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Trees and ultrametric Möbius structures

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Abstract: We define the concept of an ultrametric Möbius space (Z,M) and show that the boundary at infinity of a nonelementary geodesically complete tree is naturally an ultrametric Möbius space. In addition, we construct to a given ultrametric Möbius space (Z,M) a nonelementary geodesically complete tree, unique up to isometry, with (Z,M) being its boundary at infinity. This yields a one-to-one correspondence.

Trees and ultrametric Möbius structures

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Abstract

We define the concept of an ultrametric Möbius space and use this to characterize nonelementary geodesically complete trees.

Keywords Trees, Möbius structures, ultrametrics

Mathematics Subject Classification 53C35, 53C23

1 Introduction

In this paper we define the concept of an ultrametric Möbius space $(Z, \mathcal{M})$, where Z is some set with cardinality $|Z| \geq 3$ and $\mathcal{M}$ an ultrametric Möbius structure (see section 2). These Möbius spaces describe in a natural way the asymptotic geometry of trees. To a metric tree (X, d) one can associate in a natural way a boundary at infinity $\partial_\infty X$. We call a tree (X, d) geodesically complete, if every geodesic segment can be extended to a complete geodesic line. We call a tree nonelementary, if $|\partial_\infty X| \geq 3$. The boundary $\partial_\infty X$ of a nonelementary tree (X, d) carries a canonical ultrametric Möbius structure $\mathcal{M}_X$. If $F : (X, d) \to (X', d')$ is an isometric embedding of trees, then F extends in a natural way to a Möbius map $f : (\partial_\infty X, \mathcal{M}_X) \to (\partial_\infty X, \mathcal{M}_{X'})$.

We show that the association $(X, d) \mapsto (\partial_\infty X, \mathcal{M}_X)$ defines an equivalence between the category of isometry classes of nonelementary geodesically complete trees and the category of Möbius classes of ultrametric Möbius spaces. Therefore we will prove the following facts:

1. Let $(Z, \mathcal{M})$ be a complete ultrametric Möbius space. Then there exists a unique nonelementary geodesically complete tree (X, d), such that $(\partial_\infty X, \mathcal{M}_X)$ is Möbius equivalent to $(Z, \mathcal{M})$.

2. Let (X, d) and (X', d') be nonelementary geodesically complete trees. Then for any Möbius embedding $f : (\partial_\infty X, \mathcal{M}_X) \to (\partial_\infty X', \mathcal{M}_{X'})$, there exists a unique isometric embedding $F : (X, d) \to (X', d')$ such that the extension of F to the boundary coincides with f. If f is in addition surjective, then F is an isometry.

These results are in the spirit of [Hu] ,who investigated the case of trees with a basepoint. Probably all of the results of this paper are known in some way but usually formulated in a different language. To our opinion the viewpoint using ultrametric Möbius structures is new and worth to be formulated. Our paper is also inspired by the work of [Bi]. In this paper the author associates to a certain Möbius space $(Z, \mathcal{M})$ a filling (X, d) in the more general context of CAT(-1) spaces. The idea in [Bi] is to define the filling as the subset $\mathcal{M}_1^a \subset \mathcal{M}_X$ of antipodal diameter 1 metrics in $\mathcal{M}_X$ with a canonical metric $d_{\mathcal{M}_1^a}$ on it (compare section 5). Indeed one obtains the following fact (which is a slight generalization of a result in [Bi]):

3. Let $(Z, \mathcal{M})$ be a complete ultrametric Möbius space, $\mathcal{M}_1^a \subset \mathcal{M}_X$ the subset of antipodal diameter 1 metrics and (X, d) the tree from fact 1. Then $i : (X, d) \to (\mathcal{M}_1^a, d_{\mathcal{M}_1^a})$, $x \mapsto \rho_x$ is an isometry. Here ρ_x is the Bourdon metric at x.

The structure of the paper is as follows. In section 2 we introduce the concept of ultrametric Möbius spaces, in section 3 we recall some facts about metric trees. In section 4 we prove facts 1 and 2 and in the final section 5 we show fact 3.

2 Ultrametric Möbius spaces

2.1 Möbius Structure

Let Z be a set with cardinality $|Z| \geq 3$. An *extended metric* on Z is a map $\rho : Z \times Z \to [0, \infty]$, such that there exists a set $\Omega(\rho) \subset Z$ with cardinality $|\Omega(d)| \in \{0, 1\}$, such that ρ restricted to the set $Z \setminus \Omega(\rho)$ is a metric (taking only values in $[0, \infty)$) and such that $\rho(z, \omega) = \infty$ for all $z \in Z \setminus \Omega(\rho)$, $\omega \in \Omega(d)$. Furthermore $d(\omega, \omega) = 0$.
If $\Omega(\rho)$ is not empty, we sometimes denote $\omega \in \Omega(\rho)$ simply as ∞ and call it the (infinitely) *remote point* of (Z, ρ). We often write also $\{\omega\}$ for the set $\Omega(\rho)$ and Z_ω for the set $Z \setminus \{\omega\}$.

We consider on (Z, ρ) the topology with the basis consisting of all open distance balls $B_r(z)$ around points in $z \in Z_\omega$ and the complements D^C of all closed distance balls $D = \overline{B}_r(z)$. Note that (Z, ρ) is a Hausdorff space.

We call an extended metric space *complete*, if first every Cauchy-sequence in Z_ω converges and secondly if the infinitely remote point ω exists in case that Z_ω is unbounded. For example the real line $(\mathbb{R}, \rho)$, with its standard metric is *not* complete as an extended metric space, while $(\mathbb{R} \cup \{\infty\}, \rho)$ is complete.

One can also characterize completeness using the following notation.

Definition 2.1. Let (Z, ρ) be an extended metric space. A sequence $(z_i) \subset Z$ is a *Cauchy-sequence in* (Z, ρ), if one the following is true:

1. For an arbitrary point $o \in Z_\omega$ there exists $C > 0$ and $n \in \mathbb{N}$, such that $\rho(o, z_i) \leq C$ for all $i \geq n$ and the sequence $(z_i)_{i \geq n}$ is a Cauchy-sequence in Z_ω in the classical sense.

2. For an arbitrary point $o \in Z_\omega$ and every $C \in \mathbb{R}$, there is an $n \in \mathbb{N}$, such that $\rho(o, z_i) \geq C$ for all $i \geq n$.

Then completeness of (Z, ρ) is equivalent to the convergence of Cauchy-sequences. A Cauchy-sequence of the second type is converging to the remote point ω.

We say that a quadruple $(x, y, z, w) \in Z^4$ is *admissible*, if no entry occurs three or four times in the quadruple. We denote with $Q \subset Z^4$ the set of admissible quadruples. We define the *cross ratio triple* as the map crt $: Q \to \Sigma \subset \mathbb{R}P^2$ which maps admissible quadruples to points in the real projective plane defined by

$$\mathrm{crt}_\rho(x, y, z, w) = (\rho(x,y)\rho(z,w) : \rho(x,z)\rho(y,w) : \rho(x,w)\rho(y,z)),$$

here Σ is the subset of points $(a : b : c) \in \mathbb{R}P^2$, where all entries a, b, c are nonnegative or all entries are non-positive. Note that Σ can be identified with the standard 2-simplex, $\{(a, b, c) \,|\, a, b, c \geq 0, \ a + b + c = 1\}$.

We use the standard conventions for the calculation with ∞. If ∞ occurs once in Q, say $w = \infty$, then $\mathrm{crt}_\rho(x, y, z, \infty) = (\rho(x,y) : \rho(x,z) : \rho(y,z))$. If ∞ occurs twice , say $z = w = \infty$ then $\mathrm{crt}_\rho(x, y, \infty, \infty) = (0 : 1 : 1)$.

The cross ratio triple is a more symmetric way to define the classical cross ratio $[., ., ., .]_\rho : Q \to [0, \infty]$ which we define as

$$[x, y, z, w]_\rho := \frac{\rho(x,z)\rho(y,w)}{\rho(x,y)\rho(z,w)}.$$

It is not difficult to check that $\mathrm{crt}_\rho : Q \to \Sigma$ is continuous, where Q and Σ carry the obvious topologies induced by (Z, ρ) and $\mathbb{R}P^2$. Thus, if $(x_i, y_i, z_i, w_i) \in Q$ for $i \in \mathbb{N}$ and assume $x_i \to x, \ldots, w_i \to w$, where $(x, y, z, w) \in Q$ then $\mathrm{crt}_\rho(x_i, y_i, z_i, w_i) \to \mathrm{crt}_\rho(x, y, z, w)$.

One can characterize convergence in (Z, ρ) and the Cauchy-sequence property in terms of the cross ratio by the following Lemma (where we omit the proof) :

Lemma 2.2. *Let (Z, ρ) be an extended metric space. A sequence $(z_i) \subset Z$ converges to $z \in Z$, if and only if there are distinct points $a, b \in Z$, s.t. $\mathrm{crt}_\rho(z, z_j, a, b) \to (0 : 1 : 1)$.*
A sequence $(z_i) \subset Z$ is a Cauchy-sequence in (Z, ρ), if and only if there are distinct points $a, b \in Z$, s.t. $\mathrm{crt}_\rho(z_i, z_j, a, b) \to (0 : 1 : 1)$ for $i, j \to \infty$.

A map $f : (Z, \rho) \to (Z', \rho')$ between two extended metric spaces is called *Möbius*, if f is injective and for all admissible quadruples (x, y, z, w) of Z,

$$\mathrm{crt}_{\rho'}(f(x), f(y), f(z), f(w)) = \mathrm{crt}_\rho(x, y, z, w).$$

Now Lemma 2.2 implies that Möbius maps are continuos and Cauchy-sequences are mapped to Cauchy-sequences.

Two extended metric spaces (Z, ρ) and (Z', ρ') are *Möbius equivalent*, if there exists a bijective Möbius map $f : Z \to Z'$. In this case also f^{-1} is a Möbius map and f is in particular a homeomorphism. Furthermore, by Lemma 2.2 (Z, ρ) is complete if and only if (Z', ρ') is.

We say that two extended metrics ρ and ρ' on a set Z are *Möbius equivalent*, if the identity map $\mathrm{id} : (Z, \rho) \to (Z, \rho')$ is a Möbius map. Möbius equivalent metrics define the same topology on Z and (Z, ρ) is complete if and only (Z, ρ') is.

A *Möbius structure* on a set Z is a nonempty set $\mathcal{M}$ of extended metrics on Z, which are pairwise Möbius equivalent and which is maximal with respect to that property. Thus $\mathcal{M}$ is just an equivalence class of Möbius equivalent extended metrics on Z.

Given an extended metric ρ on Z, we denote by $[\rho]$ the equivalence class of all Möbius equivalent extended metrics. Then $[\rho]$ is a Möbius structure on Z.

A *Möbius space* is a pair $(Z, \mathcal{M})$ of a set Z with cardinality $|Z| \geq 3$ and a Möbius structure $\mathcal{M}$ on Z. If $(Z, \mathcal{M})$ is a Möbius space and (x, y, z, w) an admissible quadruple, then the cross ratio triple is independent of the metric and hence $\mathrm{crt}(x, y, z, w) := \mathrm{crt}_\mathcal{M}(x, y, z, w) := \mathrm{crt}_\rho(x, y, z, w)$ is well defined, where $\rho \in \mathcal{M}$.

A Möbius space $(Z, \mathcal{M})$ is a topological space, since $\rho, \rho' \in \mathcal{M}$ define the same topology on Z. A Möbius space is *complete*, if (Z, ρ) is complete for $\rho \in \mathcal{M}$.

A map $f : (Z, \mathcal{M}) \to (Z', \mathcal{M}')$ between two Möbius spaces is called *Möbius*, if it is injective and preserves the cross ratio triple. The spaces are *Möbius equivalent* if there exists a bijective Möbius map between them.

Similar as for metric spaces, one can define the completion of a Möbius space $(Z, \mathcal{M})$. Define the set $\overline{Z}$ to be the set of equivalence classes of Cauchy-sequences, where two Cauchy-sequences (x_i) and (y_i) are called equivalent, if $\mathrm{crt}(x_i, y_i, a, b) \to (0 : 1 : 1)$ for some distinct a, b. If $\rho \in \mathcal{M}$ then define an extended metric $\overline{\rho}$ on $\overline{Z}$ by $\overline{\rho}((x_i), (y_i)) := \lim \rho(x_i, y_i)$, here (x_i) and (y_i) are nonequivalent Cauchy-sequences. The completion of $(Z, \mathcal{M})$ is unique up to Möbius equivalence.

2.2   Ultrametric Möbius structures

A point $(a : b : c) \in \Sigma \subset \mathbb{R}P^2$, $a, b, c \geq 0$ is called *ultrametric*, if the two largest of the numbers a, b, c coincide. The set of ultrametric points in Σ

is *Y-shaped* and consists out of the three affine segments from the central point $(1:1:1)$ to the three points $(0:1:1), (1:0:1), (1:1:0)$.

A Möbius space $(Z, \mathcal{M})$ is called *ultrametric*, if for all admissible quadruples (x, y, z, w) the cross ratio triple $\mathrm{crt}(x, y, z, w)$ is ultrametric.

As usual, one calls an extended metric ρ on a set Z *ultrametric*, if for distinct $x, y, z \in Z_\omega$ the point $(\rho(x, y) : \rho(x, z) : \rho(y, z))$ is ultrametric.

If (Z, ρ) is an extended ultrametric space and $\mathcal{M} = [\rho]$, then $(Z, \mathcal{M})$ is an ultrametric Möbius space. This follows e.g. from [BS] Lemma 5.1.2. On the other side, if $(Z, \mathcal{M})$ is ultrametric, then not necessarily all $\rho \in \mathcal{M}$ are ultrametrics. Consider for example the set $Z = \{x, y, z, w\}$ and let $\rho(x, y) = t$, $\rho(z, w) = 1/t$, $\rho(x, z) = s$, $\rho(y, w) = 1/s$, $\rho(x, w) = r$, $\rho(y, z) = 1/r$, where t, s, r are numbers close to one. The only relevant cross ratio triple is $\mathrm{crt}_\rho(x, y, z, w) = (1:1:1)$ which is ultrametric. All these metrics are Möbius equivalent but only for $s = t = r = 1$ this is an ultrametric.

However in two cases one obtains ultrametrics:

Lemma 2.3. *Let $(Z, \mathcal{M})$ be an ultrametric Möbius space and $\rho \in \mathcal{M}$ such that a remote point $\omega \in \Omega(\rho)$ exists. Then ρ is an ultrametric.*

Proof. Consider without loss of generality $x, y, z \in Z_\omega$. Then

$$crt_\rho(x, y, z, \omega) = (\rho(x, y) : \rho(x, z) : \rho(y, z)).$$

As crt_ρ is ultrametric it follows that ρ is an ultrametric. $\square$

A metric ρ on Z is called *antipodal diameter 1 metric*, if $\mathrm{diam}(Z, \rho) = 1$ and every $z \in Z$ has an antipodal point $z' \in Z$, i.e. $\rho(z, z') = 1$.

Lemma 2.4. *Let $(Z, \mathcal{M})$ be an ultrametric Möbius space and $\rho \in \mathcal{M}$ an antipodal diameter 1 metric, then ρ is an ultrametric.*

Proof. Assume the contrary, then there are $x, y, z \in Z$ such that $\rho(x, y) > \rho(x, z) \geq \rho(y, z)$. Let $z' \in Z$ with $\rho(z, z') = 1$. Then

$$\rho(x, y)\rho(z, z') > \rho(x, z)\rho(y, z') \quad \wedge \quad \rho(x, y)\rho(z, z') > \rho(y, z)\rho(x, z'),$$

which contradicts the fact that the cross ratio triple $\mathrm{crt}(x, y, z, z')$ is ultrametric. $\square$

An ultrametric Möbius space is in ptolemaic in the sense of [FS], which implies:

Lemma 2.5. *Let $(Z, \mathcal{M})$ be an ultrametric Möbius space and let $\omega \in Z$, then there exists $\rho \in \mathcal{M}$, such that $\Omega(\rho) = \{\omega\}$.*

3   Trees

For this paper a *tree* is a metric space (X, d) with the following two properties:

1. X is *uniquely geodesic*: if $x, y \in X$, then there exists a unique isometric map $c : [0, \ell] \to X$, $\ell = d(x, y)$ with $c(0) = x$ and $c(\ell) = y$. The image of c is denoted by $[x, y]$ and called the (geodesic) *segment* between x and y. The points x and y are called the *endpoints* of $[x, y]$. Note that $[x, y] = [y, x]$.

2. If two geodesic segments $[x, y]$ and $[y, z]$ have only one point in common, which is an endpoint of each segment, then their union is also a segment; i.e.: if $[x, y] \cap [y, z] = \{y\}$ then $[x, z] = [x, y] \cup [y, z]$.

For the distance in X we also use the notation $|xy| := d(x, y)$.

A tree is called *geodesically complete*, if every nontrivial segment is contained in a geodesic line, i.e. if every isometric map $[0, \ell] \to X$, $\ell > 0$ extends to an isometric map $\mathbb{R} \to X$.

Trees are $\mathrm{CAT}(-1)$ spaces in the sense of comparison geometry (see e.g. [BS]) and are in particular geodesic Gromov hyperbolic spaces. Actually they are 0-hyperbolic and $\mathrm{CAT}(\kappa)$ for all $\kappa \in \mathbb{R}$. We now collect some well known facts about trees.

If $x, y, z \in X$ are three points, then there exists a unique point $w := \mathrm{Trip}(x, y, z) \in X$ with $\{w\} = [x, y] \cap [y, z] \cap [x, z]$. The point w is called the *tripod* of x, y, z.

We have $|xw| = (z|y)_x$, $|yw| = (x|z)_y$ and $|zw| = (x|y)_z$, where we use the so called *Gromov product* $(x|y)_z := \frac{1}{2}(|zx| + |zy| - |xy|)$.

To define the boundary at infinity we call a sequence (x_i) in X a *Gromov sequence*, if for some basepoint $o \in X$, $(x_i|x_j)_o \to \infty$ for $i, j \to \infty$. Two Gromov sequences $(x_i), (y_i)$ are called equivalent, if $(x_i|y_i)_o \to \infty$. The equivalence classes are the points of the boundary $\partial_\infty X$.

We call a tree X *non-elementary*, if the cardinality $|\partial_\infty X| \geq 3$. Thus (up to isometry) the only nontrivial elementary geodesically complete tree is the euclidean line $\mathbb{R}$.

Given a point $a \in \partial_\infty X$ and a point $x \in X$, there exists a unique geodesic ray (i.e. a unique isometric map) $c : [0, \infty) \to X$, such that $c(0) = x$ and $c(\infty) = a$ (which means that the sequence $c(i)$, $i \in \mathbb{N}$, represents a). We denote the image of this ray as $[x, a)$.

Given two distinct points $a, b \in \partial_\infty X$ and $x \in X$, there exists a unique point $u \in X$ such that $[x, a) \cap [x, b) = [x, u]$. Then $(a, b) := [u, a) \cup [u, b)$ is the image of an isometric embedding $\mathbb{R} \to X$ and called the line between a and b. If $(a_i), (b_i)$ are sequences in X with $a_i \to a$ and $b_i \to b$, then $(a_i|b_i)_x = |xu|$ for i sufficiently large. We define $(a|b)_x := \lim(a_i|b_i)_x = |xu|$.

For $x, y \in X$ and $a \in \partial_\infty X$ we also define $(a|x)_y := \lim(a_i|x)_y$, where (a_i) is a sequence in X converging to a.

If $a \in \partial_\infty X$, the Busemann function $B_a : X \times X \to \mathbb{R}$ is defined as

$$B_a(x, y) = \lim(|xa_i| - |ya_i|) = (a|y)_x - (a|x)_y$$

where (a_i) is a sequence converging to a. All these limits are well defined.

For $x \in X$ and $a, b \in \partial_\infty X$ let $\rho_x(a, b) := e^{-(a|b)_x}$. The ρ_x defines the *Bourdon metric* (compare [Bou]) on $\partial_\infty X$ with basepoint $x \in X$. This is an ultrametric. Since $(a|b)_x - (a|b)_y = \frac{1}{2}(B_a(x, y) + B_b(x, y))$, we have

$$\rho_y(a, b) = \lambda(a)\lambda(b)\rho_x(a, b)$$

where $\lambda = \lambda_{x,y} : \partial_\infty X \to \mathbb{R}$ is $\lambda(a) = e^{\frac{1}{2}B_a(x,y)}$.

In particular the metrics ρ_x and ρ_y are Möbius equivalent and they define (in the case that X is nonelementary) the same Möbius structure on $\partial_\infty X$.

We denote this Möbius structure by $\mathcal{M}_X$ and call it the *canonical Möbius structure* on X. This Möbius structure is ultrametric.

For $\omega \in \partial_\infty X$ and $o \in X$ we define for $a, b \in \partial_\infty X \setminus \{\omega\}$

$$(a|b)_{\omega,o} := (a|b)_o - (a|\omega)_o - (b|\omega)_o.$$

Then

$$\rho_{\omega,o}(a, b) := e^{-(a|b)_{\omega,o}} = \frac{\rho_o(a, b)}{\rho_o(a, \omega)\rho_o(b, \omega)}$$

defines a metric on $\partial_\infty X \setminus \{\omega\}$ which can be considered as an extended metric on $\partial_\infty X$ with remote point ω. This metric is also contained in $\mathcal{M}_X$.

4 Geodesically complete trees versus ultrametric Möbius spaces

Theorem 4.1. *Let $(Z, \mathcal{M})$ be a complete ultrametric Möbius space. Then there is a nonelementary geodesically complete tree X, such that $(\partial_\infty X, \mathcal{M}_X)$ is Möbius equivalent to $(Z, \mathcal{M})$.*

Proof. By Lemma 2.5 we know that for an ultrametric Möbius structure $(Z, \mathcal{M})$ there is $\rho \in \mathcal{M}$ such that $\Omega(\rho) = \{\omega\}$. By Lemma 2.3 ρ is defines an ultrametric on Z_ω. We define the space

$$X := (Z_\omega \times \mathbb{R})/ \sim$$

where

$$(z_1, t_1) \sim (z_2, t_2) \iff t_1 = t_2 \ \wedge \ t_1 \leq -\ln \rho(z_1, z_2)$$

for $z_1, z_2 \in Z_\omega$ and $t_1, t_2 \in \mathbb{R}$. Since ρ is an ultrametric $\sim$ is an equivalence relation. The space X will be the "filling" of Z. Therefore we define

$$d([z_1, t_1], [z_2, t_2]) := t_1 + t_2 - 2\min(t_1, t_2, -\ln\rho(z_1, z_2)).$$

Similar to section 6 of [Hu] one can show that d is a well defined metric. We first show that (X, d) has a natural bicombing, i.e. given $[z_1, t_1], [z_2, t_2] \in X$, there is a *natural geodesic* between these points, which can be described as follows:
Assume w.l.o.g. $t_1 < t_2$. If $t_1 \leq -\ln\rho(z_1, z_2)$, the natural geodesic joining $[z_1, t_1]$ to $[z_2, t_2]$ is $\gamma : [t_1, t_2] \to X$, $\gamma(s) := [z_2, s]$. If $t_1 > -\ln\rho(z_1, z_2)$, let $a := t_1 + \ln\rho(z_1, z_2) + t_2 + \ln\rho(z_1, z_2)$. We define the natural geodesic joining $[z_1, t_1]$ to $[z_2, t_2]$ to be

$$\gamma : [0, a] \to X \quad \gamma(s) := \begin{cases} [z_1, t_1 - s] & s \leq t_1 + \ln\rho(z_1, z_2) \\ [z_2, -\ln\rho(z_1, z_2) + s] & s > t_1 + \ln\rho(z_1, z_2) \end{cases}$$

It is straight forward to check that γ is a geodesic with $\gamma(0) = [z_1, t_1]$ and $\gamma(a) = [z_2, t_2]$. The bicombing implies in particular that X is geodesic. For any three points $[z_1, t_1], [z_2, t_2], [z_3, t_3] \in X$ there is a geodesic triangle formed by natural geodesics. Using the ultrametric property of ρ we can assume w.l.o.g. $\rho(z_1, z_2) \leq \rho(z_1, z_3) = \rho(z_2, z_3)$. By definition of the natural geodesics it follows that this geodesic triangle is 0-thin in the sense of Gromov and the tripod is $[z_1, -\ln\rho(z_1, z_2)]$. In particular for any three points $x, y, z \in X$, there exists a tripod u which is contained in all natural geodesics between the three points. Thus the equality $d(x, z) + d(z, y) = d(x, y)$ holds only if $z = u$ and hence z lies on the natural geodesic from x to y. This implies that the natural geodesics are (up to parametrization) the unique geodesics. Thus X is uniquely geodesics and satisfies the first defining property of a tree. By definition of the natural geodesics it is easy to check that also the second defining property of a tree is satisfied.

In order to prove that X is geodesically complete, we will show that any two points lie on a geodesic line. As both types of natural geodesics can be extended to bi-infinite lines, namely γ is either of the form $\gamma(t) = [z, t]$ for some $z \in Z_\omega$ or $\gamma(t) = [z_1, -t]$ for $t < -\ln\rho(z_1, z_2)$ and $\gamma(t) = [z_2, t]$ for $t \geq -\ln\rho(z_1, z_2)$, X is geodesically complete.

We will show that the geodesic boundary of X equals Z. We define a map $\iota : Z \to \partial_\infty X$ in the following way: For $z \in Z_\omega$ let $\iota(z) := \gamma_z(\infty)$ where $\gamma_z(t) := [z, t]$. Furthermore $\iota(\omega) := \overline{\gamma}_z(\infty)$ where $\overline{\gamma}_z(t) := [z, -t]$ and $z \in Z_\omega$ is arbitrary. Note that for different $z \in Z_\omega$ the rays $\overline{\gamma}_z$ are equivalent in terms of the geodesic boundary, as $[z_1, -t] = [z_2, -t]$ for $-t \leq -\ln\rho(z_1, z_2)$. It is clear that the map ι is injective.

To show surjectivity let $v' \in \partial_\infty X \setminus \{\iota(\omega)\}$ be given. Let $\gamma : [0, \infty) \to X$ be the unit speed ray with $\gamma(0) = [u, 0]$, where $u \in Z_\omega$ is some basepoint, and

$\gamma(\infty) = v'$. We know that two points can be joined by the unique natural geodesic as described above. Since $v' \neq \iota(\omega)$ we see that there is $s_0 \in [0, \infty)$, such that $\gamma(s_0) = [z_{s_0}, 0]$ and for $s > s_0$ it is $\gamma(s) = [z_s, s - s_0]$ for some points $z_s \in Z_\omega$. Consider now numbers $s' > s > s_0$. We have $d(\gamma(s'), \gamma(s)) = s' - s$ as well as $d(\gamma(s'), \gamma(s)) = s' + s - 2s_0 - \min\{s - s_0, -\ln \rho(z_s, z_{s'})\}$. Hence $-\ln \rho(z_s, z_{s'}) \geq s - s_0$. By completeness of (Z, ρ) this implies that the points z_s converge in Z_ω for $s \to \infty$. Let $z_s \to v \in Z_\omega$ for $s \to \infty$. Then $[z_s, t] = [v, t]$ for t sufficiently large. Thus $\gamma(s) = [v, s - s_0]$ for s large which implies that $\iota(v) = v'$.

Finally we show that the Möbius structures coincide. Choose a basepoint $u \in Z_\omega$. We actually show that $\iota : (Z, \rho) \to (\partial_\infty X, \rho_{\omega', [u,0]})$ is an isometry, where $\rho_{\omega', [u,0]} = e^{-(\cdot|\cdot)_{\omega', [u,0]}}$ is the metric in the canonical Möbius structure of $\partial_\infty X$ with remote point $\omega' = \iota(\omega)$.
For $a, b \in Z_\omega$ we compute, where we use $a' = \iota(a)$, $b' = \iota(b)$,

$$
\begin{aligned}
(a'|b')_{\omega', [u,0]} &= (a'|b')_{[u,0]} - (a'|\omega')_{[u,0]} - (b'|\omega')_{[u,0]} \\
&= \lim_{t \to \infty} \left(\, ([a,t]|[b,t])_{[u,0]} - ([a,t]|[u,-t])_{[u,0]} - ([b,t]|[u,-t])_{[u,0]} \, \right) \\
&= \lim_{t \to \infty} \frac{1}{2}(2t - d([a,t],[b,t])) \\
&= \lim_{t \to \infty} \frac{1}{2}(2t - (2t - 2\min\{t, -\ln \rho(a,b)\})) \\
&= -\ln \rho(a,b)
\end{aligned}
$$

which implies $\rho_{\omega', [u,0]}(a', b') = \rho(a, b)$.

$\square$

Lemma 4.2. *Let (X, d) be a tree and let $a, b, c_1, c_2 \in \partial_\infty X$ points, such that a, b, c_1 and a, b, c_2 are distinct. Let $u_i = Trip(a, b, c_i)$, then*

$$|u_1 u_2| = \pm \ln([a, c_1, c_2, b]),$$

while we have a positive sign if $u_2 \in [u_1, b)$ and a negative if $u_1 \in [u_2, a)$.

Proof. Let $o \in (a, b)$ s.t. $u_1, u_2 \in [o, b)$ and ρ_o denotes the Bourdon metric with some basepoint $o \in X$. Let $\mu_1 = |ou_1|$ and $\mu_2 = |ou_2|$, then it is $\rho_o(a, c_1) = 1$, $\rho_o(a, c_2) = 1$, $\rho_o(b, c_1) = e^{-\mu_1}$, $\rho_o(b, c_2) = e^{-\mu_2}$ and hence

$$
\begin{aligned}
[a, c_1, c_2, b] &= \frac{\rho_o(a, c_2)\rho_o(c_1, b)}{\rho_o(b, c_2)\rho_o(c_1, a)} = e^{-\mu_1 + \mu_2}. \\
\implies \quad \ln([a, c_1, c_2, b]) &= \mu_2 - \mu_1.
\end{aligned}
$$

If $\mu_1 \leq \mu_2$, $\ln([a, c_1, c_2, b]) = |u_1 u_2|$ and if $\mu_1 \geq \mu_2$, $\ln([a, c_1, c_2, b]) = -|u_1 u_2|$, which proves the claim. $\square$

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Theorem 4.3. *Let X be a nonelementary geodesically complete tree and let X' be an arbitrary tree with $|\partial_\infty X'| \geq 3$. Then for any Moebius embedding $f : \partial_\infty X \to \partial_\infty X'$ w.r.t. the natural Moebius structures, there exists a unique isometric embedding $F : X \to X'$. s.t. the extension of F to the boundary coincides with f.*

If f is in addition surjective and X' geodesically complete, then F is an isometry.

Proof. First we define a map $F : X \to X'$. For a given $x \in X$ there are, since X is nonelementary and geodesically complete, points $a, b \in \partial_\infty X$, s.t. $x \in (a, b)$. Let $c \in \partial_\infty X \backslash \{a, b\}$ and $\gamma_{a,b,c}$ the unit speed geodesic with $\gamma(-\infty) = a$, $\gamma(\infty) = b$ and $\gamma(0) = Trip(a, b, c)$. Then $x = \gamma(t_x)$ for $t_x = \pm|Trip(a, b, c)x|$, while the sign is positive if $x \in [Trip(a, b, c), b)$ and negative if $x \in (a, Trip(a, b, c)]$. Let $a' = f(a)$, $b' = b$ and $c' = c$, we define $F_{a,b,c}(x) := \gamma'_{a',b',c'}(t_x)$ for $\gamma'_{a',b',c'}$ the unit speed geodesic with $\gamma'(-\infty) = a'$, $\gamma'(\infty) = b'$ and $\gamma'(0) = Trip(a', b', c')$. We have to show that $F_{a,b,c}$ is independent of different choices of a, b and c. Let $a_1, b_1, c_1, a_2, b_2, c_2 \in \partial_\infty X$, s.t. $x \in (a_1, b_1)$, $x \in (a_2, b_2)$ and the three points a_i, b_i, c_i are mutually distinct for $i = 1, 2$. In three steps we will show the following

Claim: $F_{a_1,b_1,c_1}(x) = F_{a_2,b_2,c_2}(x)$.

Step 1: $F_{a,b,c}(x) = F_{b,a,c}(x)$

By construction it follows $\gamma_{b,a,c}(-t) = \gamma_{a,b,c}(t)$, the absolute value of t_x stays the same, but the sign of t_x changes by interchanging a and b in the above construction. In particular this implies $F_{a,b,c}(x) = F_{b,a,c}(x)$.

Step 2: If $x \in (a, b) \cap (a, b')$ then $F_{(a,b,b')}(x) = F_{(a,b',b)}(x)$ and if $x \in (a, b) \cap (a', b)$ then $F_{(a,b,a')}(x) = F_{(a',b,a)}(x)$

Note that $\gamma_{a,b,b'}$ and $\gamma_{a,b',b}$ coincide on $(-\infty, 0]$ and $\gamma_{a,b,a'}$ coincides with $\gamma_{a',b,a}$ on $[0, \infty)$.

Step 3: $F_{a,b,c_1}(x) = F_{a,b,c_2}(x)$ in the case that $c_1, c_2 \in \partial_\infty X \setminus \{a, b\}$

Let $u_i = Trip(a, b, c_i)$ and $t_x(a, b, c_i) = \pm|Trip(a, b, c_i)x|$ be the signed distances as in the construction. By construction it's

$$t_x(a, b, c_1) = t_x(a, b, c_2) - |u_1 u_2| \quad \text{if} \quad u_2 \in [u_1, b)$$
$$t_x(a, b, c_1) = t_x(a, b, c_2) + |u_1 u_2| \quad \text{if} \quad u_1 \in [u_2, b).$$

Let $u'_i = Trip(a', b', c'_i)$ and let $\pm|F_{a,b,c_2}(x)u'_1|$ be the signed distance being positive if $F_{a,b,c_2}(x) \in [u'_1, b')$ and negative if $u'_1 \in [F_{a,b,c_2}(x), b')$. Due to elementary arguments it follows

$$\pm|F_{a,b,c_2}(x)u'_1| = t_x(a, b, c_2) - |u'_1 u'_2| \quad \text{if} \quad u'_2 \in [u'_1, b')$$
$$\pm|F_{a,b,c_2}(x)u'_1| = t_x(a, b, c_2) + |u'_1 u'_2| \quad \text{if} \quad u'_1 \in [u'_2, b').$$

We have $[a, c_1, c_2, b] = [a', c'_1, c'_2, b']$, as f is a Möbius map. Using this and Lemma 4.2, it follows that $\pm|F_{a,b,c_2}(x)u'_1| = t_x(a, b, c_1)$. By construction $t_x(a, b, c_1) = \pm|F_{a,b,c_1}(x)u'_1|$. Thus $F_{a,b,c_1}(x)$ and $F_{a,b,c_2}(x)$ lie on the

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geodesic (a', b') with the same signed distance to the point u_1', hence it follows $F_{a,b,c_1}(x) = F_{a,b,c_2}(x)$.

Using step 1 and step 3 the claim follows in the case that $\{a_1, b_1\} = \{a_2, b_2\}$. Thus (by eventually interchanging a_2 and b_2) we can assume that $b_1 \neq b_2$. Consider first the case $a_1 = a_2$. Using step 2 and step 3 we get

$$F_{a_1,b_1,c_1}(x) = F_{a_1,b_1,b_2}(x) = F_{a,b_2,b_1}(x) = F_{a_1,b_2,c_2}(x)$$

and hence the claim. Let us now assume that a_1, a_2, b_1, b_2 are all distinct. Let $u = Trip(a_1, b_1, a_2)$ and $v = Trip(a_1, b_1, b_2)$. By eventually interchanging a_2 and b_2, we can assume that the order of the points on the line (a_1, b_1) are $a_1 < u < v < b_1$. Note that then $x \in (a_i, b_j)$ for every choice of i, j. Using the various equalities we obtain:

$$F_{a_1,b_1,c_1}(x) = F_{a_1,b_1,b_2}(x) = F_{a_1,b_2,b_1}(x) = F_{b_2,a_1,b_1}(x)$$
$$= F_{b_2,a_1,a_2}(x) = F_{b_2,a_2,a_1}(x) = F_{a_2,b_2,a_1}(x) = F_{a_2,b_2,c_2}(x)$$

In particular we have a well defined map $F : X \to X'$ by $F(x) := F_{a,b,c}(x)$, $a, b, c \in \partial_\infty X$ distinct and $x \in (a, b)$.

We show that F is an isometry. Let $x, y \in X$ be arbitrary, due to geodesically completeness there are $a, b \in \partial_\infty X$, s.t. $x, y \in (a, b)$. As the distance $|xy|$ is determined by the signed distances of x and y to $Trip(a, b, c)$ for some $c \in \partial_\infty X \backslash \{a, b\}$ and $F(x), F(y)$ have the same signed distances to $Trip(a', b', c')$ by construction, it follows that $|xy| = |F(x)F(y)|$.

It follows immediately from the construction that for distinct points $a, b \in \partial_\infty X$, F maps the line (a, b) to the line $(f(a), f(b))$. Thus restricted to the boundary F equals f.

We show that such an F is unique. Any isometry $\hat{F} : X \to X'$ with $\hat{F}_{|\partial_\infty X} = f$ maps the line (a, b) for $a, b \in \partial_\infty X$ to the line $(f(a), f(b))$. Let $\hat{F}$ and F be such isometries, then they differ on (a, b) by a translation. If we take $c \in \partial_\infty X \setminus \{a, b\}$ it is $Trip(a, b, c) = (a, b) \cap (b, c) \cap (a, c)$, which shows that $F(Trip(a, b, c)) = Trip(f(a), f(b), f(c)) = \hat{F}(Trip(a, b, c))$. In particular $F_{|(a,b)} = \hat{F}_{|(a,b)}$. But as the geodesic (a, b) was arbitrary and X is geodesically complete it follows $F = \hat{F}$.

The last step is to show that F is surjective if f is surjective and X' is geodesically complete. Let $x' \in X'$ be arbitrary, then there is by geodesically completeness $a', b' \in \partial_\infty X'$, s.t. $x' \in (a', b')$. Let $c' \in \partial_\infty X \backslash \{a', b'\}$ then x' is determined by the signed distance to $Trip(a', b', c')$. In particular this implies that the point $x \in (f^{-1}(a'), f^{-1}(b'))$ with the same signed distance to $Trip(f^{-1}(a'), f^{-1}(b'), f^{-1}(c'))$ is mapped to x' by construction. We note that f^{-1} exists, as f is as a Möbius map injective and assumed to be surjective. $\square$

Corollary 4.4. *The tree constructed in theorem 4.1 is unique up to isometry.*

Proof. Let $X, X^{'}$ be two such trees. As the boundaries with the natural Möbius structures both equal $(Z, \mathcal{M})$ (up to Möbius isomorphism), there is a Möbius isomorphism $f : \partial_\infty X \to \partial_\infty X^{'}$. If we apply the theorem above the result follows. $\square$

5 Antipodal diameter 1 metrics

In this section we recall some facts from [Bi] and give a slight generalization of one of its result.

Let $(Z, \mathcal{M})$ be an arbitrary Möbius space. By $\mathcal{M}_1^a$ we denote the subset of antipodal, diameter 1 metrics. The set $\mathcal{M}_1^a$ may be empty.

Biswas has showed in [Bi] that there is a natural metric on $\mathcal{M}_1^a$. We note that he assumed Z to be compact, but for the definition of this metric that is not necessary. We recall some of the notation and the definition of the metric.

Let $(Z, \mathcal{M})$ be an arbitrary Möbius structure and $\mathcal{M}_1^a$ the subset of antipodal diameter one metrics. For $\rho_1, \rho_2 \in \mathcal{M}_1^a$ and $\xi \in Z$ we define

$$\frac{d\rho_1}{d\rho_2}(\xi) := \frac{\rho_1(\eta, \eta^{'})}{\rho_1(\xi, \eta)\rho_1(\xi, \eta^{'})} \frac{\rho_2(\xi, \eta)\rho_2(\xi, \eta^{'})}{\rho_2(\eta, \eta^{'})},$$

with $\eta, \eta^{'} \in Z \setminus \{\xi\}$ arbitrary but distinct. One can show that this definition is independent of the choice of $\eta, \eta^{'} \in Z \setminus \{\xi\}$. Similar as in [Bi] one can show that

$$d_{\mathcal{M}_1^a}(\rho_1, \rho_2) := \sup_{\zeta \in Z} \ln \frac{d\rho_1}{d\rho_2}(\zeta)$$

is a metric on $\mathcal{M}_1^a$. Furthermore he shows that for a CAT(-1) space X and ρ_x, ρ_y Bourdon metrics, we have $\frac{d\rho_y}{d\rho_x} = \lambda_{x,y}^2$. By the definition of λ it follows $\ln \frac{d\rho_y}{d\rho_x}(\zeta) = B_\zeta(x, y)$.

Theorem 5.1. *Let $(Z, \mathcal{M})$ be a ultrametric Möbius space, and let (X, d) be a geodesically complete tree such that $(\partial_\infty X, \mathcal{M}_X) = (Z, \mathcal{M})$. Let $i : (X, d) \to (\mathcal{M}_1^a, d_{\mathcal{M}_1^a})$ be the map sending x to the Bourdon metric ρ_x, then i is an isometry.*

Proof. The first step is to show that i is bijective. First observe that the geodesically completeness of (X, d) implies that for every $x \in X$, the metric ρ_x is antipodal and of diameter 1. This implies that the image i lies in $\mathcal{M}_1^a$. As Z has cardinality greater than two, it easily follows that $\rho_x \neq \rho_y$ for $x \neq y$, which means that i is injective.

We show that i is surjective. Let $\rho \in \mathcal{M}_1^a$ be given. Let $a, b \in \partial_\infty X$, s.t. $\rho(a, b) = 1$ and $c \in \partial_\infty X$ arbitrary, but distinct from a and b. By lemma 2.4 ρ is an ultrametric, therefore $\rho(a, c) = 1$ or $\rho(b, c) = 1$. W.l.o.g. we assume

$\rho(a, c) = 1$. Let x be in $[Trip(a, b, c), a)$, s.t. $|xTrip(a, b, c)| = -\ln(\rho(b, c))$, then $(\rho_x)_{|\{a,b,c\}} = \rho_{|\{a,b,c\}}$. For $d \in \partial_\infty X \setminus \{a, b, c\}$ and $\mu := \rho(c, b) = \rho_x(c, b)$, we have by the Möbius equivalence of ρ and ρ_x

$$(\rho(c, d) : \rho(b, d) : \mu\rho(a, d)) = crt_\rho(c, d, a, b) = crt_{\rho_x}(c, d, a, b)$$
$$= (\rho_x(c, d) : \rho_x(b, d) : \mu\rho_x(a, d)).$$

Thus there is a factor $\nu > 0$ such that we have the vector equality

$$\nu\ (\rho(c, d), \rho(b, d), \mu\rho(a, d)) = (\rho_x(c, d), \rho_x(b, d), \mu\rho_x(a, d)).$$

We will show that $\nu = 1$. As ρ and ρ_x are ultrametrics we have $\rho(b, d) = 1$ or $\rho(a, d) = 1$, as well as $\rho_x(b, d) = 1$ or $\rho_x(a, d) = 1$. If $\rho_x(a, d) = 1 \wedge \rho(a, d) = 1$ then $\nu = 1$ and in the same way $\rho_x(b, d) = 1 \wedge \rho(b, d) = 1$ implies $\nu = 1$. One easily sees that in the remaining cases

$$\rho_x(a, d) = 1\ \wedge\ \rho(a, d) < 1\ \wedge\ \rho(b, d) = 1\ \wedge\ \rho_x(b, d) < 1$$

and

$$\rho_x(a, d) < 1\ \wedge\ \rho(a, d) = 1\ \wedge\ \rho(b, d) < 1\ \wedge\ \rho_x(b, d) = 1$$

the vector equality from above can not be satisfied. Thus $\nu = 1$, which implies that $(\rho_x)_{|\{a,b,c,d\}} = \rho_{|\{a,b,c,d\}}$. In particular this means that the construction of x is independent of the choice c or d. Furthermore it follows that for arbitrary $c, d \in \partial_\infty X$ we have $\rho_x(c, d) = \rho(c, d)$. Hence ρ is the Bourdon metric ρ_x.

Finally we have to show $d(x, y) = d_{\mathcal{M}_1^a}(\rho_x, \rho_y)$. For given points $x, y \in X$ we can extend the geodesic segment $[x, y]$ to a geodesic ray $[x, \xi)$ where $\xi \in \partial_\infty X$. Thus $d(x, y) = B_\xi(x, y)$. Since $B_\zeta(x, y) \leq d(x, y)$ for all $\zeta \in \partial_\infty X$ we have

$$d_{\mathcal{M}_1^a}(\rho_x, \rho_y) = \sup_{\zeta \in \partial_\infty X} \ln \frac{d\rho_y}{d\rho_x}(\zeta) = \sup_{\zeta \in \partial_\infty X} B_\zeta(x, y) = B_\xi(x, y) = d(x, y).$$

$\square$

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