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A Fully Discrete Galerkin Method for Abel-type Integral Equations

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Abstract

In this paper, we present a Galerkin method for Abel-type integral equation with a general class of kernel. Stability and quasi-optimal convergence estimates are derived in fractional-order Sobolev norms. The fully-discrete Galerkin method is defined by employing simple tensor-Gauss quadrature. We develop a corresponding perturbation analysis which allows to keep the number of quadrature points small. Numerical experiments have been performed which illustrate the sharpness of the theoretical estimates and the sensitivity of the solution with respect to some parameters in the equation.

Keywords: Abel's integral equation, Galerkin method, tensor-Gauss quadrature

Mathematics Subject Classification (2000): 45E10, 65R20, 65D32

1 Introduction

A variety of practical physical models, e.g., in thermal tomography, spectroscopy, astrophysics, can be modelled by Abel-type integral equations provided the problem enjoys symmetries which allow to reduce the equation to a one-dimensional equation (cf. [14], [18]). In this paper, we present a fully discrete Galerkin method for the numerical solution of Abel-type integral equations. The theory and discretization of Abel's integral equation have been investigated by many authors and a variety of methods are proposed which include, e.g., product integration method [6] and approximation by implicit interpolation [4]. In [2] the

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solution is approximated by means of an adaptive Huber method which is a kind of product integration method. A Nyström-type method which is based on the trapezoidal rule is analyzed for the Abel integral equation in [11]. Recently, the composite trapezoidal method is applied for weakly singular Volterra integral equations of the first kind [16]. Furthermore, the piecewise polynomial discontinuous Galerkin approximation of a first-kind Volterra integral equation of convolution kernel type for a smooth kernel is studied in [5]. The stability and robustness of the collocation method for Abel integral equation have been discussed in [9]. Less research has been devoted to Galerkin discretizations and to the development of a stability and convergence theory in energy spaces which are fractional-order Sobolev spaces.

For general Abel-type kernel functions, the arising double integration for computing the coefficients in the Galerkin system matrix, in general, cannot be evaluated analytically and numerical quadrature has to be employed. In our paper, we present a fully discrete Galerkin method via numerical quadrature and present a convergence analysis in energy norms for the exact Galerkin method as well as for the fully discrete version by employing a perturbation theory.

The paper is structured as follows. In Section 2, we introduce the class of Abel-type integral equations which we will investigate and present its variational formulation in appropriate fractional-order Sobolev spaces. We will show well-posedness of the problem by proving continuity and ellipticity of the arising sesquilinear form and employ the Lax-Milgram lemma. We employ a lemma from [10] for the ellipticity of the original Abel integral equation and then derive the result for generalized types of integral kernels by a perturbation argument.

In Section 3, we present the Galerkin discretization and introduce the one-dimensional polynomial finite element spaces as test and trial spaces. Section 4 is devoted to the stability and convergence analysis of the Galerkin method. Then, we introduce a simple numerical quadrature method based on tensor-Gauss formulae and develop a perturbation analysis which allows to keep the number of quadrature points small.

In Section 5, we report on numerical experiments which illustrate that the theoretically derived convergence estimates are sharp and also the estimates for the number of quadrature nodes is close to optimal. Finally, we vary the parameter $\alpha \in (0,1)$ in Abel's integral equation which determines the strength of singularity of the kernel function to study numerical its effect on the solution.

In the concluding Section 6 we briefly summarize our results.

2 Abel-type Integral Equations

2.1 Setting

In this paper, we consider Abel-type integral equations of the following abstract form: Let $\Omega = (0,1)$. For a given function $g : \Omega \to \mathbb{R}$, parameter $\alpha \in (0,1)$, and kernel function $K : \Omega \times \Omega \to \mathbb{R}$ we are seeking the solution f of the equation

$$A_{K,\alpha}(f)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} K(x,y) f(y) dy = g(x), \quad \text{for all } x \in \Omega, \quad (1)$$

where $A_{K,\alpha}$ is the Abel-type integral operator and $\Gamma(\alpha)$ denotes Euler's Gamma function. To formulate the appropriate function spaces so that (1) is well posed we first recall the definition of fractional-order Sobolev spaces. As usual the standard Lebesgue spaces are denoted by $L^p(\Omega)$ and their norm by $\|\cdot\|_{L^p(\Omega)}$. For $p = 2$, the scalar product is denoted by $(u,v) = \int_\Omega u\overline{v}$ and the norm by $\|\cdot\| = (\cdot,\cdot)^{1/2}$.

For $\ell \in \mathbb{R}$, let $\lfloor \ell \rfloor$ denote the largest integer for which $\lfloor \ell \rfloor \leqslant \ell$ and define $\lambda \in [0,1[$ by $\ell = \lfloor \ell \rfloor + \lambda$. For $\ell \in \mathbb{R}_{>0} \backslash \mathbb{N}$, we introduce the scalar product

$$\begin{aligned}(\varphi,\psi)_{H^\ell(\Omega)} &:= \sum_{\alpha \leqslant \lfloor \ell \rfloor} (D^\alpha \varphi, D^\alpha \psi) \qquad (2)\\ &+ \sum_{\alpha \leqslant \lfloor \ell \rfloor} \int_{\Omega\times\Omega} \frac{(D^\alpha\varphi(x) - D^\alpha\varphi(y))\,\overline{(D^\alpha\psi(x) - D^\alpha\psi(y))}}{|x-y|^{1+2\lambda}} dxdy\end{aligned}$$

and the norm $\|\varphi\|_{H^\ell(\Omega)} := (\varphi,\varphi)^{1/2}_{H^\ell(\Omega)}$. For $\ell \in \mathbb{N}$, the second term in (2) is skipped. Then the Sobolev space $H^\ell(\Omega)$ is given by

$$H^\ell(\Omega) := \left\{ u \in L^2(\Omega) \mid \forall 0 \le m \le \lfloor \ell \rfloor \quad u^{(m)} \in L^2(\Omega) \quad \text{and} \quad \|u\|_{H^\ell(\Omega)} < \infty \right\}.$$

The dual space of $H^\ell(\Omega)$ is denoted by $H^{-\ell}(\Omega)$ and is equipped with the scalar product

$$\|u\|_{H^{-\ell}(\Omega)} := \sup_{v \in H^\ell(\Omega)} \frac{(u,v)}{\|v\|_{H^\ell(\Omega)}}, \qquad (3)$$

where $(\cdot,\cdot)$ denotes the continuous extension of the L^2 scalar product to the anti-dual pairing $\langle \cdot, \bar{\cdot} \rangle$ in $H^{-\ell}(\Omega) \times H^\ell(\Omega)$.

2.2 Variational Formulation of Abel's Integral Equation

We consider Abel's integral equation (1) as an operator equation in $H^{\alpha/2}(\Omega)$: For given $g \in H^{\alpha/2}(\Omega)$, we are seeking $f \in H^{-\alpha/2}(\Omega)$ such that

$$A_{K,\alpha} f = g. \qquad (4)$$

We multiply (4) by test functions $\varphi \in H^{-\alpha/2}(\Omega)$ and integrate over Ω to get the variational problem:

Find $f \in H^{-\alpha/2}(\Omega)$ such that

$$a_{K,\alpha}(f,\varphi) := (A_{K,\alpha} f, \varphi) = (g,\varphi) =: G(\varphi), \quad \forall \varphi \in H^{-\alpha/2}(\Omega). \qquad (5)$$

2.3 Well-posedness of the Variational Problem

In this section we prove the well-posedness of (5) under certain assumptions on the kernel K: In Section 2.3.1, we recall the ellipticity of the sesquilinear form $a_{K,\alpha}$ for $K = 1$ and extend this result to more general kernels in Section 2.4, i.e., there exists a constant $\gamma > 0$ such that

$$\operatorname{Re} a_{K,\alpha}(u,u) \geqslant \gamma \|u\|^2_{H^{-\alpha/2}(\Omega)} \quad \forall u \in H^{-\alpha/2}(\Omega)\,. \tag{6}$$

In Section 2.3.2 we prove the continuity of $a_{K,\alpha}$ for $K = 1$ based on results in [15] and extend this results for a larger class of kernels in Section 2.4. This allows to apply the classical Lax-Milgram lemma (cf. [13]) to deduce the well-posedness.

Theorem 1 (Lax-Milgram) *Let H be a Hilbert space. Let the sesquilinear form $a : H \times H \to \mathbb{C}$ be H-elliptic and continuous. Then the variational problem*

$$\textit{find } u \in H \quad \textit{such that} \quad a(u,v) = F(v) \quad \forall v \in H$$

has a unique solution $u \in H$ for all $F \in H'$ which satisfies

$$\|u\|_H \leqslant \frac{1}{\gamma}\|F\|_{H'}. \tag{7}$$

2.3.1 Ellipticity

We first recall the $H^{-\alpha/2}$-ellipticity for the case $K = 1$ and generalize this result to a much larger class of kernel functions $K \neq 1$ by a perturbation argument.

Theorem 2 *We consider $A_{K,\alpha}$ in (1) for $K = 1$ and $\alpha \in (0,1)$. The operator $A_{1,\alpha}$ is $H^{-\alpha/2}(\Omega)$-elliptic:*

$$\operatorname{Re}(A_{1,\alpha} f, f) \geqslant \gamma \|f\|^2_{H^{-\alpha/2}(\Omega)} \quad \forall f \in H^{-\alpha/2}(\Omega) \quad \textit{with } \gamma := \cos\left(\frac{\pi\alpha}{2}\right). \tag{8}$$

Proof. In [10, Lem. 2.3], it is shown that the inequality (8) holds for $f \in L^2(\Omega)$. Since $L^2(\Omega)$ is dense in $H^{-\alpha/2}(\Omega)$ (cf. [17, Prop. 2.4.2]) this also holds for all $f \in H^{-\alpha/2}(\Omega)$. ■

2.3.2 Continuity

In this section we prove the continuity of $a_{1,\alpha}$ as in (5). For the continuity we will follow the proof in [15] and start with some preliminaries. Let

$$\mu_n = \left(n - \frac{1}{2}\right)\pi, \quad \phi_n(t) = \sqrt{2}\cos\mu_n t, \text{ for } 0 \leqslant t \leqslant 1, n \in \mathbb{N}_{\geqslant 1}. \tag{9}$$

Remark 3 *Note that* $\left(\mu_n^2, \phi_n\right)_{n\geq 1}$ *is the set of all eigenpairs of the boundary-value problem,*

$$\begin{aligned} &(1) \qquad (D^2u)(t) = -\lambda u(t), \quad 0 < t < 1, \\ &(2) \qquad \frac{du}{dt}(0) = u(1) = 0. \end{aligned} \tag{10}$$

It is known that $(\phi_n)_{n\in\mathbb{N}_{\geqslant 1}}$ *is complete in* $L^2(\Omega)$ *and we may assume that the eigenfunctions* $(\phi_n)_{n\geq 1}$ *form an orthonormal basis in* $L^2(\Omega)$ *(cf. [15, Eq. (2.1)]).*

For $\beta \in \mathbb{R}$, we may introduce as in [15, Eq. (2.2)] a Hilbert scale X_β by employing this basis. In span $\{\phi_n : n \geq 1\}$ we define the scalar product and norm by

$$(u,v)_{X_\beta} := \sum_{n=1}^{\infty} \mu_n^{2\beta}(u,\phi_n)\overline{(v,\phi_n)}, \quad \|u\|_{X_\beta} := (u,u)_{X_\beta}^{1/2}, \quad \forall u,v \in \operatorname{span}\{\phi_n : n \geq 1\}. \tag{11}$$

We introduce an operator $S : L^2(\Omega) \to L^2(\Omega)$ by

$$(Su)(t) = \sum_{n=1}^{\infty} \mu_n^{-1}(u,\phi_n)\phi_n(t), \quad u \in L^2(\Omega). \tag{12}$$

For the proofs of the following two lemmata, we refer to [15, Lem. 7, 8].

Lemma 4 *The fractional power* S^α *of* S *is given by*

$$S^\alpha u = \sum_{n=1}^{\infty} \mu_n^{-\alpha}(u,\phi_n)\phi_n, \quad u \in L^2([0,1]), \alpha \in \mathbb{R}. \tag{13}$$

The next lemma gives further insights on the space X_β.

Lemma 5

- $X_\beta = H^\beta(\Omega)$, $0 \leqslant \beta < \frac{1}{2}$,
- $X_{1/2} = \{u \in H^{1/2}(\Omega) : \int_0^1 (1-t)^{-1}|u(t)|^2 dt < \infty\}$,
- $X_\beta = \{u \in H^\beta(\Omega) : u(1) = 0\}$, $\frac{1}{2} < \beta \leqslant 1$.

Moreover there exists a constant $C = C(\beta) > 0$ *such that*

$$C^{-1}\|u\|_{X_\beta} \leqslant \|u\|_{H^\beta(\Omega)} \leqslant C\|u\|_{X_\beta}, \quad u \in X_\beta, \tag{14}$$

if $\beta \in [0,1]$ *and* $\beta \neq 1/2$*.*

This lemma is used in the proof of the continuity for $A_{K,\alpha}$.

Theorem 6 *Let the kernel function $K = K(x, y)$ satisfy the following conditions:*

- *K is continuous on $D = \{(x, y) \in \mathbb{R}^2 : 0 \leqslant y \leqslant x \leqslant 1\}$,*
- *$K(x, x) = 1$ for $0 \leqslant x \leqslant 1$,*
- *there exists a decreasing function $k \in L^2(\Omega)$ such that $\left|\frac{\partial K}{\partial_y}(x, y)\right| \leqslant k(y)$ for $0 < y \leqslant x \leqslant 1$.*

Abel's integral operator $A_{K,\alpha}$ is a continuous mapping from $H^{-\alpha/2}(\Omega)$ into $H^{\alpha/2}(\Omega)$.

Proof. We show that the operator from $H^{-\alpha/2}(\Omega)$ to $H^{\alpha/2}(\Omega)$ is bounded so we can conclude its continuity. From [15, Thm. 1], we know that there exists a constant $C = C(\alpha)$ for $\alpha \in (0, 1)$ such that

$$C^{-1}\|u\|_{X_{-\alpha}} \leqslant \|A_{K,\alpha}u\| \leqslant C\|u\|_{X_{-\alpha}}, \quad u \in L^2(\Omega). \tag{15}$$

Next, we investigate the self-adjoint compact operator $(A^*_{K,\alpha}A_{K,\alpha})^{1/2}$: $L^2(\Omega) \to L^2(\Omega)$ with its eigenvalues $(s_n(A_{K,\alpha}))_{n\in\mathbb{N}_{\geqslant 1}}$. From [15, Thm. 2], we conclude that there exists a constant $C = C(\alpha)$ such that

$$C^{-1}n^{-\alpha} \leqslant s_n(A_{K,\alpha}) \leqslant Cn^{-\alpha}. \tag{16}$$

From (11) we obtain $(\phi_n, \phi_m)_{X_{\alpha/2}} = \delta_{n,m}\mu_n^2 = \delta_{n,m}\|\phi_n\|^2_{X_{\alpha/2}}$. Hence

$$\begin{aligned}
\|A_{K,\alpha}u\|^2_{X_{\alpha/2}} = (A_{K,\alpha}u, A_{K,\alpha}u)_{X_{\alpha/2}} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} u_n\overline{u_m}(A_{K,\alpha}\phi_n, A_{K,\alpha}\phi_m)_{X_{\alpha/2}} \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} u_n\overline{u_m}s_n(A_{K,\alpha})s_m(A_{K,\alpha})(\phi_n, \phi_m)_{X_{\alpha/2}} \\
&= \sum_{n=1}^{\infty} |u_n|^2 s_n(A_{K,\alpha})^2\|\phi_n\|^2_{X_{\alpha/2}}.
\end{aligned}$$

From (11) we deduce $\|\phi_n\|^2_{X_{\alpha/2}} = \mu_n^\alpha$ so that

$$\|A_{K,\alpha}u\|^2_{X_{\alpha/2}} = \sum_{n=1}^{\infty} |u_n|^2 s_n(A_{K,\alpha})^2\mu_n^\alpha.$$

We combine this with the estimate of s_n as in (16) and the definition of μ_n to obtain

$$\begin{aligned}
\|A_{K,\alpha}u\|^2_{X_{\alpha/2}} &\leqslant C\sum_{n=1}^{\infty} |u_n|^2 n^{-2\alpha}\mu_n^\alpha = C\sum_{n=1}^{\infty} |u_n|^2 n^{-2\alpha}(\pi(n - 1/2))^\alpha \\
&\leqslant C\pi^{2\alpha}\sum_{n=1}^{\infty} |u_n|^2(\pi(n - 1/2))^{-\alpha} = C\pi^{2\alpha}\sum_{n=1}^{\infty} |u_n|^2\mu_n^{-\alpha}.
\end{aligned}$$

The last ingredient for the proof is

$$\begin{aligned}\|u\|_{X_{-\alpha/2}}^2 &= \|S^{\alpha/2}u\|^2 = \|\sum_{n=1}^{\infty}\mu_n^{-\alpha/2}(\sum_{m=1}^{\infty}u_m\phi_m,\phi_n)\phi_n\|^2 \\ &= \|\sum_{n=1}^{\infty}\mu_n^{-\alpha/2}u_n\phi_n\|^2 = \left(\sum_{n=1}^{\infty}\mu_n^{-\alpha/2}u_n\phi_n,\sum_{m=1}^{\infty}\mu_m^{-\alpha/2}u_m\phi_m\right) \\ &= \sum_{n=1}^{\infty}\mu_n^{-\alpha}|u_n|^2(\phi_n,\phi_n) = \sum_{n=1}^{\infty}\mu_n^{-\alpha}|u_n|^2.\end{aligned}$$

The combination of these relations leads to

$$\|A_{K,\alpha}u\|_{X_{\alpha/2}}^2 \leqslant C\pi^{2\alpha}\|u\|_{X_{-\alpha/2}}^2.$$

From [15, Chap. 3] it follows

$$H_0^{\beta}(\Omega) = H^{\beta}(\Omega) \text{ and } X_{-\beta} = H^{-\beta}(\Omega) \qquad \forall\, 0 \leqslant \beta < \frac{1}{2}.$$

Since $\alpha/2 \in (0, \frac{1}{2})$ we conclude from Lemma 5 that

$$\|A_{K,\alpha}u\|_{H^{\alpha/2}(\Omega)} \leqslant C_c\|u\|_{H^{-\alpha/2}(\Omega)} \quad \forall u \in H^{-\alpha/2}(\Omega) \tag{17}$$

holds. ∎

2.4 Ellipticity and Continuity for $K \neq 1$

The following lemma is needed for the proof of the main result of this section and we refer for a proof to [3]. For $0 \le s \le \infty$ and $1 \le p \le \infty$, let $W^{s,p}(\Omega)$ denote the usual Sobolev space as defined, e.g., in [1], equipped with the norm $\|\cdot\|_{W^{s,p}(\Omega)}$.

Lemma 7 *Let $1 < p < \infty$, $0 < s < \infty$, $0 < q < \infty$, $0 < \theta < 1$, $1 < t < \infty$ be such that*

$$\frac{1}{q} + \frac{\theta}{t} = \frac{1}{p}. \tag{18}$$

For $f \in W^{s,t}(\Omega) \cap L^{\infty}(\Omega)$, $g \in W^{\theta s,p}(\Omega) \cap L^{r}(\Omega)$, we have $fg \in W^{\theta s,p}(\Omega)$ and

$$\|fg\|_{W^{\theta s,p}} \le C\left(\|f\|_{L^\infty}\|g\|_{W^{\theta s,p}} + \|g\|_{L^q}\|f\|_{W^{s,t}}^{\theta}\|f\|_{L^\infty}^{1-\theta}\right). \tag{19}$$

For $s \in [0,1]$, we introduce the intervals

$$I_s := \begin{cases} \left[1, \dfrac{2}{1-2s}\right] & 0 \le s < 1/2, \\ [1,\infty[& s = 1/2, \\ [1,\infty] & 1/2 < s \le 1 \end{cases}$$

which are relevant for the embedding properties of Sobolev spaces for the interval $\Omega = (0,1)$:

$$H^s(\Omega) \hookrightarrow L^q(\Omega) \qquad \forall s \in [0,1] \quad \forall q \in I_s. \tag{20}$$

We apply Lemma 7 to prove norm equivalences for products of functions in Sobolev spaces. For this we first define a class of multipliers.

Definition 8 *Let $s \in [-1,1]$ and $\Omega = (0,1)$. A function $\kappa : \Omega \to \mathbb{R}$ belongs to the* multiplier class $\mathcal{M}(s)$ *if $\kappa \in L^\infty(\Omega)$ and $\operatorname*{ess\,inf}_{x\in\Omega} \kappa(x) =: \kappa_{\min} > 0$ and*

$$\kappa \text{ and } \frac{1}{\kappa} \in \begin{cases} H^1(\Omega) & |s| = 1, \\ W^{\frac{s}{1-\varepsilon}, 2-\frac{2\varepsilon}{2\varepsilon-1}}(\Omega) & 0 < |s| < 1 \quad \text{for some } \varepsilon \in \begin{cases} (0,1/2) & \text{if } 1/2 \leq |s| < 1, \\ \left[\frac{1}{2} - |s|, 1/2\right) & \text{if } 0 < |s| < 1/2. \end{cases} \end{cases}$$

Lemma 9 *Let $s \in [-1,1]$ and assume that $\kappa \in \mathcal{M}(s)$. Then there exist constants $c_{\kappa,s}$ and $C_{\kappa,s}$ such that*

$$c_{\kappa,s}\|f\|_{H^s(\Omega)} \leq \|\kappa f\|_{H^s(\Omega)} \leq C_{\kappa,s}\|f\|_{H^s(\Omega)}, \quad \forall f \in H^s(\Omega), \quad \forall s \in [-1,1]. \tag{21}$$

Proof. For $s = 0$, the inequality (21) is easily obtained by Hölder's inequality

$$\kappa_{\min} \|f\| \leq \|\kappa f\| \leq \|\kappa\|_{L^\infty(\Omega)} \|f\|.$$

For $s = 1$ we employ Leibniz' rule

$$\|\kappa f\|_{H^1(\Omega)} \leq \|\kappa' f\| + \|\kappa f'\| \leq \|\kappa' f\| + \|\kappa\|_{L^\infty(\Omega)} \|f'\|.$$

A Hölder's inequality for $q^{-1} + (q')^{-1} = 1$ leads to

$$\|\kappa' f\| \leq \|\kappa'\|_{L^{2q'}(\Omega)} \|f\|_{L^{2q}(\Omega)}.$$

We choose $q = \infty$ and $q' = 1$ and use (20) to derive $\|\kappa' f\| \leq C \|\kappa'\| \|f\|_{H^1(\Omega)}$. This is the upper bound with $C_{\kappa,1} = \|\kappa\|_{L^\infty(\Omega)} + C\|\kappa'\|$. To get the lower bound we start with

$$\|f\|_{H^1(\Omega)} = \|\frac{1}{\kappa}\kappa f\|_{H^1(\Omega)} \leq \left\| \left(\frac{1}{\kappa}\right)' \kappa f \right\| + \left\| \frac{1}{\kappa} \right\|_{L^\infty(\Omega)} \|(\kappa f)'\| \leq \left\| \left(\frac{1}{\kappa}\right)' \kappa f \right\| + \frac{1}{\kappa_{\min}} \|(\kappa f)'\|.$$

We use the previous result to obtain

$$\left\| \left(\frac{1}{\kappa}\right)' \kappa f \right\| \leq C \left\| \frac{\kappa'}{\kappa^2} \right\| \|\kappa f\|_{H^1(\Omega)} \leq \frac{C}{\kappa_{\min}^2} \|\kappa'\| \|\kappa f\|_{H^1(\Omega)}.$$

Hence, the lower bound follows from

$$\|f\|_{H^1(\Omega)} \leq c_{\kappa,1}^{-1} \|\kappa f\|_{H^1(\Omega)} \quad \text{with} \quad c_{\kappa,1} := \left(1 + C\frac{\|\kappa'\|_{L^2(\Omega)}}{\kappa_{\min}}\right)^{-1} \kappa_{\min}.$$

Next, we consider the case $1/2 \leq s < 1$. For some $0 < \varepsilon < 1/2$, we substitute in Lemma 7: $f \leftarrow \kappa$, $g \leftarrow f$, $p \leftarrow 2$, $q \leftarrow \varepsilon^{-1}$, $\theta \leftarrow 1-\varepsilon$, $t \leftarrow 2 - \frac{2\varepsilon}{2\varepsilon-1}$, $s \leftarrow \frac{s}{1-\varepsilon}$ and obtain

$$\|\kappa f\|_{H^s(\Omega)} \leq C \left(\|\kappa\|_{L^\infty(\Omega)} \|f\|_{H^s(\Omega)} + \|f\|_{L^{1/\varepsilon}} \|\kappa\|^{1-\varepsilon}_{W^{\frac{s}{1-\varepsilon}, 2-\frac{2\varepsilon}{2\varepsilon-1}}(\Omega)} \|\kappa\|^{\varepsilon}_{L^\infty(\Omega)} \right). \tag{22}$$

Sobolev's embedding theorem implies that there is a constant C_ε such that $\|f\|_{L^{1/\varepsilon}(\Omega)} \leq C_\varepsilon \|f\|_{H^s(\Omega)}$. Hence, the upper estimate holds with

$$C_{\kappa,s} := C \left(\|\kappa\|_{L^\infty(\Omega)} + C_\varepsilon \|\kappa\|^{1-\varepsilon}_{W^{\frac{s}{1-\varepsilon}, 2-\frac{2\varepsilon}{2\varepsilon-1}}(\Omega)} \|\kappa\|^{\varepsilon}_{L^\infty(\Omega)} \right) \quad \forall 1/2 \leq s < 1. \tag{23}$$

For the lower bound, we obtain

$$\|f\|_{H^s(\Omega)} = \|\frac{1}{\kappa}\kappa f\|_{H^s(\Omega)} \leq C_{\frac{1}{\kappa},s} \|\kappa f\|_{H^s(\Omega)} \quad \text{so that} \quad c_{\kappa,s} := C^{-1}_{\frac{1}{\kappa},s}. \tag{24}$$

Next, we consider the case $0 < s < \frac{1}{2}$ and observe

$$H^s(\Omega) \hookrightarrow L^r(\Omega), \quad \text{for} \quad 1 \leq r \leq \frac{2}{1-2s}.$$

Hence we have to restrict ε in (22) to $\frac{1}{2} - s \leq \varepsilon < 1/2$. The constant $C_{\kappa,s}$ has the same form (23) while ε therein must be chosen from the reduced range. The same holds for the lower bound: it is of the same form (24) while ε therein must be chosen from the reduced range.

It remains to prove the estimate for the $H^{-s}(\Omega)$. For $0 < s \leq 1$, we have

$$\begin{aligned} \|\kappa f\|_{H^{-s}(\Omega)} &= \sup_{\omega \in H^s(\Omega)\backslash\{0\}} \frac{|(\kappa f, \omega)_{L^2(\Omega)}|}{\|\omega\|_{H^s(\Omega)}} = \sup_{\omega \in H^s(\Omega)\backslash\{0\}} \frac{|(f, \kappa\omega)_{L^2(\Omega)}|}{\|\omega\|_{H^s(\Omega)}} \\ &\leq \|f\|_{H^{-s}(\Omega)} \sup_{\omega \in H^s(\Omega)\backslash\{0\}} \frac{\|\kappa\omega\|_{H^s(\Omega)}}{\|\omega\|_{H^s(\Omega)}} \leq C_{\kappa,s} \|f\|_{H^{-s}(\Omega)}. \end{aligned}$$

To get the lower bound, we notice that

$$\begin{aligned} \|f\|_{H^{-s}(\Omega)} &= \|\frac{1}{\kappa}\kappa f\|_{H^{-s}(\Omega)} = \sup_{\omega \in H^s(\Omega)\backslash\{0\}} \frac{|(\kappa f, \frac{1}{\kappa}\omega)_{L^2(\Omega)}|}{\|\omega\|_{H^s(\Omega)}} \\ &\leq \|\kappa f\|_{H^{-s}(\Omega)} \sup_{\omega \in H^s(\Omega)\backslash\{0\}} \frac{\|\frac{1}{\kappa}\omega\|_{H^s(\Omega)}}{\|\omega\|_{H^s(\Omega)}} \leq C_{\frac{1}{\kappa},s} \|\kappa f\|_{H^{-s}(\Omega)}. \end{aligned}$$

■

This norm equivalence allows us to generalize the class of kernel functions in Abel's integral operator.

Definition 10 *Let $s \in [-1, 1]$ and $\Omega = (0, 1)$. The class of s-admissible kernel functions $\mathcal{A}(s)$ consists of functions $K : \Omega \times \Omega \to \mathbb{R}$ such that there exists a sequence $(\psi_n)_n$ of functions in the multiplier class $\mathcal{M}(s)$ such that the following conditions hold:*

1. *The kernel of integral equation has the representation*

$$K(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}d_{n,m}\psi_n(x)\psi_m(y)$$

for some coefficients $(d_{n,m})_{n,m=1}^{\infty}$.

2. *The constants* $C_{n,s}$, $c_{n,s}$ *in the multiplier estimates*

$$c_{n,s}\|f\|_{H^s(\Omega)}\leq\|\psi_n f\|_{H^s(\Omega)}\leq C_{n,s}\|f\|_{H^s(\Omega)}\quad\forall f\in H^s(\Omega)\tag{25}$$

satisfy:

   (a) *there is a positive constant* $C_s<\infty$ *such that*

$$\sum_{n,m=1}^{\infty}d_{n,m}C_{n,s}C_{m,s}\leq C_s^2,$$

   (b) *there exists a positive constant* $\tilde{\gamma}>0$ *such that*

$$\gamma\sum_{n=1}^{\infty}d_{n,n}c_{n,s}^2-C_c\sum_{\substack{n,m=1\\n\neq m}}^{\infty}d_{n,m}C_{n,s}C_{m,s}\geq\tilde{\gamma}>0,$$

   where γ *is as in (8) and* C_c *as in (17).*

Theorem 11 *Suppose that the kernel function* K *in Abel's integral equation is* s*-admissible for* $s=-\alpha/2$. *Then* $a_{K,\alpha}(f,g)$ *is a continuous and* $H^{-\alpha/2}(\Omega)$*-elliptic sesquilinear form.*

Proof. Let $s=-\alpha/2$. We have

$$\begin{aligned}|a_{K,\alpha}(f,g)|&=\sum_{n,m=1}^{\infty}|d_{n,m}a_{1,\alpha}(\psi_m f,\psi_n g)|\leq C_c\sum_{n,m=1}^{\infty}d_{n,m}\|\psi_m f\|_{H^s(\Omega)}\|\psi_n g\|_{H^s(\Omega)}\\&\leq C_c\left(\sum_{n,m=1}^{\infty}d_{n,m}C_{n,s}C_{m,s}\right)\|f\|_{H^s(\Omega)}\|g\|_{H^s(\Omega)}\\&\leq\tilde{C}_c\|f\|_{H^s(\Omega)}\|g\|_{H^s(\Omega)}\quad\text{with}\quad\tilde{C}_c:=C_cC_s^2.\end{aligned}$$

To obtain the $H^s(\Omega)$-ellipticity, we observe that

$$a_{K,\alpha}(f,f)=\sum_{n,m=1}^{\infty}d_{n,m}a_{1,\alpha}(\psi_n f,\psi_m f)=\sum_{n=1}^{\infty}d_{n,n}a_{1,\alpha}(\psi_n f,\psi_n f)+\sum_{\substack{n,m=1\\n\neq m}}^{\infty}d_{n,m}a_{1,\alpha}(\psi_n f,\psi_m f).$$

We employ the coercivity and continuity of $a_{1,\alpha}$ to derive

$$\operatorname{Re}a_{K,\alpha}(f,f)\geq\gamma\sum_{n=1}^{\infty}d_{n,n}\|\psi_n f\|_{H^s(\Omega)}^2-C_c\sum_{\substack{n,m=1\\n\neq m}}^{\infty}d_{n,m}\left|a_{1,\alpha}(\psi_n f,\psi_m f)\right|.$$

The estimates for the multipliers ψ_n lead to

$$\operatorname{Re} a_{K,\alpha}(f,f) \geq \left(\gamma \sum_{n=1}^{\infty} d_{n,n} c_{n,s}^2 - C_c \sum_{\substack{n,m=1 \\ n \neq m}}^{\infty} d_{n,m} C_{n,s} C_{m,s} \right) \|f\|_{H^s(\Omega)}^2 .$$

The assumptions on the summability of the constants $c_{n,s}$, $C_{n,s}$ lead to the assertion. ■

3 Discretization of Abel-type Integral Equations

In order to solve the Abel-type integral equation (4), (5) numerically we discretize the continuous problem (5) by a Galerkin finite element method. For this we introduce the piecewise polynomial finite element spaces. Let a set of mesh points $\mathcal{N} = (x_i)_{i=0}^{N}$ be given

$$0 = x_0 < x_1 < \ldots < x_N = 1$$

which induces a mesh $\mathcal{T} = \{\tau_i : 1 \leq i \leq N\}$ on Ω, where $\tau_i = [x_{i-1}, x_i]$. The length of a subinterval $\tau \in \mathcal{T}$ is denoted by h_τ and the maximal mesh width by $h := \max\{h_\tau : \tau \in \mathcal{T}\}$. The variation of the lengths of neighboring intervals is controlled by the constant

$$C_{\mathcal{T}} := \max \left\{ \frac{h_\tau}{h_\sigma} : \forall \tau, \sigma \in \mathcal{T} \text{ with } \tau \cap \sigma \neq \emptyset \right\}.$$

The piecewise polynomial function space of degree $m \in \mathbb{N}_0$ on $[0,1]$ is given by

$$S_{\mathcal{T}}^m := \begin{cases} \{v \in L^\infty(\Omega) : v|_\tau \in \mathbb{P}_0(\tau), \forall \tau \in \mathcal{T}\} & m = 0, \\ \{v \in C(\Omega) : v|_\tau \in \mathbb{P}_m(\tau_i), \forall \tau \in \mathcal{T}\} & m \geq 1. \end{cases} \tag{26}$$

Here $\mathbb{P}_m(\tau)$ denote the space of all univariate polynomials on τ of maximal degree m. The nodal points are given by

$$\mathcal{N}_m := \begin{cases} \left\{ \frac{x_i + x_{i-1}}{2} : 1 \leq i \leq N \right\} & m = 0, \\ \left\{ \xi_{i,j} := x_{i-1} + j \frac{x_i - x_{i-1}}{m} \quad 1 \leq i \leq N, \, 0 \leq j \leq m-1 \right\} \cup \{1\} & m \geq 1 \end{cases}$$

so that the dimension of $S_{\mathcal{T}}^m$ is $M := N$ for $m = 0$ and $M := Nm + 1$ for $m \geq 1$. We choose the usual Lagrange basis functions $b_i^{(m)}$ of $S_{\mathcal{T}}^m$ and write b_i short for $b_i^{(m)}$ if the polynomial degree is clear from the context. The Galerkin method is given by replacing the infinite-dimensional space $H^{-\alpha/2}(\Omega)$ in (5) by the finite dimensional subspace $S := S_{\mathcal{T}}^m$:

$$\text{Find } f_S \in S \text{ such that} \qquad a_{K,\alpha}(f_S, \varphi) = G(\varphi) \quad \forall \varphi \in S. \tag{27}$$

For the computation of f_S one introduces the representation of (27) with respect to the basis $(b_i)_{i=0}^M$ of S. Let

$$a_{ij} = (A_{K,\alpha} b_j, b_i) = \Gamma(\alpha)^{-1} \int_0^1 b_i(x) \int_0^x (x-y)^{\alpha-1} K(x,y) b_j(y) dy dx,$$

$$r_i = (g, b_i) = \int_0^1 b_i g, \tag{28}$$

with the *system matrix* $\mathbf{A} = (a_{ij})_{i,j=1}^M \in \mathbb{R}^{M\times M}$ and the *right-hand side vector* $\mathbf{r} = (r_i)_{i=1}^M \in \mathbb{R}^M$. Then, the basis representation of (27) is: Find $\mathbf{f}_S \in \mathbb{R}^M$ such that

$$\mathbf{A}\mathbf{f}_S = \mathbf{r} \tag{29}$$

and the solution of (27) is given by $f_S = \sum_{i=1}^M f_{S,i} b_i$.

4 Convergence Analysis

4.1 Discretization Error

In order to estimate the discretization error of the Galerkin discretization we employ Céa's lemma.

Theorem 12 (Céa) *In Abel's integral operator $A_{K,\alpha}$, let $\alpha \in (0,1)$ and $K \in \mathcal{A}\left(-\frac{\alpha}{2}\right)$. For some $G \in H^{\alpha/2}(\Omega)$, let $f \in H^{-\alpha/2}(\Omega)$ be the exact solution of (5). Then also the Galerkin discretization (27) has a unique solution $f_S \in S$ which satisfies the quasi-optimal error estimate*

$$\|f - f_S\|_{H^{-\alpha/2}(\Omega)} \leqslant \frac{\tilde{C}_c}{\tilde{\gamma}} \inf_{v\in S} \|f - v\|_{H^{-\alpha/2}(\Omega)}, \tag{30}$$

where $\tilde{\gamma}, \tilde{C}_c$ are the ellipticity and continuity constants of the form $a_{K,\alpha}$ as in Theorem 11.

To derive convergence *rates* we investigate the error term $\inf_{v\in S_{\mathcal{T}}^m} \|f - v\|_{H^{-\alpha/2}(\Omega)}$. It is well known from interpolation theory that for sufficiently smooth solution $f \in H^{m+1}(\Omega)$ it holds

$$\|f - f_S\|_{H^{-\alpha/2}(\Omega)} \leqslant \frac{\tilde{C}_c}{\tilde{\gamma}} \min_{v\in S_{\mathcal{T}}^m} \|f - v\|_{H^{-\alpha/2}(\Omega)} \leqslant \frac{\tilde{C}_c}{\tilde{\gamma}} C h^{m+1+\alpha/2} \|f\|_{H^{m+1}(\Omega)}.$$

4.2 Perturbation (Quadrature)

To derive a fully discrete method we apply numerical quadrature to approximate the integrals in the system matrix. We develop the quadrature error analysis for analytic K, more precisely, we assume that there exists constants C_K and Λ_K such that

$$\|K\|_{C^n(\Omega\times\Omega)} \leq C_K \Lambda_K^n n! \qquad \forall n \in \mathbb{N}_0. \tag{31}$$

To reduce technicalities we assume that $\Lambda_K \geq 2$. We introduce the function $w_{i,j}(x,y) := K(x,y)(x-y)^{\alpha-1} b_i(x) b_j(y)$ and reformulate the integral for a matrix entry $a_{ij} = I(w_{i,j})$:

$$I(w_{i,j}) = \frac{1}{\Gamma(\alpha)} \int_0^1 \int_0^x w_{i,j}(x,y) dy dx = \frac{1}{\Gamma(\alpha)} \sum_{\tau \subset \text{supp}(b_i)} \sum_{\sigma \subset \text{supp}(b_j)} \int_\tau \int_{\sigma \cap [0,x]} w_{i,j}(x,y) dy dx \tag{32}$$

with $\alpha \in (0,1)$. Since all the above functions are real we may omit the complex conjugation on the second argument in the L^2 scalar product. We compute the integral in (32) in different ways depending on the relative location of τ compared to σ:

- if $\tau = \sigma$ we employ simplex coordinates (cf. Section 4.2.2),
- if τ lies to the left of σ it is easy to see that integral value is 0,
- in the case that τ lies to the right of σ we apply tensor-Gauss quadrature (cf. Section 4.2.1).

4.2.1 The Case $\tau \neq \sigma$

The $n \times n$-tensor-Gauss quadrature for a function f on an interval $\tau \times \sigma$ is given by

$$(Q_\tau^n \otimes Q_\sigma^n)(f) = \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-1} \omega_k^{\tau,G} \omega_\ell^{\sigma,G} f(\xi_k^{\tau,G}, \xi_\ell^{\tau,G}),$$

with $\omega_k^{\tau,G}, \xi_k^{\tau,G}$ denoting the weights and abscissae for Q_τ^n, the n-point Gauss-Legendre quadrature method scaled to τ. The exact integral value is given by

$$(I_\tau \otimes I_\sigma)(w_{i,j}) = \int_\tau \int_\sigma (x-y)^{\alpha-1} K(x,y) b_i(x) b_j(y) dy dx.$$

We have the following error estimate [17, Chap. 5.3.1.2]:

$$|E^n f| = |(I_\tau \otimes I_\sigma - Q_\tau^n \otimes Q_\sigma^n)(f)| \leqslant \max_{t \in \tau} |(I_\sigma - Q_\sigma^n) f(t,\cdot)| + \max_{t \in \sigma} |(I_\tau - Q_\tau^n) f(\cdot,t)| \tag{33}$$

$$= \max_{t \in \tau} |E_\sigma^n f(t,\cdot)| + \max_{t \in \sigma} |E_\tau^n f(\cdot,t)|,$$

where I_τ and I_σ are the integrals with respect to the first and second variable, respectively. We first investigate the error term $\max_{t \in \tau_i} |E_\sigma^n f(t,\cdot)|$ by the following lemma.

Lemma 13 *Let b_j, $j \in \{1,\ldots M\}$, be the Lagrange basis of $S_{\mathcal{T}}^m$ and let K satisfy (31). Assume that the number of quadrature points satisfy $2n > m$. Let $\sigma \subset \text{supp}(b_j)$ be such that $\text{dist}(\sigma, x) > 0$ for a fixed $x \in [0,1] \backslash \overline{\sigma}$. We have*

$$\left| (I_\sigma - Q_\sigma^n) \left((x-\cdot)^{\alpha-1} K(x,\cdot) b_j \right) \right| \leq \frac{2 C_K \, \mathrm{e} \, \Lambda_K}{\text{dist}^{1-\alpha}(x,\sigma)} \left(\frac{\Lambda_K}{2} \frac{h_\sigma}{\text{dist}(x,\sigma)} \right)^{2n} \|b_j\|_{C^m(\sigma)}. \tag{34}$$

Proof. Since the degree of exactness of a n-point Gaussian quadrature method is $2n-1$ it holds (cf. [12])

$$\left\|(I_\sigma - Q_\sigma^n)\left((x-\cdot)^{\alpha-1}K(x,\cdot)b_j\right)\right\|_{L^\infty(\sigma)} \leqslant 2 \inf_{p\in\mathbb{P}_{2n-1}} \left\|(x-\cdot)^{\alpha-1}K(x,\cdot)b_j - p\right\|_{L^\infty(\sigma)}. \tag{35}$$

We compute the $2n^{th}$ derivative $\partial^{2n}\left((x-\cdot)^{\alpha-1}K(x,\cdot)b_j\right)$ to estimate the remainder in a Taylor expansion

$$\left|\partial^{2n}\left((x-\cdot)^{\alpha-1}K(x,\cdot)b_j\right)\right| = \left|\sum_{k=0}^{2n}\binom{2n}{k}\left(\partial^k b_j\right)\partial^{2n-k}\left((x-\cdot)^{\alpha-1}K(x,\cdot)\right)\right|.$$

For $\mu \in \{0,\dots,2n\}$, the derivative in the last term can be estimated by

$$\begin{aligned}
\left\|\partial^\mu\left((x-\cdot)^{\alpha-1}K(x,\cdot)\right)\right\|_{L^\infty(\sigma)} &= \left\|\sum_{\nu=0}^{\mu}\binom{\mu}{\nu}\partial^\nu K(x,\cdot)\partial^{\mu-\nu}(x-\cdot)^{\alpha-1}\right\|_{L^\infty(\sigma)} \\
&\leqslant \left\|\sum_{\nu=0}^{\mu}\binom{\mu}{\nu}\partial^\nu K(x,\cdot)(\mu-\nu)!(x-\cdot)^{\alpha-1-(\mu-\nu)}\right\|_{L^\infty(\sigma)} \\
&\leqslant \left\|\mu!\,(x-\cdot)^{\alpha-1-\mu}\sum_{\nu=0}^{\mu}\frac{|\partial^\nu K(x,\cdot)|}{\nu!}\right\|_{L^\infty(\sigma)},
\end{aligned}$$

where we used $\operatorname{dist}^\nu(\sigma,x) \le 1$. Since

$$\sum_{\nu=0}^{\mu}\frac{\|\partial^\nu K(x,\cdot)\|_{L^\infty(\sigma)}}{\nu!} \le \sum_{\nu=0}^{\mu}\frac{C_K\nu!\Lambda_K^\nu}{\nu!} \le C_K\Lambda_K^{\mu+1},$$

and $\sum_{k=0}^{\infty}\frac{1}{k!} = \mathrm{e}$, we obtain

$$\begin{aligned}
\left\|\partial^{2n}\left((x-\cdot)^{\alpha-1}K(x,\cdot)b_j\right)\right\|_{L^\infty(\sigma)} &\leqslant \sum_{k=0}^{2n}\frac{(2n)!}{k!}\left\|\left|\partial^k b_j\right|(x-\cdot)^{\alpha-1-2n}\sum_{\nu=0}^{2n-k}\frac{|\partial^\nu K(x,\cdot)|}{\nu!}\right\|_{L^\infty(\sigma)} \\
&\leqslant C_K\,\mathrm{e}\,\Lambda_K^{2n+1}(2n)!\|b_j\|_{C^m(\sigma)}\operatorname{dist}(x,\sigma)^{\alpha-1-2n}.
\end{aligned}$$

This leads to

$$\begin{aligned}
&\left\|(I_\sigma - Q_\sigma^n)\left((x-\cdot)^{\alpha-1}K(x,\cdot)b_j\right)\right\|_{L^\infty(\sigma)} \\
&\qquad\leqslant 2\frac{(h_\sigma/2)^{2n}}{(2n)!}\sum_{k=0}^{2n}\frac{(2n)!}{k!}\left\|\left(\partial^k b_j\right)(x-\cdot)^{\alpha-1-(2n-k)}\sum_{\nu=0}^{2n-k}\frac{|\partial^\nu K(x,\cdot)|}{\nu!}\right\|_{L^\infty(\sigma)} \\
&\qquad\le 2\left(\frac{h_\sigma}{2}\right)^{2n}\|b_j\|_{C^m(\sigma)}C_K\Lambda_K^{2n+1}\operatorname{dist}^{\alpha-1-2n}(x,\sigma)\sum_{k=0}^{2n}\frac{1}{k!} \\
&\qquad\le \frac{2C_K\,\mathrm{e}\,\Lambda_K}{\operatorname{dist}^{1-\alpha}(x,\sigma)}\left(\frac{h_\sigma\Lambda_K}{2\operatorname{dist}(x,\sigma)}\right)^{2n}\|b_j\|_{C^m(\sigma)}
\end{aligned}$$

and we arrive at (34). ∎

We now investigate the error term $\max_{t\in\sigma}|E_\tau^n f(\cdot,t)|$ by the following lemma.

Lemma 14 *Let b_i, $i \in \{1,\dots M\}$, be the Lagrange basis for $S_{\mathcal{T}}^m$ and let K satisfy (31). Assume $2n > m$. Let $\tau \subset \operatorname{supp}(b_i)$ and assume that $\operatorname{dist}(\tau, y) > 0$ for a fixed $y \in [0,1]\backslash\overline{\tau}$. We have*

$$\left|(I_\tau - Q_\tau^n)\left((\cdot - y)^{\alpha-1} K(\cdot, y) b_i\right)\right| \leq \frac{2C_K \,\mathrm{e}\, \Lambda_K}{\operatorname{dist}^{1-\alpha}(x,\tau)} \left(\frac{\Lambda_K}{2} \frac{h_\tau}{\operatorname{dist}(y,\tau)}\right)^{2n} \|b_i\|_{C^m(\tau)}. \tag{36}$$

The proof is a repetition of the arguments used in the proof of Lemma 13 and skipped here. The proof of the following theorem follows from Lemma 13 and 14 via a straightforward tensor argument (cf. (33)).

Theorem 15 *Let b_i, b_j, $i,j \in \{1,\dots M\}$, be Lagrange basis functions of the piecewise polynomial space $S_{\mathcal{T}}^m$ and let K satisfy (31). Assume $2n > m$. For $\tau \subset \operatorname{supp}(b_i), \sigma \subset \operatorname{supp}(b_j)$ with $\operatorname{dist}(\tau,\sigma) > 0$ it holds*

$$|(I_\tau \otimes I_\sigma - Q_\tau^n \otimes Q_\sigma^n)(w_{i,j})| \leq \frac{2C_K \,\mathrm{e}\, \Lambda_K}{\operatorname{dist}^{1-\alpha}(\tau,\sigma)} \left(\frac{\Lambda_K}{2} \frac{\max\{h_\tau, h_\sigma\}}{\operatorname{dist}(\tau,\sigma)}\right)^{2n} \left(\|b_i\|_{C^m(\tau)} + \|b_j\|_{C^m(\sigma)}\right). \tag{37}$$

Remark 16 *The upper bound for the quadrature error in Theorem 15 grows if* $\max\{h_\tau, h_\sigma\} > \frac{2}{\Lambda_K} \operatorname{dist}(\tau,\sigma)$. *This is an artifact of the proof. If one employs the theory of derivative-free error quadrature error estimates via complex analysis of analytic integrands (cf. [8], [19]) one obtains exponential convergence as long as* $\operatorname{dist}(\tau,\sigma) > 0$. *Here, we employed the simpler classical theory in order to reduce technicalities. For the numerical experiments we have used the tensor Gauss-Legendre formulae for all pairs of intervals τ, σ which have positive distance.*

4.2.2 The Case $\tau = \sigma$

In order to approximate the integral

$$\int_\tau \int_{\tau \cap [0,x]} (x-y)^{\alpha-1} K(x,y) b_i(x) b_j(y)\, dy dx$$

we first transform it to the unit cube via simplex coordinates $(\xi, \eta) \to (\xi, \xi\eta)$. The determinant of the Jacobean equals ξ. Let $\tau = [a, a + h_\tau]$; then

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_a^{a+h_\tau} \int_a^x (x-y)^{\alpha-1} K(x,y) b_i(x) b_j(y)\, dy dx \\
&\quad = \frac{1}{\Gamma(\alpha)} \int_0^{h_\tau} \int_0^x (x-y)^{\alpha-1} K(x+a, y+a) b_i(x+a) b_j(y+a)\, dy dx \\
&\quad = \frac{1}{\Gamma(\alpha)} \int_0^{h_\tau} \int_0^1 \xi^\alpha (1-\eta)^{\alpha-1} K(\xi + a, \xi\eta + a) b_i(\xi + a) b_j(\xi\eta + a)\, d\eta d\xi \\
&\quad = \frac{h_\tau^{\alpha+1}}{\Gamma(\alpha)} \int_0^1 \int_0^1 \xi^\alpha (1-\eta)^{\alpha-1} K(h_\tau\xi + a, h_\tau\xi\eta + a) b_i(h_\tau\xi + a) b_j(h_\tau\xi\eta + a)\, d\eta d\xi.
\end{aligned}$$

For the ξ-integration we employ Gauss-Jacobi quadrature with weight ξ^α and for the η-integration we employ Gauss-Jacobi quadrature with weight $(1-\eta)^{\alpha-1}$. For an error analysis for these method we refer to [17, Section 5.3.2].

4.2.3 Right-Hand Side

The right-hand side of the Galerkin method can be approximated with an n-point Gaussian quadrature method:

$$r_i = \sum_{\tau\subset \mathrm{supp}(b_i)} \int_\tau g b_i \approx \sum_{\tau\subset \mathrm{supp}(b_i)} \sum_{k=0}^{n-1} \omega_k^{\tau,G} g(\xi_k^{\tau,G}) b_i(\xi_k^{\tau,G}) \tag{38}$$

The error of the right-hand side is estimated in the following lemma. We assume that the right-hand side g is analytic, more precisely, there exist constants C_g and $\Lambda_g \geq 2$ such that

$$\|g\|_{C^n(\Omega)} \leq C_g \Lambda_g^n n! \qquad \forall n \in \mathbb{N}_0. \tag{39}$$

Lemma 17 *Let b_i, $i \in \{1,\dots,M\}$, denote the Lagrange basis of $S_\mathcal{T}^m$. Assume $2n > m$ and g satisfies (39). Let $\tau \subset \mathrm{supp}(b_i)$. The error of the n-point Gauss quadrature method is given by*

$$|(I_\tau - Q_\tau^n)(b_i g)| \leqslant 2\Lambda_g \left(\frac{\Lambda_g}{2} h_\tau\right)^{2n} \|b_i\|_{C^m(\tau)}. \tag{40}$$

Proof. We compute the $2n$-th derivative of the integrand as in the previous section:

$$\begin{aligned}
\left\|\partial^{2n}(b_i g)\right\|_{L^\infty(\tau)} &= \left\|\sum_{k=0}^{2n} \binom{2n}{k} \left(\partial^k b_i\right)\left(\partial^{2n-k} g\right)\right\|_{L^\infty(\tau)} \\
&= \left\| b_i g^{(2n)} + \ldots + \binom{2n}{m} b_i^{(m)} g^{(2n-m)} \right\|_{L^\infty(\tau)} \leqslant (2n)! \|b_i\|_{C^m(\tau)} \sum_{k=0}^{2n} \frac{\|g\|_{C^k(\tau)}}{k!} \\
&\leq C_g (2n)! \|b_i\|_{C^m(\tau)} \Lambda_g^{2n+1}
\end{aligned}$$

This allows us to estimate the error via a Taylor argument

$$|(I_\tau - Q_\tau^n)(b_i g)| \leqslant 2C_g (2n)! \|b_i\|_{C^m(\tau)} \Lambda_g^{2n+1} \left(\frac{(h_\tau/2)^{2n}}{(2n)!}\right) = 2\Lambda_g \left(\frac{\Lambda_g}{2} h_\tau\right)^{2n} \|b_i\|_{C^m(\tau)}.$$

■

4.3 The Influence of the Quadrature on the Discretization Error

The approximation of the entries $\tilde{a}_{i,j}$, $\tilde{r}_i$ of the system matrix and of the right-hand side vector by numerical quadrature leads to a "perturbed" linear system of equations

$$\tilde{\mathbf{A}}\tilde{\mathbf{f}}_S = \tilde{\mathbf{r}}, \tag{41}$$

which can be translated to the following variational problem: Find $\tilde{f}_S \in S$ such that

$$\tilde{a}_{K,\alpha}(\tilde{f}_S, v) = \tilde{G}(v) \quad \forall v \in S. \tag{42}$$

Here, $\tilde{a}_{K,\alpha} : S \times S \to \mathbb{R}$ and $\tilde{G} : S \to \mathbb{R}$ are defined for $u = \sum_{i=1}^{M} \alpha_i b_i$ and $v = \sum_{i=1}^{M} \beta_i b_i$ by

$$\tilde{a}_{K,\alpha}(u,v) = \sum_{i,j=1}^{M} \beta_i \tilde{a}_{i,j} \alpha_j \quad \text{and} \quad \tilde{G}(v) = \sum_{i=1}^{M} \tilde{r}_i \beta_i.$$

We will analyze the error $\|f - \tilde{f}_S\|_{H^{-\alpha/2}(\Omega)}$ and the influence of the quadrature on the discretization error. To reduce technicalities we assume that the mesh is quasi-uniform, i.e., there exists a constant C_{qu} such that

$$C_{\text{qu}} := \max\left\{\frac{h}{h_\tau} : \tau \in \mathcal{T}\right\}.$$

First we rewrite the sesquilinear form $a(u,v)$ for $u, v \in S_{\mathcal{T}}^m, m \in \mathbb{N}_0$

$$\begin{aligned} a_{K,\alpha}(u,v) &= \frac{1}{\Gamma(\alpha)} \int_0^1 u(x) \int_0^x (x-y)^{\alpha-1} K(x,y) v(y) dy dx \qquad (43) \\ &= \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{M} \sum_{j=1}^{i} u_i v_j \sum_{\tau \subset \operatorname{supp}(b_i)} \sum_{\sigma \subset \operatorname{supp}(b_j)} \int_\tau \int_{\sigma \cap [0,x]} b_i(x) \frac{K(x,y) b_j(y)}{(x-y)^{1-\alpha}} dy dx. \end{aligned}$$

This leads to the definition $I_{\tau\times\sigma}^{i,j} := \int_\tau \int_{\sigma\cap[0,x]} b_i(x) b_j(y) K(x,y)(x-y)^{\alpha-1} dy dx$. The quadrature approximation of $I_{\tau\times\sigma}^{i,j}$ is denoted by $Q_{\tau\times\sigma}^{i,j}$. The associated error is $E_{\tau\times\sigma}^{i,j} := I_{\tau\times\sigma}^{i,j} - Q_{\tau\times\sigma}^{i,j}$. We set

$$I_{\tau\times\sigma}(u,v) := \sum_{i=1}^{M} \sum_{j=1}^{i} u_i v_j I_{\tau\times\sigma}^{i,j} \tag{44}$$

and similarly we define $Q_{\tau\times\sigma}(u,v)$ and $E_{\tau\times\sigma}(u,v)$. This motivates the following definition:

$$\mathcal{I}_\tau := \{i : \tau \subset \operatorname{supp} b_i\}.$$

Lemma 18 *Let* $E_{\tau\times\sigma}^{\max} := \max_{(i,j)\in\mathcal{I}_\tau\times\mathcal{I}_\sigma, j\leqslant i} |E_{\tau\times\sigma}^{i,j}|$. *There exists a constant* C_m *which only depends on the local polynomial degree* m *such that for all* $u, v \in S_{\mathcal{T}}^m$ *it holds*

$$|E_{\tau\times\sigma}(u,v)| \leqslant C_m h^{-1} \|u\|_{L^2(\sigma)} \|v\|_{L^2(\tau)}.$$

Proof. Let $u = \sum_{i\in\mathcal{I}_\tau} \alpha_i \, b_i|_\tau$ and $v = \sum_{i\in\mathcal{I}_\sigma} \beta_i \, b_i|_\sigma$. We set $\boldsymbol{\alpha} = (\alpha_i)_{i=\mathcal{I}_\tau}$ and $\boldsymbol{\beta} = (\beta_i)_{i=\mathcal{I}_\sigma}$ and denote by $\|\boldsymbol{\alpha}\|$ the Euclidean norm of a vector. We estimate the error by adding the local contributions and employing discrete Cauchy-Schwarz inequalities

$$\begin{aligned}|E_{\tau\times\sigma}(u,v)| &= \left|\sum_{(i,j)\in\mathcal{I}_\tau\times\mathcal{I}_\sigma, j\leqslant i}\alpha_i\beta_j E_{\tau\times\sigma}^{i,j}\right| \leqslant |E_{\tau\times\sigma}^{\max}|\sum_{i\in\mathcal{I}_\tau}|\alpha_i|\sum_{j\in\mathcal{I}_\sigma}|\beta_j| \\ &\leqslant (m+1)\,|E_{\tau\times\sigma}^{\max}|\|\boldsymbol{\alpha}\|\|\boldsymbol{\beta}\|. \end{aligned} \tag{45}$$

Well known scaling inequalities for one-dimensional finite element functions (cf. [7]) lead to $\|\boldsymbol{\alpha}\| \leq \hat{C}_m h_\tau^{-1/2}\|u\|_{L^2(\tau)}$ and similarly for $\boldsymbol{\beta}$. The combination with (45) leads to the assertion with $C_m = (m+1)\,C_{\mathrm{qu}}\hat{C}_m^2$. ■

The following corollary is similar to [17, Thm. 5.3.29]. We use the fact that one-dimensional finite elements satisfy the inverse inequality

$$\|u\| \leq C_{\mathrm{inv}} h^{-s}\|u\|_{H^{-s}(\Omega)} \qquad \forall u\in S_{\mathcal{T}}^m \quad \forall s\in\left[0,\frac{1}{2}\right). \tag{46}$$

The constant C_{inv} depends on s, m, $C_{\mathcal{T}}$, and C_{qu} (see [7]).

Corollary 19 *Let* $E_{K,\alpha}^{\max} := C_m C_{\mathrm{qu}} C_{\mathrm{inv}}^2 \max_{\tau,\sigma\in\mathcal{T}} E_{\tau\times\sigma}^{\max}$ *and let* $a_{K,\alpha}(\cdot,\cdot)$ *be the sesquilinear form as in (43) and* $\tilde{a}_{K,\alpha}(\cdot,\cdot)$ *the perturbed sesquilinear form in (42). For all* $u,v\in S_{\mathcal{T}}^m$*, it holds*

$$|a_{K,\alpha}(u,v)-\tilde{a}_{K,\alpha}(u,v)| \leqslant E_{K,\alpha}^{\max} h^{-2-\alpha}\|u\|_{H^{-\alpha/2}(\Omega)}\|v\|_{H^{-\alpha/2}(\Omega)}. \tag{47}$$

Proof. The quasi-uniformity of the mesh implies that $N \leq C_{\mathrm{qu}}h^{-1}$. The quadrature error of the sesquilinear form can be estimated by

$$\begin{aligned}|a_{K,\alpha}(u,v)-\tilde{a}_{K,\alpha}(u,v)| &\leqslant \sum_{\tau,\sigma\in\mathcal{T}}|E_{\tau\times\sigma}(u,v)| \leqslant C_m h^{-1}\sum_{\tau,\sigma\in\mathcal{T}}E_{\tau\times\sigma}^{\max}\|u\|_{L^2(\tau)}\|v\|_{L^2(\sigma)} \\ &\leqslant C_m\left(\max_{\tau,\sigma\in\mathcal{T}}E_{\tau\times\sigma}^{\max}\right)h^{-1}\sum_{\tau\in\mathcal{T}}\|u\|_{L^2(\tau)}\sum_{\sigma\in\mathcal{T}}\|v\|_{L^2(\sigma)} \leqslant C_m\left(\max_{\tau,\sigma\in\mathcal{T}}|E_{\tau\times\sigma}^{\max}|\right)h^{-1}N\,\|u\|\,\|v\| \\ &= C_m C_{\mathrm{qu}}\left(\max_{\tau,\sigma\in\mathcal{T}}E_{\tau\times\sigma}^{\max}\right)h^{-2}\|u\|\|v\| \overset{(46)}{\leqslant} E_{K,\alpha}^{\max}h^{-2-\alpha}\|u\|_{H^{-\alpha/2}(\Omega)}\|v\|_{H^{-\alpha/2}(\Omega)}.\end{aligned}$$

■

Remark 20 *From Corollary 19 and the continuity of* $a_{K,\alpha}$ *it follows that* $\tilde{a}_{K,\alpha}$ *is continuous in* $S_{\mathcal{T}}^m$*.*

Proof. A triangle inequality leads to

$$\begin{aligned}|\tilde{a}_{K,\alpha}(u,v)| &\leqslant |a_{K,\alpha}(u,v)| + |\tilde{a}_{K,\alpha}(u,v)-a_{K,\alpha}(u,v)| \\ &\leqslant \hat{C}_c\|u\|_{H^{-\alpha/2}(\Omega)}\|v\|_{H^{-\alpha/2}(\Omega)} \qquad \text{with } \hat{C}_c := E_{K,\alpha}^{\max}h^{-2-\alpha}+C_c.\end{aligned} \tag{48}$$

■

For the functional $G(v)$ as in (27), $v \in S^m_{\mathcal{T}}$, $m \in \mathbb{N}_0$, it holds

$$G(v) = \int_0^1 gv = \sum_{i=1}^{M} \sum_{\tau \subset \operatorname{supp}(b_i)} \beta_i \int_\tau g b_i. \tag{49}$$

We proceed similar as for $a(\cdot,\cdot)$ and define $I^i_\tau := \int_\tau g b_i$ and the quadrature operator Q^i_τ in the same fashion so that the error is $E^i_\tau = I^i_\tau - Q^i_\tau$. We set $I_\tau(v) := \sum_{i\in\mathcal{I}_\tau} \beta_i I^i_\tau$ and define $Q_\tau(v)$ and $E_\tau(v)$ analogously. With $E^{\max}_\tau = \max_{i\in\mathcal{I}_\tau} |E^i_\tau|$ we get

$$|E_\tau(v)| = |\sum_{i\in\mathcal{I}_\tau} \beta_i E^i_\tau| \leqslant \sqrt{m+1} E^{\max}_\tau \|\boldsymbol{\beta}\| \leqslant E^{\max}_\tau \hat{C}_m \sqrt{\frac{m+1}{h_\tau}} \|v\|_{L^2(\tau)}.$$

Corollary 21 *Let* $E^{\max}_G := \sqrt{C_m C_{\text{qu}}} C_{\text{inv}} \max_{\tau\in\mathcal{T}} E^{\max}_\tau$. *Let* $G(\cdot)$ *be the functional in (49) and* $\tilde{G}(\cdot)$ *the perturbed functional as in (42). For all* $v \in S^m_{\mathcal{T}}$ *it holds*

$$|G(v) - \tilde{G}(v)| \leqslant E^{\max}_G h^{-(1+\alpha)/2} \|v\|_{H^{-\alpha/2}(\Omega)}. \tag{50}$$

Proof. Again we localize the difference $G - \tilde{G}$ and obtain

$$\begin{aligned} |G(v) - \tilde{G}(v)| &\leqslant \sum_{\tau\in\mathcal{T}} |E_\tau(v)| \leqslant \sum_{\tau\in\mathcal{T}} E^{\max}_\tau \sqrt{C_m} h^{-1/2} \|v\|_{L^2(\tau)} \\ &\leqslant \sqrt{C_m} h^{-1/2} \left(\max_{\tau\in\mathcal{T}} E^{\max}_\tau\right) \sum_{\tau\in\mathcal{T}} \|v\|_{L^2(\tau)} \leqslant \sqrt{C_m Q_{\text{qu}}} \left(\max_{\tau\in\mathcal{T}} E^{\max}_\tau\right) h^{-1} \|v\|. \end{aligned}$$

An inverse inequality for v leads to the assertion. ■

We will now investigate the error of the perturbed Galerkin method. The proof of the next theorem is based on the first Strang lemma.

Theorem 22 *Let the assumptions in Theorem 11 be satisfied. Assume that the quadrature method is sufficiently accurate such that*

$$E^{\max}_{K,\alpha} h^{-2-\alpha} \leq \frac{\tilde{\gamma}}{2}. \tag{51}$$

Then, the perturbed Galerkin method (42) is $H^{-\alpha/2}(\Omega)$*-elliptic and the fully discrete equations (42) have a unique solution* $\tilde{f}_S$ *that satisfies the error estimate*

$$\|f - \tilde{f}_S\|_{H^{-\alpha/2}(\Omega)} \leqslant C \left(\min_{w\in S} \left(\|f - w\|_{H^{-\alpha/2}(\Omega)} + E^{\max}_{K,\alpha} h^{-2-\alpha} \|w\|_{H^{-\alpha/2}(\Omega)} \right) + E^{\max}_G h^{-(1+\alpha)/2} \right), \tag{52}$$

for some $C > 0$.

Proof. Theorem 11 implies that the exact sesquilinear form $a_{K,\alpha}$ is $H^{-\alpha/2}(\Omega)$-elliptic and we derive the property for the perturbed version $\tilde{a}_{K,\alpha}$ next. From Corollary 19 we conclude that

$$|a_{K,\alpha}(u,v) - \tilde{a}_{K,\alpha}(u,v)| \leqslant E_{K,\alpha}^{\max} h^{-2-\alpha} \|u\|_{H^{-\alpha/2}(\Omega)} \|v\|_{H^{-\alpha/2}(\Omega)}.$$

This implies

$$\begin{aligned}\operatorname{Re} \tilde{a}_{K,\alpha}(u,u) &\geqslant \operatorname{Re} a_{K,\alpha}(u,u) - \operatorname{Re}(a_{K,\alpha}(u,u) - \tilde{a}_{K,\alpha}(u,u)) \\ &\geqslant \tilde{\gamma} \|u\|^2_{H^{-\alpha/2}(\Omega)} - |a_{K,\alpha}(u,u) - \tilde{a}_{K,\alpha}(u,u)| \geqslant (\tilde{\gamma} - E_{K,\alpha}^{\max} h^{-2-\alpha}) \|u\|^2_{H^{-\alpha/2}(\Omega)}.\end{aligned}$$

Hence, condition (51) implies the $H^{-\alpha/2}(\Omega)$-ellipticity of the perturbed sesquilinear form. From the Lax-Milgram lemma we conclude that (42) has a unique solution $\tilde{f}_S$. Next we prove the error estimate (52)

$$\begin{aligned}\|f - \tilde{f}_S\|_{H^{-\alpha/2}(\Omega)} &\leqslant \|f - f_S\|_{H^{-\alpha/2}(\Omega)} + \|f_S - \tilde{f}_S\|_{H^{-\alpha/2}(\Omega)} \qquad (53) \\ &\leqslant \|f - f_S\|_{H^{-\alpha/2}(\Omega)} + \frac{2}{\tilde{\gamma}} \frac{\operatorname{Re} \tilde{a}_{K,\alpha}(f_S - \tilde{f}_S, f_S - \tilde{f}_S)}{\|f_S - \tilde{f}_S\|_{H^{-\alpha/2}(\Omega)}} \\ &\leqslant \|f - f_S\|_{H^{-\alpha/2}(\Omega)} + \frac{2}{\tilde{\gamma}} \sup_{v \in S\setminus\{0\}} \frac{|\tilde{a}_{K,\alpha}(f_S - \tilde{f}_S, v)|}{\|v\|_{H^{-\alpha/2}(\Omega)}} \\ &= \|f - f_S\|_{H^{-\alpha/2}(\Omega)} + \frac{2}{\tilde{\gamma}} \sup_{v \in S\setminus\{0\}} \frac{|\tilde{a}_{K,\alpha}(f_S, v) - \tilde{G}(v)|}{\|v\|_{H^{-\alpha/2}(\Omega)}} \\ &\leqslant \|f - f_S\|_{H^{-\alpha/2}(\Omega)} + \frac{2}{\tilde{\gamma}} \sup_{v \in S\setminus\{0\}} \frac{|\tilde{a}_{K,\alpha}(f_S, v) - a_{K,\alpha}(f_S, v)| + |G(v) - \tilde{G}(v)|}{\|v\|_{H^{-\alpha/2}(\Omega)}}.\end{aligned}$$

We consider the difference $|\tilde{a}_{K,\alpha}(f_S, v) - a_{K,\alpha}(f_S, v)|$ and obtain by using the continuity of $a_{K,\alpha}$ and $\tilde{a}_{K,\alpha}$ as well as the consistency (47) that

$$\begin{aligned}|\tilde{a}_{K,\alpha}(f_S, v) - a_{K,\alpha}(f_S, v)| &\leqslant |\tilde{a}_{K,\alpha}(f_S - w, v)| + |\tilde{a}_{K,\alpha}(w, v) - a_{K,\alpha}(w, v)| + |a_{K,\alpha}(w - f_S, v)| \\ &\leqslant \left(\tilde{C}_c + \frac{\tilde{\gamma}}{2}\right) \|f_S - w\|_{H^{-\alpha/2}(\Omega)} \|v\|_{H^{-\alpha/2}(\Omega)} + E_{K,\alpha}^{\max} h^{-2-\alpha} \|w\|_{H^{-\alpha/2}(\Omega)} \|v\|_{H^{-\alpha/2}(\Omega)} \\ &\quad + \tilde{C}_c \|w - f_S\|_{H^{-\alpha/2}(\Omega)} \|v\|_{H^{-\alpha/2}(\Omega)}\end{aligned}$$

holds for all $w \in S$. This leads to

$$\sup_{v \in S\setminus\{0\}} \frac{|\tilde{a}(f_S, v) - a(f_S, v)|}{\|v\|_{H^{-\alpha/2}(\Omega)}} \leqslant \min_{w \in S} \left(C \|f_S - w\|_{H^{-\alpha/2}(\Omega)} + E_{K,\alpha}^{\max} h^{-2-\alpha} \|w\|_{H^{-\alpha/2}(\Omega)} \right)$$

with $C := 2\tilde{C}_c + \frac{\tilde{\gamma}}{2}$. The combination with (30) and (50) leads to

$$\begin{aligned}\|f - \tilde{f}_S\|_{H^{-\alpha/2}(\Omega)} &\leqslant C \min_{w \in S} \Big(\|f - w\|_{H^{-\alpha/2}(\Omega)} \\ &\quad + \frac{2}{\tilde{\gamma}} (\|f - w\|_{H^{-\alpha/2}(\Omega)} + E_{K,\alpha}^{\max} h^{-2-\alpha} \|w\|_{H^{-\alpha/2}(\Omega)} + \sup_{v \in S\setminus\{0\}} \frac{|G(v) - \tilde{G}(v)|}{\|v\|_{H^{-\alpha/2}(\Omega)}}) \Big) \\ &\leqslant C \left(\min_{w \in S} (\|f - w\|_{H^{-\alpha/2}(\Omega)} + E_{K,\alpha}^{\max} h^{-2-\alpha} \|w\|_{H^{-\alpha/2}(\Omega)}) + E_G^{\max} h^{-(1+\alpha)/2} \right).\end{aligned}$$

■

4.4 Choice of the Quadrature Order

The results of Sections 4.2 and 4.3 allow to determine the appropriate quadrature order n so that the perturbed Galerkin method converges at the same rate as the unperturbed Galerkin method.

Theorem 23 *Let the assumption of Theorem 11 be satisfied. Assume that the kernel function K satisfies (31) and that the right-hand side g fulfills condition (39). Let the quadrature order in (37) for the integrals over those $\tau \times \sigma$ which satisfy*

$$\operatorname{dist}(\tau, \sigma) \geq \Lambda_K \max\{h_\tau, h_\sigma\}$$

be chosen according to

$$n_1 = \left\lceil (m + 2 + \alpha/4) \frac{\log\left(\frac{1}{h}\right)}{\log\left(\frac{2}{\Lambda_K} \frac{\operatorname{dist}(\tau,\sigma)}{\max\{h_\tau,h_\sigma\}}\right)} \right\rceil. \qquad (54)$$

Assume that the remaining singular and near-singular integrals are evaluated exactly.

Then the fully discrete Galerkin method is stable. Let the number of quadrature points for the right-hand side be chosen as

$$n_2 = \left\lceil m + \frac{\alpha}{2} + \frac{3}{4} \right\rceil.$$

If the exact solution is $H^{m+1}(\Omega)$, then, the perturbed Galerkin method converges at the same rate as the Galerkin method.

Remark 24 *The assumption that all singular integrals are evaluated exactly is related to the fact that we have omitted the error analysis for the singular integrals (cf. Remark 16). By using techniques as developed in [17] this case can be handled while the technicalities are increased.*

Proof. We employ Lemma 15 and standard inverse estimates for finite element functions, i.e., there exists a constant C_{inv} such that $\|b_i\|_{C^m(\tau)} \leq C_{\mathrm{inv}} h_\tau^{-m}$ (we may use the same notation as in (46) by a suitable adjustment of C_{inv}). Thus, we obtain

$$E_{\tau\times\sigma}^{\max} \leq 2C_m C_K C_{\mathrm{inv}}\, \mathrm{e}\, \Lambda_K h_\tau^{-m} \max_{\tau,\sigma\in\mathcal{T}} \left(\frac{1}{\operatorname{dist}^{1-\alpha}(\tau,\sigma)} \left(\frac{\Lambda_K}{2} \frac{\max\{h_\tau, h_\sigma\}}{\operatorname{dist}(\tau,\sigma)} \right)^{2n_1} \right).$$

The assumption $\operatorname{dist}(\tau,\sigma) \geq \Lambda_K \max\{h_\tau, h_\sigma\} \geq \Lambda_K C_{\mathrm{qu}}^{-1} h$ implies

$$E_{\tau\times\sigma}^{\max} \leq 2C_m C_K C_{\mathrm{inv}} C_{\mathrm{qu}}^{1-\alpha}\, \mathrm{e}\, \Lambda_K^{\alpha} h^{\alpha-1-m} \left(\frac{\Lambda_K}{2} \frac{\max\{h_\tau, h_\sigma\}}{\operatorname{dist}(\tau,\sigma)} \right)^{2n_1}.$$

From (51) we deduce that the fully discrete Galerkin method is stable if

$$2C_m C_K C_{\mathrm{inv}} C_{\mathrm{qu}}^{1-\alpha}\, \mathrm{e}\, \Lambda_K^{\alpha} h^{-3-m} \left(\frac{\Lambda_K}{2} \frac{\max\{h_\tau, h_\sigma\}}{\operatorname{dist}(\tau,\sigma)} \right)^{2n_1} \leq \frac{\tilde{\gamma}}{2}.$$

To obtain an optimal convergence order we obtain from (52) the stronger condition

$$2C_mC_KC_{\text{inv}}C_{\text{qu}}^{1-\alpha}\,\mathrm{e}\,\Lambda_K^{\alpha}h^{-3-m}\left(\frac{\Lambda_K}{2}\frac{\max\{h_\tau,h_\sigma\}}{\operatorname{dist}(\tau,\sigma)}\right)^{2n_1}\leq Ch^{m+1+\alpha/2}.$$

We solve this last condition for n_1 and get

$$n_1=\left\lceil(m+2+\alpha/4)\,\frac{\log\left(\frac{1}{h}\right)}{\log\left(\frac{2}{\Lambda_K}\frac{\operatorname{dist}(\tau,\sigma)}{\max\{h_\tau,h_\sigma\}}\right)}\right\rceil. \tag{55}$$

Finally, we have to determine the number n_2 of Gauss points for the approximation of the right-hand side. To preserve the optimal convergence order, the last term in (52) has to be bounded by $Ch^{m+1+\alpha/2}$. The combination with Corollary 21 leads to the condition

$$\sqrt{C_mC_{\text{qu}}}C_{\text{inv}}\left(\max_{\tau\in\mathcal{T}}E_\tau^{\max}\right)h^{-(1+\alpha)/2}\leq Ch^{m+1+\alpha/2}.$$

The quantity $E_\tau^{\max}$ can be estimated via (40) so that

$$\sqrt{C_mC_{\text{qu}}}C_{\text{inv}}^2 2\Lambda_g\left(\frac{\Lambda_g}{2}h_\tau\right)^{2n_2}h^{-(1+\alpha)/2-m}\leq Ch^{m+1+\alpha/2}$$

is a sufficient condition. We solve this for n_2 and obtain

$$n_2\leq\left(\frac{2m+3/2+\alpha}{2}\right)\frac{\log\frac{1}{h}}{\log\frac{2}{\Lambda_g}+\log\frac{1}{h_\tau}}\leq\frac{2m+3/2+\alpha}{2}.$$

■

5 Numerical Experiments

The numerical experiments are used to investigate the sharpness of our theoretical error estimates. For this we study how the quadrature method influences the convergence rate. The tests are done for the equation

$$g(x)=\Gamma(1/2)^{-1}\int_0^x\frac{f(y)}{\sqrt{x-y}}dy, \tag{56}$$

with $K=1$ and $g(x)=\frac{16}{15\Gamma(1/2)}x^{5/2}$. The solution is given by $f(y)=y^2$. The singular integrals are evaluated analytically. For the discretization we have used a uniform mesh with width $h=N^{-1}$.

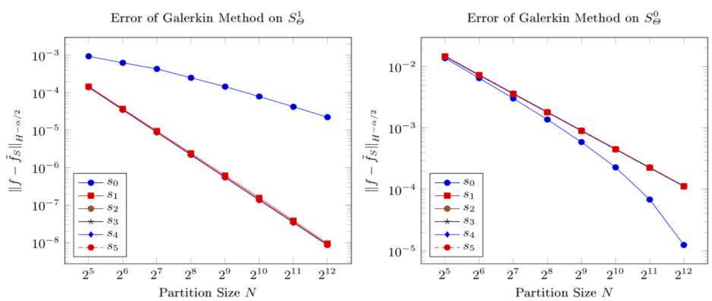

Figure 1: Plots of the error in logarithmic scale

5.1 Changing the Quadrature Order

Let $s_i := m + i + \alpha/4$ so that the prefactor in (54) in front of the logarithmic terms equals s_2. We have varied $i \in \{0, 1, \ldots, 5\}$ to numerically validate the sharpness of estimate (54) for the number of quadrature points. We will inspect its influence on the error $\|f - \tilde{f}_S\|_{H^{-\alpha/2}(\Omega)}$ for different values of s_i and for different numbers N of subintervals in the partition $\mathcal{T}$.

In Figure 1 one can see that the desired convergence rate for the Galerkin method in $S^1_{\mathcal{T}}$ is achieved for the number of Gauss points chosen as

$$n_1 = \left\lceil \frac{s_1}{2} \frac{\log(1/h)}{\log\left(\frac{2\,\mathrm{dist}(\tau,\sigma)}{h}\right)} \right\rceil. \tag{57}$$

The desired convergence rate for the Galerkin method in $S^0_{\mathcal{T}}$ is achieved by choosing

$$n_1 = \left\lceil \frac{s_0}{2} \frac{\log(1/h)}{\log\left(\frac{2\,\mathrm{dist}(\tau,\sigma)}{h}\right)} \right\rceil. \tag{58}$$

Fixed Quadrature Order:

In the previous experiment we adapted the quadrature order depending on $\mathrm{dist}(\tau, \sigma)$. Here we fix a quadrature order for the whole experiment. The results are depicted in Figure 2 and clearly show that a low quadrature order for $S^1_{\mathcal{T}}$ significantly pollutes the convergence rate.

5.2 Dependence of the Solution of Abel's Integral Equation on the Order of Singularity α

In this experiment, we investigate the sensitivity of our method on the value for $\alpha \in (0, 1)$. We choose the function $g(x) = 16/5x^{5/2}$ and solve the equation

$$g(x) = \Gamma(\alpha)^{-1} \int_0^x (x - y)^{\alpha - 1} f(y) dy. \tag{59}$$

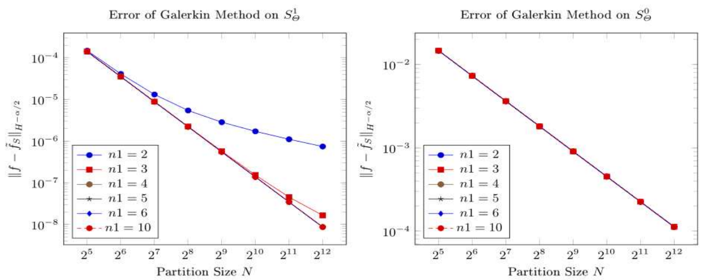

Figure 2: Plots of the error in logarithmic scale

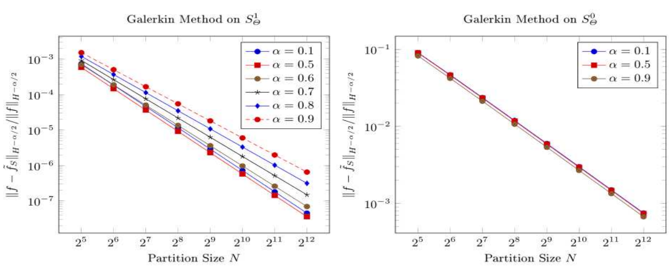

Figure 3: Plots of the relative error in logarithmic scale

We set $\alpha = \frac{k}{10}$, for $k \in \{1, \dots, 9\}$, and investigate the error in Figure 3.

For a fixed partition size $N = 2^{10}$ we display the relative error depending on the order of singularity α and see the influence in Figure 4.

6 Conclusion

Abel-type integral equation and its approximation have a long history and many researchers have worked on it. This paper generalizes the idea of Eggermont's approach in [10] and develops an error analysis for Galerkin discretizations in energy norms for generalized Abel-type kernel functions. We propose Gauss-type quadrature methods for the approximation of the entries of the system matrix and of the right-hand side. The local and global error analysis allows to keep the number of quadrature nodes fairly small. In the numerical experiments, the Galerkin method based on piecewise polynomials of constant and first order are compared and the dependency of error on the singularity is studied.

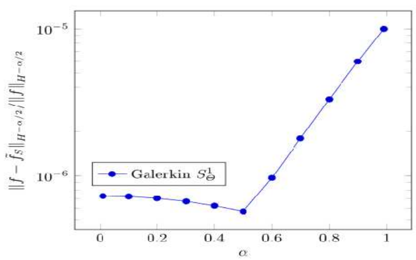

Figure 4: Plot of the relative error for fixed $N = 2^{10}$

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