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On the wellposedness of the defocusing mKdV equation below L^2

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Abstract

We prove that the renormalized defocusing mKdV equation on the circle is locally in time C^0 -wellposed on the Fourier Lebesgue space \mathcal{FL}^p for any $2 < p < \infty$. The result implies that the defocusing mKdV equation itself is illposed on these spaces since the renormalizing phase factor becomes infinite. The proof is based on the fact that the mKdV equation is an integrable PDE whose Hamiltonian is in the NLS hierarchy. A key ingredient is a novel way of representing the bi-infinite sequence of frequencies of the renormalized defocusing mKdV equation, allowing to analytically extend them to \mathcal{FL}^p for any $2 \leq p < \infty$ and to deduce asymptotics for $n \rightarrow \pm\infty$.

Keywords. mKdV equation, frequency map, well-posedness, ill-posedness

2000 AMS Subject Classification. 37K10 (primary) 35Q53, 35D05 (secondary)

1 Introduction

Consider the defocusing modified Korteweg-de Vries (mKdV) equation on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

$$\partial_t u = -\partial_x^3 u + 6u^2 \partial_x u \quad (1)$$

and its renormalized version

$$\partial_t u = -\partial_x^3 u + 6\left(u^2 - \int_0^1 u^2 dx\right) \partial_x u \quad (2)$$

referred to as the mKdV_# equation.

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According to [11], for any initial datum $u \in H^s \equiv H^s(\mathbb{T}, \mathbb{R})$ with $s \geq 0$, there exists a unique, global in time solution $u(t, x) = u(t, x, q)$ of (1), $u \in C(\mathbb{R}, H^s)$. In particular, for any time $t \in \mathbb{R}$ and $T > 0$, the nonlinear evolution operator

$$\mathcal{S}^t = \mathcal{S}(t, \cdot): H^s \rightarrow H^s, \quad v \mapsto u(t, \cdot, v),$$

and the uniquely defined solution map

$$\mathcal{S}: H^s \rightarrow C([-T, T], H^s), \quad v \mapsto u(\cdot, \cdot, v),$$

are well defined and continuous.

To investigate the wellposedness of mKdV and mKdV_# below L^2 , we introduce for any $1 \leq p \leq \infty$ the *Fourier Lebesgue spaces* $\mathcal{F}\ell^p$ consisting of 1-periodic distributions whose Fourier coefficients are in ℓ^p

$$\mathcal{F}\ell^p = \{v \in S'(\mathbb{T}, \mathbb{R}) : (\hat{v}_n) \in \ell^p\}, \quad \|v\|_{\mathcal{F}\ell^p} := \|(\hat{v})\|_{\ell^p},$$

where \hat{v}_n denotes the n th Fourier coefficient of v in $S'(\mathbb{T}, \mathbb{R})$.

For any $1 \leq p \leq \infty$, the space $\mathcal{F}\ell^p$ is a real Banach space. Moreover, for any $2 < p < \infty$, we have the following continuous embeddings

$$\mathcal{F}\ell^1 \hookrightarrow C^0 \hookrightarrow L^2 = \mathcal{F}\ell^2 \hookrightarrow L^{p'} \hookrightarrow \left\{ \begin{array}{l} L^1 \hookrightarrow \mathcal{M} \\ L^{p'} \hookrightarrow \mathcal{F}\ell^p \end{array} \right\} \hookrightarrow \mathcal{F}\ell^\infty,$$

where p' denotes the Lebesgue exponent conjugated to p and \mathcal{M} denotes the space of finite Borel measures on \mathbb{T} . The space $\mathcal{F}\ell^1$ is called the *Wiener algebra* and $\mathcal{F}\ell^\infty$ is the space of *pseudo measures*. We point out that for $2 < p < \infty$, the space $\mathcal{F}\ell^p$ is much larger than $L^{p'}$. In particular, it contains elements which are not measures but more singular distributions.

A continuous curve $\gamma: (a, b) \rightarrow \mathcal{F}\ell^p$, $\gamma(0) = v$, is called a *solution* of an equation such as the mKdV equation in $\mathcal{F}\ell^p$ with initial datum v if and only if for any sequence of C^∞ -potentials $(v_k)_{k \geq 1}$ converging to v in $\mathcal{F}\ell^p$, the corresponding sequence $(\mathcal{S}(t, v_k))_{k \geq 1}$ of solutions of (1) with initial data v_k converges to $\gamma(t)$ in $\mathcal{F}\ell^p$ for any $t \in (a, b)$. mKdV is said to be *locally in time C^0 -wellposed* if for any initial datum $w \in \mathcal{F}\ell^p$ there exists a neighborhood U and a time $T > 0$ so that the initial value problem (1) for any initial value $v \in U$ admits a solution $\mathcal{S}(\cdot, v)$ in the aforementioned sense which is defined on the time interval $[-T, T]$, and the solution map $\mathcal{S}: U \rightarrow C^0([-T, T], \mathcal{F}\ell^p)$ is continuous. mKdV is said to be *globally in time C^0 -wellposed on an open subset $U \subset \mathcal{F}\ell^p$* if for any initial datum $v \in U$ the initial value problem (1) admits a solution $\mathcal{S}(\cdot, v)$ in the aforementioned sense which is globally defined in time and the solution map $\mathcal{S}: U \rightarrow C([-T, T], \mathcal{F}\ell^p)$ is continuous for every $T > 0$. mKdV is said to be *globally in time (uniformly/ C^k / C^ω) wellposed on $U \subset \mathcal{F}\ell^p$* if the solution map $\mathcal{S}: U \rightarrow C([-T, T], \mathcal{F}\ell^p)$ is (uniformly continuous/ C^k / C^ω) for every $T > 0$.

Theorem 1.1 *The mKdV_# equation (2) is locally in time C^0 -wellposed in \mathcal{F}^p for any $2 < p < \infty$ and it is globally in time C^0 -wellposed in \mathcal{F}^p in a neighborhood of 0. As a consequence, for any $2 < p < \infty$, the solution map \mathcal{S} of the mKdV equation does not extend continuously to initial data in $\mathcal{F}^p \setminus L^2$.*

×

Our method also allows to prove partial corresponding results for the focusing modified Korteweg-de Vries (mKdV⁻) equation on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

$$\partial_t u = -\partial_x^3 u - 6u^2 \partial_x u$$

and its renormalized version

$$\partial_t u = -\partial_x^3 u - 6 \left(u^2 - \int_0^1 u^2 dx \right) \partial_x u$$

referred to as mKdV_#⁻ equation. More precisely, for any $2 < p < \infty$, the mKdV_#⁻ equation is globally in time C^0 -wellposed in \mathcal{F}^p for initial data in a neighborhood of 0, whereas the solution map of the mKdV⁻ equation does not extend continuously to small initial data in $\mathcal{F}^p \setminus L^2$ for any $2 < p < \infty$.

Finally, using the Miura map one can deduce from corresponding results in [9] for the KdV equation that for any $0 < s < 1/2$, the mKdV equation is nowhere locally uniformly C^0 -wellposed on the submanifold

$$\left\{ v \in H^s(\mathbb{T}) : \int_{\mathbb{T}} v^2 dx = d \right\}$$

for any $d > 0$.

Related results The wellposedness of the mKdV equation on \mathbb{T} has been extensively studied - cf. e.g. [12, Introduction 1] for an account on the many results obtained so far. In particular, based on the seminal work of Bourgain [1], it was shown in [4] that the mKdV equation is globally uniformly C^0 -wellposed on the submanifolds

$$M_d^s = \left\{ v \in H^s(\mathbb{T}) : \int_{\mathbb{T}} v^2 dx = d \right\}$$

with $s \geq 1/2$ and $d > 0$ arbitrary. In [11] it was proved that the mKdV equation is globally C^0 -wellposed in $H^s(\mathbb{T})$ for any $s \geq 0$. The solutions constructed in [11] are shown to be weak (in an appropriate sense) by [12] (cf. also [14, 16] for related results). See also [7, 15] for results on the local wellposedness of the mKdV equation in Fourier Lebesgue spaces of higher regularity. On the other hand, it was pointed out in [2, Section 6] that the mKdV equation is not C^3 -wellposed in $H^s(\mathbb{T})$ for $s < 1/2$, whereas in [3] it is proved that it is not uniformly C^0 -wellposed in $H^s(\mathbb{T})$ for $-1 < s < 1/2$. More recently, Molinet [12] proved that the mKdV equation is ill-posed on $H^s(\mathbb{T})$ for any

$s < 0$ in the sense that for any $T > 0$, the solution map $(C^\infty(\mathbb{T}), \|\cdot\|_s) \rightarrow \mathcal{D}'((0, T) \times \mathbb{T})$ is discontinuous at any nonconstant initial datum $v \in H^\infty(\mathbb{T})$. Here, $(C^\infty(\mathbb{T}), \|\cdot\|_s)$ denotes the vector space $C^\infty(\mathbb{T})$ endowed with the Sobolev norm $\|\cdot\|_s$.

Method of proof The mKdV equation is closely related to the *NLS system*

$$\begin{aligned} i\partial_t \varphi_- &= \partial_{\varphi_+} H = -\partial_{xx} \varphi_- + 2\varphi_+ \varphi_-^2, \\ i\partial_t \varphi_+ &= -\partial_{\varphi_-} H = \partial_{xx} \varphi_+ - 2\varphi_- \varphi_+^2. \end{aligned} \quad (3)$$

This system can be viewed as a Hamiltonian PDE with Hamiltonian

$$H_{NLS}(\varphi) = \int_{\mathbb{T}} (\partial_x \varphi_+ \partial_x \varphi_- + \varphi_+^2 \varphi_-^2) dx \quad (4)$$

on the phase space $H_c^s := H^s(\mathbb{T}, \mathbb{C}) \times H^s(\mathbb{T}, \mathbb{C})$, $s \geq 0$, with Poisson bracket

$$\{F, G\} := -i \int_{\mathbb{T}} (\partial_{\varphi_-} F \partial_{\varphi_+} G - \partial_{\varphi_+} F \partial_{\varphi_-} G) dx. \quad (5)$$

Here φ_- , φ_+ denote the two components of $\varphi = (\varphi_-, \varphi_+) \in H_c^s$, and $\partial_{\varphi_-} F$, $\partial_{\varphi_+} F$ denote the two components of the L^2 -gradient $\partial F = (\partial_{\varphi_-} F, \partial_{\varphi_+} F)$ of a C^1 -functional F on H_c^s .

The Hamiltonian $H_{NLS}(\varphi)$ admits for any $s \geq 0$ the invariant real subspaces

$$H_r^s := \{\varphi \in H_c^s : \varphi_+ = \overline{\varphi_-}\}, \quad H_i^s := \{\varphi \in H_c^s : \varphi_+ = -\overline{\varphi_-}\}.$$

Elements of H_r^s are called potentials of *real type*.

When (3) is restricted to H_r^s , with $\varphi = (v, \bar{v})$, one obtains the *defocusing NLS* (dNLS) equation

$$i\partial_t v = i\{v, H_{NLS}\} = -\partial_{xx} v + 2|v|^2 v, \quad H_{NLS}(v, \bar{v}) = \int_{\mathbb{T}} (|v_x|^2 + |v|^4) dx.$$

Similarly, when (3) is restricted to H_i^m , with $\varphi = (iv, i\bar{v})$, one obtains the *focusing NLS* equation

$$i\partial_t v = i\{v, H_{NLS}\} = -\partial_{xx} v - 2|v|^2 v, \quad H_{NLS}(iv, i\bar{v}) = - \int_{\mathbb{T}} (|v_x|^2 - |v|^4) dx.$$

The NLS system (3) admits an infinite sequence of recursively defined pairwise Poisson commuting integrals referred to as *NLS hierarchy*,

$$\begin{aligned} H_1(\varphi) &= \int_{\mathbb{T}} \varphi_- \varphi_+ dx, \\ H_2(\varphi) &= \frac{i}{2} \int_{\mathbb{T}} (\varphi_+ \partial_x \varphi_- - \varphi_- \partial_x \varphi_+) dx, \\ H_{NLS}(\varphi) = H_3(\varphi) &= \int_{\mathbb{T}} (\partial_x \varphi_- \partial_x \varphi_+ + \varphi_-^2 \varphi_+^2) dx, \\ H_4(\varphi) &= i \int_{\mathbb{T}} (\varphi_- \partial_{xxx} \varphi_+ - 3\varphi_-^2 \varphi_+ \partial_x \varphi_+) dx, \quad \dots \end{aligned}$$

In comparison with [6], the n th Hamiltonian of the NLS hierarchy (for $n \geq 2$) is multiplied by $(-i)^{n+1}$ to make the corresponding Hamiltonian flow real-valued for real-valued φ .

The Hamiltonian H_4 gives rise to the *mKdV system*

$$\begin{aligned}\partial_t \varphi_- &= \{\varphi_-, H_4\} = -i \partial_{\varphi_+} H_4 = -\partial_{xxx} \varphi_- + 6\varphi_- \varphi_+ \partial_x \varphi_-, \\ \partial_t \varphi_+ &= \{\varphi_+, H_4\} = i \partial_{\varphi_-} H_4 = -\partial_{xxx} \varphi_+ + 6\varphi_+ \varphi_- \partial_x \varphi_+.\end{aligned}\tag{6}$$

This system admits, for any $s \geq 0$, the real invariant subspaces

$$\mathcal{E}_r^s := \{\varphi \in H_r^s : \varphi_+ = \varphi_-\}, \quad \mathcal{E}_i^s := \{\varphi \in H_i^s : \varphi_+ = \varphi_-\}.$$

When (6) is restricted to \mathcal{E}_r^s , with $\varphi = (u, u)$ and u real-valued, one obtains the defocusing mKdV equation

$$\partial_t u = -i \partial_{\varphi_+} H_4|_{(u,u)} = -\partial_{xxx} u + 6u^2 \partial_x u.$$

Similarly, when (6) is restricted to \mathcal{E}_i^s , with $\varphi = (iu, iu)$ and u real-valued, one obtains the focusing mKdV equation

$$i \partial_t u = -i \partial_{\varphi_+} H_4|_{(iu,iu)} = i(-\partial_{xxx} u - 6u^2 \partial_x u)$$

or $\partial_t u = -\partial_{xxx} u - 6u^2 \partial_x u$.

The NLS hierarchy is completely integrable in the strongest possible sense meaning that by [6] it admits global Birkhoff coordinates on H_r^s for $s \geq 0$. They can be extended to the Fourier Lebesgue spaces. To give a precise statement, we introduce for any $2 \leq p < \infty$ the model space

$$\ell_r^p := \{(z_-, z_+) \in \ell_c^p : z_+ = \overline{z_-}\}, \quad \ell_c^p := \ell^p(\mathbb{Z}, \mathbb{C}) \times \ell^p(\mathbb{Z}, \mathbb{C}),$$

and the phase space

$$\mathcal{F}\ell_r^p := \{\varphi = (\varphi_-, \varphi_+) \in \mathcal{F}\ell_c^p : \varphi_+ = \overline{\varphi_-}\}, \quad \mathcal{F}\ell_c^p := \mathcal{F}\ell^p(\mathbb{T}, \mathbb{C}) \times \mathcal{F}\ell^p(\mathbb{T}, \mathbb{C}).$$

Theorem 1.2 *For any $2 \leq p < \infty$, there exists a complex neighborhood $\mathcal{W}^p \subset \mathcal{F}\ell_c^p$ of $\mathcal{F}\ell_r^p$ and an analytic map $\Phi_p: \mathcal{W}^p \rightarrow \ell_c^p$ with $\Phi_p(0) = 0$ so that the following holds:*

- (i) $\Phi_p: \mathcal{W}^p \rightarrow \ell_c^p$ is real analytic,
- (ii) for any $2 \leq p, q < \infty$, the maps Φ_p and Φ_q coincide on $\mathcal{W}^p \cap \mathcal{W}^q$,
- (iii) Φ_p is canonical,
- (iv) the map $\Phi_p: \mathcal{F}\ell_r^p \rightarrow \ell_r^p$ is one-to-one, a local diffeomorphism at every point of $\mathcal{F}\ell_r^p$, and the image of this map is open and dense in ℓ_r^p .
- (v) If $p = 2$, then the map Φ_p coincides with the one constructed in [6]. In particular, it is onto and hence a bi-real analytic diffeomorphism. \times

For $1 \leq q \leq \infty$ and $s \in \mathbb{R}$, let $\ell_+^{s,q}$ denote the positive quadrant of $\ell^{s,q}(\mathbb{Z}, \mathbb{R})$ given by

$$\ell_+^{s,q} = \{I = (I_n)_{n \in \mathbb{Z}} \in \ell^{s,q}(\mathbb{Z}, \mathbb{R}) : I_n \geq 0\}. \quad (7)$$

The Hamiltonian H_4 is a real analytic function of φ on $H_r^{3/2}$ and hence a real analytic function of the actions on $\ell_+^{3,1}$ – see [6]. The corresponding frequencies

$$\omega_n^{(4)} := \partial_{I_n} H_4, \quad n \in \mathbb{Z},$$

thus are real analytic functions of the actions on $\ell_+^{3,1}$.

The equations of motion of the system (6), when expressed in Birkhoff coordinates on $\ell_r^{s,2}$ with $s \geq 3/2$, are

$$\partial_t z_n^- = -i\omega_n^{(4)} z_n^-, \quad \partial_t z_n^+ = i\omega_n^{(4)} z_n^+, \quad n \in \mathbb{Z}.$$

On $\ell_+^{3,1}$ the frequencies $\omega_n^{(4)}$ have an asymptotic expansion for $|n| \rightarrow \infty$ of the form

$$\omega_n^{(4)} = (2n\pi)^3 + 6H_2 + 12n\pi H_1 + O(n^{-1}) \quad (8)$$

where according to [5, 6]

$$H_1 = \sum_{n \in \mathbb{Z}} I_n, \quad H_2 = \sum_{n \in \mathbb{Z}} (2n\pi) I_n.$$

In order to state our results on the analytic extensions of $\omega_n^{(4)}$, $n \in \mathbb{Z}$, we need to normalize the frequencies as follows

$$\omega_n^{(4)*} = \omega_n^{(4)} - (2n\pi)^3 - 6H_2 - 12n\pi H_1. \quad (9)$$

By a slight abuse of notation, in the sequel, we will often view the frequencies as functions of the potential. From the considerations above it follows that $\omega^{(4)*} = (\omega_n^{(4)*})_{n \in \mathbb{Z}} : H_r^{3/2} \rightarrow \ell_{\mathbb{R}}^{-3,\infty}$, referred to as the frequency map, is real analytic.

Theorem 1.3 *The map $\omega^{(4)*}$ is defined on $H_r^0 = \mathcal{F}\ell_r^2$, takes values in $\bigcap_{r>1} \ell^{-1,r}$, and $\omega^{(4)*} : \mathcal{F}\ell_r^2 \rightarrow \ell^{-1,r}$ is real analytic for any $r > 1$. Moreover, for any $p > 2$ the map $\omega^{(4)*}$ admits a real analytic extension $\omega^{(4)*} : \mathcal{F}\ell_r^p \rightarrow \ell_{\mathbb{C}}^{-1,p/2}$ with asymptotics*

$$\omega_n^{(4)*} + 12n\pi I_n = n(\ell_n^{p/3} + \ell_n^{1+}),$$

which hold locally uniformly on $\mathcal{F}\ell_r^p$. Here $\ell_n^{p/3}$ denotes a generic sequence $(a_m)_{m \in \mathbb{Z}}$ with $\sum_{m \in \mathbb{Z}} |a_m|^{p/3} < \infty$ and ℓ_n^{1+} denotes a generic sequence $(b_m)_{m \in \mathbb{Z}}$ with $\sum_{m \in \mathbb{Z}} |b_m|^r < \infty$ for any $r > 1$. \times

Theorem 1.1 then follows by observing that the frequencies $\omega_n^\#$ of the mKdV $_\#$ equation are obtained from the restriction of $\omega_n^{(4)*}$ to $\mathcal{E}_r^p := \{\varphi = (u, u) \in \mathcal{F}\ell_r^p\}$ as follows,

$$\omega_n^\# := (2n\pi)^3 + \omega_n^{(4)*} \Big|_{\mathcal{E}_r^p}.$$

See Section 7 for details.

2 Preliminaries

Let us briefly recall the main properties of the Lax-pair formulation of the NLS system. For a potential $\varphi = (\varphi_-, \varphi_+) \in \mathcal{F}\ell_c^p$, $2 \leq p < \infty$, consider the Zakharov-Shabat operator

$$L(\varphi) := \begin{pmatrix} i & \\ & -i \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} & \varphi_- \\ \varphi_+ & \end{pmatrix}$$

on the interval $[0, 2]$ with periodic boundary conditions. The spectral theory of this operator is well known in the case $p = 2$ and has been extended to the case $p > 2$ in [13]. One can show that the spectrum is discrete, and consists of a sequence of pairs of complex eigenvalues $\lambda_n^+(\varphi)$ and $\lambda_n^-(\varphi)$, when listed with algebraic multiplicities, such that

$$\lambda_n^\pm(\varphi) = n\pi + \ell_n^p, \quad n \in \mathbb{Z}.$$

Here ℓ_n^p , $n \in \mathbb{Z}$, denotes a generic ℓ^p -sequence. We may order the eigenvalues lexicographically – first by their real part, and second by their imaginary part – to represent them as a bi-infinite sequence of complex eigenvalues

$$\cdots \preceq \lambda_{n-1}^+ \preceq \lambda_n^- \preceq \lambda_n^+ \preceq \lambda_{n+1}^- \preceq \cdots.$$

By a slight abuse of terminology, we call the periodic eigenvalues of $L(\varphi)$ the periodic spectrum of φ . Further we introduce the gap lengths

$$\gamma_n(\varphi) := \lambda_n^+ - \lambda_n^- = \ell_n^p, \quad n \in \mathbb{Z},$$

and the mid points

$$\tau_n(\varphi) := \frac{\lambda_n^+ + \lambda_n^-}{2} = n\pi + \ell_n^p, \quad n \in \mathbb{Z}.$$

In the case of an L_c^2 potential, the periodic spectrum can be described in terms of the fundamental solution of $L(\varphi)$ which is denoted by $M(x, \lambda, \varphi)$. In more detail, when one introduces the discriminant

$$\Delta(\lambda, \varphi) := \text{tr } M(1, \lambda, \varphi),$$

then the periodic spectrum of φ is precisely the zero set of the entire function $\Delta^2(\lambda, \varphi) - 4$, and we have the product representation

$$\Delta^2(\lambda, \varphi) - 4 = -4 \prod_{n \in \mathbb{Z}} \frac{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}{\pi_n^2}, \quad \pi_n := \begin{cases} n\pi, & n \neq 0, \\ 1, & n = 0. \end{cases}$$

Further, $\Delta(\lambda_{2n}^+, \varphi) = \Delta(\lambda_{2n}^-, \varphi) = 2$, $\Delta(\lambda_{2n+1}^+, \varphi) = \Delta(\lambda_{2n+1}^-, \varphi) = -2$, for any $n \in \mathbb{Z}$, and

$$\Delta(\lambda, \varphi) - 2 = - \prod_{n \in \mathbb{Z}} \frac{(\lambda_{2n}^+ - \lambda)(\lambda_{2n}^- - \lambda)}{\pi_{2n}^2}. \quad (10)$$

In the case of an $\mathcal{F}\ell_c^p$ potential with $2 < p < \infty$, we *define* the discriminant by (10) so that we do not need to introduce the fundamental solution for such potentials. One can further show that in this case the λ -derivative $\Delta^\bullet := \partial_\lambda \Delta$, whose zeros are denoted by λ_n^\bullet , $n \in \mathbb{Z}$, and are listed so that they have the asymptotics $\lambda_n^\bullet = n\pi + \ell_n^p$, admits the product representation

$$\Delta^\bullet(\lambda, \varphi) = 2 \prod_{n \in \mathbb{Z}} \frac{\lambda_n^\bullet - \lambda}{\pi_n},$$

as is known from the L_c^2 case. The asymptotics of λ_n^\bullet and τ_n imply that $\lambda_n^\bullet - \tau_n = \ell_n^p$. As it turns out, λ_n^\bullet is closer even to τ_n , namely one has

$$\lambda_n^\bullet = \tau_n + \gamma_n^2 \ell_n^p, \quad (11)$$

locally uniformly on $\mathcal{F}\ell_c^p$ for any $2 \leq p < \infty$.

We also consider the Dirichlet spectrum of the Zakharov-Shabat operator – see [6] for the definition in the case $p = 2$ and [13] for an extension to the case $2 < p < \infty$. This spectrum, referred to as the *Dirichlet spectrum of φ* , is discrete and given by a sequence of eigenvalues $(\mu_n)_{n \in \mathbb{Z}}$, counted with multiplicities and ordered lexicographically so that

$$\cdots \preceq \mu_{n-1} \preceq \mu_n \preceq \mu_{n+1} \preceq \cdots, \quad \mu_n = n\pi + \ell_n^p.$$

If the potential φ is of real type, that is $\varphi_- = \overline{\varphi_+}$, then periodic and Dirichlet spectra are real, and the lexicographical ordering coincides with the ordering of real numbers

$$\cdots \leq \lambda_{n-1}^+ < \lambda_n^- \leq \lambda_n^\bullet, \mu_n \leq \lambda_n^+ < \lambda_{n+1}^- \leq \cdots.$$

For each $\varphi \in \mathcal{F}\ell_r^p$, there exists an open and connected neighborhood V_φ of φ within $\mathcal{F}\ell_c^p$ and disjoint closed discs $(U_n)_{n \in \mathbb{Z}}$, centered on the real axis, so that the following holds

(S1) for any $n \in \mathbb{Z}$, $G_n := [\lambda_n^-, \lambda_n^+]$, μ_n , and λ_n^\bullet are contained in the interior of U_n for every $\psi \in V_\varphi$;

(S2) there exists a constant $c \geq 1$ such that for all $n, m \in \mathbb{Z}$ with $m \neq n$

$$c^{-1}|m - n| \leq \text{dist}(U_n, U_m) \leq c|m - n|; \quad (12)$$

(S3) there exists an integer $N_\varphi \geq 1$ so that

$$U_n = D_n := \{\lambda \in \mathbb{C} : |\lambda - n\pi| \leq \pi/5\}, \quad |n| \geq N_\varphi. \quad (13)$$

Such discs $(U_n)_{n \in \mathbb{Z}}$ are called isolating neighborhoods in the sequel. The union of all V_φ , $\varphi \in \mathcal{F}\ell_r^p$, defines an open and connected neighborhood of $\mathcal{F}\ell_r^p$ within

$\mathcal{F}\ell_c^p$ and is denoted by \mathcal{W}^p . In the sequel, we denote for any $\varphi \in \mathcal{W}^p$ an open and connected neighborhood V_φ satisfying the above conditions.

Following [6], for $\varphi \in \mathcal{W}^p$, one can define action variables for the NLS equation by

$$I_n = \frac{1}{\pi} \int_{\Gamma_n} \frac{\lambda \Delta^\bullet(\lambda)}{\sqrt[{\epsilon}]{\Delta^2(\lambda) - 4}} d\lambda, \quad n \in \mathbb{Z}. \quad (14)$$

Here $\Gamma_n \subset U_n$ denotes any counter clockwise oriented circuit around and sufficiently close to G_n , and the *canonical root* $\sqrt[{\epsilon}]{\Delta^2(\lambda) - 4}$ is defined on $\mathbb{C} \setminus \bigcup_{\gamma_n \neq 0} G_n$, where, for φ of real type, the sign of the root is determined by

$$i \sqrt[{\epsilon}]{\Delta^2(\lambda) - 4} > 0, \quad \lambda_0^+ < \lambda < \lambda_1^- ,$$

and for $\varphi \in \mathcal{W}^p$ it is defined by continuous extension. According to [13, 6], for any given $k \in \mathbb{Z}$ the action I_k is a real analytic function on \mathcal{W}^p .

The Dirichlet eigenvalues and the discriminant can be used to construct the angles $\theta_k(\varphi)$, $k \in \mathbb{Z}$, which are conjugated to the actions $I_n(\varphi)$, $n \in \mathbb{Z}$. The angle θ_k is defined modulo 2π on $\mathcal{W}^p \setminus Z_k$ and is a real analytic function on $\mathcal{W}^p \setminus Z_k$ when considered modulo π , where

$$Z_k \equiv Z_k^p := \{\psi \in \mathcal{W}^p : \gamma_k^2(\psi) = 0\}. \quad (15)$$

Moreover, the following commutator relations hold for any $m, n \in \mathbb{Z}$

$$\{I_m, I_n\} = 0, \quad \{\theta_n, I_m\} = \delta_{nm}, \quad \{\theta_m, \theta_n\} = 0, \quad (16)$$

whenever the bracket is defined.

For any $\varphi \in \mathcal{F}\ell_r^p \setminus Z_k$ with $k \in \mathbb{Z}$ define

$$z_k^-(\varphi) := \sqrt[{\epsilon}]{I_k(\varphi)} e^{-i\theta_k(\varphi)}, \quad z_k^+(\varphi) := \sqrt[{\epsilon}]{I_k(\varphi)} e^{i\theta_k(\varphi)}. \quad (17)$$

(We point out that in [6] the meaning of z_k^\pm is a different one.) It is shown in [13, 6] that the mappings $\mathcal{F}\ell_r^p \setminus Z_k \rightarrow \mathbb{C}$, $\varphi \mapsto z_k^\pm(\varphi)$ analytically extend to a neighborhood \mathcal{W}^p , (after possibly shrinking \mathcal{W}^p if necessary). The *Birkhoff map* is then defined as follows

$$\Phi_p : \mathcal{W}^p \rightarrow \ell_c^p \quad \varphi \mapsto \Phi_p(\varphi) := (z_k^-(\varphi), z_k^+(\varphi))_{k \in \mathbb{Z}} \quad (18)$$

Its main properties are stated in Theorem 1.2.

3 The abelian integral F

It is convenient to define for any $\varphi \in \mathcal{W}^p$, $2 \leq p < \infty$, the *standard roots*

$$w_n(\lambda) = \sqrt[s]{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}, \quad \lambda \in \mathbb{C} \setminus G_n, \quad n \geq 1,$$

by the condition

$$w_n(\lambda) = (\tau_n - \lambda) \sqrt[4]{1 - \gamma_n^2/4(\tau_n - \lambda)^2}, \quad \tau_n = (\lambda_n^- + \lambda_n^+)/2. \quad (19)$$

Here $\sqrt[4]{}$ denotes the principal branch of the square root on the complex plane minus the ray $(-\infty, 0]$. The standard root is analytic in λ on $\mathbb{C} \setminus G_n$ and in (λ, ψ) on $(\mathbb{C} \setminus \overline{U_n}) \times V_\varphi$, and for all $n, m \geq 1$ with $n \neq m$ where V_φ and c are as in (S1)-(S3) above,

$$\inf_{\lambda \in \overline{U_n}} |w_m(\lambda)| \geq c^{-1}|n - m|. \quad (20)$$

If $\gamma_n = 0$, then $w_n(\lambda) = (\tau_n - \lambda)$ is an entire function of λ . On the other hand, if $\gamma_n \neq 0$, then w_n extends continuously to both sides of G_n , denoted by G_n^\pm ,

$$G_n^\pm = \{\lambda_t^\pm = \tau_n + (t \pm i0)\gamma_n/2 : -1 \leq t \leq 1\}, \quad (21)$$

and we have

$$w_n(\lambda_t^\pm) = \mp i \frac{\gamma_n}{2} \sqrt[4]{1 - t^2}, \quad -1 \leq t \leq 1. \quad (22)$$

Lemma 3.1 (i) Suppose $\gamma_n \neq 0$ and f is continuous on G_n , then

$$\sup_{\lambda \in G_n^+ \cup G_n^-} \left| \frac{1}{\pi} \int_{\lambda_n^-}^{\lambda} \frac{f(z)}{w_n(z)} dz \right| \leq \max_{\lambda \in G_n} |f(\lambda)|.$$

(ii) Suppose f is analytic in a neighborhood of G_n containing Γ_n , then

$$\frac{1}{2\pi} \left| \int_{\Gamma_n} \frac{f(\lambda)}{w_n(\lambda)} d\lambda \right| \leq \max_{\lambda \in G_n} |f(\lambda)|.$$

(iii) For any $n, m \in \mathbb{Z}$ we have

$$\frac{1}{2\pi i} \int_{\Gamma_m} \frac{d\lambda}{w_n(\lambda)} = -\delta_{m,n}. \quad \times$$

Proof. (i) We choose the parametrization λ_t^\pm of G_n^\pm , defined in (21), to obtain for $-1 \leq t \leq 1$,

$$\int_{\lambda_n^-}^{\lambda_t^\pm} \frac{f(z)}{w_n(z)} dz = \pm i \int_{-1}^t \frac{f(\lambda_r^\pm)}{\sqrt[4]{1 - r^2}} dr.$$

Since $\int_{-1}^1 \frac{1}{\sqrt[4]{1 - r^2}} dr = \pi$, the claim follows immediately.

(ii) If $\gamma_n = 0$, then $w_n(\lambda) = \tau_n - \lambda$ and the claim follows from Cauchy's theorem. On the other hand, if $\gamma_n \neq 0$, then we may apply the previous lemma.

(iii) First suppose $n \neq m$. Then w_n is analytic on U_m and therefore $\frac{1}{2\pi i} \int_{\Gamma_m} \frac{d\lambda}{w_n(\lambda)} = 0$. Now suppose $m = n$. If $\gamma_n = 0$, then $w_n(\lambda) = \tau_n - \lambda$ and the claim follows from Cauchy's theorem. On the other hand, if $\gamma_n \neq 0$, then in view of (22)

$$\frac{1}{2\pi i} \int_{\Gamma_n} \frac{d\lambda}{w_n(\lambda)} = -\frac{2}{2\pi} \int_{-1}^1 \frac{dt}{\sqrt{1 - t^2}} = -1. \quad \blacksquare$$

The *canonical root* $\sqrt[c]{\Delta^2(\lambda) - 4}$ can be written in terms of standard roots as follows

$$\sqrt[c]{\Delta^2(\lambda) - 4} := 2i \prod_{m \in \mathbb{Z}} \frac{w_m(\lambda)}{\pi_m} \quad (23)$$

and is analytic in λ on $\mathbb{C} \setminus \bigcup_{\gamma_m \neq 0} G_m$ and in (λ, ψ) on $(\mathbb{C} \setminus \bigcup_{m \in \mathbb{Z}} \overline{U_m}) \times V_\varphi$. By Corollary B.2, the quotient

$$\frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} = -i \prod_{m \in \mathbb{Z}} \frac{\lambda_m^\bullet - \lambda}{w_m(\lambda)}, \quad (24)$$

is analytic in (λ, ψ) on $(\mathbb{C} \setminus \bigcup_{m \in \mathbb{Z}} \overline{U_m}) \times V_\varphi$, and analytic in λ on $\mathbb{C} \setminus \bigcup_{\gamma_m \neq 0} G_m$.

We call a path in the complex plane *admissible* for $\varphi \in \mathcal{F}_c^p$ if, except possibly at its endpoints, it does not intersect any gap $G_m(\varphi)$, $m \in \mathbb{Z}$.

Lemma 3.2 *For each $\varphi \in \mathcal{W}^p$, $2 \leq p < \infty$, the following holds:*

(i) *The function $\frac{\Delta^\bullet(\lambda, \psi)}{\sqrt[c]{\Delta^2(\lambda, \psi) - 4}}$ extends analytically to $\mathbb{C} \setminus \bigcup_{\gamma_m \neq 0} G_m$;*

(ii) *for any admissible path from λ_n^- to λ_n^+ in U_n ,*

$$\int_{\lambda_n^-}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda, \psi)}{\sqrt[c]{\Delta^2(\lambda, \psi) - 4}} d\lambda = 0.$$

In particular, for any closed circuit Γ_n in U_n around G_n ,

$$\int_{\Gamma_n} \frac{\Delta^\bullet(\lambda, \psi)}{\sqrt[c]{\Delta^2(\lambda, \psi) - 4}} d\lambda = 0. \quad \times$$

Proof. Standard – see e.g. [13]. ■

By Lemma 3.2, for any $\varphi \in \mathcal{W}^p$, $2 \leq p < \infty$, the quotient (24) is analytic in both variables (λ, ψ) on $(\mathbb{C} \setminus \bigcup_{m \in \mathbb{Z}} \overline{U_m}) \times V_\varphi$ and analytic in λ on $\mathbb{C} \setminus \bigcup_{\gamma_m \neq 0} G_m$. We proceed by defining on the same domain for any $n \in \mathbb{Z}$ the primitive

$$F_n(\lambda, \psi) := \frac{1}{2} \left(\int_{\lambda_n^-(\psi)}^{\lambda} \frac{\Delta^\bullet(\mu, \psi)}{\sqrt[c]{\Delta^2(\mu, \psi) - 4}} d\mu + \int_{\lambda_n^+(\psi)}^{\lambda} \frac{\Delta^\bullet(\mu, \psi)}{\sqrt[c]{\Delta^2(\mu, \psi) - 4}} d\mu \right),$$

where the paths of integration are chosen to be admissible. These improper integrals exist, since for $\gamma_n = 0$ the integrand is analytic on U_n , while for $\gamma_n \neq 0$ it is of the form $1/\sqrt{\lambda_n^\pm - \lambda}$ locally around λ_n^\pm . Moreover, $\int_{\lambda_n^-}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} d\lambda = 0$ by Lemma 3.2, hence the definition of F_n is independent of the chosen admissible path and one has

$$F_n(\lambda) = \int_{\lambda_n^-(\psi)}^{\lambda} \frac{\Delta^\bullet(\mu)}{\sqrt[c]{\Delta^2(\mu) - 4}} d\mu = \int_{\lambda_n^+(\psi)}^{\lambda} \frac{\Delta^\bullet(\mu)}{\sqrt[c]{\Delta^2(\mu) - 4}} d\mu.$$

Even though the eigenvalues λ_n^\pm are, due to their lexicographical ordering, not even continuous on \mathcal{W}^p , the mappings F_n turn out to be analytic.

Lemma 3.3 For every $\varphi \in \mathcal{W}^p$, $2 \leq p < \infty$, the following holds:

(i) F_n is analytic in both variables (λ, ψ) on $(\mathbb{C} \setminus \bigcup_{m \in \mathbb{Z}} \overline{U_m}) \times V_\varphi$ with gradient

$$\partial F_n(\lambda) = \frac{\partial \Delta(\lambda)}{\sqrt[n]{\Delta^2(\lambda) - 4}},$$

and $F_n(\lambda) \equiv F_n(\lambda, \varphi)$ is analytic in λ on $\mathbb{C} \setminus \bigcup_{\gamma_m \neq 0} G_m$.

(ii) $F_0(\lambda) = F_n(\lambda) - in\pi$ on $\mathbb{C} \setminus \bigcup_{\gamma_m \neq 0} G_m$. In particular, F_0 extends continuously to all points λ_m^\pm , $m \in \mathbb{Z}$, and one has

$$F_0(\lambda_m^+) = F_0(\lambda_m^-) = -im\pi, \quad m \in \mathbb{Z}.$$

(iii) For any $n \in \mathbb{Z}$, $F_n^2(\lambda)$ is analytic on U_n .

(iv) Locally uniformly in φ and uniformly as $|n| \rightarrow \infty$,

$$\sup_{\lambda \in G_n^+ \cup G_n^-} |F_n(\lambda)| = O(\gamma_n).$$

(v) $I_n = -\frac{1}{\pi} \int_{\Gamma_n} F(\lambda) d\lambda$ for every $n \in \mathbb{Z}$.

(vi) If φ is of real type, then for any $\lambda \in G_n$

$$F_n(\lambda \pm i0) = \pm \cosh^{-1} \frac{(-1)^n \Delta(\lambda)}{2}.$$

(vii) At the zero potential one has $F_n(\lambda, 0) = -i\lambda + in\pi$. \times

Proof. (i) The proof of the analyticity of F_n on $(\mathbb{C} \setminus \bigcup_{m \in \mathbb{Z}} \overline{U_m}) \times V_\varphi$ is standard but a bit technical and can be found in [13, Appendix F]. The analyticity of $F_n(\lambda) = F_n(\lambda, \varphi)$ on $\mathbb{C} \setminus \bigcup_{\gamma_m \neq 0} G_m$ follows immediately from the properties of $\frac{\Delta^\bullet(\lambda)}{\sqrt[n]{\Delta^2(\lambda) - 4}}$ obtained in Lemma 3.2.

To obtain the formula for the gradient, we first consider the case of φ being of real type, Then $(-1)^n \Delta(\lambda, \varphi) \geq 2$ on G_n and hence

$$\min_{\lambda_n^- \leq \lambda \leq \lambda_n^+} (-1)^n \Delta(\lambda, \varphi) - \sqrt[n]{\Delta^2(\lambda, \varphi) - 4} > 0.$$

Thus, after possibly shrinking V_φ , we can choose a circuit Γ_n , which is contained in U_n , and an open neighborhood U'_n of Γ_n so that Γ_n circles around G_n , $\overline{U'_n} \subset U_m \setminus G_m$ for any potential in V_φ , and the real part of $(-1)^n \left(\Delta(\lambda, \psi) + \sqrt[n]{\Delta^2(\lambda, \psi) - 4} \right)$ is strictly positive on U'_n . In consequence, the principal branch of the logarithm

$$l_n(\lambda, \psi) = \log \frac{(-1)^n}{2} \left(\Delta(\lambda, \psi) + \sqrt[n]{\Delta^2(\lambda, \psi) - 4} \right)$$

is analytic on $U'_n \times V_\varphi$. Clearly, $\partial_\lambda l_n = \frac{\Delta^\bullet(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda)-4}}$ and $l_n(\lambda_n^\pm) \equiv 0$, hence $F_n = l_n$. Taking the gradient of the above identity yields on $U'_n \times V_\varphi$

$$\partial F_n = \partial l_n(\lambda) = \frac{\partial \Delta(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda)-4}}.$$

Since both hand sides of this identity are analytic in both variables on $(\mathbb{C} \setminus \bigcup_{m \in \mathbb{Z}} \overline{U}_m) \times V_\varphi$ and analytic in λ on $\mathbb{C} \setminus \bigcup_{\gamma_m \neq 0} G_m$, the formula for the gradient extends to these domains by the identity theorem.

(ii) Note that $F_0(\lambda) = F_n(\lambda) + \int_{\lambda_0^+}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda)-4}} d\lambda$. Since $\int_{\lambda_n^-}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda)-4}} d\lambda = 0$ for any $n \in \mathbb{Z}$ by Lemma 3.2, $F_n(\lambda_n^+) = F_n(\lambda_n^-) = 0$. Moreover,

$$\int_{\lambda_0^+}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda)-4}} d\lambda = \sum_{k=0}^{n-1} \int_{\lambda_k^+}^{\lambda_{k+1}^-} \frac{\Delta^\bullet(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda)-4}} d\lambda, \quad n \geq 1,$$

while for $n \leq -1$

$$\int_{\lambda_0^+}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda)-4}} d\lambda = - \sum_{k=n}^{-1} \int_{\lambda_k^+}^{\lambda_{k+1}^-} \frac{\Delta^\bullet(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda)-4}} d\lambda, \quad n \leq -1.$$

Therefore, it is to compute $\int_{\lambda_k^+}^{\lambda_{k+1}^-} \frac{\Delta^\bullet(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda)-4}} d\lambda$ for $k \in \mathbb{Z}$. To do this, first consider the case where φ is of real type. In this case $i(-1)^k \sqrt[\epsilon]{\Delta^2(\lambda)-4} > 0$ for $\lambda_k^+ < \lambda < \lambda_{k+1}^-$ - c.f. [6, Section 5] - so

$$\begin{aligned} \int_{\lambda_k^+}^{\lambda_{k+1}^-} \frac{\Delta^\bullet(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda)-4}} d\lambda &= i(-1)^k \int_{\lambda_k^+}^{\lambda_{k+1}^-} \frac{\Delta^\bullet(\lambda)}{\sqrt[4]{4-\Delta^2(\lambda)}} d\lambda \\ &= i(-1)^k \sin^{-1} \frac{\Delta(\lambda)}{2} \Big|_{\lambda_k^+}^{\lambda_{k+1}^-} = -i\pi, \end{aligned}$$

and hence $\int_{\lambda_0^+}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[4]{4-\Delta^2(\lambda)}} d\lambda = -in\pi$ for any $n \in \mathbb{Z}$. The function $\int_{\lambda_0^+}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda)-4}} d\lambda = F_0(\lambda) - F_n(\lambda)$ is analytic on V_φ by item (i) and vanishes by the preceding argument on the real subspace $V_\varphi \cap \mathcal{F}\ell_r^p$. Therefore, $\int_{\lambda_0^+}^{\lambda_n^+} \frac{\Delta^\bullet(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda)-4}} d\lambda = -in\pi$ holds true on all of V_φ in view of the identity theorem.

(iii) In view of item (i) it remains to show that F_n^2 admits also for $\gamma_n \neq 0$ an analytic extension from $U_n \setminus G_n$ to all of U_n . Write (24) in the form

$$\frac{\Delta^\bullet(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda)-4}} = -i \frac{\lambda_n^\bullet - \lambda}{w_n(\lambda)} \chi_n(\lambda), \quad \chi_n(\lambda) = \prod_{m \neq n} \frac{\lambda_m^\bullet - \lambda}{w_m(\lambda)}. \quad (25)$$

The functionals χ_n , $n \in \mathbb{Z}$, are analytic on U_n by Corollary B.2. Moreover, the roots $w_n(\lambda)$, $n \in \mathbb{Z}$, admit opposite signs on opposite sides of G_n . Therefore, in view of $F_n(\lambda) = \int_{\lambda_n^+}^{\lambda} \frac{\Delta^\bullet(\mu)}{\sqrt[\epsilon]{\Delta^2(\mu)-4}} d\mu$, for any $\lambda \in G_n$,

$$F_n|_{G_n^+}(\lambda) = -F_n|_{G_n^-}(\lambda).$$

Consequently, F_n^2 is continuous and hence analytic on all of U_n .

(iv) If $\gamma_n = 0$, then $G_n = \{\lambda_n^\pm\}$ and $F(\lambda_n^\pm) = 0$ so there is nothing to show. Thus suppose $\gamma_n \neq 0$. We have $\lambda_n^\bullet = \tau_n + \gamma_n^2 \ell_n^p$ and $\sup_{\lambda \in U_n} |\chi_n(\lambda) - 1| = \ell_n^p$ locally uniformly on V_φ by (11) and Lemma B.2. In view of (25) it follows with Lemma 3.1 that

$$\sup_{\lambda \in G_n^- \cup G_n^+} |F_n(\lambda)| \leq O\left(\sup_{\lambda \in G_n} |\lambda_n^\bullet - \lambda|\right) = O(\gamma_n),$$

uniformly on V and for all $n \in \mathbb{Z}$.

(v) The claim follows from (14) using integration by parts.

(vi) If φ is of real type, then for any $\lambda \in G_n$

$$F_n(\lambda \pm i0) = \pm \int_{\lambda_n^-}^{\lambda} \frac{(-1)^n \Delta^\bullet(\mu)}{\sqrt[n]{\Delta^2(\mu) - 4}} d\mu = \pm \cosh^{-1} \frac{(-1)^n \Delta(\lambda)}{2}.$$

(vii) At the zero potential, $\left. \frac{\Delta^\bullet(\lambda)}{\sqrt[n]{\Delta^2(\lambda) - 4}} \right|_{\varphi=0} = -i$, which follows directly from the product representation (24). Integration thus yields $F_n(\lambda, 0) = -i(\lambda - n\pi)$. ■

To simplify notation, write $F(\lambda) \equiv F_0(\lambda)$ and note that $F(\lambda) = F_n(\lambda) - in\pi$ for any $n \in \mathbb{Z}$.

Lemma 3.4 *Suppose φ is a finite gap potential of real type, then F is analytic outside a disc of finite radius centered at the origin and admits the Laurent expansion*

$$F(\lambda) = -i\lambda + i \sum_{n \geq 1} \frac{H_n}{(2\lambda)^n}, \quad (26)$$

where H_n denotes the n th Hamiltonian of the NLS hierarchy. ×

Proof. By the preceding lemma $F(\lambda)$ is analytic on $\mathbb{C} \setminus \bigcup_{\gamma_m \neq 0} G_m$. Suppose φ is a finite gap potential of real type, then $F(\lambda)$ is analytic outside a disc of finite radius. Moreover, the product expansion (24) of $\frac{\Delta^\bullet(\lambda)}{\sqrt[n]{\Delta^2(\lambda) - 4}}$ is finite, whence one verifies directly that $F(\lambda) = O(\lambda)$ uniformly as $|\lambda| \rightarrow \infty$. Therefore, F is meromorphic with a pole at infinity of order at most one, and it suffices to determine the Laurent expansion of F along an arbitrary sequence ν_n with $|\nu_n| \rightarrow \infty$.

Since φ is assumed to be of real type, given any $n \in \mathbb{Z}$, the restriction of the function $(-1)^{n+1} \Delta(\lambda)$ to the interval $[\lambda_n^+, \lambda_{n+1}^-]$ is strictly increasing from -2 to 2 . The canonical root for $\lambda_n^+ < \lambda < \lambda_{n+1}^-$ is given by

$$\sqrt[n]{\Delta^2(\lambda) - 4} = (-1)^{n+1} i \sqrt[4]{4 - \Delta^2(\lambda)}.$$

Furthermore, one computes for $\lambda_n^+ < \lambda < \lambda_{n+1}^-$ that

$$\partial_\lambda \left(-i \sin^{-1} \left((-1)^{n+1} \frac{\Delta(\lambda)}{2} \right) \right) = i \frac{(-1)^n \Delta \bullet(\lambda)/2}{\sqrt[4]{1 - \Delta^2(\lambda)/4}} = \frac{\Delta \bullet(\lambda)}{\sqrt[4]{\Delta^2(\lambda) - 4}}.$$

Hence for any $\lambda_n^+ \leq \lambda \leq \lambda_{n+1}^-$,

$$\begin{aligned} F(\lambda) &= F(\lambda_n^+) + \int_{\lambda_n^+}^{\lambda} \frac{\Delta \bullet(\mu)}{\sqrt[4]{\Delta^2(\mu) - 4}} d\mu \\ &= -in\pi + \left[-i \sin^{-1} \left((-1)^{n+1} \frac{\Delta(\mu)}{2} \right) \right]_{\lambda_n^+}^{\lambda} \\ &= -i(n+1/2)\pi - i \sin^{-1} \left((-1)^{n+1} \frac{\Delta(\lambda)}{2} \right). \end{aligned}$$

Let $\nu_n = (n+1/2)\pi$, then by [6, Theorem 4.8]¹ for any $N \geq 1$

$$\Delta(\nu_n) = 2 \cos i\sigma_N(\nu_n) + O(\nu_n^{-N}), \quad \sigma_N(\lambda) = -i\lambda + i \sum_{n=1}^N \frac{H_n}{(2\lambda)^n},$$

as $|n| \rightarrow \infty$. Using that $\partial_z \sin^{-1}(z) = 1/\sqrt[4]{1-z^2}$ one gets by the mean value theorem

$$\left| \sin^{-1} \left((-1)^{n+1} \frac{\Delta(\nu_n)}{2} \right) - \sin^{-1} \left((-1)^{n+1} \cos i\sigma_N(\nu_n) \right) \right| = O(\nu_n^{-N}),$$

and hence

$$F(\nu_n) = -i\nu_n - i \sin^{-1} \left((-1)^{n+1} \cos i\sigma_N(\nu_n) \right) + O(\nu_n^{-N}), \quad n \rightarrow \infty.$$

Finally, writing $(-1)^{n+1} = -\sin \nu_n$ one gets by the addition theorem for the sine

$$(-1)^{n+1} \cos i\sigma_N(\nu_n) = \sin(i\sigma_N(\nu_n) - \nu_n),$$

and hence

$$-i \sin^{-1} \left((-1)^{n+1} \cos i\sigma_N(\nu_n) \right) = \sigma_N(\nu_n) + i\nu_n.$$

This gives $F(\nu_n) = \sigma_N(\nu_n) + O(\nu_n^{-N})$, hence the Laurent coefficients of F can be determined from σ_N . ■

The expansion (26) of $F(\lambda)$ yields one of $F^4(\lambda)$ which we record for later use.

Corollary 3.5 *Suppose φ is a finite gap potential of real type, then F^4 is analytic outside a disc of finite radius and admits the Laurent expansion*

$$F^4(\lambda) = \lambda^4 - 2H_1\lambda^2 - H_2\lambda - \frac{1}{2}(H_3 - 3H_1^2) - \frac{1}{4}(H_4 - 6H_1H_2)\frac{1}{\lambda} + O(\lambda^{-2}). \quad \times$$

¹Note that in comparison to [6] we multiplied for $n \geq 2$ the n th Hamiltonian with the factor $(-i)^{n+1}$.

We conclude this section by refining the asymptotics $\sup_{\lambda \in G_n^+ \cup G_n^-} |F_n(\lambda)| = O(\gamma_n)$ of Lemma 3.3.

Lemma 3.6 *For any $2 \leq p < \infty$, locally uniformly on \mathcal{W}^p*

$$\sup_{\lambda \in G_n^+ \cup G_n^-} |F_n(\lambda) - iw_n(\lambda)| = \gamma_n(\ell_n^{p/2} + \ell_n^{1+}).$$

At the zero potential, the right hand side of the latter identity vanishes, $F_n(\lambda) = iw_n(\lambda) = -i\lambda + in\pi$. \times

Proof. With (25) write F_n in the form

$$F_n(\lambda) = \int_{\lambda_n^-}^{\lambda} \frac{\Delta^\bullet(\mu)}{c\sqrt{\Delta^2(\mu) - 4}} d\mu = -i \int_{\lambda_n^-}^{\lambda} \frac{\lambda_n^\bullet - \mu}{w_n(\mu)} \chi_n(\mu) d\mu, \quad \chi_n(\lambda) = \prod_{m \neq n} \frac{\lambda_m^\bullet - \lambda}{w_m(\lambda)}.$$

By (11), $\lambda_n^\bullet - \tau_n = \gamma_n^2 \ell_n^p$, hence Lemma B.3 gives $\sup_{\lambda \in U_n} |\chi_n(\lambda) - 1| = \ell_n^{p/2} + \ell_n^{1+}$. As an immediate consequence we obtain from Lemma 3.1 that

$$\sup_{\lambda \in G_n^+ \cup G_n^-} \left| F_n(\lambda) - (-i) \int_{\lambda_n^-}^{\lambda} \frac{\lambda_n^\bullet - \mu}{w_n(\mu)} d\mu \right| \leq \max_{\lambda \in G_n^+ \cup G_n^-} |(\lambda_n^\bullet - \lambda)(\chi_n(\lambda) - 1)| = \gamma_n(\ell_n^{p/2} + \ell_n^{1+}).$$

One further checks that $\partial_\lambda w_n(\lambda) = -\frac{\tau_n - \lambda}{w_n(\lambda)}$ for $\lambda \notin G_n$, hence

$$-i \int_{\lambda_n^-}^{\lambda} \frac{\lambda_n^\bullet - \xi}{w_n(\xi)} d\xi = iw_n(\lambda) + i(\tau_n - \lambda_n^\bullet) \int_{\lambda_n^-}^{\lambda} \frac{1}{w_n(\xi)} d\xi.$$

If $\gamma_n = 0$, then $\tau_n = \lambda_n^\bullet$ and the claim is evident. On the other hand, if $\gamma_n \neq 0$, then Lemma 3.1 gives $\sup_{\lambda \in G_n^- \cup G_n^+} \left| \int_{\lambda_n^-}^{\lambda} \frac{1}{w_n(\xi)} d\xi \right| \leq \pi$ and the claim follows with the estimate $\tau_n - \lambda_n^\bullet = \gamma_n^2 \ell_n^p$. \blacksquare

4 Refined estimate of $\lambda_m^\bullet - \sigma_m^n$

For $\varphi \in \mathcal{W}^p$, $2 \leq p < \infty$, we denote by ψ_n , $n \in \mathbb{Z}$, the entire function of the form

$$\psi_n(\lambda) = -\frac{2}{\pi_n} \prod_{k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k}, \quad \sigma_k^n = k\pi + \ell_k^p, \quad (27)$$

which is characterized by the property that

$$\frac{1}{2\pi} \int_{\Gamma_k} \frac{\psi_n(\lambda)}{c\sqrt{\Delta^2(\lambda) - 4}} d\lambda = \delta_{nk}. \quad (28)$$

The roots of ψ_n are precisely the complex numbers σ_k^n , $k \neq n$, and can be shown to satisfy

$$\sigma_k^n - \tau_k = \gamma_k^2 \ell_k^p [n] \quad (29)$$

uniformly in n and locally uniformly on \mathcal{W}^p – see [13]. The expression $\ell_k^p[n]$, $k \in \mathbb{Z}$, means that there exists a sequence α_k^n , $k \in \mathbb{Z}$, whose elements may depend on $n \in \mathbb{Z}$, such that $\sigma_k^n = \tau_k + \gamma_k^2 \alpha_k^n$ where

$$\sum_{k \in \mathbb{Z}} |\alpha_k^n|^p \leq C, \quad (30)$$

and $C > 0$ can be chosen uniformly in n and locally uniformly on \mathcal{W}^p . Further, it turns out to be convenient to set

$$\sigma_n^n := \lambda_n^\bullet. \quad (31)$$

According to (29) and (11)

$$\sigma_k^n - \lambda_k^\bullet = (\sigma_k^n - \tau_k) + (\tau_k - \lambda_k^\bullet) = \gamma_k^2 \ell_k^p[n], \quad k \neq n. \quad (32)$$

The purpose of this subsection is to improve on these estimates.

Proposition 4.1 *Locally uniformly on \mathcal{W}^p , $2 < p < \infty$,*

$$\sigma_k^n - \lambda_k^\bullet = \gamma_k \ell_k^{1+}[n].$$

In more detail, there exists a sequence α_k^n so that $\sigma_k^n = \lambda_k^\bullet + \gamma_k \alpha_k^n$ where for any $q > 1$

$$\sum_{k \in \mathbb{Z}} |\alpha_k^n|^q \leq C_q,$$

and $C_q > 0$ can be chosen uniformly in n and locally uniformly on \mathcal{W}^p . \times

Remark. Note that in the case $p = 2$, the estimate (32) implies that $\sigma_k^n - \lambda_k^\bullet = \gamma_k \ell_k^1[n]$. \dashv

Proof. In view of (27) and (23) for any $n, k \geq 1$

$$\frac{\psi_n(\lambda)}{\sqrt[4]{\Delta^2(\lambda) - 4}} = \frac{i}{w_n(\lambda)} \frac{\sigma_k^n - \lambda}{\sigma_n^n - \lambda} \zeta_k(\lambda), \quad \zeta_k(\lambda) = \prod_{m \neq k} \frac{\sigma_m^n - \lambda}{w_m(\lambda)}, \quad (33)$$

where the function ζ_k is analytic on U_k . By (28), the roots σ_k^n , $k \neq n$, of ψ_n are characterized by the equation

$$0 = \int_{\Gamma_k} \frac{\psi_n(\lambda)}{\sqrt[4]{\Delta^2(\lambda) - 4}} d\lambda = i \int_{\Gamma_k} \frac{\sigma_k^n - \lambda}{\sigma_n^n - \lambda} \frac{\zeta_k(\lambda)}{w_k(\lambda)} d\lambda, \quad k \neq n. \quad (34)$$

It implies that

$$(\sigma_k^n - \lambda_k^\bullet) \int_{\Gamma_k} \frac{1}{\sigma_n^n - \lambda} \frac{\zeta_k(\lambda)}{w_k(\lambda)} d\lambda = \int_{\Gamma_k} \frac{\lambda - \lambda_k^\bullet}{\sigma_n^n - \lambda} \frac{\zeta_k(\lambda)}{w_k(\lambda)} d\lambda, \quad k \neq n. \quad (35)$$

This identity is the starting point for estimating $\sigma_k^n - \lambda_k^\bullet$. A key step in the proof of the claimed estimate is to rewrite this identity in an appropriate way. Let us multiply it by $(\sigma_n^n - \lambda_k^\bullet)$ and introduce

$$\xi_k(\lambda) = \frac{\sigma_n^n - \lambda_k^\bullet}{\sigma_n^n - \lambda} \zeta_k(\lambda). \quad (36)$$

Note that $\sigma_n^n - \lambda_k^\bullet \neq 0$ by (12). It then follows from (35) that

$$(\sigma_k^n - \lambda_k^\bullet) \int_{\Gamma_k} \frac{\xi_k(\lambda)}{w_k(\lambda)} d\lambda = \int_{\Gamma_k} \frac{(\lambda - \lambda_k^\bullet) \xi_k(\lambda)}{w_k(\lambda)} d\lambda, \quad k \neq n.$$

Since by (36), $\xi_k(\lambda) = \left(1 + \frac{\lambda - \lambda_k^\bullet}{\sigma_n^n - \lambda}\right) \zeta_k(\lambda)$, we get

$$\int_{\Gamma_k} \frac{(\lambda - \lambda_k^\bullet) \xi_k(\lambda)}{w_k(\lambda)} d\lambda = \int_{\Gamma_k} \frac{(\lambda - \lambda_k^\bullet) \zeta_k(\lambda)}{w_k(\lambda)} d\lambda + \int_{\Gamma_k} \frac{(\lambda - \lambda_k^\bullet)^2 \zeta_k(\lambda)}{(\sigma_n^n - \lambda) w_k(\lambda)} d\lambda.$$

The second term on the right hand side of the latter identity is expected to be small in comparison to the first term since $\sigma_n^n - \lambda$ is of the size of $n - k$. We proceed by writing the first term in a more convenient form. Note that the roots λ_k^\bullet , $k \in \mathbb{Z}$, of Δ^\bullet are characterized by the equation

$$0 = \int_{\Gamma_k} \frac{\Delta^\bullet(\lambda)}{\sqrt{c \Delta^2(\lambda) - 4}} d\lambda = -i \int_{\Gamma_k} \frac{\lambda_k^\bullet - \lambda}{w_k(\lambda)} \chi_k(\lambda) d\lambda, \quad k \in \mathbb{Z}, \quad (37)$$

where χ_k is given by (25). Hence

$$\int_{\Gamma_k} \frac{(\lambda - \lambda_k^\bullet) \zeta_k(\lambda)}{w_k(\lambda)} d\lambda = \int_{\Gamma_k} \frac{(\lambda - \lambda_k^\bullet) (\zeta_k(\lambda) - \chi_k(\lambda))}{w_k(\lambda)} d\lambda.$$

Altogether, identity (35) then reads

$$\begin{aligned} (\sigma_k^n - \lambda_k^\bullet) \int_{\Gamma_k} \frac{\xi_k(\lambda)}{w_k(\lambda)} d\lambda &= \int_{\Gamma_k} \frac{(\lambda - \lambda_k^\bullet) (\zeta_k(\lambda) - \chi_k(\lambda))}{w_k(\lambda)} d\lambda \\ &+ \int_{\Gamma_k} \frac{(\lambda - \lambda_k^\bullet)^2 \zeta_k(\lambda)}{(\sigma_n^n - \lambda) w_k(\lambda)} d\lambda, \quad k \neq n. \end{aligned} \quad (38)$$

The integrals in (38) are now estimated separately. First note that for $\lambda \in G_k$ we have

$$\frac{\sigma_n^n - \lambda_k^\bullet}{\sigma_n^n - \lambda} = 1 + \frac{\lambda - \lambda_k^\bullet}{\sigma_n^n - \lambda}, \quad \frac{\lambda - \lambda_k^\bullet}{\sigma_n^n - \lambda} = O\left(\frac{\gamma_k}{n - k}\right).$$

By Lemma B.3 we have $\zeta_k|_{G_k} = 1 + \ell_k^p$, implying that

$$\xi_k|_{G_k} = (1 + \ell_k^p) \zeta_k|_{G_k} = 1 + \ell_k^p.$$

Since by Lemma 3.1 (iii), $\frac{1}{2\pi i} \int_{\Gamma_k} \frac{1}{w_k(\lambda)} d\lambda = -1$, we then conclude

$$\frac{1}{2\pi i} \int_{\Gamma_k} \frac{\xi_k(\lambda)}{w_k(\lambda)} d\lambda = -1 + \ell_k^p. \quad (39)$$

Concerning the integral $\int_{\Gamma_k} \frac{(\lambda - \lambda_k^\bullet)(\zeta_k(\lambda) - \chi_k(\lambda))}{w_k(\lambda)} d\lambda$, it follows from Lemma 3.1 that

$$\left| \frac{1}{2\pi} \int_{\Gamma_k} \frac{(\lambda - \lambda_k^\bullet)(\zeta_k(\lambda) - \chi_k(\lambda))}{w_k(\lambda)} d\lambda \right| \leq |\gamma_k| |\zeta_k(\lambda) - \chi_k(\lambda)|_{G_k}. \quad (40)$$

Similarly, for the integral $\int_{\Gamma_k} \frac{(\lambda - \lambda_k^\bullet)^2 \zeta_k(\lambda)}{(\sigma_n^n - \lambda)w_k(\lambda)} d\lambda$, we get

$$\left| \frac{1}{2\pi} \int_{\Gamma_k} \frac{(\lambda - \lambda_k^\bullet)^2 \zeta_k(\lambda)}{(\sigma_n^n - \lambda)w_k(\lambda)} d\lambda \right| \leq \left| \frac{(\lambda - \lambda_k^\bullet)^2}{\sigma_n^n - \lambda} \zeta_k(\lambda) \right|_{G_k} = O\left(\frac{\gamma_k^2}{n-k}\right). \quad (41)$$

Since $\frac{1}{n-k} = \ell_k^{1+}[n]$ we conclude

$$O\left(\frac{\gamma_k^2}{n-k}\right) = \gamma_k \ell_k^1[n]. \quad (42)$$

Inserting estimates (39)-(42) into (38) yields

$$(\sigma_k^n - \lambda_k^\bullet)(1 + \ell_k^p) = O(\gamma_k |\zeta_k(\lambda) - \chi_k(\lambda)|_{G_k}) + \gamma_k \ell_k^1[n]. \quad (43)$$

To estimate $|\zeta_k(\lambda) - \chi_k(\lambda)|_{G_k}$, write the product expansions (33) and (25), respectively, as $\zeta_k(\lambda) = f_k(\lambda, \tilde{\alpha}^1)$ and $\chi_k(\lambda) = f_k(\lambda, \tilde{\alpha}^0)$, where we have set $\alpha^1 = (\sigma_m^n)$, $\alpha^0 = (\lambda_m^\bullet)$, and

$$f_k(\lambda, \tilde{\alpha}) = \prod_{m \neq k} \frac{\alpha_m - \lambda}{w_m(\lambda)}, \quad \tilde{\alpha}_n = \alpha_n - n\pi.$$

By Corollary B.2, the function f_k is analytic on $(\mathbb{C} \setminus \bigcup_{m \neq k} G_k) \times \ell_{\mathbb{C}}^p$ and by Lemma B.3 satisfies the estimate $|f_k(\lambda, \tilde{\alpha}) - 1|_{G_k} = 1 + \ell_k^p$ locally uniformly on $\ell_{\mathbb{C}}^p$. Thus we may write for any $\lambda \in G_k$,

$$\zeta_k(\lambda) - \chi_k(\lambda) = f_k(\lambda, \tilde{\alpha}^1) - f_k(\lambda, \tilde{\alpha}^0) = \int_0^1 \sum_{m \neq k} \partial_{\alpha_m} f_k(\lambda, \tilde{\alpha}^t) (\sigma_m^n - \lambda_m^\bullet) dt,$$

where $\alpha^t := (\alpha_m^t) = ((1-t)\sigma_m^n + t\lambda_m^\bullet)$. Since for any $m \neq k$ one has

$$\partial_{\alpha_m} f_k(\lambda, \tilde{\alpha}) = \frac{1}{\alpha_m - \lambda} f_k(\lambda, \tilde{\alpha}),$$

we conclude

$$\zeta_k(\lambda) - \chi_k(\lambda) = \int_0^1 f_k(\lambda, \tilde{\alpha}^t) \sum_{m \neq k} \frac{\sigma_m^n - \lambda_m^\bullet}{\alpha_m^t - \lambda} dt.$$

By Lemma B.3, we can choose $M > 0$ so that $\sup_{0 \leq t \leq 1} |f_k(\lambda, \tilde{\alpha}^t)|_{G_k} \leq M$ for all $k \geq 1$. Moreover, by the mean value theorem there exists a sequence $(\nu_k) \subset \mathbb{C}$ with $\nu_k \in G_k$ such that

$$|\zeta_k(\lambda) - \chi_k(\lambda)|_{G_k} \leq M \int_0^1 \left| \sum_{m \neq k} \frac{\sigma_m^n - \lambda_m^\bullet}{\alpha_m^t - \nu_k} \right| dt. \quad (44)$$

Since U_m is a disc and hence convex, we have $\alpha_m^t \in U_m$ for all $m \geq 1$. Moreover, $\nu_k \in G_k \subset U_k$ for all $k \geq 1$. Thus by (12), there exists $c > 0$ so that

$$|\alpha_m^t - \nu_k| \geq c|m - k|, \quad m \neq k, \quad 0 \leq t \leq 1.$$

Write the estimate (32) in the form $\sigma_m^n - \lambda_m^\bullet = \gamma_m \ell_m^{q_1}[n]$ with $q_1 = p/2$ and suppose that for some $j \geq 1$

$$\sigma_m^n - \lambda_m^\bullet = \gamma_m \ell_m^{q_j}[n], \quad 1 < q_j \leq p/2,$$

then $\sigma_m^n - \lambda_m^\bullet = \ell_m^{r_j}[n]$ with $r_j = \frac{p}{p+q_j}q_j$. It follows with Lemma A.2 from (44) that

$$|\zeta_k(\lambda) - \chi_k(\lambda)|_{G_k} = \gamma_k(\ell_k^{r_j} + \ell_k^{1+}).$$

We conclude with (43) that for all k sufficiently large

$$\sigma_k^n - \lambda_k^\bullet = \gamma_k(\ell_k^{r_j} + \ell_k^{1+}) + \gamma_k \ell_k^1[n] = \gamma_k(\ell_k^{q_{j+1}} + \ell_k^{1+}) \quad (45)$$

where

$$q_{j+1} = r_j = \frac{p}{p+q_j}q_j = \frac{p}{j+1}.$$

Thus, after finitely many iterations, $q_{j+1} \leq 1$ and hence $\sigma_k^n - \lambda_k^\bullet = \gamma_k \ell_k^{1+}$ as claimed. By going through the arguments of the proof, one verifies that the estimate holds locally uniformly on \mathcal{W}^p . ■

5 Proof of Theorem 1.3

In this section we derive, a formula for the frequencies $\omega_n^{(4)}$, $n \in \mathbb{Z}$, of the mKdV system (6) which we then use to study their asymptotics as $|n| \rightarrow \infty$. Our starting point is the following identity for the n th frequency

$$\omega_n^{(4)} = \{H_4, \theta_n\},$$

which a priori holds on $H_c^{3/2} \cap (\mathcal{W}^2 \setminus Z_n)$.

It turns out to be convenient to introduce for any integers $n, k \in \mathbb{Z}$ and $m \geq 0$ the moments

$$\Omega_{nk}^{(m)} := \int_{\Gamma_k} \frac{F_k^m(\lambda) \psi_n(\lambda)}{\sqrt[2]{\Delta^2(\lambda) - 4}} d\lambda.$$

We recall from Section 4 the product representation of the quotient $\psi_n(\lambda)/\sqrt[2]{\Delta^2(\lambda) - 4}$ given by (33)

$$\frac{\psi_n(\lambda)}{\sqrt[2]{\Delta^2(\lambda) - 4}} = i \frac{\zeta_n(\lambda)}{w_n(\lambda)} = \frac{\sigma_k^n - \lambda}{w_k(\lambda)} \zeta_k^n(\lambda), \quad \zeta_k^n(\lambda) = \frac{i}{\sigma_k^n - \lambda} \zeta_k(\lambda), \quad (46)$$

where we recall that $\sigma_n^n = \lambda_n^\bullet$, and $\zeta_k(\lambda) = \prod_{m \neq k} \frac{\sigma_m^n - \lambda}{w_m(\lambda)}$. Note that for any $k \in \mathbb{Z}$, the functions ζ_k and ζ_k^n are analytic on U_k .

Lemma 5.1 *Given any $2 \leq p < \infty$, the moments $\Omega_{nk}^{(m)}: \mathcal{W}^p \rightarrow \mathbb{C}$ have the following properties:*

- (i) $\Omega_{nk}^{(0)} = 2\pi\delta_{nk}$, for all $n, k \in \mathbb{Z}$. \times
- (ii) Each moment $\Omega_{nk}^{(m)}$, $n, k \in \mathbb{Z}$, $m \geq 1$, is analytic on \mathcal{W}^p .
- (iii) $\Omega_{nk}^{(2l+1)} = 0$ for all $n, k \in \mathbb{Z}$ and $l \geq 0$.
- (iv) If $\gamma_k = 0$, then $\Omega_{nk}^{(m)} = 0$ for all $n \in \mathbb{Z}$ and $m \geq 1$.

Proof. (i): The identity follows from the characterization (28) of the functions ψ_n .

(ii): Let $\varphi \in \mathcal{W}^p$. We choose circuits Γ_k , $k \in \mathbb{Z}$, and open neighborhoods U'_k of Γ_k such that Γ_k circles around G_k and $\overline{U'_k} \subset U_k \setminus G_k$ for any potential in V_φ with V_φ given as in Section 2. In view of (23), Lemma (27), and Lemma 3.3, the integrand $\frac{F_k^m(\lambda)\psi_n(\lambda)}{\sqrt[\varepsilon]{\Delta^2(\lambda)-4}}$ is analytic on $U'_k \times V_\varphi$ for any $k \in \mathbb{Z}$. Consequently, $\Omega_{nk}^{(m)}$ is analytic on V_φ .

(iii): The function F_k and the canonical root each extend continuously to the two sides G_k^\pm of the gap G_k and take opposite signs there. Consequently, for any $l \geq 0$ the quotient $F_k^{2l+1}(\lambda)/\sqrt[\varepsilon]{\Delta^2(\lambda)-4}$ extends continuously from $U_k \setminus G_k$ to U_k and hence is analytic on all of U_k . Together with the fact that ψ_n is an entire function, we conclude that $\Omega_{nk}^{(2l+1)} = 0$ for all $n, k \in \mathbb{Z}$.

(iv): In view of item (iii) it remains to consider the case $m = 2l$, $l \geq 1$, and $n, k \in \mathbb{Z}$ with $\gamma_k = 0$. Suppose $k \neq n$. Since $\sigma_k^n = \tau_k$ by (29), and $w_k(\lambda) = \tau_k - \lambda$ in view of (19), by (33) the quotient $\psi_n(\lambda)/\sqrt[\varepsilon]{\Delta^2(\lambda)-4}$ equals $\zeta_k^n(\lambda)$ and hence is analytic on U_k . Since by Lemma 3.3 also F_k^{2l} is analytic on U_k , we conclude $\Omega_{nk}^{(2l)} = 0$. Now suppose $k = n$. Since ζ_n and F^{2l} are analytic on U_n and $w_n(\lambda) = \tau_n - \lambda$, we have in view of (33) and Cauchy's Theorem that $\Omega_{nn}^{(2)} = 2\pi F_n^{2l}(\tau_n)\zeta_n(\tau_n)$. Since $\gamma_n = 0$ we find by Lemma 3.3 that $F_n(\tau_n) = F_n(\lambda_n^\pm) = 0$ proving the claim. \blacksquare

Lemma 5.2 *For any finite gap potential of real type and any $n \in \mathbb{Z}$*

$$\omega_n^{(4)*} = \omega_n^{(4)} - 6H_2 - 12n\pi H_1 - (2n\pi)^3 = -12 \sum_{k \in \mathbb{Z}} k \Omega_{nk}^{(2)}. \quad \times \quad (47)$$

Proof. Suppose φ is a finite gap potential, then there exists $K \geq 1$ so that $\gamma_k(\varphi) = 0$ for $|k| > K$. By Corollary 3.5, the function $F^4(\lambda)$ is analytic outside a sufficiently large circle C_r , which encloses all open gaps G_k , $|k| \leq K$, and admits the Laurent expansion

$$F^4(\lambda) = \lambda^4 - 2H_1\lambda^2 - H_2\lambda - \frac{1}{2}(H_3 - 3H_1^2) - \frac{1}{4}(H_4 - 6H_1H_2)\frac{1}{\lambda} + O(\lambda^{-2})$$

Therefore, by the Cauchy Theorem

$$-\frac{1}{4}(H_4 - 6H_1H_2) = \frac{1}{2\pi i} \int_{C_r} F^4(\lambda) \, d\lambda.$$

Consider any $n \in \mathbb{Z}$ with $\gamma_n(\varphi) \neq 0$. Then $\theta_n \bmod \pi$ is analytic near φ . Since $H_1 = \sum_{m \in \mathbb{Z}} I_m$ and $H_2 = \sum_{m \in \mathbb{Z}} (2m\pi)I_m$, one computes that

$$\begin{aligned} \omega_n^{(4)} - 6H_2 - 12n\pi H_1 &= \{H_4 - 6H_1H_2, \theta_n\} \\ &= \frac{4i}{2\pi} \int_{C_r} \{F^4(\lambda), \theta_n\} \, d\lambda \\ &= \frac{16i}{2\pi} \int_{C_r} \frac{F^3(\lambda)\{\Delta(\lambda), \theta_n\}}{\sqrt[3]{\Delta^2(\lambda) - 4}} \, d\lambda. \end{aligned}$$

Using $-2\{\Delta(\lambda), \theta_n\} = \psi_n(\lambda) - \text{c.f. [10, Lemma 18.2]}$ – one thus obtains

$$\omega_n^{(4)} - 6H_2 - 12n\pi H_1 = -\frac{8i}{2\pi} \int_{C_r} \frac{F^3(\lambda)\psi_n(\lambda)}{\sqrt[3]{\Delta^2(\lambda) - 4}} \, d\lambda.$$

On the one hand $F^3(\lambda)$ is analytic on $\mathbb{C} \setminus \bigcup_{\gamma_k \neq 0} G_k$, while on the other hand for any $k \in \mathbb{Z}$ with $\gamma_k = 0$ one has $\sigma_k^n = \tau_k$ and $w_k(\lambda) = \tau_k - \lambda$ so that in view of the product representations (23) and (27) the integrand extends analytically to U_k . Consequently, the integrand is analytic on $\mathbb{C} \setminus \bigcup_{|k| \leq K} G_k$ and one obtains by contour deformation

$$\omega_n^{(4)} - 6H_2 - 12n\pi H_1 = -\frac{8i}{2\pi} \sum_{|k| \leq K} \int_{\Gamma_k} \frac{F^3(\lambda)\psi_n(\lambda)}{\sqrt[3]{\Delta^2(\lambda) - 4}} \, d\lambda.$$

Proceeding by expanding $F(\lambda)^3 = (F_k(\lambda) - ik\pi)^3 = F_k^3(\lambda) - 3i(k\pi)F_k^2(\lambda) - 3(k\pi)^2F_k(\lambda) + i(k\pi)^3$ and using that $\Omega_{nk}^{(3)} \equiv \Omega_{nk}^{(1)} \equiv 0$ by Lemma 5.1 we thus get

$$\begin{aligned} \omega_n^{(4)} - 6H_2 - 12n\pi H_1 &= -\frac{8}{2\pi} \sum_{|k| \leq K} \left(3(k\pi)\Omega_{nk}^{(2)} - (k\pi)^3\Omega_{nk}^{(0)} \right) \\ &= \sum_{k \in \mathbb{Z}} \left(-12k\Omega_{nk}^{(2)} + (2k\pi)^3\delta_{kn} \right), \end{aligned}$$

where we used in the second line that $\Omega_{nk}^{(2)} = 0$ for all $k > K$. This shows that (47) holds for all n with $\gamma_n(\varphi) \neq 0$.

Now consider any $n \in \mathbb{Z}$ with $\gamma_n(\varphi) = 0$, and denote $A = \{k \in \mathbb{Z} : \gamma_k(\varphi) \neq 0\}$. By [6] we can choose a sequence of finite gap potentials φ_l in H_r^1 with $\gamma_k(\varphi_l) = \gamma_k(\varphi)$ for $k \neq n$, $\gamma_n(\varphi_l) \neq 0$, and $\varphi_l \rightarrow \varphi$ in H_r^1 . In particular, $A^{(l)} := \{j \in \mathbb{Z} : \gamma_j(\varphi_l) \neq 0\}$ is given by $A \cup \{n\} =: A^\natural$ for any $l \geq 1$. Since each $\Omega_{nk}^{(2)}$, $k \in \mathbb{Z}$, is continuous, indeed analytic, on \mathcal{W}^p , and A^\natural is finite and independent of l , it follows that

$$\sum_{k \in \mathbb{Z}} k\Omega_{nk}^{(2)}(\varphi_l) = \sum_{k \in A^\natural} k\Omega_{nk}^{(2)}(\varphi_l) \xrightarrow{l \rightarrow \infty} \sum_{k \in \mathbb{Z}} k\Omega_{nk}^{(2)}(\varphi) = \sum_{k \in A^\natural} k\Omega_{nk}^{(2)}(\varphi).$$

On the other hand, ω_n is continuous at $\varphi \in H_r^2$ which shows that (47) holds for all $n \in \mathbb{Z}$. ■

We proceed by deriving decay estimates for $\Omega_{nk}^{(2)}$.

Lemma 5.3 *Locally uniformly on \mathcal{W}^p , $2 \leq p < \infty$,*

$$\Omega_{nk}^{(2)} = \frac{\gamma_k^3 \ell_k^{1+}[n]}{n-k}, \quad k \neq n, \quad \Omega_{nn}^{(2)} = \frac{\gamma_n^2}{4} \left(\pi + \ell_n^{p/2} + \ell_n^{1+} \right).$$

In more detail, there exists a sequence α_k^n , $k \in \mathbb{Z}$, so that $\Omega_{nk}^{(2)} = \frac{\gamma_k^3}{n-k} \alpha_k^n$ where for any $q > 1$

$$\sum_{k \in \mathbb{Z}} |\alpha_k^n|^q \leq C_q,$$

and $C_q > 0$ can be chosen uniformly in n and locally uniformly on \mathcal{W}^p . ×

Proof. We begin with the case $k \neq n$ and prove the estimate

$$(\sigma_n^n - \tau_k) \Omega_{nk}^{(2)} = \gamma_k^3 \ell_k^1[n] \quad (48)$$

using Proposition 4.1. Since by (12), $|\sigma_n^n - \tau_k| \geq c|n-k|$, the claimed estimate $(n-k)\Omega_{nk}^{(2)} = \gamma_k^3 \ell_k^1[n]$ then follows immediately. To prove (48) note that

$$(\sigma_n^n - \tau_k) \Omega_{nk}^{(2)} = \int_{\Gamma_k} \frac{F_k^2(\lambda) (\sigma_n^n - \tau_k) \psi_n(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda) - 4}} d\lambda.$$

Our goal is to compare the two product expansions

$$\frac{(\sigma_n^n - \lambda) \psi_n(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda) - 4}} = i \prod_{m \in \mathbb{Z}} \frac{\sigma_m^n - \lambda}{w_m(\lambda)}, \quad \frac{\Delta^\bullet(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda) - 4}} = -i \prod_{m \in \mathbb{Z}} \frac{\lambda_m^\bullet - \lambda}{w_m(\lambda)}$$

By Lemma 3.3 the function $F_k^3(\lambda)$ is analytic on $U_k \setminus G_k$ and we compute

$$\partial_\lambda \left(\frac{1}{3} F_k^3(\lambda) \right) = \frac{F_k^2(\lambda) \Delta^\bullet(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda) - 4}}.$$

Therefore, in view of existence of the primitive

$$\int_{\Gamma_k} \frac{F_k^2(\lambda) \Delta^\bullet(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda) - 4}} d\lambda = 0.$$

Thus the moment $\Omega_{nk}^{(2)}$ may be written in the form

$$(\sigma_n^n - \tau_k) \Omega_{nk}^{(2)} = \int_{\Gamma_k} \frac{F_k^2(\lambda) ((\sigma_n^n - \tau_k) \psi_n(\lambda) + \Delta^\bullet(\lambda))}{\sqrt[\epsilon]{\Delta^2(\lambda) - 4}} d\lambda = I_k^\flat + I_k^\sharp,$$

where the terms I_k^\flat and I_k^\sharp are defined by

$$I_k^\flat = \int_{\Gamma_k} \frac{F_k^2(\lambda) ((\sigma_n^n - \lambda) \psi_n(\lambda) + \Delta^\bullet(\lambda))}{\sqrt[\epsilon]{\Delta^2(\lambda) - 4}} d\lambda,$$

$$I_k^\sharp = \int_{\Gamma_k} \frac{F_k^2(\lambda) (\lambda - \tau_k) \psi_n(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda) - 4}} d\lambda.$$

We first consider the term I_k^{\natural} . For $k \neq n$ we have by (46)

$$\frac{\psi_n(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda) - 4}} = \frac{\sigma_k^n - \lambda}{w_k(\lambda)} \zeta_k^n(\lambda).$$

By Lemma B.3 one has

$$(n - k)\zeta_k^n(\lambda) = \frac{i}{\pi} + \ell_k^{p/2}[n] + \ell_k^{1+}[n].$$

Since $|F_k|_{G_k} = O(\gamma_k)$ by Lemma 3.3 and $|\sigma_k^n - \lambda|_{G_k} = O(\gamma_k)$, we obtain with Lemma 3.1 that

$$\begin{aligned} I_k^{\natural} &= \int_{\Gamma_k} \frac{F_k^2(\lambda)(\lambda - \tau_k)\psi_n(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda) - 4}} d\lambda = \int_{\Gamma_k} \frac{F_k^2(\lambda)(\lambda - \tau_k)(\sigma_k^n - \lambda)\zeta_k^n(\lambda)}{w_k(\lambda)} d\lambda \\ &= \frac{\gamma_k^4}{n - k} \left(\frac{i}{\pi} + \ell_k^{p/2}[n] + \ell_k^{1+}[n] \right). \end{aligned}$$

Clearly, $I_k^{\natural} = \gamma_k^4 \ell_k^{1+}[n]$.

It remains to consider the term I_k^{\flat} . To this end, recall the product expansions

$$\Delta^{\bullet}(\lambda) = 2 \prod_{m \in \mathbb{Z}} \frac{\lambda_m^{\bullet} - \lambda}{\pi_m}, \quad \psi_n(\lambda) = -\frac{2}{\pi_n} \prod_{m \neq n} \frac{\sigma_m^n - \lambda}{\pi_m}, \quad \sqrt[\epsilon]{\Delta^2(\lambda) - 4} = 2i \prod_{m \in \mathbb{Z}} \frac{w_m(\lambda)}{\pi_m}.$$

We may thus write

$$-i \frac{(\sigma_n^n - \lambda)\psi_n(\lambda) + \Delta^{\bullet}(\lambda)}{\sqrt[\epsilon]{\Delta^2(\lambda) - 4}} = f(\lambda, \tilde{\alpha}^1) - f(\lambda, \tilde{\alpha}^0),$$

where $\alpha^1 = (\sigma_m^n)_{m \in \mathbb{Z}}$, $\alpha^0 = (\lambda_m^{\bullet})_{m \in \mathbb{Z}}$, and

$$f(\lambda, \tilde{\alpha}) := \frac{\alpha_k - \lambda}{w_k(\lambda)} f_k(\lambda, \tilde{\alpha}), \quad f_k(\lambda, \tilde{\alpha}) := \prod_{m \neq k} \frac{\alpha_m - \lambda}{w_m(\lambda)}, \quad \alpha_m = m\pi + \tilde{\alpha}_m.$$

By Corollary B.2 the functions $f_k: (\mathbb{C} \setminus \bigcup_{m \neq k} G_m) \times \ell_{\mathbb{C}}^p \rightarrow \mathbb{C}$ and $f: (\mathbb{C} \setminus \bigcup_{m \in \mathbb{Z}} G_m) \times \ell_{\mathbb{C}}^p \rightarrow \mathbb{C}$ are analytic. One further computes that

$$\partial_{\tilde{\alpha}_m} f(\lambda, \tilde{\alpha}) = \frac{\alpha_k - \lambda}{\alpha_m - \lambda} \frac{f_k(\lambda, \tilde{\alpha})}{w_k(\lambda)}, \quad m \neq k, \quad \partial_{\tilde{\alpha}_k} f(\lambda, \tilde{\alpha}) = \frac{f_k(\lambda, \tilde{\alpha})}{w_k(\lambda)}.$$

Further, for $0 \leq s \leq 1$, let $\alpha^s = (\alpha_m^s) = ((1 - s)\sigma_m^n + s\lambda_m^{\bullet})$, then

$$\begin{aligned} f(\lambda, \tilde{\alpha}^1) - f(\lambda, \tilde{\alpha}^0) &= \int_0^1 \sum_m \partial_{\alpha_m} f(\lambda, \tilde{\alpha}^s) (\sigma_m^n - \lambda_m^{\bullet}) ds \\ &= \int_0^1 \left(\sum_{m \neq k} \frac{\sigma_m^n - \lambda_m^{\bullet}}{\alpha_m^s - \lambda} \right) \frac{(\alpha_k^s - \lambda) f_k(\lambda, \tilde{\alpha}^s)}{w_k(\lambda)} ds \\ &\quad + (\sigma_k^n - \lambda_k^{\bullet}) \int_0^1 \frac{f_k(\lambda, \tilde{\alpha}^s)}{w_k(\lambda)} ds. \end{aligned}$$

Since first $|f_k(\lambda, \tilde{\alpha}^s)|_{G_k}$ is bounded uniformly in k and $0 \leq s \leq 1$ by Lemma B.3, second $|\alpha_k^s - \lambda|_{G_k} = O(\gamma_k)$ uniformly in k and $0 \leq s \leq 1$, third by (12),

$$\inf_{\lambda \in G_k} |\alpha_m^s - \lambda| \geq c|m - k|, \quad m \neq k, \quad 0 \leq s \leq 1,$$

fourth $\sigma_m^n - \lambda_m^\bullet = \gamma_m \ell_m^{1+}[n]$ by Proposition 4.1, and fifth $|F_k| = O(\gamma_k)$ by Lemma 3.3, we obtain with Lemma A.2 and Lemma 3.1 that

$$I_k^\flat = \int_{\Gamma_k} \frac{F_k^2(\lambda)((\sigma_m^n - \lambda)\psi_n(\lambda) + \Delta^\bullet(\lambda))}{\sqrt[3]{\Delta^2(\lambda) - 4}} d\lambda = \gamma_k^3 \ell_k^{1+}[n].$$

We thus have shown the claimed estimate (48).

It remains to consider the case $k = n$. Since $|F_n(\lambda)|_{G_n}, |w_n(\lambda)|_{G_n} = O(|\gamma_n|)$, it then follows from Lemma 3.6 that

$$|F_n^2(\lambda) + w_n^2(\lambda)|_{G_n} = \gamma_n^2(\ell_n^{p/2} + \ell_n^{1+}).$$

Moreover, $\zeta_n(\lambda)|_{U_n} = 1 + \ell_n^{p/2} + \ell_n^{1+}$ by Lemma B.3. We thus write

$$\begin{aligned} \Omega_{nn}^{(2)} &= i \int_{\Gamma_n} \frac{F_n^2(\lambda)\zeta_n(\lambda)}{w_n(\lambda)} d\lambda \\ &= i \int_{\Gamma_n} \frac{(-w_n^2(\lambda) + F_n^2(\lambda) + w_n^2(\lambda))(1 + \zeta_n(\lambda) - 1)}{w_n(\lambda)} d\lambda, \end{aligned}$$

and may apply Lemma 3.1 to obtain the estimate

$$\begin{aligned} &\frac{1}{2\pi} \left| \Omega_{nn}^{(2)} + i \int_{\Gamma_n} w_n(\lambda) d\lambda \right| \\ &\leq |F_n^2(\lambda) + w_n^2(\lambda)|_{G_n} + |w_n^2(\lambda)|_{G_n} |\zeta_n(\lambda) - 1|_{G_n} + |F_n^2(\lambda) + w_n^2(\lambda)|_{G_n} |\zeta_n(\lambda) - 1|_{G_n} \\ &= \gamma_n^2(\ell_n^{p/2} + \ell_n^{1+}). \end{aligned}$$

To compute the integral $\int_{\Gamma_n} w_n(\lambda) d\lambda$ note that if $\gamma_n = 0$ then $w_n(\lambda) = (\tau_n - \lambda)$ and hence $\int_{\Gamma_n} w_n(\lambda) d\lambda = 0$. On the other hand, if $\gamma_n \neq 0$, then we may use (22) to compute

$$\int_{\Gamma_n} w_n(\lambda) d\lambda = i \frac{\gamma_n^2}{2} \int_{-1}^1 \sqrt[3]{1 - t^2} dt = i\pi \frac{\gamma_n^2}{4}.$$

Consequently,

$$\Omega_{nn}^{(2)} = \frac{\gamma_n^2}{4} (\pi + \ell_n^{p/2} + \ell_n^{1+}). \quad \blacksquare$$

Theorem 5.4 (i) For any $n \in \mathbb{Z}$, the sum $-12 \sum_{k \in \mathbb{Z}} k \Omega_{nk}^{(2)}$ converges absolutely and locally uniformly on \mathcal{W}^p , $2 \leq p < \infty$, to an analytic function which is an analytic extension of $\omega_n^{(4)\star}$ given by (9).

(ii) For any $2 < p < \infty$, $\omega^{(4)\star} = (\omega_n^{(4)\star})_{n \in \mathbb{Z}}: \mathcal{W}^p \rightarrow \ell_{\mathbb{C}}^{-1,p/2}$ is a real analytic map which satisfies

$$\omega_n^{(4)\star} + (12n\pi)I_n = n(\ell_n^{p/3} + \ell_n^{1+})$$

locally uniformly on \mathcal{W}^p . For $p = 2$, $\omega^{(4)\star}: \mathcal{W}^2 \rightarrow \ell_{\mathbb{C}}^{-1,r}$ is real-analytic for any $r > 1$. \times

Remark. Let $\mathcal{V}^{p/2} = \{I = (I_n(\varphi))_{n \in \mathbb{Z}} : \varphi \in \mathcal{W}^p\}$ for $2 \leq p < \infty$. Then $\mathcal{V}^{p/2}$ is an open and dense subset of $\ell^{p/2}$ which contains ℓ^1 . Arguing as in the proof of [6, Theorem 20.3], one sees that for any $2 \leq p < \infty$, the frequency $\omega_n^{(4)\star}$ is a real analytic function of the actions on $\mathcal{V}^{p/2}$. \dashv

Proof. (i): In view of Lemma 5.3 for $k \neq n$

$$k\Omega_{nk}^{(2)} = (k-n)\Omega_{nk}^{(2)} + n\Omega_{nk}^{(2)} = \gamma_k^3 \ell_k^{1+}[n] + \frac{n}{n-k} \gamma_k^3 \ell_k^{1+}[n] \quad (49)$$

locally uniformly on \mathcal{W}^p . In particular, the sum $-12 \sum_{k \in \mathbb{Z}} k\Omega_{nk}^{(2)}$ is absolutely and locally uniformly convergent to an analytic function on \mathcal{W}^p for any $2 \leq p < \infty$. Moreover, the identity $\omega_n^{(4)\star} = -12 \sum_{k \in \mathbb{Z}} k\Omega_{nk}^{(2)}$, $n \in \mathbb{Z}$, holds for any real valued finite gap potential by Lemma 5.2. Consequently, $-12 \sum_{k \in \mathbb{Z}} k\Omega_{nk}^{(2)}$ is the unique analytic extension of $\omega_n^{(4)\star}$ from the set of finite gap potentials to \mathcal{W}^p , $2 \leq p < \infty$.

(ii): By (49) and Lemma A.1 we have

$$\sum_{k \neq n} k\Omega_{nk}^{(2)} = \sum_{k \neq n} \gamma_k^3 \ell_k^{1+}[n] + n \sum_{k \neq n} \frac{\gamma_k^3 \ell_k^{1+}[n]}{n-k} = \ell_n^\infty + n(\ell_n^{p/3} + \ell_n^{1+}).$$

Moreover, $n\Omega_{nn}^{(2)} = n \frac{\gamma_n^2}{4} (\pi + \ell_n^{p/2} + \ell_n^{1+})$ by Lemma 5.3, while

$$\frac{4I_n}{\gamma_n^2} = 1 + \ell_n^{p/2} + \ell_n^{1+},$$

by [13, Theorem 11.2] and hence

$$\frac{\gamma_n^2}{4} = I_n + \ell_n^{p/4} + \ell_n^1,$$

so that

$$\omega_n^{(4)\star} = -12n\pi I_n + n(\ell_n^{p/3} + \ell_n^{1+})$$

locally uniformly on \mathcal{W}^p . In particular, it follows that

$$\omega_n^{(4)\star} = n(\ell_n^{p/2} + \ell_n^{1+}). \quad \blacksquare$$

Proof of Theorem 1.3. The statement of the theorem is a direct consequence of Theorem 5.4. \blacksquare

6 Symmetries of the mKdV frequencies

The phase space $H^{3/2}(\mathbb{T}, \mathbb{R})$ of the mKdV and mKdV_# equation corresponds to the subspace $\mathcal{E}_r^2 \cap H_r^{3/2}$ of $H_r^{3/2}$ when they are viewed as equations in the NLS hierarchy. Here, for $2 \leq p < \infty$

$$\mathcal{E}_r^p = \{\varphi = (u, u) \in \mathcal{F}\ell_r^p\}.$$

As a first step towards the proof of Theorem 1.1, we describe this subspace in Birkhoff coordinates. It was shown in [5] that the space \mathcal{E}_r^p in the case $p = 2$ is characterized by the elements $\varphi \in \mathcal{F}\ell_r^2$ satisfying

$$I_{-k}(\varphi) = I_k(\varphi) \quad \forall k \in \mathbb{Z}, \quad \theta_{-k}(\varphi) = -\theta_k(\varphi) \quad \forall k \in \mathbb{Z} \text{ with } I_k(\varphi) \neq 0.$$

The same characterization holds for $2 < p < \infty$, since \mathcal{E}_r^2 is dense in \mathcal{E}_r^p and the Birkhoff map extends real analytically to \mathcal{W}^p . As a consequence, $z_n^- = \sqrt{I_n}e^{-i\theta_n}$ and $z_n^+ = \sqrt{I_n}e^{i\theta_n}$ defined on $\mathcal{F}\ell_r^p \setminus Z_n$ satisfy for any $n \in \mathbb{Z}$,

$$z_{-n}^-(\varphi) = z_n^+(\varphi), \quad z_n^+(\varphi) = \overline{z_{-n}^-(\varphi)}, \quad \forall \varphi \in \mathcal{E}_r^p. \quad (50)$$

In particular, the Hamiltonian

$$H_2 = \sum_{m \in \mathbb{Z}} (2m\pi) I_m$$

vanishes when restricted to $\mathcal{E}_r^2 \cap H_r^{1/2}$.

Theorem 5.4 then can be used to prove the following

Corollary 6.1 (i) *On \mathcal{E}_r^2 , the frequencies $\omega_n^{(4)}$, $n \in \mathbb{Z}$, are well defined and real analytic. For any $n \in \mathbb{Z}$, the restriction $\omega_n^{(4)}|_{\mathcal{E}_r^2}$ is the n th mKdV frequency which we denote by ω_n .*

(ii) *On \mathcal{E}_r^2 ,*

$$\omega_{-n} = -\omega_n, \quad \omega_{-n}^{(4)\star} = -\omega_n^{(4)\star}, \quad n \in \mathbb{Z},$$

and hence on \mathcal{E}_r^p , $2 \leq p < \infty$,

$$\omega_{-n}^{(4)\star} = -\omega_n^{(4)\star}, \quad n \in \mathbb{Z}. \quad \times$$

Proof. (i) At any finite gap potential of real type we have by (47)

$$\omega_n^{(4)} = (2n\pi)^3 + 6H_2 + 12n\pi H_1 + \omega_n^{(4)\star},$$

and the right hand side is real analytic on $H_r^{1/2}$. Moreover, on $\mathcal{E}_r^2 \cap H_r^{1/2}$ the Hamiltonian H_2 vanishes identically. Since H_1 is well defined and real analytic on $\mathcal{F}\ell_r^2$ (cf. [6]) and by Theorem 5.4, $\omega_n^{(4)\star}$, $n \in \mathbb{Z}$, are also well defined and real analytic on $\mathcal{F}\ell_r^2$, $\omega_n^{(4)}$ is well defined on \mathcal{E}_r^2 and real analytic there.

(ii) Let ϕ_t be a local flow for the vector field X_{H_4} . The mKdV flow leaves the space $\mathcal{E}_r^2 \cap H_r^{3/2}$ invariant, that is, $\phi_0 \in \mathcal{E}_r^2 \cap H_r^{3/2}$ implies that $\phi_t \in \mathcal{E}_r^2 \cap H_r^{3/2}$ for all $t \in \mathbb{R}$. Consequently, we compute on $\mathcal{E}_r^2 \cap H_c^{3/2} \setminus Z_n$

$$\omega_{-n}^{(4)} = \left. \frac{d}{dt} \theta_{-n} \circ \phi_t \right|_{t=0} = - \left. \frac{d}{dt} \theta_n \circ \phi_t \right|_{t=0} = -\omega_n^{(4)}, \quad n \in \mathbb{Z},$$

and the identity extends to all of \mathcal{E}_r^2 by continuity. Finally, on \mathcal{E}_r^2

$$\omega_n^{(4)*} = \omega_n^{(4)} - (2n\pi)^3 - 12n\pi H_1,$$

and the right hand side is an odd function of n . The claimed identity on \mathcal{E}_r^p with $2 \leq p < \infty$ then follows by a density argument. \blacksquare

7 Proof of Theorem 1.1

In a first step we establish the results corresponding to the ones of Theorem 1.1 in Birkhoff coordinates and to this end introduce some more notation. By (50), the Birkhoff coordinates have the following symmetries on \mathcal{E}_r^p

$$z_{-n}^- = z_n^+ \quad \text{and} \quad z_n^- = \overline{z_n^+} \quad \forall n \in \mathbb{Z}.$$

For any $2 \leq p < \infty$ we therefore define the real subspaces

$$e_r^p := \{(z_-, z_+) \in \ell_r^p : z_{-n}^- = z_n^+ \forall n \in \mathbb{Z}\},$$

and note that $\Phi_p(\mathcal{E}_r^p)$ is open and dense in e_r^p whereas for $p = 2$, $\Phi_2(\mathcal{E}_r^2) = e_r^2$. On $\Phi_2(\mathcal{E}_r^2)$, the mKdV equation takes the form

$$\partial_t z_n^- = -i\omega_n z_n^-, \quad \partial_t z_n^+ = i\omega_n z_n^+, \quad \forall n \in \mathbb{Z}, \quad (51)$$

where for any $n \in \mathbb{Z}$ and $\varphi \in \mathcal{E}_r^2$

$$\omega_n(\varphi) = (2n\pi)^3 + 12n\pi H_1(\varphi) + \omega_n^{(4)*}(\varphi).$$

Similarly, the mKdV $_{\#}$ equation reads

$$\partial_t z_n^- = -i\omega_n^{\#} z_n^-, \quad \partial_t z_n^+ = i\omega_n^{\#} z_n^+, \quad \forall n \in \mathbb{Z}, \quad (52)$$

where for any $n \in \mathbb{Z}$ and $\varphi \in \mathcal{E}_r^2$

$$\omega_n^{\#}(\varphi) = (2n\pi)^3 + \omega_n^{(4)*}(\varphi).$$

By Theorem 5.4, ω_n , $\omega_n^{\#}$, $n \in \mathbb{Z}$, are well defined on \mathcal{E}_r^2 and real analytic there. The corresponding solution maps are denoted by $\mathcal{S}_{\Phi}(t, \varphi)$ and $\mathcal{S}_{\Phi}^{\#}(t, \varphi)$, respectively. In more detail, given any $\varphi \in \mathcal{E}_r^2$ let $z = (z_-, z_+) = \Phi_2(\varphi)$ and $\omega_n = \omega_n(\varphi)$, $\omega_n^{\#} = \omega_n^{\#}(\varphi)$. Then for any $t \in \mathbb{R}$,

$$\mathcal{S}_{\Phi}(t, \varphi) = (e^{-i\omega_n t} z_n^-, e^{i\omega_n t} z_n^+)_{n \in \mathbb{Z}}, \quad \mathcal{S}_{\Phi}^{\#}(t, \varphi) = (e^{-i\omega_n^{\#} t} z_n^-, e^{i\omega_n^{\#} t} z_n^+)_{n \in \mathbb{Z}}.$$

By Corollary 6.1, on \mathcal{E}_r^2 one has for any $n \in \mathbb{Z}$ that $\omega_{-n} = -\omega_n$ hence by the definition of $\omega_n^\#$, one also has

$$\omega_{-n}^\# = -\omega_n^\# \quad (53)$$

It implies that $\mathcal{S}_\Phi(\cdot, \varphi)$ and $\mathcal{S}_\Phi^\#(\cdot, \varphi)$ both leave the subspace e_r^2 invariant. By Theorem 5.4, the frequencies $\omega_n^\#$, $n \in \mathbb{Z}$, real analytically extend to \mathcal{E}_r^p for any $2 \leq p < \infty$. It allows to extend the solution operator $\mathcal{S}_\Phi^\#(\cdot, \varphi)$ as well.

Theorem 7.1 (i) For any $\varphi \in \mathcal{E}_r^p$ with $2 \leq p < \infty$, the curve

$$\mathbb{R} \rightarrow \ell_r^p, \quad t \mapsto \mathcal{S}_\Phi^\#(t, \varphi) = (e^{-i\omega_n^\# t} z_n^-, e^{i\omega_n^\# t} z_n^+)_{n \in \mathbb{Z}}$$

is continuous and takes values in e_r^p .

(ii) For any $2 \leq p < \infty$ and $T > 0$,

$$\mathcal{S}_\Phi^\# : \mathcal{E}_r^p \rightarrow C([-T, T], e_r^p)$$

is continuous.

(iii) For any $2 < p < \infty$, $n \in \mathbb{Z}$, and $t > 0$, the coordinate functions $\varphi \mapsto e^{-i\omega_n^\# t} z_n^-$ and $\varphi \mapsto e^{i\omega_n^\# t} z_n^+$ cannot be extended continuously to elements $\varphi \in \mathcal{E}_r^p \setminus \mathcal{E}_r^2$ with $z_n^+ \neq 0$ and $z_n^- \neq 0$, respectively. \times

Proof. Item (i) and (ii) follow from [9, Theorem E.1], taking into account that by Corollary 6.1, the identity (53) holds on \mathcal{E}_r^p for any $2 \leq p < \infty$, implying that according to the above arguments, $\mathcal{S}_\Phi^\#(\cdot, \varphi)$ leaves e_r^p invariant for any $\varphi \in \mathcal{E}_r^p$.

Item (iii) holds since on $\mathcal{E}_r^p \setminus \mathcal{E}_r^2$, \mathcal{H}_1 is infinite. \blacksquare

We are now in a position to prove Theorem 1.1. First let us consider the case $p = 2$. According to [11], for any initial datum $\varphi \in \mathcal{E}_r^2$, there exists a unique global in time solution $\psi(t, x) = \psi(t, x, \varphi)$ of (1) with $t \mapsto \psi(t, \cdot, \varphi)$ being a continuous curve in \mathcal{E}_r^2 . Furthermore, for any $t \in \mathbb{R}$, the flow map $\mathcal{S}(t, \cdot) : \mathcal{E}_r^2 \rightarrow \mathcal{E}_r^2$ is continuous and for any $T > 0$ the solution map

$$\mathcal{S} : \mathcal{E}_r^2 \rightarrow C([-T, T], \mathcal{E}_r^2), \quad \varphi \mapsto \psi(\cdot, \cdot, \varphi)$$

is continuous as well. Since the mKdV flow preserves the L^2 -norm of any initial datum in \mathcal{E}_r^2 , it follows that for any solution $u(t)$ of (1) with $u(0) = u_0 \in L^2(\mathbb{T}, \mathbb{R})$ the curve $t \mapsto v(\cdot, t) = u(\cdot + 6\|u\|_{L^2}^2 t, t)$ is a solution of the mKdV $_\#$ equation with the same initial datum. Therefore, on \mathcal{E}_r^2 , solutions of (1) can be easily converted into solutions of (2) and vice versa, hence for any $T > 0$, the solution map $\mathcal{S}^\#$ of the mKdV $_\#$ equation

$$\mathcal{S}^\# : \mathcal{E}_r^2 \rightarrow C([-T, T], \mathcal{E}_r^2)$$

is real analytic and »equivalent« to the solution map \mathcal{S} of the mKdV equation. In contrast to \mathcal{S} , the map $\mathcal{S}^\#$ can be continuously extended to \mathcal{E}_r^p for any $2 < p < \infty$ as follows:

Corollary 7.2 (i) *The mKdV $_\#$ equation is locally in time C^0 -wellposed in $\mathcal{F}\ell^p(\mathbb{T}, \mathbb{R})$ for any $2 < p < \infty$.*

(ii) *The mKdV $_\#$ equation is globally in time C^0 -wellposed in a neighborhood of 0 in $\mathcal{F}\ell^p(\mathbb{T}, \mathbb{R})$ for any $2 < p < \infty$.*

(iii) *For any $2 < p < \infty$ and any $t > 0$, the solution map \mathcal{S} of the mKdV equation does not extend continuously to any element of $\mathcal{F}\ell^p(\mathbb{T}, \mathbb{R}) \setminus L^2(\mathbb{T}, \mathbb{R})$.* \times

Proof. (i) Let $\varphi \in \mathcal{E}_r^p (\cong \mathcal{F}\ell^p(\mathbb{T}, \mathbb{R}))$ with $2 \leq p < \infty$. Since the range of Φ_p is open in ℓ_r^p , by Theorem 7.1, there exists $T > 0$ and a neighborhood U of φ in \mathcal{E}_r^p so that

$$\mathcal{S}_\Phi^\#([-T, T], U) \subset \Phi_p(\mathcal{E}_r^p).$$

Then the map

$$\mathcal{S}^\# : U \rightarrow C([-T, T], \mathcal{E}_r^p), \quad \psi \mapsto (t \mapsto \Phi_p^{-1}(\mathcal{S}_\Phi^\#(t, \psi)))$$

is continuous.

(ii) Since the range of Φ_p contains an open neighborhood of the origin, by Theorem 7.1 we can choose a neighborhood U_0 of 0 in \mathcal{E}_r^p so that

$$\mathcal{S}_\Phi^\#(\mathbb{R}, U_0) \subset \Phi_p(\mathcal{E}_r^p).$$

Then the map

$$\mathcal{S}^\# : U_0 \rightarrow C([-T, T], \mathcal{E}_r^p), \quad \psi \mapsto (t \mapsto \Phi_p^{-1}(\mathcal{S}_\Phi^\#(t, \psi)))$$

is continuous for any $T > 0$.

(iii) The claimed statement directly follows from item (iv) of Theorem 7.1.

■

Proof of Theorem 1.1. The theorem is a direct consequence of Corollary 7.2.

■

Remark. Since Theorem 1.2 provides Birkhoff coordinates locally around zero also for the focusing mKdV equation – cf. [8] – by the same approach as for the defocusing case, one can show that the renormalized focusing mKdV equation is globally in time C^0 -wellposed locally around zero. Therefore, the focusing mKdV equation is ill-posed in $\mathcal{E}_r^p \setminus \mathcal{E}_r^2$ for $2 < p < \infty$. \dashv

A Discrete Hilbert Transform on ℓ^p

In this appendix we recall some well known facts on the discrete Hilbert transform on $\ell^p(\mathbb{Z}, \mathbb{C})$ – see e.g. [17].

Lemma A.1 *For any $2 \leq p < \infty$, the discrete Hilbert transform*

$$H: \ell_{\mathbb{C}}^p \rightarrow \ell_{\mathbb{C}}^p, \quad (Hx)_n = \sum_{m \neq n} \frac{x_m}{m - n},$$

defines an isomorphism on $\ell_{\mathbb{C}}^p$. \times

To simplify notation we define $\sigma^0 = (n\pi)_{n \in \mathbb{Z}}$. The following bound of a modified version of the Hilbert transform is used in to estimate certain infinite products.

Lemma A.2 *Suppose $\sigma = \sigma^0 + \tilde{\sigma}$ and $\rho = \sigma^0 + \tilde{\rho}$ are sequences of complex numbers with $\tilde{\sigma}, \tilde{\rho} \in \ell_{\mathbb{C}}^{\infty}$ such that for some $C > 0$*

$$|\rho_m - \sigma_n| \geq C^{-1}|m - n|, \quad m \neq n.$$

Then for any $2 \leq p < \infty$,

$$(Ax)_n = \pi \sum_{m \neq n} \frac{x_m}{\rho_m - \sigma_n}$$

defines a bounded linear operator A on $\ell_{\mathbb{C}}^p$ whose bound depends only on $\|\tilde{\sigma}\|_{\infty}$, $\|\tilde{\rho}\|_{\infty}$, C , and p . \times

B Infinite products

In this appendix we provide several estimates of infinite products with ℓ^p coefficients. The proofs for the case $p = 2$ can be found in [6] and their generalization to the case $2 \leq p < \infty$ is straightforward – see also [13].

Recall from (19) that for $\varphi \in \mathcal{W}^p$ the standard root is defined by

$$w_n(\lambda, \varphi) = (\tau_n - \lambda) \sqrt[3]{1 - \gamma_n^2/4(\tau_n - \lambda)^2}, \quad \lambda \notin G_n,$$

and is analytic in both variables on $(\mathbb{C} \setminus \overline{U_n}) \times V_{\varphi}$, where U_n , V_{φ} , and \mathcal{W}^p have been introduced in Section 2 – see (S1)-(S3).

Lemma B.1 *For any φ in \mathcal{W}^p , $2 \leq p < \infty$, and $n \in \mathbb{Z}$,*

$$f_n(\lambda) = \frac{1}{\pi_n} \prod_{m \neq n} \frac{w_m(\lambda)}{\pi_m}$$

defines a function which is analytic in λ on $\mathbb{C} \setminus \bigcup_{m \neq n} G_m$ and analytic in both variables on $(\mathbb{C} \setminus \bigcup_{m \neq n} \overline{U_m}) \times V_{\varphi}$. Moreover, f_n does not vanish on these domains. \times

We denote $\sigma^0 = (n\pi)_{n \in \mathbb{Z}}$.

Corollary B.2 For any $\varphi \in \mathcal{W}^p$, $\tilde{\sigma} = \sigma - \sigma^0 \in \ell^p$, and $n \in \mathbb{Z}$, the function

$$\phi_n(\lambda, \tilde{\sigma}) = \prod_{m \neq n} \frac{\sigma_m - \lambda}{w_m(\lambda)}$$

is analytic on $(\mathbb{C} \setminus \bigcup_{m \neq n} \overline{U_m}) \times \ell^p \times V_\varphi$. \times

We also introduce $\tau = (\tau_n)_{n \in \mathbb{Z}}$ then $\tau - \sigma^0 \in \ell^p$ for any $\varphi \in \mathcal{W}^p$.

Lemma B.3 For any $\varphi \in \mathcal{W}^p$, $2 \leq p < \infty$, and $\sigma - \sigma^0 \in \ell^p$ with $\sigma - \tau \in \ell^q$, $1 \leq q \leq p$,

$$\sup_{\lambda \in U_n} \left| \prod_{m \neq n} \frac{\sigma_m - \lambda}{w_m(\lambda)} - 1 \right| = \ell_n^q + \ell_n^{p/2} + \ell_n^{1+},$$

locally uniformly on $\ell^p \times V_\varphi$. As a consequence,

$$\frac{\sin \lambda}{\lambda - n\pi} \left(\frac{1}{\pi_n} \prod_{m \neq n} \frac{w_m(\lambda)}{\pi_m} \right)^{-1} = 1 + \ell_n^q + \ell_n^{p/2} + \ell_n^{1+}, \quad \lambda \in U_n$$

locally uniformly on V_φ . \times

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