University of Zurich

Department of Economics

Working Paper Series

ISSN 1664-7041 (print)
ISSN 1664-705X (online)

Working Paper No. 286

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April 2018

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April 26, 2018

Abstract. The symmetric two-player Hirshleifer (1989) contest is shown to admit a unique equilibrium. The support of the equilibrium strategy is finite and includes, in particular, the zero expenditure level. We also establish a lower bound for the cardinality of the support and an upper bound for the undissipated rent.

Keywords. Contests · Mixed-strategy equilibrium · Rent dissipation · Uniqueness

JEL codes. C72, D72, D74

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1. Introduction

Mixed equilibria in contests of the generalized Tullock form, for which winning probabilities depend on the *ratio* of resources expended, have recently received much attention from theorists (Baye et al., 1994; Alcade and Dahm, 2010; Ewerhart, 2015, 2017a, 2017b; Feng and Lu, 2017). There is another appealing class of contests, however, where the winning probabilities depend instead on the *difference* of resources expended (Hirshleifer, 1989; Skaperdas, 1996; Baik, 1998; Che and Gale, 2000). In particular, Hirshleifer's framework has its merits for the analysis of military combat (Dupuy, 1987; Hirshleifer, 2000). Notwithstanding, the nature of mixed equilibria in that model has remained poorly understood.

In this paper, we prove uniqueness of the equilibrium in the symmetric two-player Hirshleifer contest, and offer a characterization of the mixed equilibrium. It is shown that the support of the symmetric equilibrium strategy is finite and includes the origin. Moreover, the cardinality of the support grows over any finite bound as the decisiveness parameter goes to infinity. Further, we show that the undissipated rent converges to zero as the decisiveness parameter goes to infinity, and that ex-post overdissipation may occur. We conclude by extending the uniqueness result to a larger class of contests.

The uniqueness result is stated in Section 2, and proven in Section 3. Section 4 characterizes the equilibrium. Rent dissipation is dealt with in Section 5. Section 6 discusses ex-post overdissipation. Alternative contest technologies are considered in Section 7.

2. Statement of the uniqueness result

The Hirshleifer contest is specified as follows. Each of two players $i \in \{1, 2\}$ expends resources $x_i \geq 0$ in an attempt to win a prize of normalized value one. Player i's payoff is given as

$$\Pi_i(x_i, x_j) = \frac{\exp(\alpha x_i)}{\exp(\alpha x_i) + \exp(\alpha x_j)} - x_i \tag{1}$$

$$= \frac{1}{1 + \exp(\alpha(x_j - x_i))} - x_i, \tag{2}$$

where $j \in \{1, 2\}$ with $j \neq i$, and $\alpha > 0$ measures the decisiveness of the difference-form contest. In particular, for $\alpha \to \infty$, payoffs converge against those of the all-pay auction.

Any bid exceeding one is strictly dominated. We therefore define a *mixed strategy* for player i as a probability measure μ_i on the Borel subsets of $[0, 1]$. The set of mixed strategies for player i will be denoted by M, where pure strategies $x_i \in [0, 1]$ are interpreted as Dirac measures, as usual. Each player i's expected payoff is well-defined for any $(\mu_i, \mu_j) \in M \times M$, and will, with some abuse of notation, be denoted by $\Pi_i(\mu_i, \mu_j)$. An *equilibrium* is a pair $\mu^* = (\mu_1^*, \mu_2^*) \in M \times M$ such that $\Pi_i(\mu_i^*, \mu_j^*) \geq \Pi_i(\mu_i, \mu_j^*)$ for any $i, j \in \{1, 2\}$ with $j \neq i$, and for any $\mu_i \in M$.

Proposition 1. *For any $\alpha > 0$, the Hirshleifer contest with parameter α has a unique equilibrium.*

3. Proof of Proposition 1

Equilibrium existence is known (cf. Hirshleifer, 1989, fn. 12). The proof of uniqueness starts from the following observation.

Lemma 1. *Let $\mu = (\mu_1, \mu_2) \in M \times M$. Then, for any $i, j \in \{1, 2\}$ with $j \neq i$, the set of maximizers $X_i(\mu) = \arg\max_{\widetilde{x}_i \in [0,1]} \Pi_i(\widetilde{x}_i, \mu_j)$ is finite.*

Proof. The proof is a straightforward adaption of Ewerhart (2015, Th. 3.2), and therefore omitted.[1] $\square$

Next, we show the following.

Lemma 2. *The set* $X^\alpha = \bigcap_{\mu^* \text{ equilibrium}} X_1(\mu^*)$ *is nonempty, and contains the support of any equilibrium strategy (for both players).*

Proof. Take an equilibrium $\mu^* = (\mu_1^*, \mu_2^*)$. Clearly, the support of μ_1^* is a subset of $X_1(\mu^*)$. Let $\mu^{**} = (\mu_1^{**}, \mu_2^{**})$ be an arbitrary equilibrium. Then, since equilibria in two-player contests are interchangeable (Ewerhart, 2017b, Appendix), (μ_1^*, μ_2^{**}) is an equilibrium. Therefore, the support of μ_1^* is a subset of $X_1(\mu_1^*, \mu_2^{**})$. But $X_1(\mu_1^*, \mu_2^{**}) = X_1(\mu^{**})$. Hence, the support of μ_1^* is contained in $X_1(\mu^{**})$ for any equilibrium μ^{**}. In particular, $X^\alpha \neq \varnothing$. The second claim follows by symmetry. $\square$

Denote by $K = |X^\alpha|$ the number of elements of X^α. Thus, $X^\alpha = \{z_1, ..., z_K\}$, where $z_1 > z_2 > ... > z_K$. Suppose first that $K = 1$. Then, the equilibrium is obviously unique. Suppose next that $K \geq 2$. Fix some equilibrium $\mu^* = (\mu_1^*, \mu_2^*)$, and let $p_j^m = \mu_i^*(\{z_m\}) \geq 0$ denote the weight assigned by μ_j^* to z_m, for $j \in \{1, 2\}$ and $m \in \{1, ..., K\}$. We know that $z_1, ..., z_K$ all deliver the equilibrium payoff Π_i^* against μ_j^*, i.e.,

$$\Pi_i^* = \left(\sum_{m=1}^{K} p_j^m \frac{\exp(\alpha z_k)}{\exp(\alpha z_k) + \exp(\alpha z_m)} \right) - z_k \qquad (k = 1, ..., K; j \neq i). \quad (3)$$

Thus, there are K equations to identify $(K+1)$ unknowns $p_j^1, ..., p_j^K$ and Π_i^*. Notably, adding the relationship $\sum_{m=1}^{K} p_j^m = 1$ does not help in general. Instead, we focus on the *largest* element of the support of player i's equilibrium strategy.[2] Since $K \geq 2$, we know that z_1 is an interior maximum.

[1] If attention is restricted to strategies that are absolutely continuous with respect to the Lebesgue measure, the use of complex-analytic methods may be circumvented (Sun, 2017).

[2] The first-named author would like to thank Larry Samuelson for this suggestion.

Hence, the first-order condition implies

$$\sum_{m=1}^{K} p_j^m \frac{\alpha \exp(\alpha z_1) \exp(\alpha z_m)}{(\exp(\alpha z_1) + \exp(\alpha z_m))^2} = 1. \tag{4}$$

Combining these $(K+1)$ equations yields

$$\begin{pmatrix} \frac{\exp(\alpha z_1)}{\exp(\alpha z_1)+\exp(\alpha z_1)} & \cdots & \frac{\exp(\alpha z_1)}{\exp(\alpha z_1)+\exp(\alpha z_K)} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\exp(\alpha z_K)}{\exp(\alpha z_K)+\exp(\alpha z_1)} & \cdots & \frac{\exp(\alpha z_K)}{\exp(\alpha z_K)+\exp(\alpha z_K)} & 1 \\ \frac{\alpha \exp(\alpha z_1) \exp(\alpha z_1)}{(\exp(\alpha z_1)+\exp(\alpha z_1))^2} & \cdots & \frac{\alpha \exp(\alpha z_1) \exp(\alpha z_K)}{(\exp(\alpha z_1)+\exp(\alpha z_K))^2} & 0 \end{pmatrix} \begin{pmatrix} p_j^1 \\ \vdots \\ p_j^K \\ -\Pi_i^* \end{pmatrix} = \begin{pmatrix} z_1 \\ \vdots \\ z_K \\ 1 \end{pmatrix}. \tag{5}$$

It turns out that (5) has at most one solution.

Lemma 3. *The square matrix on the left-hand side of (5) is invertible.*

Proof. Let $e_k = \exp(\alpha z_k)$ for $k = 1, ..., K$, and

$$A_1 = \begin{pmatrix} \frac{e_1}{e_1+e_1} & \cdots & \frac{e_1}{e_1+e_K} & 1 \\ \frac{e_2}{e_2+e_1} & \cdots & \frac{e_2}{e_2+e_K} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \frac{e_K}{e_K+e_1} & \cdots & \frac{e_K}{e_K+e_K} & 1 \\ \frac{\alpha e_1 e_1}{(e_1+e_1)^2} & \cdots & \frac{\alpha e_1 e_K}{(e_1+e_K)^2} & 0 \end{pmatrix}. \tag{6}$$

Subtracting row $k = 1$ from row k, for $k = 2, ..., K$, yields $\det A_1 = \det A_2$, where

$$A_2 = \begin{pmatrix} \frac{e_1}{e_1+e_1} & \cdots & \frac{e_1}{e_1+e_K} & 1 \\ \frac{(e_2-e_1)e_1}{(e_2+e_1)(e_1+e_1)} & \cdots & \frac{(e_2-e_1)e_K}{(e_2+e_K)(e_1+e_K)} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \frac{(e_K-e_1)e_1}{(e_K+e_1)(e_1+e_1)} & \cdots & \frac{(e_K-e_1)e_K}{(e_K+e_K)(e_1+e_K)} & 0 \\ \frac{\alpha e_1 e_1}{(e_1+e_1)^2} & \cdots & \frac{\alpha e_1 e_K}{(e_1+e_K)^2} & 0 \end{pmatrix}. \tag{7}$$

Next, we extract the factor $e_m/(e_1 + e_m) > 0$ from column m, for $m = 1, ..., K$, and the factor $(e_k - e_1) > 0$ from row k, for $k = 2, ..., K$. Further, we extract the factor $\alpha e_1 > 0$ from the last row. This yields

$$\det A_2 = \left(\prod_{1 \le m \le K} \frac{e_m}{e_1 + e_m} \right) \cdot \left(\prod_{2 \le k \le K} (e_k - e_1) \right) \cdot \alpha e_1 \cdot \det A_3, \tag{8}$$

where

$$A_3 = \begin{pmatrix} \frac{e_1}{e_1} & \frac{e_1}{e_2} & \cdots & \frac{e_1}{e_K} & 1 \\ \frac{1}{e_2+e_1} & \frac{1}{e_2+e_2} & \cdots & \frac{1}{e_2+e_K} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{e_K+e_1} & \frac{1}{e_K+e_2} & \cdots & \frac{1}{e_K+e_K} & 0 \\ \frac{1}{e_1+e_1} & \frac{1}{e_1+e_2} & \cdots & \frac{1}{e_1+e_K} & 0 \end{pmatrix}. \tag{9}$$

Finally, we exchange row 1 and row $K+1$. Therefore, $\det A_3 = -\det A_4$, where $A_4 = \{\frac{1}{e_k+e_m}\}_{k=1,...,K;m=1,...,K}$ is a Cauchy matrix (e.g., Krattenthaler, 2001) with

$$\det A_4 = \frac{\prod_{1\leq k<m\leq K}(e_k - e_m)^2}{\prod_{1\leq k\leq K,1\leq m\leq K}(e_k + e_m)} \neq 0. \tag{10}$$

This proves the lemma. □

Recall that the support of *any* equilibrium is contained in $X^\alpha = \{z_1, ..., z_K\}$. Hence, with probabilities $p_j^1, ..., p_j^K$ being unique, there can indeed be at most one equilibrium.

4. Characterization[3]

Since the Hirshleifer contest with parameter $\alpha > 0$ admits only one equilibrium, the two players necessarily use the same equilibrium strategy $\mu_\alpha \in M$. The following result characterizes μ_α.

Proposition 2. *Let $\alpha > 0$. Then, the following properties hold:*

(i) μ_α has finite support $\{y_1, ..., y_L\}$, where $y_1 > ... > y_L$, with $L \geq \frac{\alpha}{4}$.

(ii) μ_α has a mass point at the zero bid, i.e., $y_L = 0$.

(iii) there is two-sided peace (i.e., $L = 1$) if and only if $\alpha \leq 4$.

Proof. (i) By Lemma 1, the support of μ_α is finite. Denote by $q_m = \mu_\alpha(\{y_m\}) > 0$ the probability assigned to y_m, for $m \in \{1, ..., L\}$. From the

[3]This section and the next supersede the corresponding parts of earlier work by the authors (Ewerhart, 2014; Sun, 2017).

KKT conditions,

$$\frac{q_l}{4} + \sum_{\substack{m=1 \\ m \neq l}}^{L} \frac{q_m \exp(\alpha y_m) \exp(\alpha y_l)}{(\exp(\alpha y_l) + \exp(\alpha y_m))^2} \leq \frac{1}{\alpha} \qquad (l = 1, ..., L), \tag{11}$$

with equality for $l = 1, ..., L-1$, so that $q_l \leq \frac{4}{\alpha}$ for any $l = 1, ..., L$. Therefore, $L \geq \frac{\alpha}{4}$. (ii) For $L = 1$, the claim is due to Hirshleifer (1989). Suppose next that $L \geq 2$. Then, player i's expected payoff against μ_α at the *smallest* mass point y_L satisfies

$$\begin{aligned} & \frac{\partial^2 \Pi_i(y_L, \mu_\alpha)}{\partial x_i^2} \\ = & \sum_{m=1}^{L-1} \frac{q_m \alpha^2 (\exp(\alpha y_m) - \exp(\alpha y_L)) \exp(\alpha y_L) \exp(\alpha y_m)}{(\exp(\alpha y_L) + \exp(\alpha y_m))^3} && (12) \\ > & \ 0, && (13) \end{aligned}$$

which shows that y_L cannot be an interior maximum. Hence, $y_L = 0$. (iii) Hirshleifer (1989) has shown that two-sided peace is an equilibrium for $\alpha \leq 4$. For $\alpha > 4$, however, part (i) implies $L \geq 2$. □

Example ($L = 2$). Consider an equilibrium strategy μ_α that places probability $q_1 > 0$ on $y_1 > 0$, and probability $1 - q_1 > 0$ on $y_2 = 0$. Then,

$$y_1 = \frac{1}{2} - \frac{1}{1 + \exp(\alpha y_1)}, \tag{14}$$

$$q_1 = 1 - \frac{\alpha - 4}{4 \alpha y_1^2}. \tag{15}$$

This equilibrium exists numerically for $\alpha \in (4, 6.79)$.[4]

[4] The implicit value for y_1 may be characterized alternatively in terms of the r-Lambert function (Mesö and Baricz, 2017).

5. Rent dissipation

Rent dissipation is always incomplete in the Hirshleifer contest.[5] For $\alpha \to \infty$, however, the equilibrium payoff Π^* goes to zero, as the following result shows.

Proposition 3. $\Pi^* \leq \frac{2}{\alpha}$.

Proof.

$$\begin{aligned} \Pi^* &= \Pi_i(0, \mu_\alpha) & (16) \\ &= \sum_{l=1}^{L} \frac{q_l}{1+\exp(\alpha y_l)} & (17) \\ &\leq 2\sum_{l=1}^{L} \frac{q_l}{1+\exp(\alpha y_l)} \cdot \frac{\exp(\alpha y_l)}{1+\exp(\alpha y_l)} & (18) \\ &= \frac{2}{\alpha}\left(\frac{\partial \Pi_i(0, \mu_\alpha)}{\partial x_i} + 1\right). & (19) \end{aligned}$$

Since $\partial \Pi_i(0, \mu_\alpha)/\partial x_i \leq 0$, the claim follows. $\square$

Figure 1 outlines the equilibrium payoff Π^* and its upper bound as a function of α. Note that Π^*, contrary to intuition, is not globally declining. For example, if $\alpha = 6.1$ (< 6.6), the equilibrium is given by $y_1 = 0.4337$ (0.4517), $q_1 = 0.5425$ (0.5173), and $\Pi^* = 0.2646$ (< 0.2662). Thus, the increase in y_1 is more than compensated by a decline in q_1.

[5] This fact contrasts, of course, with the complete rent dissipation pervasive in sufficiently decisive contests of the ratio form (Baye et al., 1994; Alcade and Dahm, 2010; Ewerhart, 2015, 2017a).

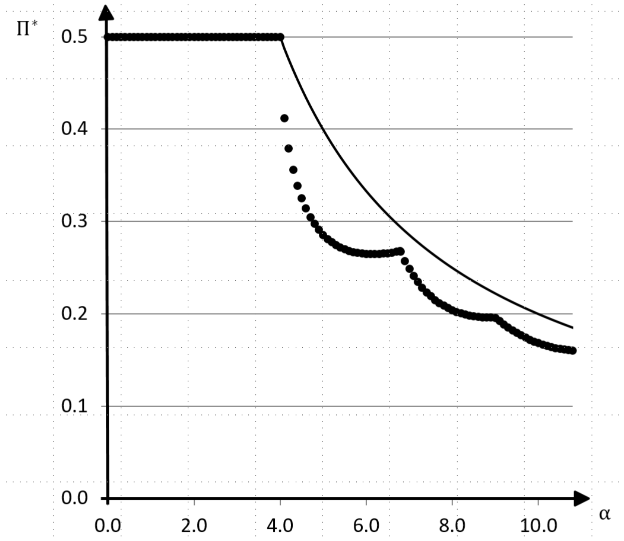

Figure 1. Undissipated rent as a function of α.

6. Ex-post overdissipation

Although rent dissipation is less severe than in the Tullock case, the Hirshleifer contest may nevertheless feature ex-post overdissipation, i.e., the sum of realized bids may exceed the value of the prize with positive probability (cf. Baye et al., 1999).

Proposition 4. $y_1 \geq \frac{3}{4} - \frac{10}{3\alpha}$.

Proof. Recall that $y_1 > ... > y_L$. Let L^* be the largest $l \in \{1, ..., L\}$ such that

$$\frac{\exp(\alpha y_1)}{\exp(\alpha y_1) + \exp(\alpha y_l)} < \frac{3}{4}, \tag{20}$$

and let

$$Q = \sum_{l=1}^{L^*} q_l. \tag{21}$$

Then, by Proposition 3,

$$\frac{2}{\alpha} \geq \Pi^* \geq \frac{3}{4}(1 - Q) + \frac{1}{2}Q - y_1. \tag{22}$$

Therefore,

$$Q \geq 3 - 4y_1 - \frac{8}{\alpha}. \tag{23}$$

But, from the first-order condition at y_1,

$$1 \geq \alpha \sum_{l=1}^{L^*} q_l \frac{\exp(\alpha y_1)\exp(\alpha y_l)}{(\exp(\alpha y_1) + \exp(\alpha y_l))^2} \geq \alpha \cdot Q \cdot \frac{3}{4} \cdot \frac{1}{4}. \tag{24}$$

Using (23) in (24) yields the claim. □

Proposition 4 implies that ex-post overdissipation occurs for any sufficiently large α. Using numerical analysis, we verified that $y_1 > 0.5$ holds for $\alpha > 7.2$.

7. Alternative contest technologies

Consider a contest technology of the form

$$\Pi_i^h(x_i, x_j) = \frac{h(x_i)}{h(x_i) + h(x_j)} - x_i, \tag{25}$$

where $h > 0$ is a positive impact function (as in Neary, 1997). Provided that h admits, in addition, a real-analytic extension to $(-\varepsilon, \infty)$, for some $\varepsilon > 0$, the uniqueness argument goes through. In that case, however, the equilibrium need no longer possess a mass point at the origin.[6]

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[6] E.g., there is no mass point at zero for $h(x_i) = \eta + x_i$ if $\eta < 1/4$ (Amegashie, 2006).

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