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# **Sharp Convergence Rates for Forward Regression in High-Dimensional Sparse Linear Models**

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# Sharp Convergence Rates for Forward Regression in High-Dimensional Sparse Linear Models

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**Abstract:** Forward regression is a statistical model selection and estimation procedure which inductively selects covariates that add predictive power into a working statistical regression model. Once a model is selected, unknown regression parameters are estimated by least squares. This paper analyzes forward regression in high-dimensional sparse linear models. Probabilistic bounds for prediction error norm and number of selected covariates are proved. The analysis in this paper gives sharp rates and does not require  $\beta$ -min or irrepresentability conditions.

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**Keywords and phrases:** forward regression, high-dimensional models, sparsity, model selection

**Import bibliographic note.** The content presented in this paper has been merged into another paper titled *Testing-Based Forward Model Selection*, with a draft available at <https://arxiv.org/abs/1512.02666>. The two previous papers were originally separate projects, but have now been merged in preparation for the publication process. No original material is submitted simultaneously to multiple peer-reviewed journals.

This paper remains posted on ArXiv for the time being in order to serve as a record of the progression of drafts posted to the internet.

## 1. Introduction

Forward regression is a statistical model selection and estimation technique that inductively selects covariates which substantially increase predictive accuracy into a working statistical model until a stopping criterion is met. Once a model is selected, unknown regression parameters are estimated by least squares. This

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\* This version is of April 11, 2018. An earlier version of this paper, *Testing-Based Forward Model Selection* [23], is being split into two papers. The current paper presents fundamental results needed for analysis of forward regression in general settings, while the other paper focuses on using hypothesis tests rather than a simple threshold to decide which covariates enter the selected model. I gratefully acknowledge helpful discussion with Christian Hansen, Tim Conley, Attendants at the ETH Zürich Seminar für Statistik Research Seminar, Attendants at the Center for Law and Economics Internal Seminar, as well as financial support of the ETH Fellowship program.

paper studies statistical properties and proves convergence rates for forward regression in high-dimensional settings.

Dealing with a high-dimensional dataset necessarily involves dimension reduction or regularization. A principal goal of research in high-dimensional statistics and econometrics is to generate predictive power that guards against false discovery and overfitting, does not erroneously equate in-sample fit to out-of-sample predictive ability, and accurately accounts for using the same data to examine many different hypotheses or models. Without dimension reduction or regularization, however, any statistical model will overfit a high dimensional dataset. Forward regression is a method for doing such regularization which is simple to implement, computationally efficient, and easy to understand mechanically.

There are several earlier analyses of forward selection. [37] gives bounds on the performance and number of selected covariates under a  $\beta$ -min condition which restricts the minimum magnitude of nonzero regression coefficients. [40] and [33] prove performance bounds for greedy algorithms under a strong irrerepresentability condition, which restricts the empirical covariance matrix of the predictors. [14] prove bounds on the relative performance in population  $R$ -squared of forward regression (relative to infeasible  $R$ -squared) when the number of variables allowed for selection is fixed.

A key difference between the analysis in this paper relative to previous analysis of forward regression is that all bounds are stated in terms of the sparse eigenvalues of the empirical Gram matrix of the covariates. No  $\beta$ -min or irrerepresentability conditions are required. Under these general conditions, this paper proves probabilistic bounds on the predictive performance which rely on a bound on the number of selected covariates. In addition, the rates derived here are sharp.

A principal idea in the proof is to track average correlation among selected covariates. The only way for many covariates to be falsely selected into the model is that they be correlated to the outcome variable. Then, by merit of being correlated to the outcome, subsets of the selected covariates must also exhibit correlation amongst each other. On the other hand, sparse eigenvalue conditions on the empirical Gram matrix put upper limits on average correlations between covariates. These two observations together imply a bound on the number of covariates which can be selected. Finally, the convergence rates for forward regression follow.

A related method is forward-backward regression, which proceeds similarly to forward regression, but allows previously selected covariates to be discarded from the working model at certain steps. The convergence rates proven in this paper match those in the analysis of a forward-backward regression in [41]. Despite the similarity between the two procedures, it is still desirable to have a good understanding of forward selection. An advantage of forward selection relative to forward-backward is computational simplicity. In addition, understanding the properties of forward selection may lead to better understanding of general early stopping procedure in statistics (see [38], [42] ) as well as other greedy algorithms (see [9], [17] ). The analysis required for forward regression requires

quite different techniques, since there is no chance to correct ‘model selection mistakes.’

There still are many other sensible approaches to high dimensional estimation and regularization. An important and common approach to generic high dimensional estimation problems is the Lasso. The Lasso minimizes a least squares criteria augmented with a penalty proportional to the  $\ell_1$  norm of the coefficient vector. For theoretical and simulation results about the performance of Lasso, see [16] [32], [19], [13], [1], [2], [7], [11], [10] [12], [13], [20], [21], [22], [24], [25], [27], [28], [32], [34], [36], [39], [4], [8], [4], among many more. In addition, [15] have shown that under restrictive conditions, Lasso and forward regression yield approximately the same solutions (see also [26]). This paper derives statistical performance bounds for forward selection which match those given by Lasso in more general circumstances.

Finally, an important potential application for forward regression is as an input for post-model-selection analysis. One example is the selection of a conditioning set, to properly control for omitted variables bias when there are many potential control variables (see [6], [35], [5]). Another example is the selection of instrumental variables for later use in a first stage regression (see [3]). Both applications require a model selection procedure with the hybrid property of both producing a good fit and returning a sparse set of covariates. The results derived in this paper are relevant for both objectives, deriving bounds for both prediction error norm as well as the size of the selected set for forward regression.

## 2. Framework

The observed data is given by  $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^n$ . The data consists of a set of covariates  $x_i \in \mathbb{R}^p$ , as well as outcome variables  $y_i \in \mathbb{R}$  for each observation  $i = 1, \dots, n$ . The data satisfy

$$y_i = x_i' \theta_0 + \varepsilon_i$$

for some unknown parameter of interest  $\theta_0 \in \mathbb{R}^p$  and unobserved disturbance terms  $\varepsilon_i \in \mathbb{R}$ . The covariates  $x_i$  are normalized so that  $\mathbb{E}_n[x_{ij}] = 0$  and  $\mathbb{E}_n[x_{ij}^2] = 1$  for every  $j = 1, \dots, p$ , where  $\mathbb{E}_n[\cdot] = \frac{1}{n} \sum_{i=1}^n (\cdot)$  denotes empirical expectation. Finally, the parameter  $\theta_0$  is sparse in the sense that the set of non-zero components of  $\theta_0$ , denoted  $S_0 = \text{supp}(\theta_0)$ , has cardinality  $s_0 < n$ . The interest in this paper is to study how well forward regression can estimate  $x_i' \theta_0$  for  $i = 1, \dots, n$ .

Define a loss function  $\ell(\theta)$

$$\ell(\theta) = \mathbb{E}_n[(y_i - x_i' \theta)^2].$$

Note that  $\ell(\theta)$  depends on  $\mathcal{D}$ , but this dependence is suppressed from the notation. Define also

$$\ell(S) = \min_{\theta: \text{supp}(\theta) \subseteq S} \ell(\theta).$$

The estimation strategy proceeds by first searching for a sparse subset  $\widehat{S} \subseteq \{1, \dots, p\}$ , with cardinality  $\widehat{s}$ , that assumes a small value of  $\ell(S)$ , followed by estimating  $\theta_0$  with least squares via

$$\widehat{\theta} \in \arg \min_{\theta: \text{supp}(\theta) \subseteq \widehat{S}} \ell(\theta).$$

This gives the construction of the estimates  $x'_i \widehat{\theta}$  for  $i = 1, \dots, n$ . The paper provides bounds for the prediction error norm defined by

$$\mathbb{E}_n[(x'_i \theta_0 - x'_i \widehat{\theta})^2]^{1/2}.$$

The set  $\widehat{S}$  is selected by forward regression. For any  $S$  define the incremental loss from the  $j$ th covariate by

$$\Delta_j \ell(S) = \ell(S \cup \{j\}) - \ell(S).$$

Consider the greedy algorithm which inductively selects the  $j$ th covariate to enter a working model if  $-\Delta_j \ell(S)$  exceeds a threshold  $t$ :

$$-\Delta_j \ell(S) > t$$

and  $\Delta_j \ell(S) \geq \Delta_k \ell(S)$  for each  $k \neq j$ . The threshold  $t$  is chosen by the user; it is the only tuning parameter required. This defines forward regression. It is summarized formally here:

Algorithm 1: Forward Regression

**Initialize.** Set  $\widehat{S} = \{\}$ .

**For**  $1 \leq k \leq p$ :

**If:**  $-\Delta_j \ell(S) > t$  for some  $j \in \{1, \dots, p\} \setminus \widehat{S}$ , then select

$$\widehat{j} \in \arg \max \{-\Delta_j \ell(S) : -\Delta_j \ell(S) > t\}.$$

**Update:**  $\widehat{S} = \widehat{S} \cup \{\widehat{j}\}$ .

**Else: Break.**

**Set:**

$$\widehat{\theta} \in \arg \min_{\theta: \text{supp}(\theta) \subseteq \widehat{S}} \ell(\theta).$$

### 3. Analysis of Forward Regression

In order to state the main theorem, a few more definitions are convenient. Define the empirical Gram matrix  $G_x$  by  $G_x = \mathbb{E}_n[x_i x'_i]$ . Let  $\varphi_{\min}(s)(G_x)$  denote the

minimum  $s$ -sparse eigenvalues given by

$$\varphi_{\min}(s)(G_x) = \min_{S \subseteq \{1, \dots, p\}; |S| \leq s} \lambda_{\min}([G_x]_{S,S})$$

where  $[G_x]_{S,S}$  is the principal submatrix of  $G_x$  corresponding to the component set  $S$ . Let

$$\mathcal{C}_1 = \sqrt{\hat{s} + s_0} \varphi_{\min}(\hat{s} + s_0)(G_x)^{-1} \left[ 2 \|\mathbb{E}_n[\varepsilon_i x'_i]\|_{\infty} + t^{1/2} \right].$$

For each positive integer  $m$ , let

$$\mathcal{C}_2(m) = 1 + 72 \times 1.783^2 \times \varphi_{\min}(m + s_0)(G_x)^{-5}.$$

The above quantities are useful for displaying results in Theorem 1. Slightly tighter but messier usable quantities than  $\mathcal{C}_1$  and  $\mathcal{C}_2(m)$  are derived in the proof. Note also that  $\mathcal{C}_1$  depends on  $\hat{s}$ .

**Theorem 1.** *Consider data  $\mathcal{D}$  with parameter  $\theta_0$ . Then under Algorithm 1 with threshold  $t$ ,*

$$\mathbb{E}_n[(x'_i \theta_0 - x'_i \hat{\theta})^2]^{1/2} \leq \mathcal{C}_1.$$

For every integer  $m \geq 0$  such that  $t^{1/2} \geq 2\varphi_{\min}(m + s_0)(G_x)^{-1} \|\mathbb{E}_n[x_i \varepsilon_i]\|_{\infty}$  and  $m \leq |\hat{S} \setminus S_0|$ , it holds that

$$m \leq \mathcal{C}_2(m) s_0.$$

The above theorem calculates explicit constants bounding the prediction error norm. It is also helpful to consider the convergence rates implied by Theorem 1 under more concrete conditions on  $\mathcal{D}$ . Next, consider the following conditions on a sequence of datasets  $\mathcal{D}_n$ . In what follows, the parameters  $\theta_0$ , the thresholds  $t$ , and distribution of the data can all depend on  $n$ .

**Condition 1** [*Model and Sparsity*].  $s_0 = o(n)$ .

**Condition 2** [*Sparse Eigenvalues*]. There is a sequence  $K_n$  such that  $s_0 = o(K_n)$ . In addition,  $\varphi_{\min}(K_n)(G_x)^{-1} = O(1)$  with probability  $1 - o(1)$ .

**Condition 3** [*Threshold and Disturbance Terms*]. The threshold satisfies  $t = O(\log p/n)$ . In addition,  $t^{1/2} \geq 2\varphi_{\min}(K_n)(G_x)^{-1} \|\mathbb{E}_n[x_i \varepsilon_i]\|_{\infty}$  with probability  $1 - o(1)$ .

**Theorem 2.** *For a sequence of datasets  $\mathcal{D}_n$  with parameters  $\theta_0$  and thresholds  $t$  satisfying Conditions 1-3, the bounds*

$$\mathbb{E}_n[(x'_i \theta_0 - x'_i \hat{\theta})^2]^{1/2} = O(\sqrt{s_0 \log p/n}),$$

$$\hat{s} \leq O(1) s_0$$

hold with probability  $1 - o(1)$ .

The theorem shows that forward regression exhibits asymptotically the same convergence rates in prediction error norm as other high-dimensional estimators like Lasso, provided an appropriate threshold  $t$  is used. In addition, forward regression selects a set with cardinality commensurate with  $s_0$ .

Condition 1 bounds the size of  $S_0$  and requires that the sparsity level is small relative to the sample size. Condition 2 is a sparse eigenvalue condition useful for proving results about high dimensional techniques like Lasso. In standard regression analysis where the number of covariates is small relative to the sample size, a conventional assumption used in establishing desirable properties of conventional estimators of  $\theta$  is that  $G_x$  has full rank. In the high-dimensional setting,  $G_x$  will be singular if  $p > n$  and may have an ill-behaved inverse even when  $p \leq n$ . However, good performance of many high-dimensional estimators only requires good behavior of certain moduli of continuity of  $G_x$ . There are multiple formalizations and moduli of continuity that can be considered here; see [7]. This analysis focuses on a simple eigenvalue condition which was used in [3]. Condition 2 could be shown to hold under more primitive conditions by adapting arguments found in [4] which build upon results in [39] and [30]; see also [29]. Condition 2 is notably weaker than previously used irrepresentability conditions. Irrepresentability conditions require that for certain sets  $S$  and  $k \notin S$ , letting  $x_{iS}$  be the subvector of  $x_i$  with components  $j \in S$ , that  $\|\mathbb{E}_n[x_{iS}x'_{iS}]^{-1}\mathbb{E}_n[x_{iS}x'_{ik}]\|_1$  is bounded, or even strictly less than 1.

Condition 3 is a regularization condition similar to regularization conditions common in the analysis of Lasso. The condition, requires  $t^{1/2}$  to dominate a multiple of the  $\|\mathbb{E}_n[x_i\varepsilon_i]\|_\infty$ . This condition is stronger than that typically encountered with Lasso, because the multiple relies on the sparse eigenvalues of  $G_x$ . To illustrate why such a condition is useful, let  $\check{x}_{ij}$  denote  $x_{ij}$  residualized away from previously selected regressors and renormalized. Then even if  $\mathbb{E}_n[x_{ij}\varepsilon_i] < t^{1/2}$ ,  $\mathbb{E}_n[\check{x}_{ij}\varepsilon_i]$  can exceed  $t^{1/2}$  resulting in more selections into the model. Nevertheless, using the multiple  $2\varphi_{\min}(K_n)(G_x)^{-1}$  which stays bounded with  $n$ , is sufficient to ensure that  $\hat{s}$  does not grow faster than  $s_0$ . From a practical standpoint, this condition also requires the user to know more about the design of the data in choosing an appropriate  $t$ . Choosing feasible thresholds which satisfy a similar condition to Condition 3 is considered in [23].

**Theorem 3.** *For a sequence of datasets  $\mathcal{D}_n$  with parameters  $\theta_0$  and thresholds  $t$  satisfying Conditions 1-3, the bounds*

$$\|\theta_0 - \hat{\theta}\|_2 = O(\sqrt{s_0 \log p/n}) \text{ and } \|\theta_0 - \hat{\theta}\|_1 = O(\sqrt{s_0^2 \log p/n})$$

*hold with probability  $1 - o(1)$ .*

Finally, two direct consequence of Theorem 2 are bounds on the deviations  $\|\hat{\theta} - \theta_0\|_1$  and  $\|\hat{\theta} - \theta_0\|_2$  of  $\hat{\theta}$  from underlying unknown parameter  $\theta_0$ . Theorem 3

above shows that deviations of  $\widehat{\theta}$  from  $\theta_0$  also achieve rates typically encountered in high-dimensional estimators like Lasso.

#### 4. Proof of Theorem 1

*Proof.* The proof of Theorem 1 is divided into seven steps. Step 1 shows the first statement of Theorem 1. Step 2 defines a useful normalization of the selected covariates. Step 3 establishes certain bounds on the average correlation between selected covariates. Steps 4-6 show that if  $\widehat{s}$  is too high, then there must exist subsets of the selected covariates over which the average correlation must exceed what is permitted by assumption on the sparse eigenvalues of the empirical Gram matrix  $G$ . Step 7 concludes by pulling together the previous six steps.

##### Step 1

This first section of the proof provides a bound on  $\mathbb{E}_n[(x'_i\theta_0 - x'_i\widehat{\theta})^2]$  which depends on  $\widehat{s}$  thereby proving the first statement of Theorem 1. First note that  $\ell(\widehat{S}) = \ell(\widehat{S} \cup S_0) + [\ell(\widehat{S}) - \ell(\widehat{S} \cup S_0)]$ . Note that  $\ell(\widehat{S}) = \ell(\widehat{\theta})$  and  $\ell(\widehat{S} \cup S_0) \leq \ell(\theta_0)$ . In addition, by Lemma 3.3 of [14],

$$\ell(\widehat{S}) - \ell(\widehat{S} \cup S_0) \leq \varphi_{\min}(\widehat{s} + s_0)(G)^{-1} \sum_{j \in S_0 \setminus \widehat{S}} (-\Delta_j \ell(\widehat{S})) \leq s_0 t \varphi_{\min}(\widehat{s} + s_0)(G)^{-1}.$$

This gives

$$\ell(\widehat{\theta}) \leq \ell(\theta_0) + s_0 t \varphi_{\min}(\widehat{s} + s_0)(G)^{-1}.$$

Expanding the above two quadratics in  $\ell(\cdot)$  gives

$$\begin{aligned} \mathbb{E}_n[(x'_i\theta_0 - x'_i\widehat{\theta})^2] &\leq |2\mathbb{E}_n[\varepsilon_i x'_i(\widehat{\theta} - \theta_0)]| + s_0 t \varphi_{\min}(\widehat{s} + s_0)(G)^{-1} \\ &\leq 2\|\mathbb{E}_n[\varepsilon_i x'_i]\|_{\infty} \|\theta_0 - \widehat{\theta}\|_1 + s_0 t \varphi_{\min}(\widehat{s} + s_0)(G)^{-1} \end{aligned}$$

To bound  $\|\theta_0 - \widehat{\theta}\|_1$ :

$$\begin{aligned} \|\theta_0 - \widehat{\theta}\|_1 &\leq \sqrt{\widehat{s} + s_0} \|\theta_0 - \widehat{\theta}\|_2 \\ &\leq \sqrt{\widehat{s} + s_0} \varphi_{\min}(\widehat{s} + s_0)(G)^{-1} \mathbb{E}_n[(x'_i\theta_0 - x'_i\widehat{\theta})^2]^{1/2}. \end{aligned}$$

Combining the above bounds and dividing by  $\mathbb{E}_n[(x'_i\theta_0 - x'_i\widehat{\theta})^2]^{1/2}$  gives

$$\begin{aligned} \mathbb{E}_n[(x'_i\theta_0 - x'_i\widehat{\theta})^2]^{1/2} &\leq 2\|\mathbb{E}_n[\varepsilon_i x'_i]\|_{\infty} \sqrt{\widehat{s} + s_0} \varphi_{\min}(\widehat{s} + s_0)(G)^{-1} \\ &\quad + \frac{s_0 t \varphi_{\min}(\widehat{s} + s_0)(G)^{-1}}{\mathbb{E}_n[(x'_i\theta_0 - x'_i\widehat{\theta})^2]^{1/2}}. \end{aligned}$$



Finally, either  $\mathbb{E}_n[(x'_i\theta_0 - x'_i\hat{\theta})^2]^{1/2} \leq \sqrt{s_0 t \varphi_{\min}(\hat{s} + s_0)(G)^{-1}}$ , in which case the first statement of Theorem 1 holds, or alternatively  $\mathbb{E}_n[(x'_i\theta_0 - x'_i\hat{\theta})^2]^{1/2} > \sqrt{s_0 t \varphi_{\min}(\hat{s} + s_0)(G)^{-1}}$ , in which case

$$\begin{aligned} \mathbb{E}_n[(x'_i\theta - x'_i\hat{\theta})^2]^{1/2} &\leq 2\|\mathbb{E}_n[\varepsilon_i x'_i]\|_\infty \sqrt{\hat{s} + s_0} \varphi_{\min}(\hat{s} + s_0)(G)^{-1} \\ &\quad + \sqrt{s_0 t \varphi_{\min}(\hat{s} + s_0)(G)^{-1}} \end{aligned}$$

and the first statement of Theorem 1 follows.

### Step 2

This section of the proof defines *true* and *false* covariates, introduces a convenient orthogonalization of all selected covariates, and associates to each false selected covariate a parameter  $\tilde{\gamma}_j$  on which the analysis is based.

Let  $x_j = [x_{1j}, \dots, x_{nj}]'$  be the vector in  $\mathbb{R}^n$  with components  $x_{ij}$  stacked vertically. Similarly, define  $\varepsilon = [\varepsilon_1, \dots, \varepsilon_n]'$  and  $y = [y_1, \dots, y_n]'$ . Let  $v_k \in \mathbb{R}^n$ ,  $k = 1, \dots, s_0$  denote *true covariates* which are defined as the the vectors  $x_j$  for  $j \in S_0$ . Define *false covariates* simply as those which do not belong to  $S_0$ .

Consider any point in time in the the forward regression algorithm when there are  $m$  false covariates selected into the model. These falsely selected covariates are denoted  $w_1, \dots, w_m$ , each in  $\mathbb{R}^n$ , ordered according to the order they were selected.

The true covariates are also ordered according to the order they are selected into the model. Any true covariates unselected after the  $m$  false covariate selection are temporarily ordered arbitrarily at the end of the list. Let  $\mathcal{M}_k$  be projection in  $\mathbb{R}^n$  onto the space orthogonal to  $\text{span}(\{v_1, \dots, v_k\})$ . Let

$$\tilde{v}_k = \frac{\mathcal{M}_{k-1} v_k}{(v'_k \mathcal{M}_{k-1} v_k)^{1/2}} \text{ for } k = 1, \dots, s_0.$$

In addition, set

$$\tilde{\varepsilon} = \frac{\mathcal{M}_{s_0} \varepsilon}{(\varepsilon' \mathcal{M}_{s_0} \varepsilon)^{1/2}}.$$

Let  $\tilde{V}_{\text{temp}} = [\tilde{v}_1, \dots, \tilde{v}_{s_0}]$ , ordered according to the temporary order. Note that there is  $\tilde{\theta} \in \mathbb{R}^{s_0}$  and  $\tilde{\theta}_\varepsilon \in \mathbb{R}$  such that

$$\tilde{V}_{\text{temp}} \tilde{\theta}_{\text{temp}} + \tilde{\theta}_\varepsilon \tilde{\varepsilon} = y.$$

At this time, reorder the true covariates. Let  $\hat{k}$  denote the index of the final true covariate selected into the model when the  $m$ -th false covariate is selected. The variables  $\tilde{v}_1, \dots, \tilde{v}_{\hat{k}}$  maintain their original order. The unselected true covariates  $\tilde{v}_{\hat{k}+1}, \dots, \tilde{v}_{s_0}$  are reordered in such a way that under the new ordering,  $\tilde{\theta}_{k,\text{temp}} \geq \tilde{\theta}_{l,\text{temp}}$  whenever  $l > k$ . Also define  $\tilde{V} = [\tilde{v}_1, \dots, \tilde{v}_{s_0}]$  consistent with the new ordering. Redefine  $\tilde{\theta}$  by  $\tilde{V} \tilde{\theta} + \tilde{\theta}_\varepsilon \tilde{\varepsilon} = y$  so that it is also consistent with the new ordering. Note that no new orthogonalization needs to be done.

For any set  $S$ , Let  $\mathcal{Q}_S$  be projection onto the space orthogonal to  $\text{span}(\{x_j, j \in S\})$ . For each selected covariate,  $w_j$ , set  $S_{\text{pre-}w_j}$  to be the set of (both true and false) covariates selected prior to  $w_j$ . Define

$$\tilde{w}_j = c_j \mathcal{Q}_{S_{\text{pre-}w_j}} w_j$$

where the normalization constants  $c_j$  are defined in the next paragraph.

Each  $\tilde{w}_j$  can be decomposed into components  $\tilde{w}_j = \tilde{r}_j + \tilde{u}_j$  with  $\tilde{r}_j \in \text{span}(\tilde{V})$  and  $\tilde{u}_j \in \text{span}(\tilde{V})^\perp$ . The normalizations  $c_j$  introduced above are then chosen so that  $\tilde{u}_j' \tilde{u}_j = 1$ .

Associates to each false covariate  $\tilde{w}_j$ , a vector  $\tilde{\gamma}_j \in \mathbb{R}^{s_0}$ , defined as the solution in  $\mathbb{R}^{s_0}$  to the following equation

$$\tilde{V} \tilde{\gamma}_j = \tilde{r}_j.$$

Set  $\tilde{\gamma}_{j\tilde{\varepsilon}} = \tilde{\varepsilon}' \tilde{w}_j$ . Assume without loss of generality that each component of  $\tilde{\theta}$  is positive (since otherwise, the true covariates can just be multiplied by  $-1$ .) Also assume without loss of generality that  $\tilde{\gamma}_j' \tilde{\theta} \geq 0$ .

### Step 3

This section provides upper bounds on quantities related to the  $\tilde{\gamma}_j$  defined above. The idea guiding the argument in the next sections is that if too many covariates  $w_j$  are selected, then on average they must be correlated with each other since they must be correlated to  $y$ . For a discussion of partial transitivity of correlation, see [31]. If the covariates are highly correlated amongst themselves, then  $\varphi_{\min}(m + s_0)(G)^{-1}$  must be very high. As a result, the sparse eigenvalues of  $G$  can be used to upper bound the number of selections. Average correlations between covariates are tracked with the aid of the quantities  $\tilde{\gamma}_j$ .

Divide the set of false covariates into two sets  $A_1$  and  $A_2$  where

$$A_1 = \left\{ j : |\tilde{\gamma}_{j\tilde{\varepsilon}}| \leq \frac{t^{1/2} n^{1/2}}{(2\varepsilon' \mathcal{M}_{s_0} \varepsilon)^{1/2}} \right\}, \quad A_2 = \left\{ j : |\tilde{\gamma}_{j\tilde{\varepsilon}}| > \frac{t^{1/2} n^{1/2}}{(2\varepsilon' \mathcal{M}_{s_0} \varepsilon)^{1/2}} \right\}.$$

Sections 3 - 5 of the proof bound the number of elements in  $A_1$ . Section 6 of the proof bounds the number of elements in  $A_2$ .

Suppose the set  $A_1$  contains  $m_1$  total false selections. Collect these false selections into  $\tilde{W} = [\tilde{w}_{j_1}, \dots, \tilde{w}_{j_{m_1}}]$ . Set  $\tilde{R} = [\tilde{r}_{j_1}, \dots, \tilde{r}_{j_{m_1}}]$ ,  $\tilde{U} = [\tilde{u}_{j_1}, \dots, \tilde{u}_{j_{m_1}}]$ . Decompose  $\tilde{W} = \tilde{R} + \tilde{U}$ . Then  $\tilde{W}' \tilde{W} = \tilde{R}' \tilde{R} + \tilde{U}' \tilde{U}$ . Since  $\text{diag}(\tilde{U}' \tilde{U}) = I$ , it follows that the average inner product between the  $\tilde{u}_j$ , given by  $\bar{\rho}$ :

$$\bar{\rho} = \frac{1}{m_1(m_1 - 1)} \sum_{j \neq l \in A_1} \tilde{u}_j' \tilde{u}_l,$$

must be bounded below by

$$\bar{\rho} \geq -\frac{1}{m_1 - 1}$$

due to the positive definiteness of  $\tilde{U}'\tilde{U}$ . This implies an upper bound on the average off-diagonal term in  $\tilde{R}'\tilde{R}$  since  $\tilde{W}'\tilde{W}$  is a diagonal matrix. Since  $\tilde{v}_k$  are orthonormal, the sum of all the elements of  $\tilde{R}'\tilde{R}$  is given by  $\|\sum_{j \in A_1} \tilde{\gamma}_j\|_2^2$ . Since  $\|\sum_{j \in A_1} \tilde{\gamma}_j\|_2^2 = \sum_{j \in A_1} \|\tilde{\gamma}_j\|_2^2 + \sum_{j \neq l \in A_1} \tilde{\gamma}_j' \tilde{\gamma}_l$  and since  $\tilde{W}'\tilde{W}$  is a diagonal matrix, it must be the case that

$$\frac{1}{m_1(m_1 - 1)} \sum_{j \neq l \in A_1} \tilde{\gamma}_j' \tilde{\gamma}_l = -\bar{\rho}.$$

Therefore,

$$\bar{\rho} = \frac{1}{m_1(m_1 - 1)} \left( \left\| \sum_{j \in A_1} \tilde{\gamma}_j \right\|_2^2 - \sum_{j \in A_1} \|\tilde{\gamma}_j\|_2^2 \right) \leq \frac{1}{m_1 - 1}.$$

This implies that

$$\left\| \sum_{j \in A_1} \tilde{\gamma}_j \right\|_2^2 \leq m_1 + \sum_{j \in A_1} \|\tilde{\gamma}_j\|_2^2.$$

Next, bound  $\max_{j \in A_1} \|\tilde{\gamma}_j\|_2^2$ . Note  $\|\tilde{\gamma}_j\|_2^2 = \|\tilde{r}_j\|_2^2$  since  $\tilde{V}$  is orthonormal. Note that  $\|\tilde{u}_j\|_2^2 / \|\tilde{w}_j\|_2^2 = 1 / \|\tilde{w}_j\|_2^2$  is lower bounded by  $\varphi_{\min}(m + s_0)(G)$ . This follows from the fact that you can associate  $\|\tilde{u}_j/c_j\|_2^2$  to an element of the inverse covariance matrix for  $w_j$  and previously selected covariates. Therefore,  $\|\tilde{r}_j\|_2^2 = \|\tilde{w}_j\|_2^2 - 1 \leq \varphi_{\min}(m + s_0)(G)^{-1} - 1$ . It follows that

$$\max_{j \in A_1} \|\tilde{\gamma}_j\|_2^2 \leq \varphi_{\min}(m + s_0)(G)^{-1} - 1.$$

This then implies that

$$\left\| \sum_{j \in A_1} \tilde{\gamma}_j \right\|_2^2 \leq m_1 \varphi_{\min}(m + s_0)(G)^{-1}.$$

The same argument as above also shows that for any choice  $e_j \in \{-1, 1\}$  of signs, it is always the case that

$$\left\| \sum_{j \in A_1} e_j \tilde{\gamma}_j \right\|_2^2 \leq m_1 \varphi_{\min}(m + s_0)(G)^{-1}.$$

(In more detail, take  $\tilde{W}_e = [\tilde{w}_{j_1} e_{j_1}, \dots, \tilde{w}_{j_{m_1}} e_{j_{m_1}}]$ , etc. and rerun the same argument.)

#### Step 4

Next search for a particular choice of signs  $\{e_j\}_{j \in A_1}$  which give a lower bound proportional to  $m_1^2/s_0$  on the above term. Note that this will imply an upper bound on  $m_1$ . For each  $k = 1, \dots, s_0$ , let  $A_{1k}$  be the set which contains those

$j \in A_1$  such that  $w_j$  is selected before  $v_k$ , but not before any other true covariate. Note that the sets  $A_1(\widehat{k}+2), \dots, A_1(s_0+1)$  are set empty if  $\widehat{k} < s_0$ . Also, empty sums are set to zero. Define the following two matrices:

$$\Gamma = \begin{bmatrix} \sum_{j \in A_{11}} \tilde{\gamma}_{j1} & \sum_{j \in A_{11}} \tilde{\gamma}_{j2} & \cdots & \sum_{j \in A_{11}} \tilde{\gamma}_{js_0} \\ 0 & \sum_{j \in A_{12}} \tilde{\gamma}_{j2} & \cdots & \sum_{j \in A_{12}} \tilde{\gamma}_{js_0} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{j \in A_{1s_0}} \tilde{\gamma}_{js_0} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{\tilde{\theta}_1}{\theta_1} & \frac{\tilde{\theta}_2}{\theta_1} & \cdots & \frac{\tilde{\theta}_{s_0}}{\theta_1} \\ \frac{\tilde{\theta}_2}{\theta_1} & \frac{\tilde{\theta}_2}{\theta_2} & \cdots & \frac{\tilde{\theta}_{s_0}}{\theta_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\tilde{\theta}_{s_0}}{\theta_1} & \frac{\tilde{\theta}_{s_0}}{\theta_2} & \cdots & \frac{\tilde{\theta}_{s_0}}{\theta_{s_0}} \end{bmatrix}$$

Note that the  $k$ th row of  $\Gamma$  is equal to  $\sum_{j \in A_{1k}} \tilde{\gamma}_k$  since the orthogonalization process had enforced  $\tilde{\gamma}_{jl} = 0$  for each  $l < k$ . Therefore, the diagonal elements of the product  $\Gamma B$  satisfy the equality

$$[\Gamma B]_{k,k} = \sum_{j \in A_k} \tilde{\gamma}'_j \tilde{\theta} / \tilde{\theta}_k.$$

Let  $C_1, C_2$  be constants such that

$$\tilde{\gamma}'_j \tilde{\theta} / \tilde{\theta}_k \geq C_1$$

for  $j \in A_{1k}$ , and

$$\tilde{\theta}_k / \tilde{\theta}_l \geq C_2$$

for  $l > k$ . These key constants are calculated explicitly in Section 5 of the proof. They imply that

$$[\Gamma B]_{k,k} \geq C_1 |A_{1k}| \quad \text{and} \quad \text{tr}(\Gamma B) \geq C_1 m_1.$$

Further observe that whenever  $\tilde{\theta}_k \geq C_2 \tilde{\theta}_l$  for each  $k, l > k$ , assuming without loss of generality that  $C_2 \leq 1$ , that  $(B + C_2^{-1}I)$  is positive semidefinite. This can be checked by constructing auxiliary random variables who have covariance matrix  $B + C_2^{-1}I$ : inductively build a covariance matrix where the  $(k+1)$ th random variable has  $\tilde{\theta}_k / \tilde{\theta}_{k-1}$  covariance with the  $k$ th random variable. Then  $B + C_2^{-1}I$  has a positive definite symmetric matrix square root so let  $D^2 = B + C_2^{-1}I$ . Therefore,  $B = (D + C_2^{-1/2}I)(D - C_2^{-1/2}I)$ . Note that the rows (and columns) of  $D$  each have norm  $\leq 1 + C_2^{-1}$  and therefore  $B$  decomposes into a product  $B = E'F$  where the rows of  $E, F$  have norms bounded by  $1 + C_2^{-1} + C_2^{-1/2}$ . Therefore, let  $C_3 = 1 + C_2^{-1} + C_2^{-1/2}$ .

Consider the set

$$\mathcal{S}_{s_0} = \{Z \in \mathbb{R}^{s_0 \times s_0} : Z_{ij} = X_i' Y_j \text{ for some } X_i, Y_j \in \mathbb{R}^{s_0}, \|X_i\|_2, \|Y_j\|_2 \leq 1\}$$

and observe that  $\bar{B} := C_3^{-1}B \in \mathcal{G}_{s_0}$ . Then this observation allows the use of Grothendieck's inequality (using the exact form described in [18]) which gives

$$\max_{Z \in \mathcal{G}_{s_0}} \text{tr}(\Gamma Z) \leq K_G^{\mathbb{R}} \|\Gamma'\|_{\infty \rightarrow 1}.$$

Here,  $K_G^{\mathbb{R}}$  is an absolute constant which is known to be less than 1.783. It does not depend on  $s_0$ . Therefore,  $C_1 m \leq \text{tr}(\Gamma B) = C_3 \text{tr}(\Gamma \bar{B}) \leq \max_{Z \in \mathcal{G}_{s_0}} \text{tr}(\Gamma Z) \leq K_G^{\mathbb{R}} \|\Gamma'\|_{\infty \rightarrow 1}$ , which implies

$$(K_G^{\mathbb{R}})^{-1} C_3^{-1} C_1 m_1 \leq \|\Gamma'\|_{\infty \rightarrow 1}.$$

Therefore, there is  $\nu \in \{-1, 1\}^{s_0}$  such that  $\|\nu' \Gamma\|_1 \geq (K_G^{\mathbb{R}})^{-1} C_3^{-1} C_1 m_1$ . For this particular choice of  $\nu$ , it follows that

$$\|\nu' \Gamma\|_2 \geq s_0^{-1/2} (K_G^{\mathbb{R}})^{-1} C_3^{-1} C_1 m_1.$$

Then by definition of  $\Gamma$ ,  $\|\nu' \Gamma\|_2^2 = \|\sum_{k=1}^{s_0} \sum_{j \in A_{1k}} \nu_k \tilde{\gamma}_j\|_2^2$ . In Section 3, it was noted that  $\|\sum_{j=1}^{m_1} e_j \tilde{\gamma}_j\|_2^2 \leq m_1 \varphi_{\min}(m + s_0) (G)^{-1}$  for any choice of signs  $e_j \in \{-1, 1\}^{m_1}$ . It follows that

$$s_0^{-1} (K_G^{\mathbb{R}})^{-2} C_3^{-2} C_1^2 m_1^2 \leq m_1 \varphi_{\min}(m + s_0) (G)^{-1}$$

which yields the conclusion

$$m_1 \leq \varphi_{\min}(m + s_0) (G)^{-1} C_1^{-2} C_3^2 (K_G^{\mathbb{R}})^2 s_0.$$

### Step 5

It is left to calculate  $C_1, C_2$  which lower bound  $\tilde{\gamma}'_j \tilde{\theta} / \tilde{\theta}_k$  for  $j \in A_{1k}$  and  $\tilde{\theta}_k / \tilde{\theta}_l$  for  $l > k$ . A simple derivation can be made to show that the incremental decrease in empirical loss from the  $j$ th false selection is

$$-\Delta_j \ell(S_{\text{pre-}w_j}) = \frac{1}{n} y' \tilde{w}_j (\tilde{w}'_j \tilde{w}_j)^{-1} \tilde{w}'_j y = \frac{1}{n} \frac{1}{\tilde{w}'_j \tilde{w}_j} (\tilde{\theta}' \tilde{\gamma}_j + \tilde{\theta}'_{\varepsilon} \tilde{\gamma}_j \tilde{\varepsilon})^2$$

Note the slight abuse of notation in  $-\Delta_j(S_{\text{pre-}w_j})$  signifying change in loss under inclusion of  $w_j$  rather than  $x_j$ . Next,

$$(\tilde{\theta}' \tilde{\gamma}_j + \tilde{\theta}'_{\varepsilon} \tilde{\gamma}_j \tilde{\varepsilon})^2 \leq 2(\tilde{\theta}' \tilde{\gamma}_j)^2 + 2(\tilde{\theta}'_{\varepsilon} \tilde{\gamma}_j \tilde{\varepsilon})^2$$

Since  $\tilde{\theta}_{\varepsilon} = (\varepsilon' \mathcal{M}_{s_0} \varepsilon)^{1/2}$ ,  $\tilde{w}'_j \tilde{w}_j \geq 1$ , and  $j \in A_1$  it follows that

$$\frac{1}{n} \frac{1}{\tilde{w}'_j \tilde{w}_j} (\tilde{\theta}'_{\varepsilon} \tilde{\gamma}_j \tilde{\varepsilon})^2 \leq \frac{1}{n} \frac{1}{\tilde{w}'_j \tilde{w}_j} \tilde{\theta}_{\varepsilon}^2 \left( \frac{t^{1/2} n^{1/2}}{2(\varepsilon' \mathcal{M}_{s_0} \varepsilon)^{1/2}} \right)^2 \leq \frac{t}{4}.$$

This implies

$$\frac{1}{2} (-\Delta_j \ell(S_{\text{pre-}w_j})) \leq \frac{1}{n} \frac{1}{\tilde{w}'_j \tilde{w}_j} (\tilde{\theta}' \tilde{\gamma}_j)^2 + \frac{t}{4}.$$

By the condition that the false  $j$  is selected, it holds that  $-\Delta_j \ell(S_{\text{pre-}w_j}) > t$  and so  $\frac{1}{4}(-\Delta_j \ell(S_{\text{pre-}w_j})) > \frac{t}{4}$  which implies that

$$\frac{1}{2}(-\Delta_j \ell(S_{\text{pre-}w_j})) - \frac{t}{4} \geq \frac{1}{4}(-\Delta_j \ell(S_{\text{pre-}w_j})).$$

Finally, this yields that

$$\frac{1}{n\tilde{w}'_j \tilde{w}_j} (\tilde{\gamma}'_j \tilde{\theta})^2 \geq \frac{1}{4}(-\Delta_j \ell(S_{\text{pre-}w_j})).$$

By the fact that  $w_j$  was selected ahead of  $v_k$  it holds that

$$-\Delta_j \ell(S_{\text{pre-}w_j}) \geq -\Delta_k \ell(S_{\text{pre-}w_j}).$$

Therefore, further bound the righthand side. Let  $\tilde{z}_k$  be the projection of  $\tilde{v}_k$  onto the space orthogonal to all previously selected (true and false) covariates. Then

$$-\Delta_k \ell(S_{\text{pre-}w_j}) \geq \frac{1}{n} \tilde{z}'_k \tilde{z}_k \tilde{\theta}_k^2.$$

Furthermore,  $\tilde{z}'_k \tilde{z}_k \geq \varphi_{\min}(m + s_0)(G)^2$ . This is seen by noting that  $\tilde{z}_k$  results in the composition of two projections onto a span of covariates of size bounded by  $m + s_0$ .

This gives

$$\frac{1}{n\tilde{w}'_j \tilde{w}_j} (\tilde{\gamma}'_j \tilde{\theta})^2 \geq \frac{1}{4} \frac{1}{n} \varphi_{\min}(m + s_0)(G)^2 \tilde{\theta}_k^2.$$

Using the fact that  $\tilde{w}'_j \tilde{w}_j \geq 1$  implies that

$$(\tilde{\gamma}'_j \tilde{\theta})^2 / \tilde{\theta}_k^2 \geq \frac{1}{4} \varphi_{\min}(m + s_0)(G)^2.$$

Now suppose no true variables remain when  $j$  is selected, then  $\tilde{w}'_j \tilde{w}_j = \tilde{u}'_j \tilde{u}_j = 1$ . Therefore,

$$-\Delta_j \ell(S_{\text{pre-}w_j}) = \frac{1}{n} \tilde{\gamma}_{j\tilde{\varepsilon}}^2 \tilde{\theta}_{\tilde{\varepsilon}}^2 \geq t$$

Note that  $\tilde{\theta}$  is given by  $\tilde{\theta}_{\tilde{\varepsilon}} = \tilde{\varepsilon}' y = \varepsilon' \mathcal{M}_{s_0} y / (\varepsilon' \mathcal{M}_{s_0} \varepsilon)^{1/2} = (\varepsilon' \mathcal{M}_{s_0} \varepsilon)^{1/2}$ . Therefore,

$$\tilde{\gamma}_{j\tilde{\varepsilon}}^2 \geq t \frac{n}{\varepsilon' \mathcal{M}_{s_0} \varepsilon}.$$

This implies that  $j \in A_2$ . Therefore, set  $C_1 = \frac{1}{2} \varphi_{\min}(m + s_0)(G)$ .

Next, construct  $C_2$ . For each selected true covariate,  $v_k$ , set  $S_{\text{pre-}v_k}$  to be the set of (both true and false) covariates selected prior to  $v_k$ . Note that

$$\tilde{\theta}_k^2 = -\Delta_k \ell(\{v_1, \dots, v_{k-1}\}) \geq -\Delta_k \ell(S_{\text{pre-}v_k})$$

since  $\{v_1, \dots, v_{k-1}\} \subseteq S_{\text{pre-}v_k}$ . In addition, if  $v_k$  is selected before  $v_l$  (or  $v_l$  is not selected), then

$$-\Delta_k \ell(S_{\text{pre-}v_k}) \geq -\Delta_l \ell(S_{\text{pre-}v_k}) \geq \tilde{z}'_l \tilde{z}_l \tilde{\theta}_l^2 \geq \varphi_{\min}(\hat{s} + s_0)(G)^2 \tilde{\theta}_l^2.$$

Therefore, taking

$$C_2 = \varphi_{\min}(m + s_0)(G)$$

implies that  $\tilde{\theta}_k / \tilde{\theta}_l \geq C_2$  for any  $l > k$ .

### Step 6

In this section, the number of elements of  $A_2$  is bounded. Recall that the criteria for  $j \in A_2$  is that  $|\tilde{\gamma}_{j\tilde{\varepsilon}}| > \frac{t^{1/2}n^{1/2}}{(2\varepsilon'\mathcal{M}_{s_0}\varepsilon)^{1/2}}$ . Note also that  $\tilde{\gamma}_{j\tilde{\varepsilon}}$  is found by the coefficient in the expression

$$\tilde{\gamma}_{j\tilde{\varepsilon}} = \tilde{\varepsilon}'\tilde{w}_j = \varepsilon' \frac{1}{(\varepsilon'\mathcal{M}_{s_0}\varepsilon)^{1/2}} \mathcal{M}_{s_0}\tilde{w}_j$$

Next, let  $H$  be the matrix  $H = [v_1, \dots, v_{s_0}, w_1, \dots, w_m]$ . Note that

$$\frac{1}{(\varepsilon'\mathcal{M}_{s_0}\varepsilon)^{1/2}} \mathcal{M}_{s_0}\tilde{w}_j \in \text{span}(H)$$

Which implies that the above expression is unchanged when premultiplied by  $H(H'H)^{-1}H'$ . Therefore,

$$\tilde{\gamma}_{j\tilde{\varepsilon}} = \varepsilon' H(H'H)^{-1}H' \frac{1}{(\varepsilon'\mathcal{M}_{s_0}\varepsilon)^{1/2}} \mathcal{M}_{s_0}\tilde{w}_j.$$

Let  $\mu_j$  be the  $+1$  for each  $j \in A_2$  such that  $\tilde{\gamma}_{j\tilde{\varepsilon}} > 0$  and  $-1$  for each  $j \in A_2$  such that  $\tilde{\gamma}_{j\tilde{\varepsilon}} < 0$ . By the fact that  $j \in A_2$ ,  $\tilde{\gamma}_{j\tilde{\varepsilon}}\mu_j > \frac{t^{1/2}n^{1/2}}{(2\varepsilon'\mathcal{M}_{s_0}\varepsilon)^{1/2}}$ , summing over  $j \in A_2$  gives

$$\sum_{j \in A_2} \varepsilon' H(H'H)^{-1}H' \frac{1}{(\varepsilon'\mathcal{M}_{s_0}\varepsilon)^{1/2}} \mathcal{M}_{s_0}\tilde{w}_j\mu_j > m_2 \frac{t^{1/2}n^{1/2}}{(2\varepsilon'\mathcal{M}_{s_0}\varepsilon)^{1/2}}$$

This implies that

$$\left\| (H'H)^{-1}H' \frac{1}{(\varepsilon'\mathcal{M}_{s_0}\varepsilon)^{1/2}} \sum_{j \in A_2} \mathcal{M}_{s_0}\tilde{w}_j\mu_j \right\|_1 \|\varepsilon'H\|_\infty > m_2 \frac{t^{1/2}n^{1/2}}{(2\varepsilon'\mathcal{M}_{s_0}\varepsilon)^{1/2}}$$

Which further implies that

$$\sqrt{m + s_0} \left\| (H'H)^{-1}H' \frac{1}{(\varepsilon'\mathcal{M}_{s_0}\varepsilon)^{1/2}} \sum_{j \in A_2} \mathcal{M}_{s_0}\tilde{w}_j\mu_j \right\|_2 \|\varepsilon'H\|_\infty > m_2 \frac{t^{1/2}n^{1/2}}{(2\varepsilon'\mathcal{M}_{s_0}\varepsilon)^{1/2}}$$

Next, further upper bound the  $\|\cdot\|_2$  term on the left side above by

$$\left\| (H'H)^{-1}H' \frac{1}{(\varepsilon'\mathcal{M}_{s_0}\varepsilon)^{1/2}} \sum_{j \in A_2} \mathcal{M}_{s_0}\tilde{w}_j\mu_j \right\|_2$$

$$\leq \frac{n^{-1/2}}{(\varepsilon' \mathcal{M}_{s_0} \varepsilon)^{1/2}} \varphi_{\min}(s_0 + m)(G)^{-1/2} \|\mathcal{M}_{s_0} \sum_{j \in A_2} \tilde{w}_j \mu_j\|_2$$

next, by the fact that  $\mathcal{M}_{s_0}$  is a projection (hence non-expansive) and  $\tilde{w}_j$  are mutually orthogonal,

$$\leq \frac{n^{-1/2}}{(\varepsilon' \mathcal{M}_{s_0} \varepsilon)^{1/2}} \varphi_{\min}(s_0 + m)(G)^{-1/2} \sqrt{\sum_{j \in A_2} \|\tilde{w}_j \mu_j\|_2^2}.$$

In Section 3, it was shown that  $\max_j \|\tilde{w}_j\|_2^2 \leq \varphi_{\min}(s_0 + m)(G)^{-1}$ . Therefore, putting the above inequalities together,

$$\frac{n^{-1/2}}{(\varepsilon' \mathcal{M}_{s_0} \varepsilon)^{1/2}} \sqrt{m + s_0} \varphi_{\min}(m + s_0)(G)^{-1} \sqrt{m_2} \|\varepsilon' H\|_{\infty} > m_2 \frac{t^{1/2} n^{1/2}}{(2\varepsilon' \mathcal{M}_{s_0} \varepsilon)^{1/2}}.$$

This implies that

$$\begin{aligned} m_2 &< \frac{1}{n} \frac{2}{t} (\varepsilon' \mathcal{M}_{s_0} \varepsilon)(m + s_0) \frac{\|\varepsilon' H\|_{\infty}^2}{\varepsilon' \mathcal{M}_{s_0} \varepsilon} \varphi_{\min}(m + s_0)(G)^{-2} \\ &\leq 2(m + s_0) \frac{\|\mathbb{E}_n[x_i \varepsilon_i]\|_{\infty}^2}{t} \varphi_{\min}(m + s_0)(G)^{-2}. \end{aligned}$$

Under the assumed condition that  $t^{1/2} \geq 2\|\mathbb{E}_n[x_i \varepsilon_i]\|_{\infty} \varphi_{\min}(m + s_0)(G)^{-1}$ , it follows that

$$m_2 \leq \frac{1}{2}(m + s_0).$$

By substituting  $m = m_1 + m_2$  gives  $m_2 \leq m_1 + s_0$ .

### Step 7

This section concludes the proof of the second statement of the theorem by bringing together all of the facts proven in Steps 3-6. Combining  $m_1 \leq \varphi_{\min}(m + s_0)(G)^{-1} C_1^{-2} C_3^2 (K_G^{\mathbb{R}})^2 s_0$  and  $m_2 \leq m_1 + s_0$  gives

$$m \leq \left[ 2\varphi_{\min}(m + s_0)(G)^{-1} C_1^{-2} C_3^2 K_G^{\mathbb{R}^2} + 1 \right] s_0.$$

In addition,

$$\begin{aligned} C_1 &= \frac{1}{2} \varphi_{\min}(m + s_0)(G), \\ C_2 &= \varphi_{\min}(m + s_0)(G), \\ C_3 &= (1 + \varphi_{\min}(m + s_0)(G)^{-1/2} + \varphi_{\min}(m + s_0)(G)^{-1}), \end{aligned}$$

and  $K_G^{\mathbb{R}} < 1.783$ . Therefore,

$$\begin{aligned} m &\leq \left[ 1 + 8 \times 1.783^2 \times \varphi_{\min}(m + s_0)(G)^{-3} \right. \\ &\quad \left. \times (1 + \varphi_{\min}(m + s_0)(G)^{-1/2} + \varphi_{\min}(m + s_0)(G)^{-1})^2 \right] s_0. \end{aligned}$$



Since  $C_3^2 \leq 9\varphi_{\min}(s_0 + m)(G)^{-2}$ , the expression above can be simplified at the expense of a slightly less tight constant, so that

$$m \leq [1 + 72 \times 1.783^2 \times \varphi_{\min}(m + s_0)(G)^{-5}]s_0.$$

Since this bound holds for each positive integer  $m$  of wrong selections, this concludes the proof of Theorem 1.  $\square$

## 5. Proof of Theorems 2 and 3

Theorem 2 follows by applying Theorem 1 in the following way. If  $\hat{s}$  grows faster than  $s_0$ , then there is  $m < \hat{s}$  such that  $s_0 < m < K_n$  and  $m/s_0$  exceeds  $\mathcal{C}_2(K_n) = O(1)$ , giving a contradiction. The first statement of the theorem follows from applying the bound on  $\hat{s}$ . Theorem 3 follows by  $\|\theta_0 - \hat{\theta}\|_1 \leq \sqrt{\hat{s} + s_0}\|\theta_0 - \hat{\theta}\|_2 \leq \sqrt{\hat{s} + s_0}\varphi_{\min}(\hat{s} + s_0)(G_x)^{-1}\mathbb{E}_n[(x_i'\theta_0 - x_i'\hat{\theta})^2]^{1/2}$ .

## 6. Conclusion

This paper proved convergence rates for forward selection. The rates in prediction error norm match typical rates encountered with other high-dimensional estimation techniques like Lasso. The results are derived under minimal conditions which do not require  $\beta$ -min assumptions or irrepresentability.

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