppose towards a contradiction that this does not hold for some $(a_1, \gamma) \in A_1 \times \Gamma$ and some $v \geq \underline{v}^{a_1}$. That is, we have $(v f^{a_1} + g^{a_1}) \cdot \gamma > 0$ but $f^{a_1} \cdot \gamma < 0$. Then, (A.3) implies that $(\underline{v}^{a_1} f^{a_1} + g^{a_1}) \cdot \gamma \leq 0$. Hence, we have

$$0 < (v f^{a_1} + g^{a_1}) \cdot \gamma = \underbrace{(\underline{v}^{a_1} f^{a_1} + g^{a_1}) \cdot \gamma}_{\leq 0} + \underbrace{(v - \underline{v}^{a_1}) f^{a_1} \cdot \gamma}_{\leq 0} \leq 0,$$

which leads to a contradiction.

(Only-if statement) Suppose that (A.3) is violated for some $(a_1, \gamma) \in A_1 \times \Gamma$, i.e. $(\underline{v}^{a_1} f^{a_1} + g^{a_1}) \cdot \gamma > 0$ but $f^{a_1} \cdot \gamma < 0$. Let $\xi > 0$ be small enough such that:

$$\max\{|\xi \gamma_1|, ..., |\xi \gamma_n|\} < 1/n.$$

Consider two probability distributions $\alpha_2, \alpha_2' \in \Delta(A_2)$, where $\alpha_2 \equiv \sum_{i=1}^n \frac{1}{n} \delta_{a_2^i}$, $\alpha_2' \equiv \sum_{i=1}^n (\frac{1}{n} - \xi \gamma_i) \delta_{a_2^i}$, and $\delta_{a_2^i}$ denotes the Dirac measure on $a_2^i \in A_2$. Let $\underline{\theta}$ be the smallest element in Θ, which exists since Θ is a complete lattice. By construction, when playing a_1, type $\underline{\theta}$ sender strictly prefers α_2 to α_2'. However, since $f^{a_1} \cdot \gamma < 0$, $u_1(\cdot, a_1, \alpha_2) - u_1(\cdot, a_1, \alpha_2')$ is strictly decreasing in θ. Hence, condition (A.1) is violated. $\square$

Next, consider the linear operator $\tau:\mathbb{R}^n\to\mathbb{R}^{n-1}$ with

$$\tau(w)\equiv(w_1-w_n,\ldots,w_{n-1}-w_n),\quad \forall w\in\mathbb{R}^n.$$

By construction, $\tau(w)=0$ if and only if w is a constant vector. In addition, for every $\gamma\in\Gamma$ and $w\in\mathbb{R}^n$, we have $w\cdot\gamma=\sum_{i=1}^{n-1}(w_i-w_n)\gamma_i=\tau(w)\cdot\gamma$. Our next lemma provides a further characterization of condition (A.1) via the linear mapping τ.

Lemma A4. *Suppose that u_1 has the representation (A.2). Then, u_1 satisfies condition* (A.3) *if and only if for every $a_1\in A_1$, there exist $\lambda,\mu\in[0,+\infty)$ with $(\lambda,\mu)\neq(0,0)$ such that*

$$\lambda\tau(f^{a_1})=\mu\tau(\underline{v}^{a_1}f^{a_1}+g^{a_1}). \tag{A.4}$$

PROOF. For every $w\in\mathbb{R}^n$, let us partition Γ into $\Gamma^+(w),\Gamma^-(w),\Gamma^0(w)$, such that $w\cdot\gamma>0$ (resp., $w\cdot\gamma<0$) for every $\gamma\in\Gamma^+(w)$ (resp., $\gamma\in\Gamma^-(w)$), and $\Gamma^0(w)=\Gamma\setminus(\Gamma^+(w)\cup\Gamma^-(w))$. Now we can equivalently state condition (A.3) as

$$\Gamma^+(\underline{v}^{a_1}f^{a_1}+g^{a_1})\subset\Gamma^0(f^{a_1})\cup\Gamma^+(f^{a_1}),\quad \forall a_1\in A_1. \tag{A.5}$$

(*If statement*) Pick any $a_1\in A_1$ and suppose there exist λ and μ such that (A.4) holds. If either λ or μ is 0, then since $(\lambda,\mu)\neq(0,0)$, we have either $\tau(f^{a_1})=0$ or $\tau(\underline{v}^{a_1}f^{a_1}+g^{a_1})=0$. In both cases, (A.5) is satisfied. If $\lambda\mu\neq0$, then by (A.4) we have for every $\gamma\in\Gamma^+(\underline{v}^{a_1}f^{a_1}+g^{a_1})$,

$$f^{a_1}\cdot\gamma=\tau(f^{a_1})\cdot\gamma=\frac{\mu}{\lambda}\tau(\underline{v}^{a_1}f^{a_1}+g^{a_1})\cdot\gamma=\frac{\mu}{\lambda}(\underline{v}^{a_1}f^{a_1}+g^{a_1})\cdot\gamma>0.$$

Hence, $\gamma\in\Gamma^+(f^{a_1})$.

(*Only-if statement*) Pick any $a_1\in A_1$ and consider the two $n-1$ dimensional vectors $\tau(f^{a_1})$ and $\tau(\underline{v}^{a_1}f^{a_1}+g^{a_1})$. Suppose that the required λ and μ do not exist. Then, there exists no $\kappa\geq0$ such that

$\kappa\tau(f^{a_1}) = \tau(\underline{v}^{a_1} f^{a_1} + g^{a_1})$. By Farkas' Lemma, there exists $\tilde{\gamma} \equiv (\tilde{\gamma}_1, ..., \tilde{\gamma}_{n-1}) \in \mathbb{R}^{n-1}$ such that

$$\tau(f^{a_1}) \cdot \tilde{\gamma} < 0 \text{ but } \tau(\underline{v}^{a_1} f^{a_1} + g^{a_1}) \cdot \tilde{\gamma} > 0.$$[25]

Let $\gamma \equiv (\tilde{\gamma}_1, ..., \tilde{\gamma}_{n-1}, \tilde{\gamma}_n)$, where $\tilde{\gamma}_n \equiv -\sum_{i=1}^{n-1} \tilde{\gamma}_i$. The construction of $\tilde{\gamma}$ implies that $\gamma \in \Gamma^+(\underline{v}^{a_1} f^{a_1} + g^{a_1})$ but $\gamma \in \Gamma^-(f^{a_1})$. This violates (A.5) and thus also violates (A.3). □

To conclude the proof of Proposition 1, we derive (4.12) from (A.4). According to the definition of τ, Lemma A4 implies that for every $(a_1, a_2) \in A_1 \times A_2$,

$$\lambda\left(f(a_1,a_2) - f(a_1,a_2^n)\right) = \mu\left[\left(\underline{v}^{a_1} f(a_1,a_2) + g(a_1,a_2)\right) - \left(\underline{v}^{a_1} f(a_1,a_2^n) + g(a_1,a_2^n)\right)\right],$$

or, equivalently,

$$\mu g(a_1,a_2) = (\lambda - \mu \underline{v}^{a_1}) f(a_1,a_2) + h(a_1),$$

where

$$h(a_1) = \mu(\underline{v}^{a_1} f(a_1,a_2^n) + g(a_1,a_2^n) - \lambda f(a_1,a_2^n).$$

On the one hand, if $\mu \neq 0$, let

$$\hat{v}(\theta,a_1) \equiv v(\theta,a_1) + (\lambda - \mu \underline{v}^{a_1})/\mu \text{ and } \hat{c}(\theta,a_1) \equiv c(\theta,a_1) + h(a_1),$$

which obtains representation (4.12). Note that by construction, $\min_{\theta\in\Theta} \hat{v}(\theta,a_1) = \lambda/\mu \geq 0$.

On the other hand, if $\mu = 0$, then we must have $\lambda \neq 0$ and $f(a_1,a_2) = h(a_1)/\lambda$. In this case, let

$$\hat{f}(a_1,a_2) \equiv g(a_1,a_2),\ \hat{v}(\theta,a_1) \equiv 1 \text{ and } \hat{c}(\theta,a_1) \equiv c(\theta,a_1) + h(a_1)v(\theta,a_1)/\lambda,$$

which obtains representation (4.12). □

[25] Farkas' Lemma implies the existence of $\hat{\gamma} \in \mathbb{R}^{n-1}$ such that $\tau(f^{a_1}) \cdot \hat{\gamma} \leq 0$ and $\tau(\underline{v}^{a_1} f^{a_1} + g^{a_1}) \cdot \hat{\gamma} > 0$. But given that $\tau(f^{a_1}) \neq 0$, if $\tau(f^{a_1}) \cdot \hat{\gamma} = 0$, there must exist $\tilde{\gamma} \in \mathbb{R}^{n-1}$ close to $\hat{\gamma}$ such that $\tau(f^{a_1}) \cdot \tilde{\gamma} < 0$ and $\tau(\underline{v}^{a_1} f^{a_1} + g^{a_1}) \cdot \tilde{\gamma} > 0$.

B Characterizing Quasiconcavity-Preserving

In this Appendix, we provide a characterization of the quasiconcavity-preserving property based on the primitives of the model. We first introduce a strict version of the signed-ratio monotonicity condition in Quah and Strulovici (2012).

Definition B1 (Strict Signed-Ratio Monotonicity). *A pair of functions $\gamma_\theta^{a_1}, \gamma_{\theta'}^{a_1} : I \to \mathbb{R}$ obeys strict signed-ratio monotonicity (or SSRM) if*

(1) for every i such that $\gamma_\theta^{a_1}(i) < 0$ and $\gamma_{\theta'}^{a_1}(i) > 0$, we have

$$-\frac{\gamma_\theta^{a_1}(i)}{\gamma_{\theta'}^{a_1}(i)} > -\frac{\gamma_\theta^{a_1}(j)}{\gamma_{\theta'}^{a_1}(j)} \quad \textit{for every } j > i, \textit{ and}$$

(2) for every i such that $\gamma_\theta^{a_1}(i) > 0$ and $\gamma_{\theta'}^{a_1}(i) < 0$, we have

$$-\frac{\gamma_{\theta'}^{a_1}(i)}{\gamma_\theta^{a_1}(i)} > -\frac{\gamma_{\theta'}^{a_1}(j)}{\gamma_\theta^{a_1}(j)} \quad \textit{for every } j > i.$$

The next result characterizes the quasiconcavity-preserving property in our setting, which is a straightforward extension of Theorem 1 in Quah and Strulovici (2012):

Proposition B1. *The receiver's payoff is quasiconcavity-preserving if and only if (i) $\gamma_\theta^{a_1}$ satisfies SSCP for every $(\theta, a_1) \in \Theta \times A_1$, and (ii) $\gamma_\theta^{a_1}$ and $\gamma_{\theta'}^{a_1}$ obey SSRM for every $a_1 \in A_1$ and every $\theta, \theta' \in \Theta$.*

PROOF. (*Only-if statement*) Suppose that the receiver's payoff is quasiconcavity-preserving, i.e., $\Gamma_{\tilde{\pi}}^\theta$ has the strict single-crossing property for every $(a_1, \tilde{\pi}) \in A_1 \times \Delta(\Theta)$. Then, (i) immediately follows by taking the degenerate distributions over $\Delta(\Theta)$. For (ii), pick any pair of functions $\gamma_\theta^{a_1}$ and $\gamma_{\theta'}^{a_1}$. Suppose that $\gamma_\theta^{a_1}(i) < 0$ and $\gamma_{\theta'}^{a_1}(i) > 0$. Let

$$\beta = \frac{-\gamma_\theta^{a_1}(i)/\gamma_{\theta'}^{a_1}(i)}{1 - \gamma_\theta^{a_1}(i)/\gamma_{\theta'}^{a_1}(i)},$$

so that $\beta \in (0,1)$ and $\beta\gamma_\theta^{a_1}(i) + (1-\beta)\gamma_{\theta'}^{a_1}(i) = 0$. Since $\beta\gamma_\theta^{a_1} + (1-\beta)\gamma_{\theta'}^{a_1}$ has the strict single-crossing property, we have $\beta\gamma_\theta^{a_1}(j) + (1-\beta)\gamma_{\theta'}^{a_1}(j) > 0$ for all $j > i$. Given that $\gamma_{\theta'}^{a_1}$ must satisfy SSCP and thus

$\gamma^{a_1}_{\theta'}(j) > 0$, we can further obtain

$$\frac{1-\beta}{\beta} = -\frac{\gamma^{a_1}_{\theta}(i)}{\gamma^{a_1}_{\theta'}(i)} > -\frac{\gamma^{a_1}_{\theta}(j)}{\gamma^{a_1}_{\theta'}(j)}.$$

Hence, $\gamma^{a_1}_{\theta}$ and $\gamma^{a_1}_{\theta'}$ must obey SSRM for every $a_1 \in A_1$ and every $\theta, \theta' \in \Theta$.

(*If-statement*) Let $\Theta \equiv \{\theta_1, ..., \theta_K\}$. We need to show that $\forall \mu \equiv (\mu_1, ..., \mu_K) \in [0,1]^K$ such that $\sum_{k=1}^{K} \mu_i = 1$, the function $\Gamma^{a_1}_{\mu} : I \to \mathbb{R}$ with $\Gamma^{a_1}_{\mu}(i) \equiv \sum_{k=1}^{K} \mu_k \gamma^{a_1}_{\theta_k}(i)$ satisfies the strict single-crossing property. Since SSCP is preserved under positive scalar multiplication, and if $\gamma^{a_1}_{\theta}$ and $\gamma^{a_1}_{\theta'}$ obey SSRM then so do $\beta\gamma^{a_1}_{\theta}$ and $\gamma^{a_1}_{\theta'}$ for all $\beta \geq 0$, it suffices for us to show that $\Gamma^{a_1} \equiv \sum_{k=1}^{K} \gamma^{a_1}_{\theta_k}$ satisfies SSCP.

Suppose that $\Gamma^{a_1}(i) \geq 0$. We want to show that $\Gamma^{a_1}(j) > 0$ for every $j > i$. If $\gamma^{a_1}_{\theta_k}(i) \geq 0$ for all $k = 1, ..., K$, then we are done because each $\gamma^{a_1}_{\theta_k}$ satisfies SSCP. Now suppose that $\gamma^{a_1}_{\theta_k}(i) < 0$ for some $\theta_k \in \Theta$. In this case, let us partition Θ into three subsets, Θ^+, Θ^0 and Θ^-, such that $\theta_{k'} \in \Theta^+$ if $\gamma^{a_1}_{\theta_{k'}}(i) > 0$, $\theta_{k'} \in \Theta^0$ if $\gamma^{a_1}_{\theta_{k'}}(i) = 0$, and $\theta_{k'} \in \Theta^-$ if $\gamma^{a_1}_{\theta_{k'}}(i) < 0$. Hence, we have

$$\sum_{\theta_k \in \Theta^+ \cup \Theta^-} \gamma^{a_1}_{\theta_k} = \sum_{\ell=1}^{L} \gamma^{a_1}_{\ell},$$

where each function $\gamma_\ell : I \to \mathbb{R}$ is a positive linear combination of at most two functions $\gamma^{a_1}_{\theta_k}, \gamma^{a_1}_{\theta_{k'}}$ such that $\theta_k, \theta_{k'} \in \Theta^+ \cup \Theta^-$, and $\gamma_\ell(i) \geq 0$ for all $\ell = 1, ..., L$.

To complete the proof, it now suffices to show that for every $\ell = 1, ..., L$, if $\gamma^{a_1}_{\ell} = \alpha\gamma^{a_1}_{\theta_k} + \beta\gamma^{a_1}_{\theta_{k'}}$ for some $\alpha, \beta > 0$ and $\gamma^{a_1}_{\theta_k}, \gamma^{a_1}_{\theta_{k'}}$ such that $\gamma^{a_1}_{\theta_k}(i) < 0$ and $\gamma^{a_1}_{\theta_{k'}}(i) > 0$, we would then obtain $\gamma^{a_1}_{\ell}(j) > 0$ for every $j > i$. This is true because by SSRM, we have

$$\frac{\beta}{\alpha} \geq -\frac{\gamma^{a_1}_{\theta_k}(i)}{\gamma^{a_1}_{\theta_{k'}}(i)} > -\frac{\gamma^{a_1}_{\theta_k}(j)}{\gamma^{a_1}_{\theta_{k'}}(j)} \quad \text{for every } j > i,$$

and hence $\gamma^{a_1}_{\ell}(j) = \alpha\gamma^{a_1}_{\theta_k}(j) + \beta\gamma^{a_1}_{\theta_{k'}}(j) > 0$. $\square$

C Generalized Results with Infinite A_2

In this Appendix, we generalize our monotonicity results to cases where A_1 is infinite. For simplicity, we shall assume that $A_1 \subset \mathbb{R}^n$ with $n \geq 1$ and it is a complete lattice with to the product order on the Euclidean space. With infinite A_1, a technical difficulty is that some of the actions in the support of σ_1^θ can be suboptimal. Therefore, the notion of monotonicity in Definition 2 does not apply.

For every $a_1 \in A_1$ and $\alpha_1 \in \Delta(A_1)$, let $\Pr(\alpha_1 \succ a_1)$ be the probability that the realization of α_1 is strictly higher than a_1. To accommodate the above-mentioned technical difficulty, we introduce the following weaker version of monotonicity:

Definition C1. *σ_1 is an almost surely monotone strategy if for every $\theta \succ \theta'$ and $a_1 \in supp(\sigma_1^\theta)$, we have $\Pr(\sigma_1^{\theta'} \succ a_1) = 0$. An equilibrium (σ_1, σ_2) is almost surely monotone if σ_1 is almost surely monotone.*

We establish the following result, which generalizes Theorem 1.

Theorem C1. *If $|A_2| = 2$ and the sender's payoff is monotone-supermodular, then every Nash equilibrium is almost surely monotone.*

PROOF. The proof of Theorem 1 implies the following lemma.

Lemma C1. *Given the receiver's strategy σ_2, for every $\theta \succ \theta'$ and $a_1 \succ a_1'$, if a_1' is a best response to σ_2 for type θ, then a_1 is not a best response to σ_2 for type θ'.*

For every $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ be the open ball around x with radius r. For every $\theta \succ \theta'$ and $a_1 \in \text{supp}(\sigma_1^\theta)$, we have $\sigma_1^\theta(B(a_1,r)) > 0$ for every $r > 0$. That is to say, there exists $a_1' \in B(a_1,r)$ such that a_1' is optimal for type θ. Let a_1^r be the smallest element that is above every element in $B(a_1,r)$. Lemma C1 implies that $\Pr(\sigma_1^{\theta'} \succ a_1^r) = 0$ for every $r > 0$. For every strictly positive decreasing sequence $\{r_i\}_{i=1}^\infty$ with $\lim_{i\to\infty} r_i = 0$, we have:

$$\lim_{i\to\infty}\{a_1'|a_1' \succ a_1^{r_i}\} = \{a_1'|a_1' \succ a_1\} \text{ and } \{a_1'|a_1' \succ a_1^{r_i}\} \supset \{a_1'|a_1' \succ a_1^{r_j}\} \text{ for every } i > j.$$

The monotone convergence theorem implies that:

$$\Pr(\sigma_1^{\theta'} \succ a_1) = \Pr(\sigma_1^{\theta'} \succ \lim_{i\to\infty} a_1^{r_i}) = \lim_{i\to\infty} \Pr(\sigma_1^{\theta'} \succ a_1^{r_i}) = 0. \qquad \square$$

The corresponding generalizations of Theorems 2 and 3 are similar and, therefore, omitted.

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