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Inference for Structural Impulse Responses in SVAR-GARCH Models

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Abstract

Conditional heteroskedasticity can be exploited to identify the structural vector autoregressions (SVAR) but the implications for inference on structural impulse responses have not been investigated in detail yet. We consider the conditionally heteroskedastic SVAR-GARCH model and propose a bootstrap-based inference procedure on structural impulse responses. We compare the finite-sample properties of our bootstrap method with those of two competing bootstrap methods via extensive Monte Carlo simulations. We also present a three-step estimation procedure of the parameters of the SVAR-GARCH model that promises numerical stability even in scenarios with small sample sizes and/or large dimensions.

KEY WORDS: Bootstrap; conditional heteroskedasticity; multivariate GARCH; structural impulse responses; structural vector autoregression.

JEL CLASSIFICATION NOS: C12, C13, C32

1 Introduction

Identifying the structural vector autoregressive (SVAR) model is typically one of the crucial issues in structural impulse response analysis. The existing literature offers a plethora of different identification strategies; see Kilian and Lütkepohl (2017) for an excellent overview. A recent strand of this literature exploits the presence of conditional heteroskedasticity for the identification of the SVAR model; see e.g., Normandin and Phaneuf (2004), Lanne et al. (2010), Bouakez and Normandin (2010) and Herwartz and Lütkepohl (2014).

The presence of conditional heteroskedasticity entails that the standard assumption of an independent and identically distributed (i.i.d.) error process is no longer valid and needs to be replaced by weaker assumptions on the error process, such as weak stationarity and serial uncorrelatedness; see e.g., Lütkepohl and Milunovich (2016, p.242). Unfortunately, the deviation from the common i.i.d. assumption invalidates inference on structural impulse responses which is based on standard residual-based bootstrap methods; see e.g., the methods of Runkle (1987), Kilian (1998a) and Kilian (1998b). Thus, confidence intervals that are based on these bootstrap methods may lead to wrong conclusions.

Brüggemann et al. (2016) consider a conditionally heteroskedastic VAR model. However, the asymptotic validity of their proposed moving-block bootstrap method is only proven for structural impulse responses that are identified via an recursive ordering approach; see Brüggemann et al. (2016, p.75). As of yet, it is unknown whether the validity of this moving-block bootstrap also holds when the SVAR is identified by conditional heteroskedasticity. Moreover, it turns out that there is no study that analyzes in detail the implications of identifying the SVAR by conditional heteroskedasticity for inference on structural impulse responses.

This paper takes up the just mentioned issue, that is, the construction of confidence intervals for the structural impulse responses in a conditionally heteroskedastic framework. We consider a conditionally heteroskedastic SVAR model where the conditional heteroskedasticity is driven by a multivariate generalized autoregressive conditional heteroskedastic (GARCH) process, that is, the SVAR-GARCH model.

The contribution of this paper is twofold. First, we propose a new three-step estimation procedure of the parameters of the SVAR-GARCH model. The proposed estimation procedure exhibits numerical stability even in scenarios with small sample sizes and/or large dimensional parameter spaces. In contrast, the existing estimation procedures, see e.g., Bouakez et al. (2013, 2014) and Lütkepohl and Milunovich (2016), are prone to suffer from convergence problems in these delicate scenarios because an integral part of these estimation procedures is the Newton-type optimization of a likelihood function.

Second, we propose a bootstrap-based inference procedure for the structural impulse responses in the SVAR-GARCH model. The proposed bootstrap procedure is based on resampling (with replacement) of the devolatized residuals and incorporates the specific GARCH structure of the conditional heteroskedasticity. In addition, we conduct a Monte Carlo experiment to compare the finite-sample properties of our proposed method with the finite-sample properties of the bootstrap methods of Runkle (1987) and Brüggemann et al. (2016).

The remainder of the paper is organized as follows. Section 2 reviews the SVAR-GARCH model and presents details about the estimation procedure. Section 3 proposes a new bootstrap method to obtain the bootstrap sampling distribution of the estimator of the structural impulse response. Section 4 describes the competing bootstrap methods. Section 5 describes the Monte Carlo experiment and presents the empirical results. Section 6 concludes. The Appendix provides additional details about the estimation procedure, boxplots describing the finite-sample properties of the various bootstrap methods and figures.

2 The Model

2.1 Some Preliminaries

Let $\{y_t : t \in \mathbb{Z}\}$ be an m-dimensional stochastic process following a reduced-form VAR(p) model

$$A(L)y_t = \nu + u_t \ , \tag{1}$$

where $y_t := (y_{1,t}, \ldots, y_{m,t})$, $A(L) := I_m - \sum_{i=1}^{p} A_i L^i$ is a matrix polynomial in the backshift operator L, the A_i are $m \times m$ coefficient matrices, I_m is the $m \times m$ identity matrix and $\nu \in \mathbb{R}^m$ is a deterministic intercept. The reduced-form error process $\{u_t : t \in \mathbb{Z}\}$ is *weak* white noise, that is,

$$\mathbb{E}\left[u_t\right] = 0, \quad \mathbb{E}\left[u_t u_t'\right] = \Sigma_u \quad \text{and} \quad \mathbb{E}\left[u_t u_{t+h}'\right] = 0 \ , \tag{2}$$

for $h \neq 0$ and $\Sigma_u \in \mathbb{R}^{m \times m}$ is positive definite[1]. Note that the common independence assumption for the reduced-form process $\{u_t : t \in \mathbb{Z}\}$ is replaced by the weaker assumption of zero serial correlation. Moreover, it is assumed that the VAR coefficient matrices $A_1, \ldots, A_p$ satisfy the following stability condition

$$\det\left(A(z)\right) \neq 0 \quad \text{for all } z \in \mathbb{C} \text{ with } |z| \leq 1 \ .$$

The stable reduced-form VAR(p) process in (1) exhibits an equivalent Wold vector moving average (VMA) representation

$$y_t = \mu + \Psi(L)u_t \ ,$$

where $\mu := A(1)^{-1}\nu$ denotes the unconditional expectation of y_t, $\Psi(L) := \sum_{i=0}^{\infty} \Psi_i L^i$ is an (infinite) matrix polynomial in L and the Ψ_i matrices are determined via $\Psi(z) = A(z)^{-1}$. In particular, $\Psi_0 = I_m$ and $\Psi_s = \sum_{j=1}^{p} \Psi_{s-j} A_j$ for $s \in \mathbb{N}_{>0}$.

[1] Following Francq and Raïssi (2007), a VAR process (1) with (strictly stationary and ergodic) weak white noise is called a weak VAR(p) model.

The structural VAR (SVAR) corresponding to (1) is given by

$$B(L)y_t = B_0\nu + B_0 u_t \ , \tag{3}$$

where $B(L) := B_0 A(L)$ and $B_0 \in \mathbb{R}^{m\times m}$ is a nonsingular linear mapping that transforms the reduced-form error u_t into the (instantaneously uncorrelated) structural error ε_t, that is, $\varepsilon_t := B_0 u_t$; see e.g., Kilian (2013). Thus, the structural error process $\{\varepsilon_t : t \in \mathbb{Z}\}$ satisfies

$$\mathbb{E}\left[\varepsilon_t\right] = 0, \quad \mathbb{E}\left[\varepsilon_t\varepsilon_t'\right] = \Sigma_\varepsilon \quad \text{and} \quad \mathbb{E}\left[\varepsilon_t\varepsilon_{t+h}'\right] = 0 \ , \tag{4}$$

for $h \neq 0$ and $\Sigma_\varepsilon \in \mathbb{R}^{m\times m}$ is diagonal and positive definite.

When the process $\{u_t : t \in \mathbb{Z}\}$, and hence also the process $\{\varepsilon_t : t \in \mathbb{Z}\}$, is conditionally heteroskedastic, it may be possible to identify B_0 (or equivalently B_0^{-1}) without imposing additional identifying assumptions; see e.g., Lütkepohl and Netšunajev (2017) and the references therein. In the present paper, we assume that the reduced-form process $\{u_t : t \in \mathbb{Z}\}$ is given by a conditionally heteroskedastic multivariate GARCH process. More specifically, it is assumed that $\{u_t : t \in \mathbb{Z}\}$ follows a Generalized Orthogonal GARCH (GO-GARCH) model à la van der Weide (2002); the details are provided in the next section.

2.2 GO-GARCH Model

The assumption of a GO-GARCH model à la van der Weide (2002) implies that $\{u_t : t \in \mathbb{Z}\}$ can be represented as a nonsingular linear transformation of a process $\{\varepsilon_t : t \in \mathbb{Z}\}$ consisting of m conditionally uncorrelated univariate GARCH(1,1) processes, that is,

$$u_t = B_0^{-1}\varepsilon_t \tag{5}$$

$$= B_0^{-1}H_t^{1/2}e_t \ , \tag{6}$$

where $H_t := \operatorname{diag}(\sigma_{t,1}^2, \ldots, \sigma_{t,m}^2)$, the diagonal elements of H_t evolve according to the following univariate GARCH(1,1) specification[2]

$$\sigma_{t,i}^2 = (1-\alpha_i-\beta_i) + \alpha_i\varepsilon_{t-1,i}^2 + \beta_i\sigma_{t-1,i}^2, \quad \alpha_i, \beta_i \geq 0, \ \alpha_i+\beta_i < 1 \ , \tag{7}$$

and $\{e_t : t \in \mathbb{Z}\}$ is a sequence of i.i.d. random vectors with mutually independent components $e_{t,i}$, $i = 1, \ldots, m$, having mean zero and unit variance, that is, $\mathbb{E}[e_t] = 0$ and $\mathbb{E}[e_t e_t'] = I_m$.

The structural error process $\{\varepsilon_t : t \in \mathbb{Z}\}$, consisting of conditionally uncorrelated GARCH(1,1) processes, is a martingale difference sequence and conditionally heteroskedastic with diagonal conditional variance matrix H_t. Moreover, the process satisfies

$$\mathbb{E}[\varepsilon_t] = 0, \quad \mathbb{E}[\varepsilon_t\varepsilon_t'] = I_m \quad \text{and} \quad \mathbb{E}[\varepsilon_t\varepsilon_{t+h}'] = 0 \ , \tag{8}$$

for all $h \neq 0$. The unconditional variance of ε_t is restricted to the identity matrix because of the normalization of the intercept in (7). This corresponds to the so-called B-normalization of

[2] See Bollerslev (1986) for more details about the univariate GARCH(p,q) process.

SVAR models; see e.g. Lütkepohl (2005, Section 9.1). As is evident from (8), $\{\varepsilon_t : t \in \mathbb{Z}\}$ is weakly stationary.

The reduced-form error process $\{u_t : t \in \mathbb{Z}\}$ is a nonsingular nonlinear transformation of the structural error process (which consists of separate GARCH(1,1) processes). Hence, the reduced-form error process is also a martingale difference sequence and conditionally heteroskedastic with non-diagonal conditional variance matrix $B_0^{-1} H_t B_0^{-1\prime}$. The unconditional first and second moments are given by

$$\mathbb{E}[u_t] = 0, \quad \mathbb{E}[u_t u_t'] = B_0^{-1} B_0^{-1\prime} \quad \text{and} \quad \mathbb{E}[u_t u_{t+h}'] = 0 \; , \tag{9}$$

for all $h \neq 0$. Hence, $\{u_t : t \in \mathbb{Z}\}$ is weakly stationary[3]. Moreover, (9) confirms that the assumption of a GO-GARCH model for $\{u_t : t \in \mathbb{Z}\}$ agrees with the assumption of weak white noise in Section 2.1.

As noted by van der Weide (2002), the GO-GARCH model is nested in the more general BEKK model of Engle and Kroner (1995). Therefore, Theorem 2.4 of Boussama et al. (2011, p.2336), which concerns the properties of BEKK models, establishes strict stationarity and ergodicity of $\{u_t : t \in \mathbb{Z}\}$ under the parameter restrictions in (7) and the following two additional assumptions: (i) the joint distribution of e_1 is absolutely continuous with respect to the Lebesque measure on $\mathbb{R}^m$ and (ii) the point zero is in the interior of the support of the joint distribution of e_1.

2.3 Identification

Structural impulse responses are as such partial derivatives $\partial y_{t,i}/\partial \varepsilon_{t-h,j}$, $i, j \in \{1, \ldots, m\}$, $h \in \mathbb{N}$, and hence elements of the coefficient matrices $\Psi_h B_0^{-1}$ in the structural vector moving average (VMA) representation of $\{y_t : t \in \mathbb{Z}\}$, that is,

$$y_t = \mu + \sum_{i=1}^{+\infty} \Psi_i B_0^{-1} \varepsilon_{t-i} \; , \tag{10}$$

where $\mu = A(1)^{-1}\nu$ and $\varepsilon_t = H_t^{1/2} e_t$. A meaningful structural impulse response analysis requires an identification result that makes the two factors in $B_0^{-1}\varepsilon_t$, and hence the partial derivative $\partial y_{t,i}/\partial \varepsilon_{t-h,j}$, (at least) locally unique.

Proposition 3 of Milunovich (2014, p.7) implies that, if there are at least $r \geq m - 1$ nontrivial GARCH processes[4] in the m-dimensional structural error process, the structural VMA representation is unique apart from column permutations and sign changes of B_0^{-1} and the components of ε_t. Thus, every structural VMA representation of $\{y_t : t \in \mathbb{Z}\}$ that can be

[3]Alternatively, the weak stationarity of $\{u_t : t \in \mathbb{Z}\}$ follows from the weak stationarity of $\{\varepsilon_t : t \in \mathbb{Z}\}$ and the fact that $u_t = B_0^{-1}\varepsilon_t, t \in \mathbb{Z}$.

[4] A nontrivial GARCH process exhibits a time-varying conditional variance, that is, the conditional variance equation satisfies $\tilde{\alpha} > 0 \vee \tilde{\beta} > 0$, where $\tilde{\alpha}$ and $\tilde{\beta}$ denote the parameters of the ARCH and the GARCH part, respectively.

written as

$$y_t = \mu + \sum_{i=1}^{+\infty} \Psi_i \breve{B}_0^{-1} \varepsilon_{t-i}^* \ , \tag{11}$$

where $\breve{B}_0^{-1} := B_0^{-1}DP$ and $\varepsilon_t^* := P'D^{-1}\varepsilon_t$ for some diagonal matrix $D := \text{diag}(d_1, \ldots, d_m)$ with $d_i \in \{-1, 1\}$ and some permutation matrix $P \in \mathbb{R}^{m\times m}$ is observationally equivalent to the representation in (10). The set of observationally equivalent structural VMA representations can be characterized by

$$\mathcal{E}\left(B_0^{-1}\right) := \left\{\breve{B}_0^{-1} \in \mathbb{R}^{m\times m} \mid \breve{B}_0^{-1} = B_0^{-1}DP\right\} \ , \tag{12}$$

for all matrices D and P (as defined above). In other words, the set $\mathcal{E}\left(B_0^{-1}\right)$ consists of all matrices $\breve{B}_0^{-1}$ that are obtained by permutations and sign changes of the columns of B_0^{-1}. Moreover, note that $\varepsilon_t^* = P'D^{-1}\varepsilon_t = P'D^{-1}H_t^{1/2}e_t$ (for some D and P) implies that the vectors of GARCH parameters $\alpha := (\alpha_1, \ldots, \alpha_m)'$ and $\beta := (\beta_1, \ldots, \beta_m)'$ are also locally identified because the pre-multiplication with $P'D^{-1}$ only affects the ordering of the GARCH processes. The reordered GARCH parameter vectors that correspond to $\breve{B}_0^{-1}$ are denoted by $\breve{\alpha} := P'D^{-1}\alpha$ and $\breve{\beta} := P'D^{-1}\beta$, respectively.

Remark 2.1. As outlined above, the number of nontrivial GARCH components r in ε_t is critical for the local identification of B_0^{-1}, yet unknown in any practical application. Fortunately, Lanne and Saikkonen (2007) and Lütkepohl and Milunovich (2016) provide statistical tests to test the null hypothesis of $r = r_0$ nontrivial GARCH components in ε_t versus the two-sided alternative, that is, $\mathbb{H}_0 : r = r_0$ versus $\mathbb{H}_1 : r \neq r_0$. ■

2.4 Estimation

2.4.1 Motivation

The model parameters are the reduced-form VAR parameters $\nu, A_1, \ldots, A_p$, the transformation matrix B_0^{-1} and the GARCH parameters α and β. The literature proposes different estimation procedures. Bouakez et al. (2013, 2014) use a two-step procedure, that is, in the first step, the VAR parameters $\nu, A_1, \ldots, A_p$ are estimated by multivariate least squares (LS) and in the second step, the parameters associated with the GO-GARCH model, that is, B_0^{-1}, α, β, are estimated by quasi-maximum likelihood (QML). Lütkepohl and Milunovich (2016) estimate the model parameters by a QML procedure that is to a large extent based on the procedure outlined in Lanne and Saikkonen (2007, p.64–65).

An integral part of both estimation procedures is the numerical optimization of a likelihood function. Hence, in scenarios with small sample sizes and/or large dimensions, both methods can be prone to numerical convergence problems[5]. Numerical instability is particularly problematic

[5] Hwang and Pereira (2006) report serious convergence problems of the QML estimator of a GARCH(1,1) process in small sample sizes, and hence recommend to use at least 500 observations to estimate the model parameters (by QML). The GO-GARCH model is a multivariate GARCH model and is expected to also suffer from convergence problems in small sample sizes.

in applications where the estimation procedure has to be repeatedly applied, such as bootstrap-based inference. As a result, we propose a (partially) novel estimation procedure which is numerically stable even in the delicate scenarios mentioned above.

We propose a three-step estimation procedure. The first step consists of the estimation of the reduced-form VAR parameters $\nu, A_1, \ldots, A_p$ by standard LS. The second step consists of the estimation of the parameter matrix B_0^{-1} using the method of moment (MM) estimation procedure of GO-GARCH models proposed in Boswijk and van der Weide (2011). The computation of the MM estimator of Boswijk and van der Weide does not involve any Newton-type optimization of an objective function but involves only iterated matrix rotations, and hence the procedure does not suffer from numerical convergence problems regardless of the available sample size and the dimension. The third step consists of the estimation of the vectors of GARCH parameters α and β using the least-squares estimator of Preminger and Storti (2017) component-wise. The estimator of Preminger and Storti does involve Newton-type optimization but simulation results (not reported here) suggest that, in small sample scenarios, the least-squares estimator exhibits superior convergence properties compared to the standard QML estimator of GARCH models. Moreover, the simulation results of Preminger and Storti (2017) show that their least-squares estimator is competitive in terms of the root-mean-squared error.

Remark 2.2. Kristensen and Linton (2006) proposes a closed-form estimator of the GARCH(1,1) model which is based on the corresponding ARMA(1,1) representation. The proposed closed-from estimator is attractive from a computational point of view, but the finite-sample properties are by far inferior to the properties of the QML estimator; see Kristensen and Linton (2006, p.334). ■

Remark 2.3. For simplicity, it is assumed that the true lag order $p \in \mathbb{N}$ of the VAR is known. In a practical application the true lag order is usually unknown and has to be determined from the data; see e.g. Lütkepohl (2005, Chapter 4). In that case, the first estimation step consists of the estimation of $\nu, A_1, \ldots, A_{\hat{p}}$, where $\hat{p}$ denotes the estimated lag order. ■

2.4.2 Three-Step Estimation Procedure

Estimation of the model parameters $\nu, A_1, \ldots, A_p, B_0^{-1}, \alpha$ and β in three steps.

(i) *Estimation of* $\nu, A_1, \ldots, A_p$:
Using the original series $\{y_{-p+1}, \ldots, y_0, y_1, \ldots, y_T\}$, estimate the VAR coefficients $\nu, A_1, \ldots, A_p$ by standard multivariate LS; see e.g. Lütkepohl (2005, Section 3) for details. Denote the LS estimators by $\hat{\nu}, \hat{A}_1, \ldots, \hat{A}_p$. Next, obtain the corresponding residuals $\{\hat{u}_1, \ldots, \hat{u}_T\}$ according to

$$\hat{u}_t := y_t - \hat{v} - \hat{A}_1 y_{t-1} - \ldots - \hat{A}_p y_{t-p} \ , \ t = 1, \ldots, T \ .$$

(ii) *Estimation of* B_0^{-1}:
Using the residuals $\{\hat{u}_1, \ldots, \hat{u}_T\}$ from the first step, estimate B_0^{-1} by the method of moment procedure proposed in Boswijk and van der Weide (2011); see Appendix A for details. Denote the resulting MM estimator by $\hat{B}_0^{-1}$. Next, obtain the structural residuals $\{\hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_T\}$ according to

$$\hat{\varepsilon}_t := (\hat{B}_0^{-1})^{-1} \hat{u}_t \ , \ t = 1, \ldots, T \ .$$

(iii) *Estimation of* α, β:
Using the structural residuals $\{\hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_T\}$ from the second step, estimate the GARCH parameters α and β by the component-wise application of the least squares estimator of Preminger and Storti (2017); see Appendix B for details. Denote the corresponding estimators by $\hat{\alpha} := (\hat{\alpha}_1, \ldots, \hat{\alpha}_m)'$ and $\hat{\beta} := (\hat{\beta}_1, \ldots, \hat{\beta}_m)'$, respectively.

The outlined three-step estimation procedure produces estimators of all parameters of the model, that is, $\hat{\nu}, \hat{A}_1, \ldots, \hat{A}_p, \hat{B}_0^{-1}, \hat{\alpha}, \hat{\beta}$. However, the estimation of the model parameters per se is only an intermediate step, as the objects of interest are the structural impulse responses.

It is well-known that the standard plug-in estimator of the (i,j)-th structural impulse response at propagation horizon $h \in \mathbb{N}$ is a non-linear function of the VAR slope estimators $\hat{A}_1, \ldots, \hat{A}_p$ and the matrix $\hat{B}_0^{-1}$. However, the matrix $\hat{B}_0^{-1}$ may be replaced with an equivalent matrix $\mathring{\hat{B}}_0^{-1} \in \mathcal{E}(\hat{B}_0^{-1})$ by the researcher; see e.g. Lütkepohl and Milunovich (2016, p.243) for a discussion of this issue. Thus, the estimator of the (i,j)-th structural impulse response at propagation horizon h is given by

$$\widehat{\Theta}_{ij,h} := f_{ij,h}\left(\hat{A}_1, \ldots, \hat{A}_p, \mathring{\hat{B}}_0^{-1}\right) \ . \tag{13}$$

2.4.3 Consistency of the Structural Impulse Response Estimator

The estimator $\widehat{\Theta}_{ij,h}$ is a continuous function of $\hat{A}_1, \ldots, \hat{A}_p$ and $\mathring{\hat{B}}_0^{-1}$ for every $i, j \in \{1, \ldots, m\}$ and $h \in \mathbb{N}$. Thus, by the continuous mapping theorem, the consistency of the estimators $\hat{A}_1, \ldots, \hat{A}_p$ and $\mathring{\hat{B}}_0^{-1}$ is sufficient for the consistency of $\widehat{\Theta}_{ij,h}$.

Proposition 1 of Francq and Raïssi (2007, p.458) establishes the strong consistency of the LS estimators $\hat{\nu}, \hat{A}_1, \ldots, \hat{A}_p$ of the parameters of a stable VAR(p) model under the assumption of a strictly stationary and ergodic reduced-form error process $\{u_t : t \in \mathbb{Z}\}$. The GO-GARCH model is strictly stationary and ergodic under mild regularity conditions (see Section 2.2) and hence, under these regularity conditions, it holds that

$$\text{vec}(\hat{\nu}, \hat{A}_1, \ldots, \hat{A}_p) \overset{a.s.}{\to} \text{vec}(\nu, A_1, \ldots, A_p) \quad \text{as } T \to \infty \ , \tag{14}$$

where $\overset{a.s.}{\to}$ denotes almost sure convergence.

The MM estimator $\mathring{\hat{B}}_0^{-1}$ is based on the reduced-form residuals from the first estimation step, that is, $\{\hat{u}_1, \ldots \hat{u}_T\}$, and not based on the unknown true errors $\{u_1, \ldots u_T\}$. However, the convergence (14) implies that $\hat{u}_t \overset{a.s.}{\to} u_t$ and Proposition 1 of Francq and Raïssi (2007, p.458) yields that $\widehat{\Sigma}_u := T^{-1} \sum_{t=1}^{T} \hat{u}_t \hat{u}_t' \overset{a.s.}{\to} \Sigma_u$. As a consequence, the sample moments underlying the estimator $\mathring{\hat{B}}_0^{-1}$ converge in probability to the identical population values as the estimator that is based on $\{u_1, \dots u_T\}$. Thus, based on Theorem 1 in Boswijk and van der Weide (2011, p.122), the consistency of $\mathring{\hat{B}}_0^{-1}$ is established under the following additional assumptions on the process $\{\varepsilon_t : t \in \mathbb{Z}\}$:

- $\mathbb{E}[\varepsilon_{t,i}^4] < +\infty$, $i = 1, \dots, m$.
- For some $s \in \mathbb{N}$, the autocorrelations $\rho_{ik} := \text{Corr}(\varepsilon_{t,i}^2, \varepsilon_{t-k,i}^2)$ satisfy

$$\max_{1 \le k \le s} \min_{1 < i \le j \le m} |\rho_{ik} - \rho_{jk}| > 0 \ .$$

Remark 2.4. Boswijk and van der Weide (2011) consider a setup that allows for other more general specifications of the conditional heteroskedasticity than the GO-GARCH model. For that reason, their proof of consistency is based on a more extensive set of assumptions than the one above. However, it is straightforward to see that the assumption of a GO-GARCH model allows to narrow the set of required assumptions to establish consistency of the estimator. ■

Remark 2.5. A necessary and sufficient condition for a finite fourth moment of a GARCH(1,1) process is found in He and Teräsvirta (1999, p.827). Alternatively, for the special case of $e_{t,i} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$, a simpler (necessary and sufficient) condition for $\mathbb{E}[\varepsilon_{t,i}^4] < +\infty$ is found in Bollerslev (1986, p.311). ■

The importance of the finite fourth moment assumption for the consistency of the structural impulse response estimator (13) is analyzed by means of a simulation-based root mean squared error (RMSE) analysis. More specifically, three different data generating processes (DGPs) are considered from which two DGPs satisfy all assumptions required for consistency (DGP-1a and DGP-1b) and one DGP that violates the finite fourth moment assumption (DGP-1c); see Section 5.1 for more details. The results for $\widehat{\Theta}_{11,h}, h \in \{0, \dots, 12\}$, are found in Appendix D.

The analysis highlights the importance of the assumption of $\mathbb{E}[\varepsilon_{t,i}^4] < +\infty$. For the two DGPs with $\mathbb{E}[\varepsilon_{t,i}^4] < +\infty$, the RMSE of the structural impulse response estimator is strictly decreasing in the sample size and, for very large sample sizes, close to zero at all propagation horizons h. In contrast, for the DGP that violates the assumption $\mathbb{E}[\varepsilon_{t,i}^4] < +\infty$, the RMSE of the estimator is only weakly decreasing. Moreover, for small propagation horizons $h \in \{0, 1, 2, 3\}$, the RMSE is substantially away from zero even for a very large sample size of $T = 5,000$.

3 Inference for Structural Impulse Responses

3.1 Motivation

We are interested in constructing a marginal bootstrap percentile confidence interval à la Hall (1992) for the structural impulse response $\Theta_{ij,h}$ at propagation horizon $h \in \{0, \dots, H\}$. Such a percentile confidence interval for $\Theta_{ij,h}$ with a nominal coverage probability of $(1-\alpha) \in (0,1)$ is given by

$$\mathrm{CI}_{h,(1-\alpha)} := \left[\widehat{\Theta}_{ij,h} - q^*_{h,(1-\alpha/2)}; \widehat{\Theta}_{ij,h} - q^*_{h,\alpha/2}\right] , \tag{15}$$

where $q^*_{h,(1-\alpha/2)}$ and $q^*_{h,(\alpha/2)}$ are the $(1-\alpha/2)$- and $(\alpha/2)$-quantiles, respectively, of the bootstrap sampling distribution (at propagation horizon h). Evidently, the confidence interval $\mathrm{CI}_{h,(1-\alpha)}$ is derived under the usual bootstrap analogy, that is, the bootstrap sampling distribution approximates the unknown true sampling distribution

$$\mathcal{L}\left(\widehat{\Theta}^*_{ij,h} - \widehat{\Theta}_{ij,h} \mid y_{-p+1}, \dots, y_0, y_1, \dots, y_T\right) \approx \mathcal{L}\left(\widehat{\Theta}_{ij,h} - \Theta_{ij,h}\right) , \tag{16}$$

where $\mathcal{L}(X)$ denotes the distribution of a random variable X and $\widehat{\Theta}^*_{ij,h}$ denotes the estimator computed based on artificial bootstrap data $\{y^*_{-p+1}, \dots, y^*_0, y^*_1, \dots, y^*_T\}$. Hence, a bootstrap procedure that satisfies (16) is expected to produce a confidence interval $\mathrm{CI}_{h,(1-\alpha)}$ that exhibits an actual coverage probability that is close to the nominal coverage probability $(1-\alpha)$, and therefore provides valid inference.

The deviation from i.i.d. reduced-form errors by specifying the reduced-form error process as a conditionally heteroskedastic GO-GARCH model (see Section 2.2) entails that standard bootstrap procedures, such as the procedures of Runkle (1987), Kilian (1998a) and Kilian (1998b), may not result in a valid percentile confidence interval for $\Theta_{ij,h}$, even for very large sample sizes. The reason being that these bootstrap procedures are based on resampling with replacement from the empirical distribution of the reduced-form residuals, and hence the validity of these procedures essentially relies on the underlying assumption of i.i.d. reduced-form errors.

Brüggemann et al. (2016) investigate, among other things, inference for structural impulse responses in VAR models with conditional heteroskedasticity of unknown form and propose a residual-based moving block bootstrap procedure in the spirit of Künsch (1989). However, the authors prove the validity of their proposed moving block bootstrap only for structural impulse responses which are identified via a standard recursive ordering approach[6]; see Corollary 5.2 of Brüggemann et al. (2016, p.75). Thus, their theoretical result does not directly apply to structural impulse responses that are identified via the conditional heteroskedasticity of the GO-GARCH model. The same concern applies to the results of the extensive Monte Carlo study of Brüggemann et al. (2016). Moreover, we are not aware of a study that provides theoretical or simulation-based results that are applicable in the present framework where the structural impulse responses are identified via the conditional heteroskedasticity in the error process.

We propose a nonparametric bootstrap procedure that explicitly incorporates the GO-GARCH structure of the reduced-form error process. In this way, the corresponding artificial bootstrap data resembles the data generated from the true model, and hence the resulting bootstrap sampling distribution is supposed to approximate the true sampling distribution, at least for large sample sizes. Moreover, the proposed bootstrap procedure can be viewed as

[6] This means that B_0^{-1} is given as the lower-triangular Cholesky decomposition of Σ_u.

a multivariate generalization of the bootstrap procedure for the univariate ARMA-GARCH model outlined in Shimizu (2010, p.68–70).

The proposed bootstrap procedure is straightforward to implement, as it only requires the availability of the following quantities: (i) the estimators of the model parameters, that is, $\hat{\nu}, \hat{A}_1, \ldots, \hat{A}_p, \hat{\breve{B}}_0^{-1}, \hat{\breve{\alpha}}$ and $\hat{\breve{\beta}}$; (ii) the corresponding series of residuals $\{\hat{u}_1, \ldots, \hat{u}_T\}$; and (iii) the pre-sample of the original data $\{y_{-p+1}, \ldots, y_0\}$.

3.2 Residual Bootstrap

The following bootstrap algorithm, subsequently referred to as the *residual bootstrap*, produces the marginal percentile interval at each propagation horizon $h \in \{0, \ldots, H\}$.

a) For each component $i = 1, \ldots, m$, compute the estimated conditional variances according to

$$\hat{\sigma}^2_{t,i} := (1 - \hat{\breve{\alpha}}_i - \hat{\breve{\beta}}_i) + \hat{\breve{\alpha}}_i \hat{\varepsilon}^2_{t-1,i} + \hat{\breve{\beta}}_i \hat{\sigma}^2_{t-1,i}, \; t = 1, \ldots, T \; ,$$

where the starting values are $\hat{\varepsilon}^2_{0,i} := \hat{\sigma}^2_{0,i} = (1 - \hat{\breve{\alpha}}_i - \hat{\breve{\beta}}_i)$. Next, obtain the estimated conditional variance matrices $\hat{H}_t := \operatorname{diag}(\hat{\sigma}^2_{t,1}, \ldots, \hat{\sigma}^2_{t,m})$, $t = 1, \ldots, T$.

b) Compute the devolatized residuals $\hat{e}_t := \hat{H}_t^{-1/2} \hat{\breve{B}}_0 \hat{u}_t$, $t = 1, \ldots, T$. Next, center and rescale the devolatized residuals according to

$$\breve{e}_t := \hat{\Sigma}_e^{-1/2} \left(\hat{e}_t - \bar{\hat{e}} \right) \; , \; t = 1, \ldots, T \; ,$$

where $\bar{\hat{e}} := T^{-1} \sum_{t=1}^T \hat{e}_t$ and $\hat{\Sigma}_e := T^{-1} \sum_{t=1}^T (\hat{e}_t - \bar{\hat{e}})(\hat{e}_t - \bar{\hat{e}})'$.

c) For each component $i = 1, \ldots, m$, generate the univariate bootstrap sample $\left\{ \hat{\varepsilon}^*_{1,i}, \ldots, \hat{\varepsilon}^*_{T,i} \right\}$ according to

$$\begin{aligned} \hat{\varepsilon}^*_{t,i} &:= \hat{\sigma}^*_{t,i} \hat{e}^*_{t,i} \\ \hat{\sigma}^{*2}_{t,i} &:= (1 - \hat{\breve{\alpha}}_i - \hat{\breve{\beta}}_i) + \hat{\breve{\alpha}}_i \hat{\varepsilon}^{*2}_{t-1,i} + \hat{\breve{\beta}}_i \hat{\sigma}^{*2}_{t-1,i} \; , \end{aligned}$$

where $\hat{e}^*_{t,i}$ is a random draw with replacement from the (univariate) empirical distribution of $\{\breve{e}_{t,i}\}_{t=1}^T$. The starting values are $\hat{\varepsilon}^{*2}_{0,i} := \hat{\sigma}^{*2}_{0,i} = (1 - \hat{\breve{\alpha}}_i - \hat{\breve{\beta}}_i)$. Next, obtain the series of bootstrap residuals $\{\hat{u}^*_1, \ldots, \hat{u}^*_T\}$ via $u^*_t := \hat{\breve{B}}_0^{-1} \hat{\varepsilon}^*_t$, $t = 1, \ldots, T$.

d) Generate the bootstrap sample $\{y^*_1, \ldots, y^*_T\}$ according to

$$y^*_t := \hat{\nu} + \hat{A}_1 y^*_{t-1} + \ldots + \hat{A}_p y^*_{t-p} + u^*_t \; , \; t = 1, \ldots, T \; ,$$

where the starting values are equal to the pre-sample of the original series $\{y_{-p+1}, \ldots, y_0\}$.

e) Compute the estimators $\hat{\nu}^*, \hat{A}_1^*, \ldots, \hat{A}_p^*$ and $\hat{B}_0^{-1*}$ based on $\{y_{-p+1}^*, \ldots, y_0^*, y_1^*, \ldots, y_T^*\}$. Replace $\hat{B}_0^{-1*}$ with an equivalent matrix $\mathring{\hat{B}}_0^{-1*}$ which satisfies

$$\mathring{\hat{B}}_0^{-1*} \in \underset{B \in \mathcal{E}(\hat{B}_0^{-1*})}{\arg\min} \left\| B - \mathring{\hat{B}}_0^{-1} \right\|_F ,$$

where $\|\cdot\|_F$ denotes the Frobenius norm; see Remark 3.1 for a comment. Next, compute the bootstrap impulse response $\widehat{\Theta}_{ij,h}^* := f_{ij,h}\left(\hat{A}_1^*, \ldots, \hat{A}_p^*, \mathring{\hat{B}}_0^{-1*}\right)$.

f) Repeat steps b) to d) B times and obtain the empirical bootstrap sampling distribution $\left\{\widehat{\Theta}_{ij,h,b}^* - \widehat{\Theta}_{ij,h}\right\}_{b=1}^B$ for each propagation horizon $h \in \{0, \ldots, H\}$. Determine the marginal percentile intervals by

$$\left[\widehat{\Theta}_{ij,h} - \tilde{q}_{h,(1-\alpha/2)}^*; \widehat{\Theta}_{ij,h} - \tilde{q}_{h,\alpha/2}^*\right] ,$$

where $\tilde{q}_{h,(1-\alpha/2)}^*$ and $\tilde{q}_{h,\alpha/2}^*$ are the empirical $(1-\alpha/2)$- and $(\alpha/2)$-quantiles, respectively, of $\left\{\widehat{\Theta}_{ij,h,b}^* - \widehat{\Theta}_{ij,h}\right\}_{b=1}^B$.

Remark 3.1. The replacement of $\hat{B}_0^{-1*}$ with $\mathring{\hat{B}}_0^{-1*}$ in step e) of the residual bootstrap ensures that the bootstrap structural impulse response estimator $\widehat{\Theta}_{ij,h}^*$ is computed based on the particular matrix $\mathring{\hat{B}}_0^{-1*} \in \mathcal{E}(\hat{B}_0^{-1*})$ that is closest to the matrix $\mathring{\hat{B}}_0^{-1}$ which is at the basis of the structural impulse response estimator $\widehat{\Theta}_{ij,h}$. In this way, the particular bootstrap VMA representation (characterized by $\mathring{\hat{B}}_0^{-1*}$) is selected that is most similar to the VMA representation characterized by $\mathring{\hat{B}}_0^{-1}$. ■

We also consider a modified version of the residual bootstrap procedure. The modified version, subsequently referred to as the *symmetrized residual bootstrap*, is obtained by the following modification in step c): $\hat{e}_{t,i}^*$ is a random draw with replacement from the empirical distribution of the *symmetrized series* $\{\tilde{e}_{1,i}, \ldots, \tilde{e}_{2T,i}\} := \{\pm\breve{e}_{1,i}, \ldots, \pm\breve{e}_{T,i}\}$ of length $2T$ instead of the empirical distribution of $\{\breve{e}_{1,i}, \ldots, \breve{e}_{T,i}\}$ as in the residual bootstrap; see Appendix C for details.

The modification serves the purpose to ensure that the empirical skewness of $\{\tilde{e}_{1,i}, \ldots, \tilde{e}_{2T,i}\}$ is zero. Furthermore, note that the first and the second empirical moment of $\{\tilde{e}_{1,i}, \ldots, \tilde{e}_{2T,i}\}$ is equal to zero and one, respectively, due to the preceding centering and rescaling in step b). Thus, in scenarios where the true distribution of $e_{t,i}$ is symmetric around zero (e.g. $e_{t,i} \sim \mathcal{N}(0,1)$), and hence exhibits a skewness of zero[7], the empirical distribution of $\{\tilde{e}_{1,i}, \ldots, \tilde{e}_{2T,i}\}$ matches not only the first and the second moment but also the skewness of the true distribution of $e_{t,i}$. Eventually, this modification results in improved finite-sample properties of the corresponding confidence intervals in these scenarios.

[7] Here, we tacitly assume that $\mathbb{E}[e_{t,i}^3] < +\infty$.

4 Competing Methods

In order to assess the finite-sample performance of the residual bootstrap and the symmetrized residual bootstrap, we compare their finite-sample properties with two competing procedures: (i) the standard i.i.d. bootstrap originally proposed by Runkle (1987) and (ii) the moving-block bootstrap proposed in Brüggemann et al. (2016). In the following, the two competing procedures are briefly outlined.

4.1 I.i.d. Bootstrap

a) Generate the bootstrap sample $\{y_1^*, \ldots, y_T^*\}$ according to

$$y_t^* := \hat{\nu} + \hat{A}_1 y_{t-1}^* + \ldots + \hat{A}_p y_{t-p}^* + \tilde{u}_t^* \ , \ t = 1, \ldots, T \ ,$$

where $\tilde{u}_t^*$ is a random draw with replacement from the empirical distribution of the centered and rescaled (reduced-form) residuals[8]. The starting values are equal to the pre-sample of the original series $\{y_{-p+1}, \ldots, y_0\}$.

b) identical to step e) of the residual bootstrap.

c) identical to step f) of the residual bootstrap.

4.2 Moving-Block Bootstrap

a) Choose a block length $l < T$ and let $N := \lceil T/l \rceil$ denote the number of blocks. Define $B_{i,l} := (\hat{u}_{i+1}, \ldots, \hat{u}_{i+l})\,, i = 0, \ldots, T-l$ and let $i_1, \ldots, i_N$ be i.i.d. random variables uniformly distributed on the set $\{0, 1, \ldots, T-l\}$. Obtain $\{\hat{u}_1^*, \ldots, \hat{u}_T^*\}$ by laying blocks $B_{i_1,l}, \ldots, B_{i_N,l}$ end-to-end together and discard the last $Nl - T$ observations.

b) Center $\{\hat{u}_1^*, \ldots, \hat{u}_T^*\}$ according to

$$\breve{u}_{jl+s}^* := \hat{u}_{jl+s}^* - \frac{1}{T-l+1} \sum_{r=0}^{T-l} \hat{u}_{s+r} \ , t = 1, \ldots, T \ , \tag{17}$$

for $s = 1, 2, \ldots, l$ and $j = 0, 1, 2, \ldots, N-1$.

c) Generate the bootstrap sample $\{y_1^*, \ldots, y_T^*\}$ according to

$$y_t^* := \hat{\nu} + \hat{A}_1 y_{t-1}^* + \ldots + \hat{A}_p y_{t-p}^* + \breve{u}_t^* \ , \ t = 1, \ldots, T \ ,$$

where the starting values are equal to the pre-sample of the original series $\{y_{-p+1}, \ldots, y_0\}$.

d) identical to step e) of the residual bootstrap.

e) identical to step f) of the residual bootstrap.

[8] The centering and rescaling is carried out as suggested in Stine (1987).

In the Monte Carlo simulations, the block lengths are given by $l \in \{10, 20, 50, 75, 200\}$ for sample sizes $T \in \{100, 250, 500, 1000, 5000\}$. For $T \in \{500, 5000\}$, the block lengths are as in Brüggemann et al. (2016). The sample sizes $T \in \{100, 250, 1000\}$ are not considered in their study, and hence the block lengths are found via interpolation.

5 Monte Carlo Simulation

5.1 Data Generating Processes

We consider the following bivariate model from Brüggemann et al. (2016), that is,

$$\textbf{DGP-1} \quad y_t = A_1 y_{t-1} + A_2 y_{t-2} + B_0^{-1} \varepsilon_t \ , \tag{18}$$

where

$$A_1 := \begin{pmatrix} 0.40 & 0.60 \\ -0.10 & 1.20 \end{pmatrix} \ , \ A_2 := \begin{pmatrix} -0.20 & 0.00 \\ -0.20 & -0.10 \end{pmatrix} \text{ and } B_0^{-1} := \begin{pmatrix} 1.00 & 0.00 \\ 0.50 & \sqrt{0.75} \end{pmatrix} \ .$$

The moduli of the roots of the characteristic polynomial of the VAR are given by 1.08, 2.78, and 5.98, which implies moderate persistence of the process $\{y_t : t \in \mathbb{Z}\}$. The following set of different GARCH(1, 1) specifications for the two components of $\{\varepsilon_t : t \in \mathbb{Z}\}$ are considered

a) $(\alpha_1, \beta_1)' = (0.10, 0.80)'$; $(\alpha_2, \beta_2)' = (0.20, 0.65)'$,

b) $(\alpha_1, \beta_1)' = (0.10, 0.80)'$; $(\alpha_2, \beta_2)' = (0.085, 0.90)'$,

c) $(\alpha_1, \beta_1)' = (0.095, 0.90)'$; $(\alpha_2, \beta_2)' = (0.25, 0.65)'$,

where the specific variants of DGP-1 will be denoted by DGP-1i, $i \in \{a, b, c\}$, depending on the specific choice of the GARCH specification.

The following two univariate distributions for the (mutually independent) components of the i.i.d. process $\{e_t : t \in \mathbb{Z}\}$ are considered:

- $e_{t,i} \sim \mathcal{N}(0, 1)$, standard normal distribution.
- $e_{t,i} \sim \frac{3}{5} t_5$, t-distribution with 5 degrees of freedom, scaled to have variance 1.

Under DGP-1a, the persistence of the GARCH processes, measured by $\alpha_i + \beta_i$, is given by 0.90 and 0.85, respectively which implies only moderate persistence. For both considered distributions of $e_{t,i}$, DGP1a implies a consistent estimator $\widehat{\Theta}_{ij,h}$ of the structural impulse responses. Under DGP-1b, the persistence is given by 0.90 and 0.985, respectively. Similarly to DGP-1a, DGP-1b implies a consistent estimator $\widehat{\Theta}_{ij,h}$ for both distributions of $e_{t,i}$. Under DGP-1c, the persistence of the GARCH processes is given by 0.995 and 0.90, respectively. The first component of ε_t does *not* have a finite fourth moment (for both distributions of $e_{t,i}$) and hence the assumptions underlying the consistency of $\widehat{\Theta}_{ij,h}$ are violated. Hence, this DGP is included to investigate the sensitivity of the bootstrap procedure from deviations of the finite fourth moment assumption.

5.2 Simulation Parameters and Performance Evaluation

Data samples of length $T \in \{100, 250, 500, 1000, 5000\}$ are generated and the maximum propagation horizon is $H = 12$. The nominal confidence level of the marginal confidence bands is 90%. The number of bootstrap replications is $B = 1000$ throughout and the number of Monte Carlo replications is 1000. The finite-sample performance of the confidence intervals is evaluated by the empirical coverage rate and the empirical length. In particular, the empirical empirical coverage rate is computed in the usual way as

$$EC_{ij,h} := \frac{1}{1000} \sum_{m=1}^{1000} \mathbb{1}_{\{\Theta_{ij,h} \in \text{ CI}_{ij,h,m}\}} ,$$

where $\text{CI}_{ij,h}$ denotes the marginal confidence interval for $\Theta_{ij,h}$ and $\mathbb{1}_{\{A\}}$ denotes the indicator function of an event A. The empirical length is computed as

$$L_{ij,h} := \frac{1}{1000} \sum_{m=1}^{1000} (u_{ij,h,m} - l_{ij,h,m}) \ ,$$

where $u_{ij,h,m}$ denotes the upper bound of the marginal confidence interval for $\Theta_{ij,h}$ and $l_{ij,h,m}$ denotes the corresponding lower bound.

5.3 Results

The boxplots summarizing the performance of the four different bootstrap methods across different scenarios are found in Appendices E, F and G. The tables with the simulation results (empirical coverage rates and empirical lengths) are available from the authors upon request. The main conclusions are as follows:

- Under DGP-1a with $e_{t,i} \sim \mathcal{N}(0,1)$ and $T = 100$, the empirical coverage rates of the confidence intervals based on the residual bootstrap exhibit a large dispersion among propagation horizons h and impulse responses ranging from 63.60% to 96.90%. Yet, increasing the sample size T results in a substantial reduction in the coverage bias and its dispersion. For $T = 5000$, the range of the coverage rates of the confidence intervals is given by 86.90% to 96.00%. Heavy-tailed GARCH errors, that is, $e_{t,i} \sim \frac{3}{5}t_5$, increase the coverage bias of the residual bootstrap especially for $T \in \{1000, 5000\}$.

  The higher persistence in the GARCH process of $\varepsilon_{t,2}$ in DGP-1b has an overall negative effect on the coverages rates of the intervals based on the residual bootstrap, but the negative effect is more pronounced for impulse responses where the shock occurs in the second variable, that is, $\Theta_{12,h}$ and $\Theta_{22,h}$. For $T = 100$ and $e_{t,i} \sim \mathcal{N}(0,1)$, the coverage rates range from 36.90% to 93.50%. The negative coverage bias of the residual bootstrap is decreasing in the sample size T but even with $T = 5000$ the range of the coverage rates is 77.10% to 96.40%. The heavy-tailed GARCH errors ($e_{t,i} \sim \frac{3}{5}t_5$) result in larger coverage biases for all sample sizes, where the negative effect is more pronounced than under the less persistent DGP-1a.

The non-finite fourth moment of $\varepsilon_{t,1}$ in DGP-1c has also an overall negative effect on the coverage rates of the residual bootstrap. The strongest effect is on the intervals for $\Theta_{11,0}$ and $\Theta_{11,1}$. For $T = 100$ and $e_{t,i} \sim \mathcal{N}(0,1)$, the coverage rates for $\Theta_{11,0}$ and $\Theta_{11,1}$ are 4.40% and 27.80%, respectively, whereas the remaining coverage rates range from 61.00% to 95.30%. The effect of an increase in the sample size T is ambiguous; increasing the sample size to $T \in \{250, 500\}$ results in a overall reduction of the coverage bias of the confidence intervals, but after that, a further increase to $T \in \{1000, 5000\}$ results either in nearly no improvement or even in a deterioration of the performance (compared to the $T = 500$ scenario). Similarly to DGP-1a and DGP-1b, heavy-tailed GARCH errors result in an increase in the coverage bias.

- Overall, the symmetrized residual bootstrap exhibits a very similar performance as the residual bootstrap. Hence, the symmetrized residual bootstrap is not capable of systematically outperforming the residual bootstrap, although both considered distributions of $e_{t,i}$ are symmetric around zero.

- Under DGP-1a with $e_{t,i} \sim \mathcal{N}(0,1)$ and $T = 100$, similar to the residual bootstrap, the empirical coverage rates of the confidence intervals based on the i.i.d. bootstrap exhibit a large dispersion among propagation horizons h and impulse responses ranging from 63.40% to 96.70%. Increasing the sample size T results in higher empirical coverage rates of the intervals based on the i.i.d. bootstrap. For $T = 5000$, the coverages rates range between 79.80% and 99.30%, however, the majority of confidence intervals exhibit coverages rates above the nominal level of 90%. The effect of heavy-tailed GARCH errors is drastic; the confidence intervals for 42 (out of 52) impulse responses exhibit a lower coverage rate with $T = 5000$ than with $T = 100$.

  Under DGP-1b, the performance of the i.i.d. bootstrap is basically similar to DGP-1a except that the dispersion of the coverage rates among the propagation horizons and the structural impulse responses is more pronounced. For $T = 100$ and $e_{t,i} \sim \mathcal{N}(0,1)$, the coverage rates range between 36.30% and 92.40% and with $T = 5000$, the range is still given by 62.90% to 100.00%. The effect of heavy-tailed GARCH errors is again disastrous. Even for the very large sample size $T = 5000$, the coverage rates of the intervals based on the i.i.d bootstrap vary between 32.00% and 79.20%.

  The violation of the finite fourth moment assumption (DGP-1c) exhibits an overall negative effect on the confidence intervals based on the i.i.d. bootstrap. Similar to the residual bootstrap, the intervals for $\Theta_{11,0}$ and $\Theta_{11,1}$ are the most affected with coverage rates of 3.3% and 30.50% for $T = 100$ and $e_{t,i} \sim \mathcal{N}(0,1)$. Moreover, the behavior of the coverage rates depending on the sample size is erratic; for some impulse responses the coverage rate of the corresponding interval is increasing in the sample size and for others the coverage rate is indeed decreasing in the sample size. Heavy-tailed GARCH errors result in an overall performance that is decreasing with the sample size T.

- Under DGP-1a with $e_{t,i} \sim \mathcal{N}(0,1)$ and $T = 100$, the empirical coverage rates of the confidence intervals based on the moving block bootstrap also exhibit a large dispersion among propagation horizons h and impulse responses ranging from 61.00% to 92.00%. The overall coverage bias is decreasing for sample sizes $T \in \{250, 500, 1000\}$ but a further increase to $T = 5000$ exhibits an ambiguous effect on the coverage rates of the moving block bootstrap. For $T = 5000$, the range of the coverage rates is given by 76.70% to 89.90%. Heavy-tailed GARCH errors result in a downward shift of the coverage rates of the confidence intervals based on the moving block bootstrap; for $T = 5000$, the coverage rates vary between 70.00% and 87.00%.

  The higher persistence of DGP-1b results in higher coverage biases and more dispersion. For $T = 100$ and $e_{t,i} \sim \mathcal{N}(0,1)$, the coverage rates of the intervals based on the moving-block bootstrap are between 38.40% and 89.30%. Similar to DGP-1a, the overall performance continuously improves only for $T \in \{250, 500, 1000\}$. For $T = 5000$, the coverage rates range between 75.50% and 89.10%. Heavy-tailed GARCH errors increase the bias and the dispersion of the intervals based on the moving block bootstrap.

  Under DGP-1c, the violation of the finite fourth moment assumption exhibits an overall negative effect on the intervals based on the moving block bootstrap. Similar to the residual and the i.i.d. bootstrap, the intervals for $\Theta_{11,0}$ and $\Theta_{11,1}$ are the most affected with coverage rates of 2.70% and 26.20%, respectively for $T = 100$ and $e_{t,i} \sim \mathcal{N}(0,1)$. The effect of an increasing sample size is ambiguous and depends on the particular impulse response under consideration. For $T = 5000$, the range of the coverage rates is 51.10% to 87.90%. Again, the heavy-tailed GARCH errors result in a deterioration of the performance of the confidence intervals based on the moving block bootstrap.

- The confidence intervals based on the residual bootstrap exhibits the smallest absolute deviation from the nominal level in 1188 out of the 1560 scenarios. The intervals based on the i.i.d. bootstrap and the moving block bootstrap exhibit the smallest absolute deviation in only 199/1560 and 173/1560 scenarios, respectively[9]. Hence, the residual bootstrap exhibits the best overall performance in terms of the coverage bias.

- For the small sample size $T = 100$, neither of the bootstrap methods is capable of reliably producing a bootstrap sampling distribution that constitutes a good approximation of the true sampling distribution. Hence, the resulting confidence intervals eventually understate the actual estimation uncertainty; see Appendix H for an analysis of the bootstrap sampling distributions as a function of the sample size.

- The results of the Monte Carlo simulations confirm the result from Brüggemann et al. (2016) that the presence of heteroskedasticity substantially increases the estimation uncertainty.

[9] The symmetrized residual bootstrap is omitted in this comparison because it is a modification of the residual bootstrap.

6 Conclusion

A recent strand of the literature exploits conditional heteroskedasticity to identify the structural vector autoregressions. However, the implications for inference on structural impulse responses have not been investigated in the literature yet.

In this paper, we have considered the conditionally heteroskedastic SVAR-GARCH model. We have proposed (i) an estimation procedure of the model parameters that offers numerical stability even in small sample and/or high dimension scenarios and (ii) a bootstrap procedure to construct marginal percentile confidence intervals for structural impulse responses.

By means of a Monte Carlo simulation, we have compared the finite-sample properties of our proposed bootstrap method to those of two benchmarking methods: the i.i.d. bootstrap of Runkle (1987) and the moving block bootstrap of Brüggemann et al. (2016). The confidence intervals based on our proposed bootstrap method exhibits the best overall performance. Nevertheless, the intervals may understate the estimation uncertainty by a substantial amount especially in small samples.

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A Method-of-Moment Estimator of GO-GARCH Models

The following algorithm describes the computation of the method of moment estimator $\hat{B}_0^{-1}$ outlined in Boswijk and van der Weide (2011) based on the series of reduced-form residuals $\{\hat{u}_1, \ldots, \hat{u}_T\}$.

1. Based on $\{\hat{u}_1, \ldots, \hat{u}_T\}$, estimate the unconditional variance matrix $\widehat{\Sigma}_u := T^{-1} \sum_{t=1}^{T} \hat{u}_t \hat{u}_t'$ and obtain its symmetric square root $\hat{S}$. Next, compute the standardized series $s_t := \hat{S}^{-1} \hat{u}_t$, $t = 1, \ldots, T$.

2. Obtain the matrix-valued series $S_t := s_t s_t' - I_m, t = 1, \ldots, T$, and the sample auto-covariance matrices $\hat{\Gamma}(k) := T^{-1} \sum_{t=1}^{T} S_t S_{t-k}$, $k = 1, \ldots, \tilde{k}$. Next, obtain the sample auto-correlation matrices

$$\hat{\Phi}(k) := \hat{\Gamma}(0)^{-1/2} \hat{\Gamma}(k) \hat{\Gamma}(0)^{-1/2} \ , k = 1, \ldots, \tilde{k} \ ,$$

   where $\hat{\Gamma}(0)^{-1/2}$ denotes the symmetric square root of $\hat{\Gamma}(0)^{-1}$. Next, obtain the symmetrized sample auto-correlation matrices $\tilde{\Phi}(k) := \frac{1}{2}(\hat{\Phi}(k) + \hat{\Phi}(k)')$, $k = 1, \ldots, \tilde{k}$.

3. The estimator $\hat{U}$ is then obtained by minimizing the following objective function

$$S(U) := \sum_{k=1}^{\tilde{k}} \operatorname{tr}\left(U'\tilde{\Phi}(k)U - \operatorname{diag}(U'\tilde{\Phi}(k)U)\right)' \times \operatorname{tr}\left(U'\tilde{\Phi}(k)U - \operatorname{diag}(U'\tilde{\Phi}(k)U)\right) \ ,$$

   over all orthogonal matrices U, where $\operatorname{tr}(\cdot)$ denotes the trace operator. The solution to the minimization problem is obtained via the F-G algorithm of Flury and Gautschi (1986).

4. Compute the estimator $\hat{B}_0^{-1} := \hat{S}\hat{U}$.

B Least-Squares Estimator of Univariate GARCH(1,1) Models

The following algorithm describes the computation of the least-squares estimator $(\hat{\alpha}_i, \hat{\beta}_i)'$ outlined in Preminger and Storti (2017) of the i-th GARCH process based on the series of structural errors $\{\hat{\varepsilon}_{1,i}, \ldots, \hat{\varepsilon}_{T,i}\}$.

1. Using the univariate series $\{\hat{\varepsilon}_{1,i}, \ldots, \hat{\varepsilon}_{T,i}\}$, estimate the GARCH parameters $(\alpha_i, \beta_i)'$ via the quasi-maximum tail-trimmed likelihood (QMTTL) estimator of Hill (2015, p.7); see Remark B.1. Obtain the corresponding devolatized residuals $\hat{e}_{t,i} := \hat{\varepsilon}_{1,i}/\hat{\sigma}_{t,i}, t = 1, \ldots, T$, where $\hat{\sigma}_{t,i}$ denotes the estimate based on the QMTTL estimator. Next, compute

$$\hat{c}_T := \frac{1}{T} \sum_{t=1}^{T} \log\left(\hat{e}_{t,i}^2\right) \ .$$

2. The least-squares estimator $(\hat{\alpha}_i, \hat{\beta}_i)$ of Preminger and Storti (2017) is then obtained by minimizing the following objective function

$$Q_T(\tilde{\alpha}, \tilde{\beta}; \hat{c}_T) := \frac{1}{T} \sum_{t=1}^{T} \left(\log(\hat{\varepsilon}_{t,i}^2) - \hat{c}_T - \log(\hat{\sigma}_{t,i}^2(\tilde{\alpha}, \tilde{\beta})) \right)^2 ,$$

that is, $(\hat{\alpha}_i, \hat{\beta}_i) := \underset{(\tilde{\alpha}, \tilde{\beta})' \in \Theta}{\arg\min}\, Q_T(\tilde{\alpha}, \tilde{\beta}; \hat{c}_T)$.

Remark B.1. Preminger and Storti (2017) use the standard quasi-maximum likelihood estimator $(\hat{\alpha}_i^{\text{QML}}, \hat{\beta}_i^{\text{QML}})'$ in the first step instead of the QMTTL estimator of Hill (2015). We replaced the standard QML estimator with the QMTTL estimator of Hill (2015) because the aforementioned estimator enjoys improved convergence properties in small samples compared to the standard QML estimator.

C Symmetrized Residual Bootstrap

a) identical to the residual bootstrap.

b) identical to the residual bootstrap.

c) For each component $i = 1, \dots, m$, generate the univariate bootstrap sample $\left\{ \hat{\varepsilon}_{1,i}^*, \dots, \hat{\varepsilon}_{T,i}^* \right\}$ according to

$$\begin{aligned} \hat{\varepsilon}_{t,i}^* &= \hat{\sigma}_{t,i}^* \hat{e}_{t,i}^* \\ \hat{\sigma}_{t,i}^{*2} &= (1 - \hat{\hat{\alpha}}_i - \hat{\hat{\beta}}_i) + \hat{\hat{\alpha}}_i \hat{\varepsilon}_{t-1,i}^{*2} + \hat{\hat{\beta}}_i \hat{\sigma}_{t-1,i}^{*2} , \end{aligned}$$

where $\hat{e}_{t,i}^*$ is a random draw with replacement from the univariate empirical distribution of the symmetrized series $\{\tilde{e}_{1,i}, \dots, \tilde{e}_{2T,i}\} := \{\pm \breve{e}_{1,i}, \dots, \pm \breve{e}_{T,i}\}$ of length $2T$. The starting values are $\hat{\varepsilon}_{0,i}^{*2} = \hat{\sigma}_{0,i}^{*2} = (1 - \hat{\hat{\alpha}}_i - \hat{\hat{\beta}}_i)$. Next, obtain the series of bootstrap residuals $\{\hat{u}_1^*, \dots, \hat{u}_T^*\}$ via $u_t^* = \hat{\hat{B}}_0^{-1} \hat{\varepsilon}_t^*$, $t = 1, \dots, T$.

d) identical to the residual bootstrap.

e) identical to the residual bootstrap.

f) identical to the residual bootstrap.

D RMSE

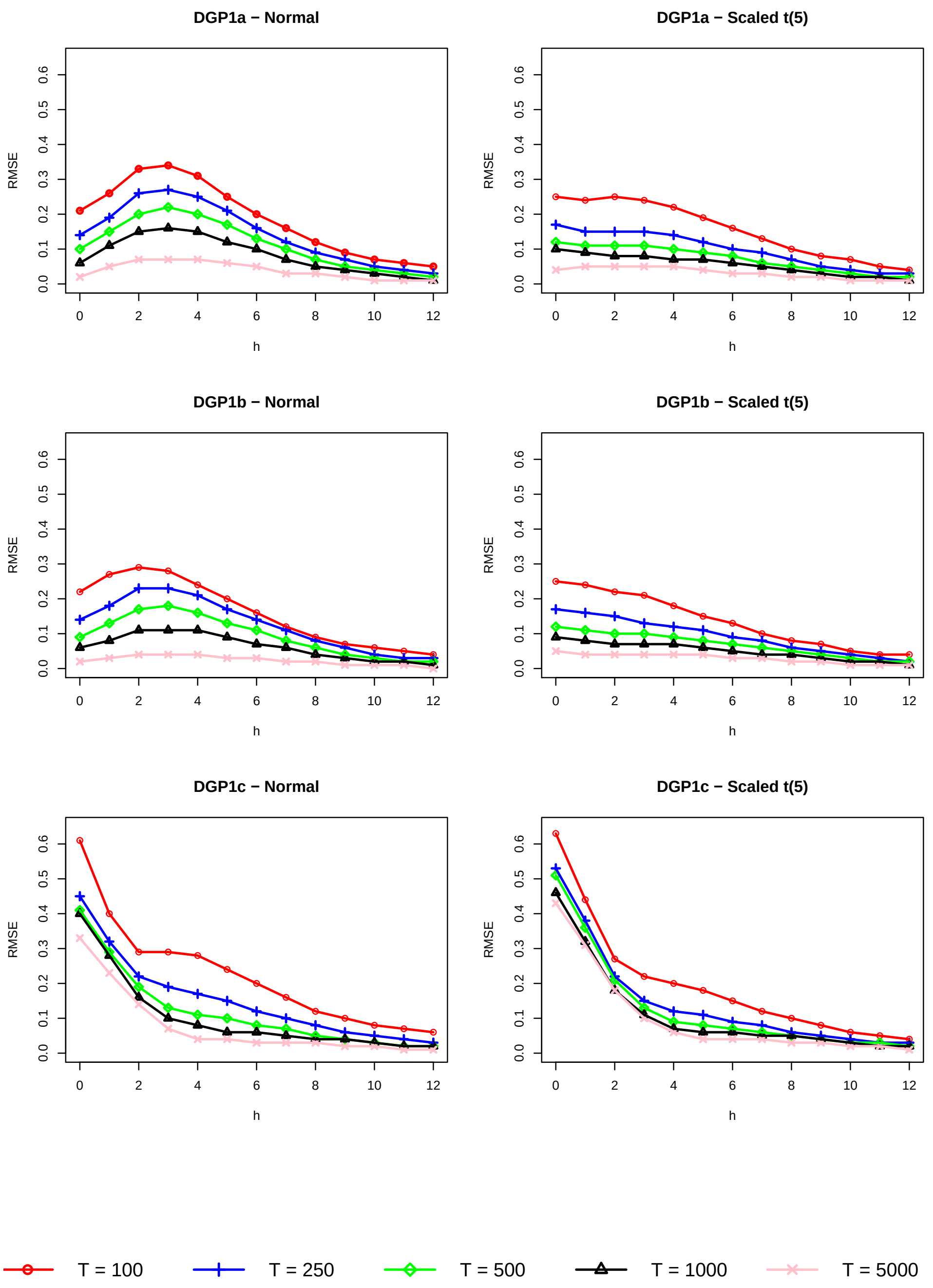

Figure D.1: Root mean squared error (RMSE) of $\widehat{\Theta}_{11,h}$ for propagation horizons $h \in \{0, \ldots, 12\}$ with $T \in \{100, 250, 500, 1000, 5000\}$ based on 10,000 Monte Carlo repetitions.

E DGP-1a

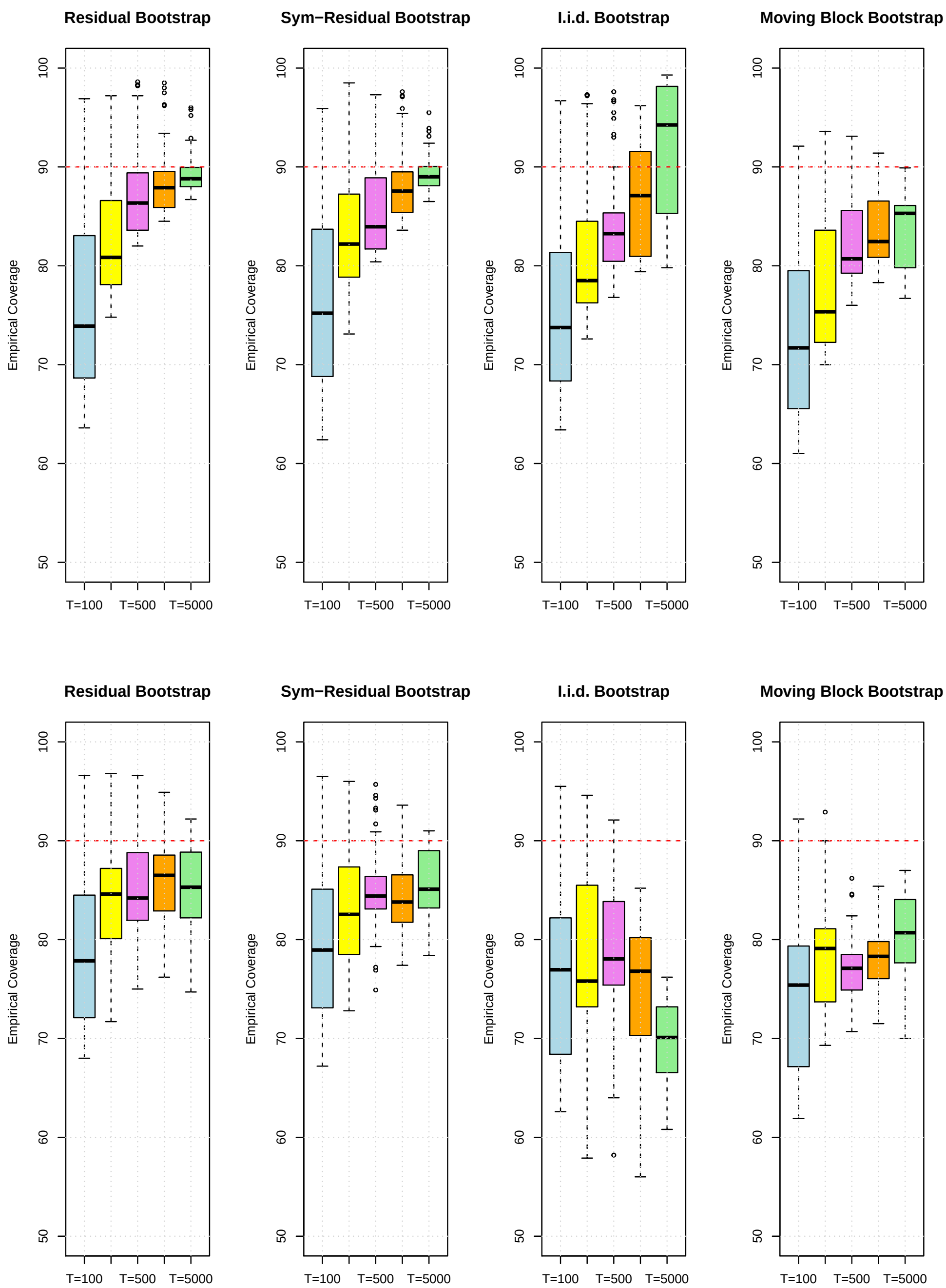

Figure E.1: Boxplots of the empirical coverages across all impulse responses and all propagation horizons (52 parameter constellations in total) of nominal 90% marginal confidence intervals for $T \in \{100, 250, 500, 1000, 5000\}$. The first row corresponds to $e_{t,i} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and the second row corresponds to $e_{t,i} \overset{\text{i.i.d.}}{\sim} \frac{3}{5}t_5$.

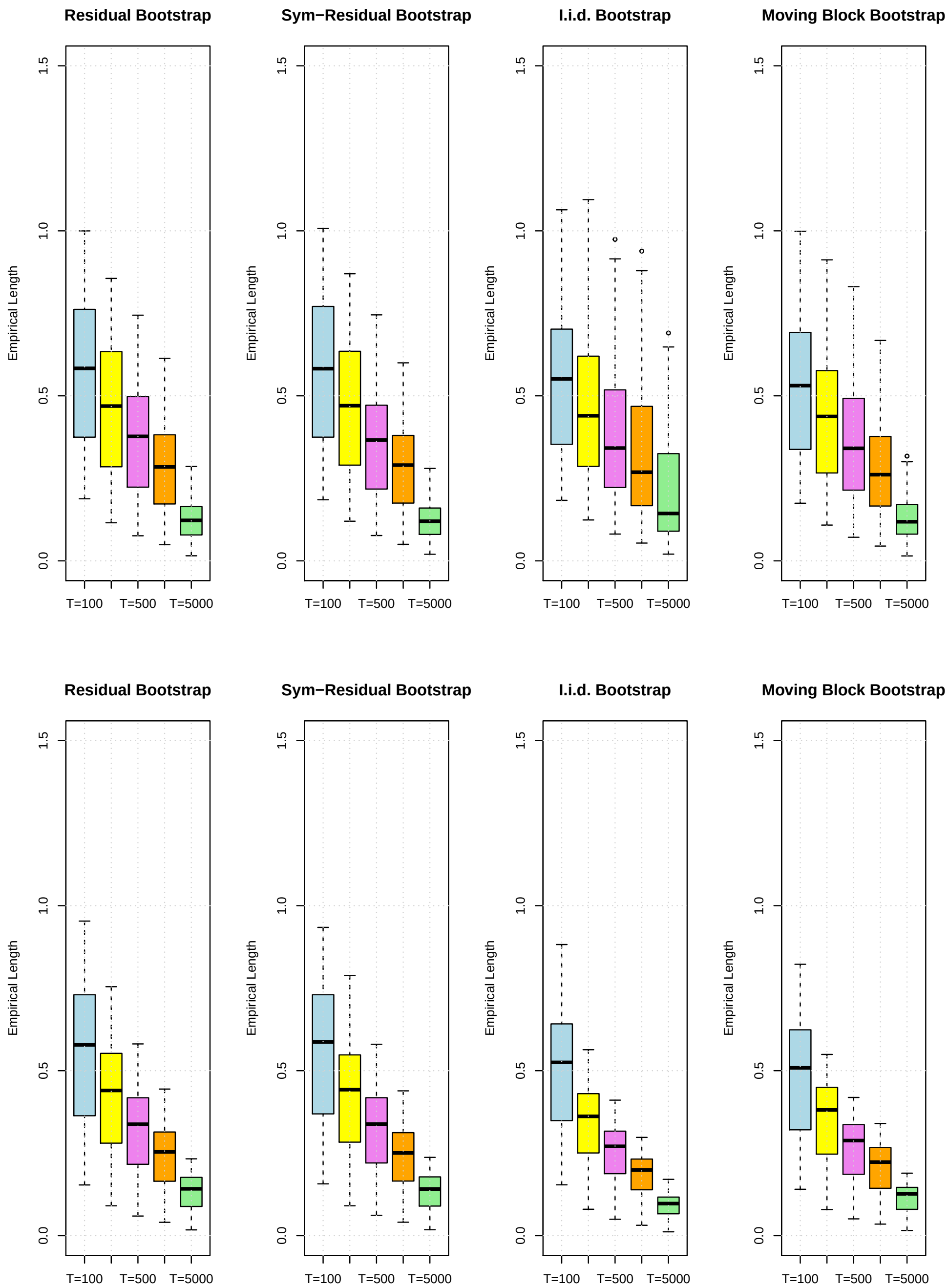

Figure E.2: Boxplots of the empirical lengths across all impulse responses and all propagation horizons (52 parameter constellations in total) of nominal 90% marginal confidence intervals for $T \in \{100, 250, 500, 1000, 5000\}$. The first row corresponds to $e_{t,i} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and the second row corresponds to $e_{t,i} \overset{\text{i.i.d.}}{\sim} \frac{3}{5}t_5$.

F DGP-1b

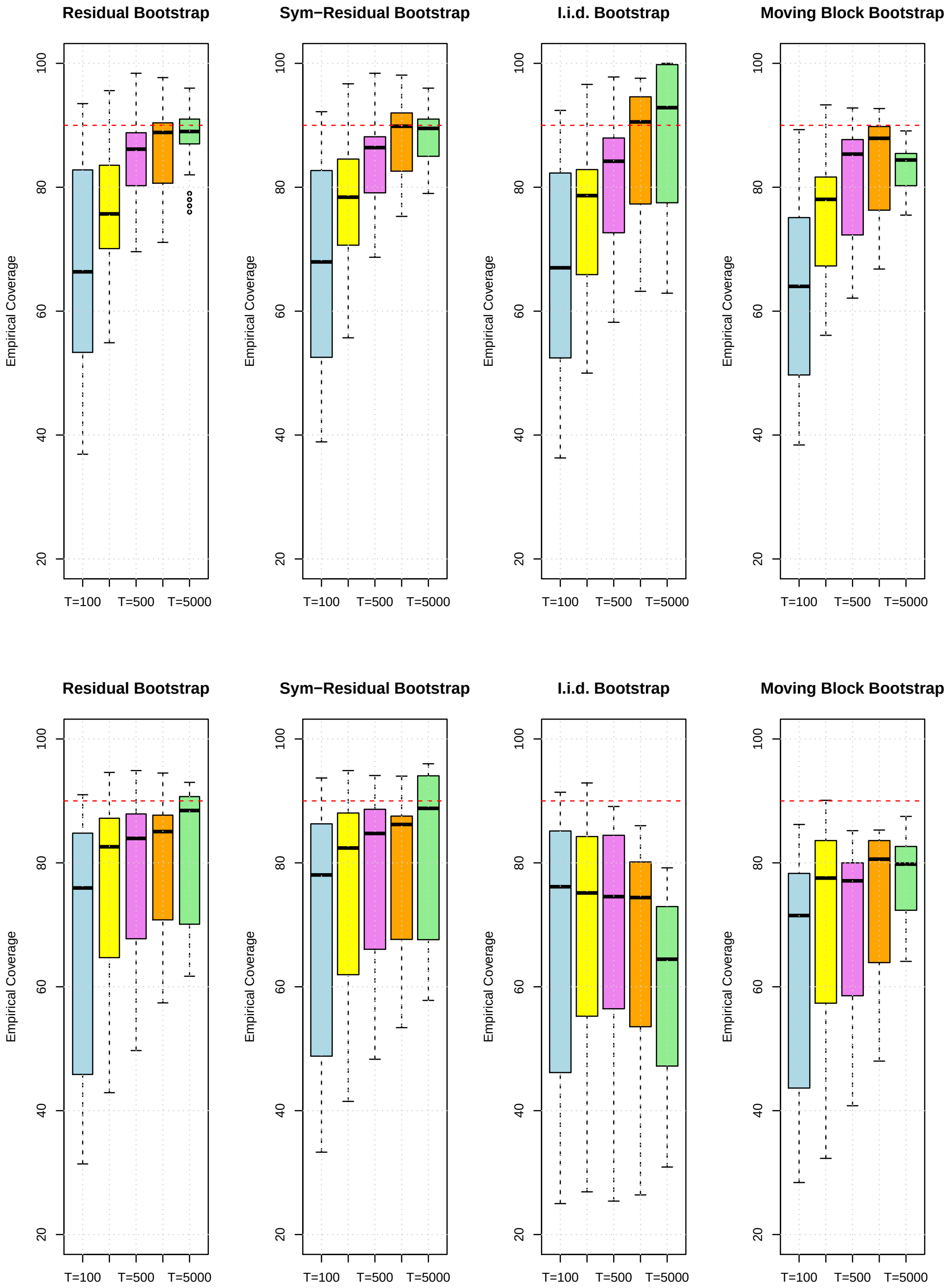

Figure F.1: Boxplots of the empirical coverages across all impulse responses and all propagation horizons (52 parameter constellations in total) of nominal 90% marginal confidence intervals for $T \in \{100, 250, 500, 1000, 5000\}$. The first row corresponds to $e_{t,i} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and the second row corresponds to $e_{t,i} \overset{\text{i.i.d.}}{\sim} \frac{3}{5}t_5$.

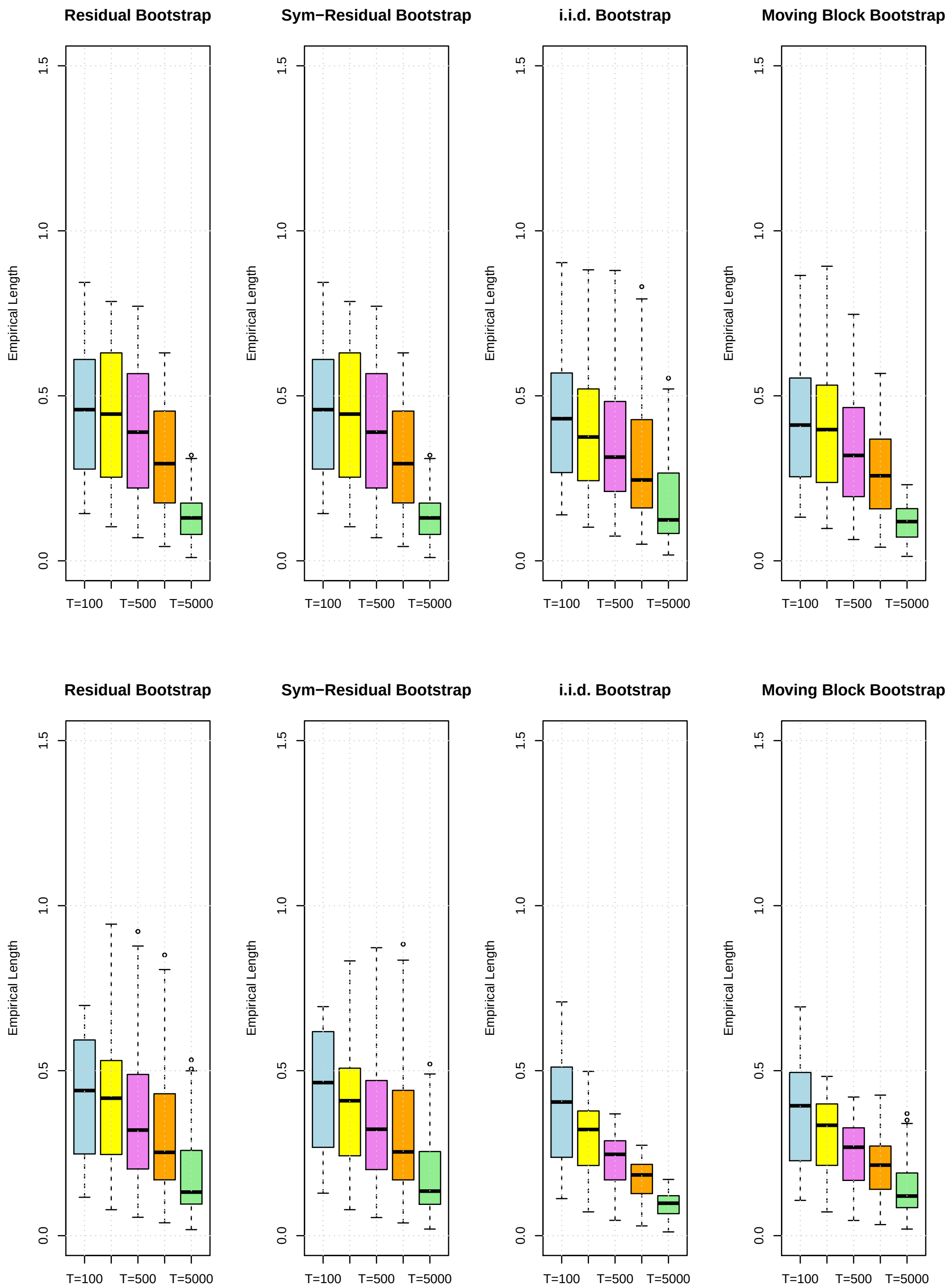

Figure F.2: Boxplots of the empirical lengths across all impulse responses and all propagation horizons (52 parameter constellations in total) of nominal 90% marginal confidence intervals for $T \in \{100, 250, 500, 1000, 5000\}$. The first row corresponds to $e_{t,i} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and the second row corresponds to $e_{t,i} \overset{\text{i.i.d.}}{\sim} \frac{3}{5}t_5$.

G DGP-1c

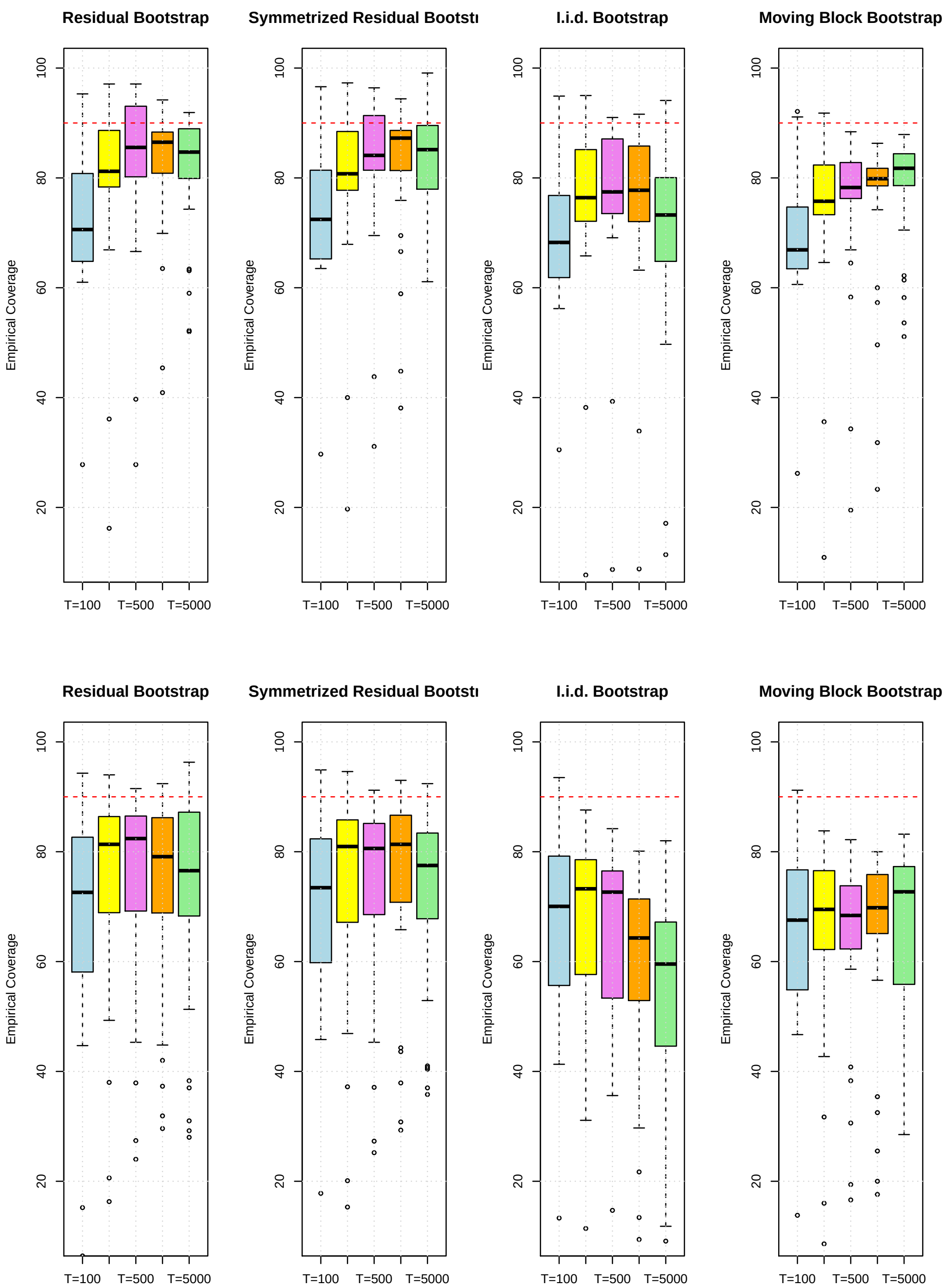

Figure G.1: Boxplots of the empirical coverages across all impulse responses and all propagation horizons (52 parameter constellations in total) of nominal 90% marginal confidence intervals for $T \in \{100, 250, 500, 1000, 5000\}$. The first row corresponds to $e_{t,i} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and the second row corresponds to $e_{t,i} \overset{\text{i.i.d.}}{\sim} \frac{3}{5}t_5$.

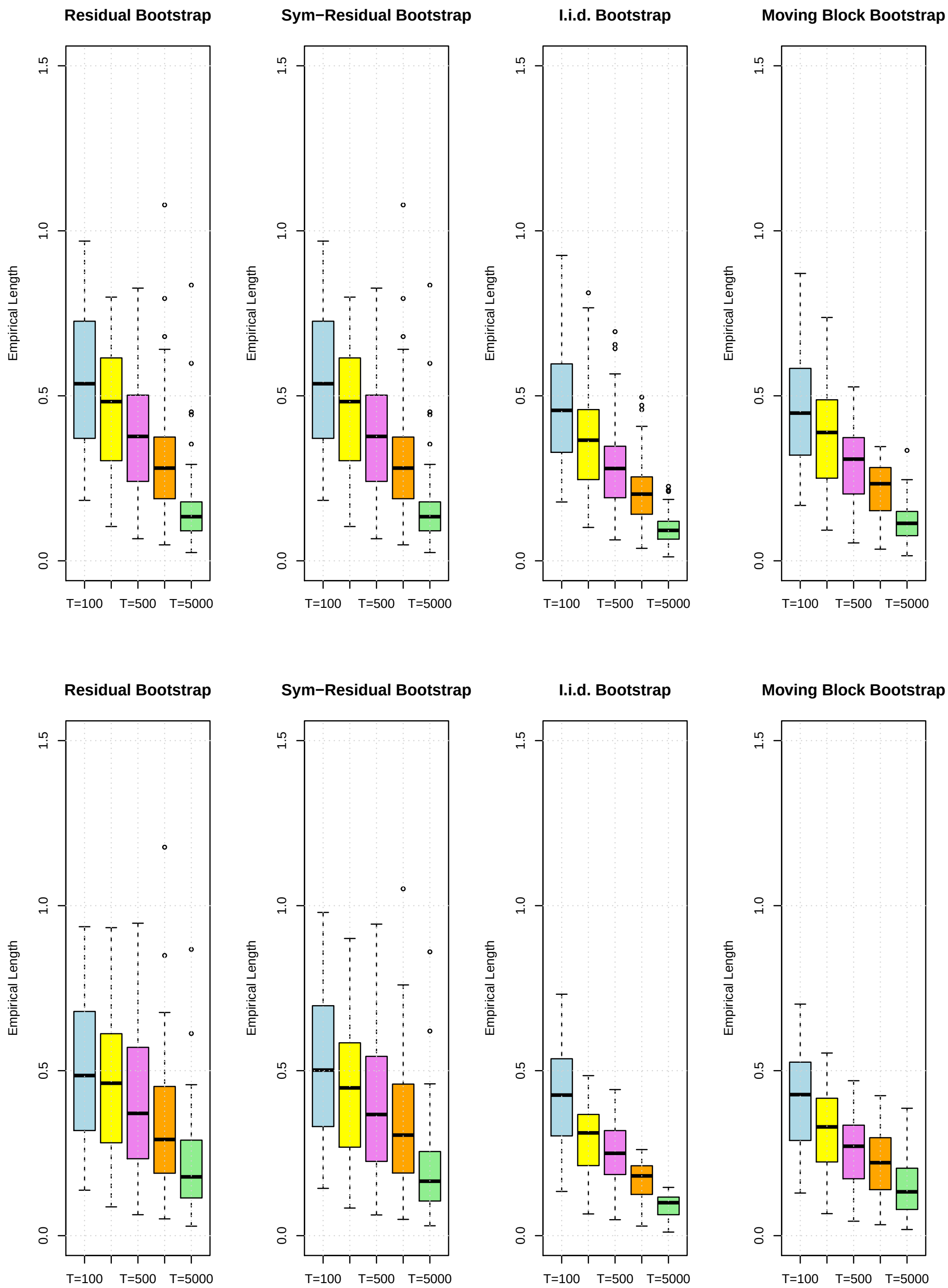

Figure G.2: Boxplots of the empirical lengths across all impulse responses and all propagation horizons (52 parameter constellations in total) of nominal 90% marginal confidence intervals for $T \in \{100, 250, 500, 1000, 5000\}$. The first row corresponds to $e_{t,i} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and the second row corresponds to $e_{t,i} \overset{\text{i.i.d.}}{\sim} \frac{3}{5}t_5$.

H Bootstrap

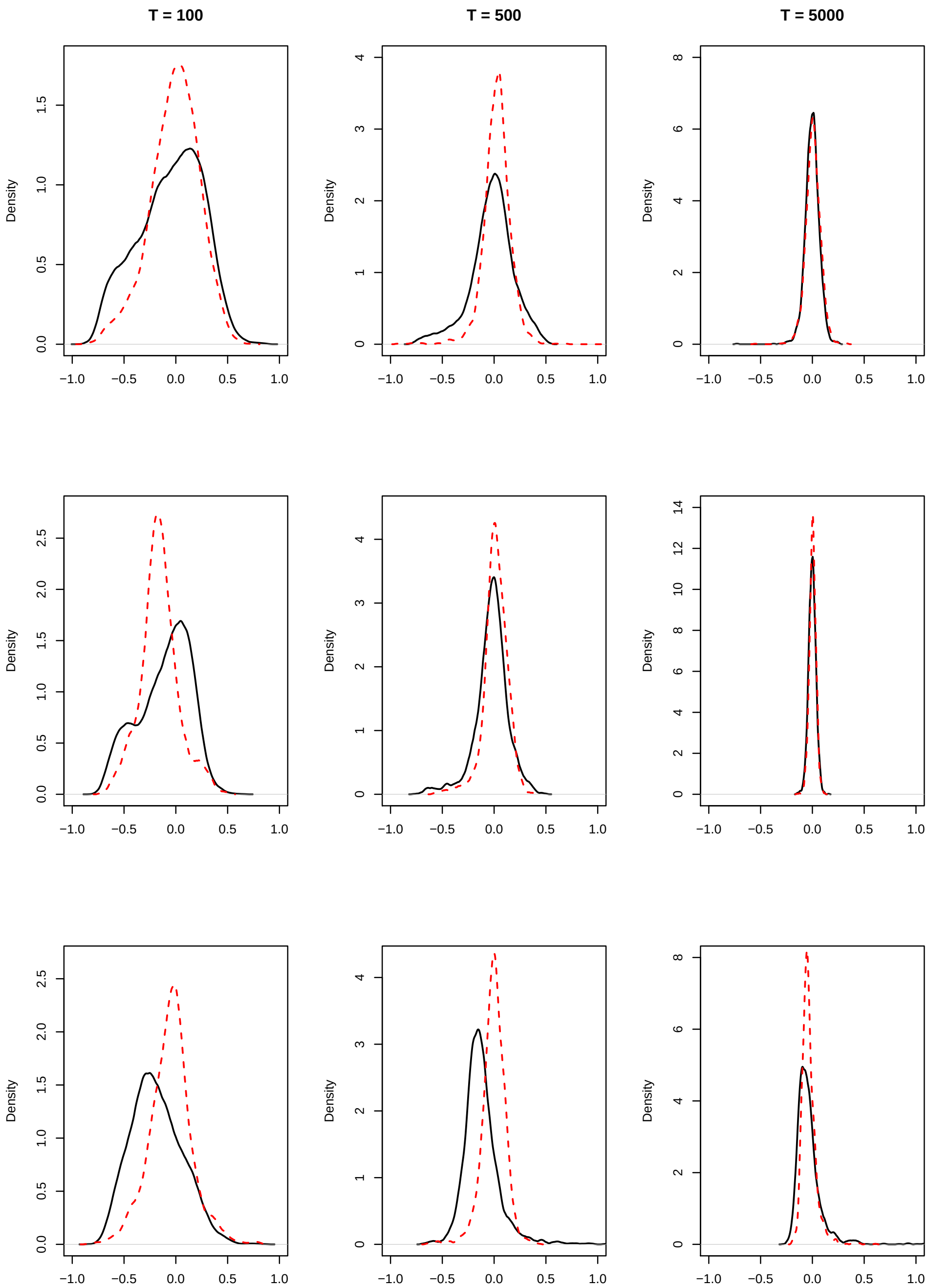

Figure H.1: The simulated density of the sampling distribution $\widehat{\Theta}_{11,3} - \Theta_{11,3}$ (solid line) versus the simulated density of the bootstrap sampling distribution $\widehat{\Theta}^*_{11,3} - \widehat{\Theta}_{11,3}$ (dashed line) using the residual bootstrap. Both densities are estimated with the Epanechnikov kernel and based on 2'000 simulated observations. The first column corresponds to DGP-1a, the second column to DGP-1b and the third column to DGP-1c.

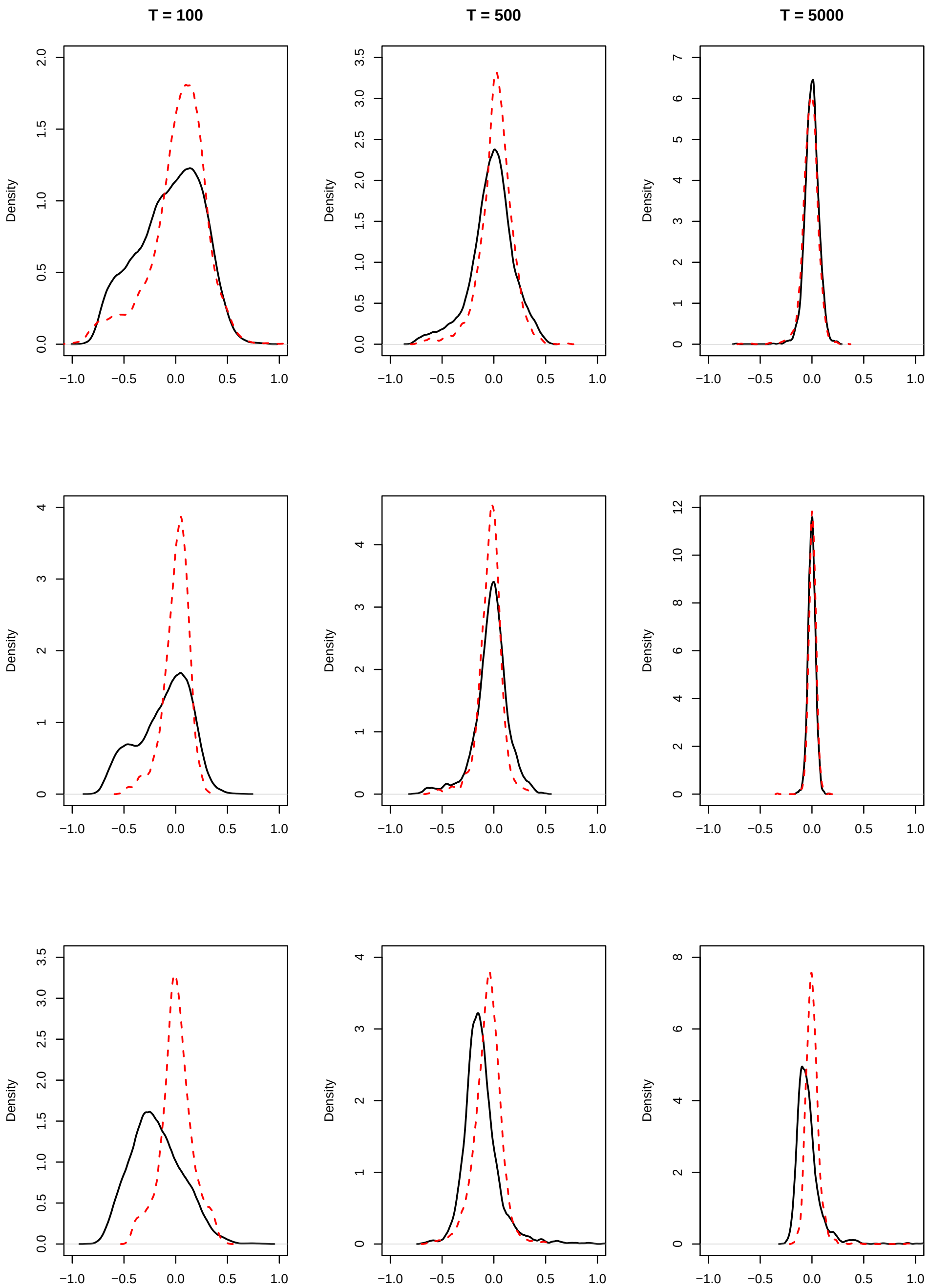

Figure H.2: The simulated density of the sampling distribution $\widehat{\Theta}_{11,3} - \Theta_{11,3}$ (solid line) versus the simulated density of the bootstrap sampling distribution $\widehat{\Theta}^*_{11,3} - \widehat{\Theta}_{11,3}$ (dashed line) using the symmetrized residual bootstrap. Both densities are estimated with the Epanechnikov kernel and based on 2'000 simulated observations. The first column corresponds to DGP-1a, the second column to DGP-1b and the third column to DGP-1c.

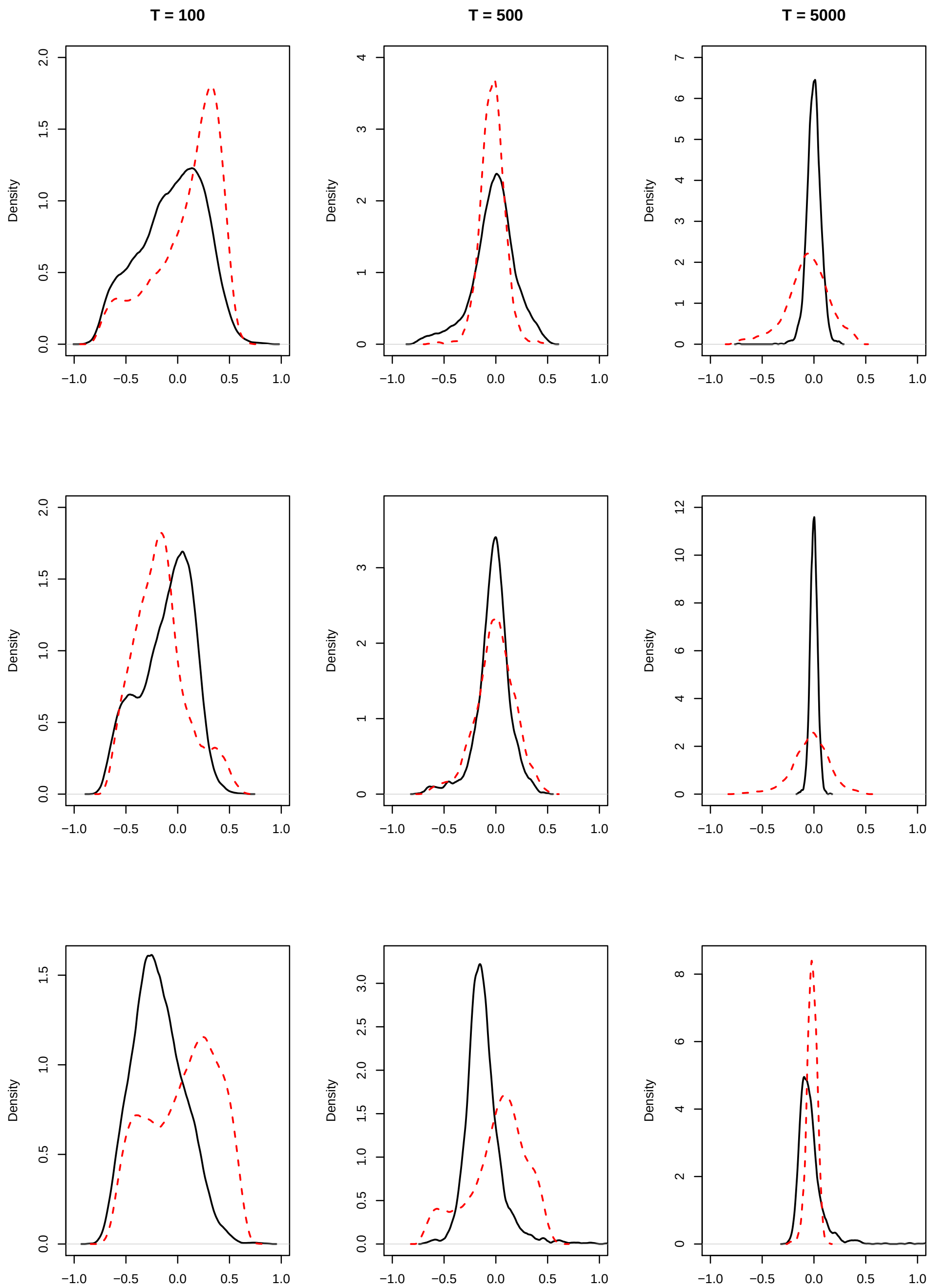

Figure H.3: The simulated density of the sampling distribution $\widehat{\Theta}_{11,3} - \Theta_{11,3}$ (solid line) versus the simulated density of the bootstrap sampling distribution $\widehat{\Theta}^{*}_{11,3} - \widehat{\Theta}_{11,3}$ (dashed line) using the i.i.d. bootstrap. Both densities are estimated with the Epanechnikov kernel and based on 2'000 simulated observations. The first column corresponds to DGP-1a, the second column to DGP-1b and the third column to DGP-1c.

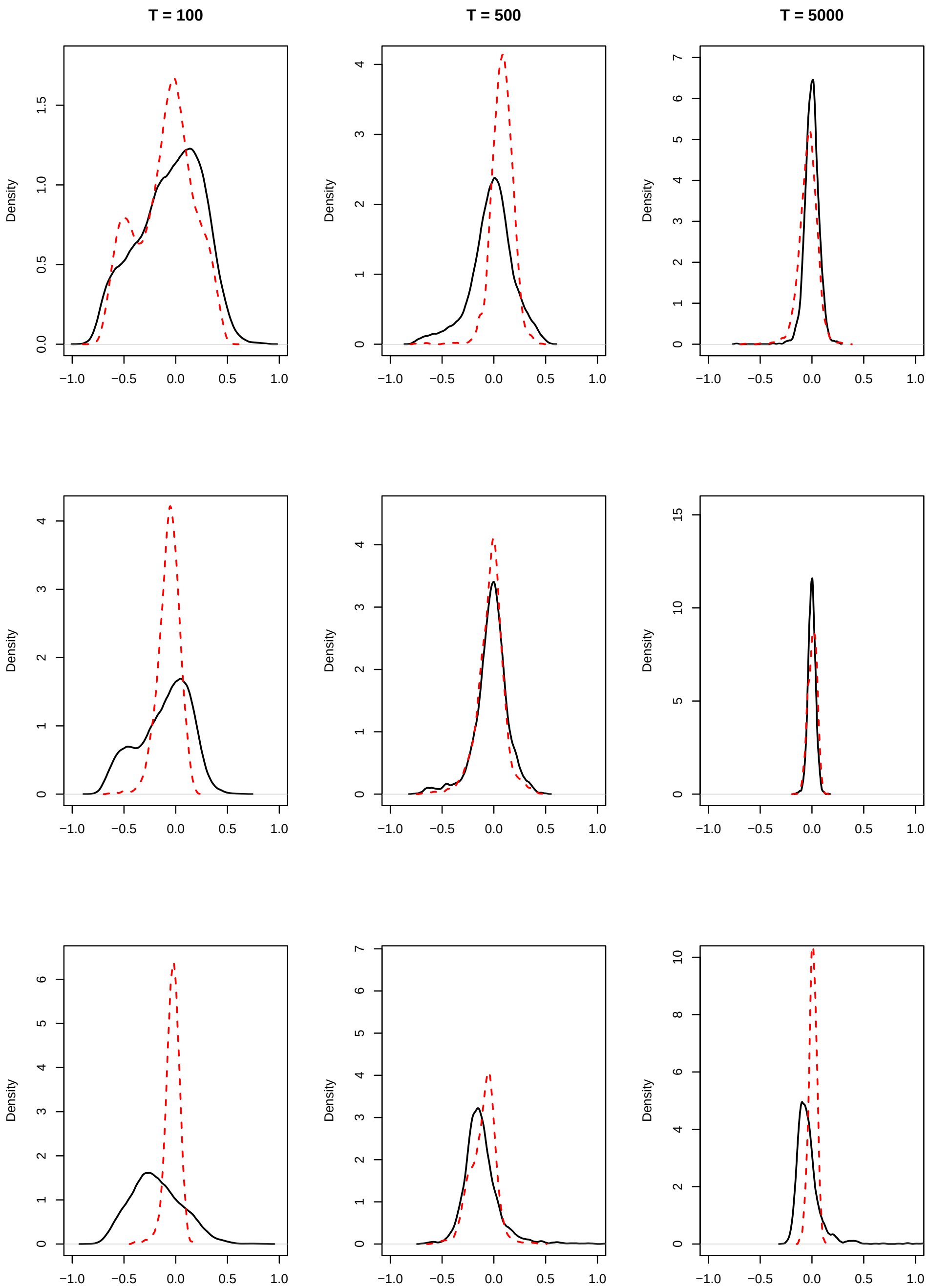

Figure H.4: The simulated density of the sampling distribution $\widehat{\Theta}_{11,3} - \Theta_{11,3}$ (solid line) versus the simulated density of the bootstrap sampling distribution $\widehat{\Theta}^{*}_{11,3} - \widehat{\Theta}_{11,3}$ (dashed line) using the moving block bootstrap. Both densities are estimated with the Epanechnikov kernel and based on 2'000 simulated observations. The first column corresponds to DGP-1a, the second column to DGP-1b and the third column to DGP-1c.