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Anchoring on Utopia: A Generalization of the Kalai-Smorodinsky Solution*

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Abstract

Many bargaining solutions anchor on disagreement, allocating gains with respect to the worst-case scenario. We propose here a solution anchoring on utopia (the ideal, maximal aspirations for all agents), but yielding feasible allocations for any number of agents. The *negotiated aspirations solution* proposes the best allocation in the direction of utopia starting at an endogenous reference point which depends on both the utopia point and bargaining power. The Kalai-Smorodinsky solution becomes a particular case if (and only if) the reference point lies on the line from utopia to disagreement. We provide a characterization for the two-agent case relying only on standard axioms or natural restrictions thereof: strong Pareto optimality, scale invariance, restricted monotonicity, and restricted concavity. A characterization for the general (n-agent) case is obtained by relaxing Pareto optimality and adding the (standard) axiom of restricted contraction independence, plus the minimal condition that utopia should be selected if available.

JEL Classification: C71 · C78

Keywords: n-Person bargaining · Utopia point · Axiomatic approach · Kalai-Smorodinsky solution

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1 Introduction

The celebrated Kalai-Smorodinsky solution (Kalai and Smorodinsky, 1975) for n-person bargaining problems captures a simple intuition. Consider the disagreement point, that is, the allocation which would result if negotiations broke down. Consider the exact opposite, the utopical (and typically unattainable) allocation where everybody would be granted his or her maximal aspirations. Starting from disagreement, increase the surplus allocated to every agent in the direction of the utopian dream, increasing each agent's outcome in a proportional way. The Kalai-Smorodinsky solution proposes to choose the best allocation obtained through this procedure while maintaining feasibility.

This approach, however, ignores bargaining power, which has played a prominent role in the bargaining literature even since Nash (1950). Thus, Kalai (1977) introduced the *proportional solutions*, which choose the best feasible point along the line starting at disagreement and proceeding "up" in a direction determined exclusively by an exogenous vector of bargaining weights. As a consequence, the utopian point plays no role whatsoever in the proportional solutions: they anchor on disagreement. Conversely, the *equal-losses solutions* of Chun (1988) choose the best feasible point along the line starting at utopia and proceeding "down" in a direction determined exclusively by the vector of bargaining weights. In those solutions, the disagreement point plays no role. However, the symmetry with respect to the proposal of Kalai (1977) is imperfect, because, as shown by Chun and Peters (1991), the equal-losses solutions can fail to prescribe a feasible allocation for three or more agents.

Thomson (1994) introduced the *weighted Kalai-Smorodinsky solutions*, which recommend the maximal point along the line starting at disagreement and proceeding up in a direction determined by both the utopia point and the vector of bargaining weights. The gradual increments starting at the disagreement point are required to be made proportionally to both the bargaining power and the utopian claims, while in the Kalai-Smorodinsky solution they would follow the line connecting disagreement to utopia. Even though the utopia point is taken into account to compute the *weighted* Kalai-Smorodinsky solution, this solution concept is clearly anchored on disagreement.

In this paper, we propose an intuitive bargaining solution anchoring on utopia, taking into account bargaining power, and delivering attainable allocations for any number of agents. The new solution fills a gap in the literature by providing a logical counterpart to (weighted) Kalai-Smorodinsky solutions which anchor on disagreement. Specifically, the proposal is as follows. Anchor on the utopian ideal. Reduce each individual aspiration, as captured by the utopia point, by the bargaining power parameter. The resulting point can be shown to be always feasible, and typically in the interior of the bargaining set. Now choose the best allocation along the line connecting the utopia point and the new reference point. This procedure can be seen as allocating losses with respect to utopia, or as allocating gains from the endogenous reference point in the direction of utopia. Implicitly, the procedure asks agents to identify their maximal aspirations and proposes the outcome of a negotiation which takes into account bargaining power, feasibility, and the desire to approach the utopia point. Hence, we call our proposal the *negotiated aspirations solution.*

To understand what this solution actually entails, we provide full axiomatic characterizations. For the two-agent case, surprisingly, the solution is characterized by a combination of well-established axioms, namely strong Pareto optimality, scale invariance (Nash, 1950), restricted monotonicity (Rosenthal, 1976), and a natural restriction of the concavity axiom (Myerson, 1981), which we introduce here and call restricted concavity. For the general case of n-agent bargaining problems, the negotiated aspirations solution is characterized by the same properties with three differences. First, weak Pareto optimality replaces the strong version. Second, restricted contraction independence (Roth, 1977) is added. Third, a minimal condition has to be added, namely the property that, if utopia were actually available (and hence there is no bargaining "problem"), the solution would choose it ("utopia fulfillment").

The negotiated aspirations solution is also conceptually related to other solutions proposed in the literature. First, the idea of minimizing losses while anchoring on the utopia point was previously explored by Yu (1973) and Freimer and Yu (1976), who studied a family of solutions indexed by a parameter p determining the shape of the indifference curves, encompassing the equal losses solution (Chun, 1988) ($p = \infty$). These solutions distribute losses from the utopia point keeping a fixed slope for every bargaining set, with independence of the maximal aspiration of each agent. In contrast, in the negotiated aspirations solution the slope depends on the utopia point, hence on the maximal aspiration of each agent. Second, the concept of a reference point driving a bargaining solution is well established, e.g., Salonen (1985, 1987) and Anbarci (1995). In particular, a number of works have proposed solutions which proceed "upwards" from a constructed reference point. The solution of Gupta and Livne (1988) is defined over "bargaining triplets" which include a problem-specific, exogenously given reference point, required to be Pareto-superior to the disagreement point but not Pareto optimal. The solution's outcome is the intersection between a ray connecting the exogenous reference point with the utopia point and the bargaining set's boundary. Balakrishnan et al (2011) propose a similar solution using a ray connecting the exogenously given reference point to a "tempered utopia," i.e. a modified utopia point (in this sense, the authors call their solution "dual" to the one by Gupta and Livne, 1988). More recently, Karagözoglu et al (2015) have proposed a two-parameter family of bargaining solutions encompassing the two concepts mentioned above. A key difference of our approach with respect to those works is that the reference point is not exogenously given as part of the problem. Rather, it is co-determined by the bargaining power of the agents and the maximal aspiration of each agent given the bargaining set, and in this sense is endogenous.

The remainder of the paper is structured as follows. Section 2 introduces the notation and defines the negotiated aspirations solution. Section 3 then discusses the basic axioms we require. Section 4 proves the characterization for the case of two agents. Section 5 presents the two last axioms and the characterization for the general case (n agents). Section 6 concludes.

2 Bargaining Problems and the Negotiated Aspirations Solution

A group of n agents, $i = 1, \ldots, n$, need to agree on a utility vector from a set of potential possibilities, $S \subseteq \mathbb{R}^n_+$. In case of disagreement, a default option or *disagreement point* $d = \mathbf{0} \in S$ will be implemented. Formally, a *bargaining problem* is a pair $(S, \mathbf{0})$ where S is a compact, convex[1] subset $S \subseteq \mathbb{R}^n_+$ (called the *bargaining set*) with $\mathbf{0} \in S$ such that the following two properties are fulfilled. The first is a non-degeneracy condition requiring that there exists $x \in S$ such that $x_i > 0$ for all $i = 1, \ldots, n$. The second, usually called $\mathbf{0}$-comprehensiveness, states that for every $x \in S$ and every $y \in \mathbb{R}^n$ with $\mathbf{0} \leq y \leq x$, it follows that $y \in S$.[2]

Let Σ_0 be the set of all bargaining problems of the form $(S, \mathbf{0})$. A *bargaining solution* (or simply a solution) is a function $f : \Sigma_0 \mapsto \mathbb{R}^n_+$ satisfying $f(S, \mathbf{0}) \in S$ for every $(S, \mathbf{0}) \in \Sigma_0$; that is, given a bargaining problem $(S, \mathbf{0})$, the solution f prescribes $f(S, \mathbf{0})$. For conciseness, since we assume $d = \mathbf{0}$, a bargaining problem $(S, \mathbf{0}) \in \Sigma_0$ will be denoted simply by S from now on.[3]

For a given $S \in \Sigma_0$, the *utopia point* $m(S) \in \mathbb{R}^n_+$ is the point where each coordinate i contains the maximum conceivable outcome for player i among individually rational alternatives.[4] Formally, for each i,

$$m_i(S) = \max \{ x_i \mid x \geq \mathbf{0} \text{ and } x \in S \}.$$

To measure asymmetries in bargaining power, let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a vector of *bargaining weights*, i.e. $\alpha \in \Delta^{n-1} = \left\{ \beta \in \mathbb{R}^n_+ \mid \sum_{i=1}^n \beta_i = 1 \right\}$. The key for the bargaining solution we propose is the endogenous reference point

$$m^\alpha(S) = (\alpha_1 m_1(S), \ldots, \alpha_n m_n(S))$$

which is always feasible for every $S \in \Sigma_0$, i.e. $m^\alpha(S) \in S$.[5] At this point, every agent achieves a fraction of his or her utopian utility which is proportional to the agents' bargaining power. That is, we start with ("anchor at") the utopian point $m(S)$, which represents the maximal aspirations of the agents, and adjust them down to $m^\alpha(S)$ taking into account bargaining power. In this sense, the point $m^\alpha(S)$ captures a starting point for "negotiated aspirations." In general, $m^\alpha(S)$ might be in the interior of S, that is,

[1] Most of the literature concentrates on bargaining over convex sets. See, however, Conley and Wilkie (1996), Hougaard and Tvede (2003), and Qin et al (2015).

[2] The vector notation $y \geq x$ means $y_i \geq x_i$ for $i = 1, \ldots, n$; $y > x$ means $y \geq x$ and $y \neq x$.

[3] The assumption $d = \mathbf{0}$ is without loss of generality if we consider solutions satisfying *translation invariance*, i.e. the property that $f(S + \{t\}, d + t) = f(S, d) + t$ for all $t \in \mathbb{R}^n$. This is because for any pair (S, d), by translation invariance $f(S, d) = f(S - \{d\}, \mathbf{0}) + d$, and $(S - \{d\}, \mathbf{0}) \in \Sigma_0$. That is, the restriction of f to Σ_0 completely determines f.

[4] An allocation x is individually rational if $x \geq d$. Since we assume $d = \mathbf{0}$ and $S \subseteq \mathbb{R}^n_+$, individual rationality is guaranteed. Peters (1986) defines utopia points without the individual rationality constraint.

[5] To see this, let z^i be an element of S where the problem $\max \{ x_i \mid x \geq \mathbf{0} \text{ and } x \in S \}$ achieves its solution, hence $z^i \geq \mathbf{0}$ and $z^i_i = m_i(S)$. Let $\hat{z}^i$ be given by $\hat{z}^i_i = m_i(S)$ and $\hat{z}^i_j = 0$ for all $j \neq i$. Since S is $\mathbf{0}$-comprehensive, it follows that $\hat{z}^i \in S$ for all i. Since S is convex, we obtain that $m^\alpha(S) = \sum_{i=1}^n \alpha_i \hat{z}^i \in S$.

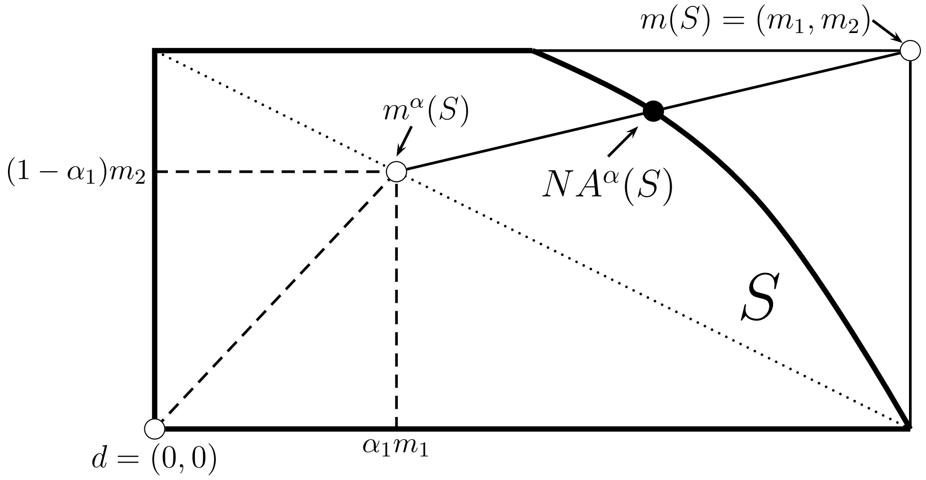

Figure 1: The negotiated aspirations solution for 2 agents, $\alpha = (\alpha_1, 1-\alpha_1)$.

it will typically be possible to achieve a point closer to utopia. The *negotiated aspirations solution* identifies the Pareto optimal point above $m^\alpha(S)$ which respects the given bargaining weights. The following definition gives the formal statement, and Figure 1 provides a graphical illustration for the case of two agents.

Definition 1. The **negotiated aspirations solution** with weights $\alpha \in \Delta^{n-1}$ is given by

$$NA^\alpha(S) = (1-\lambda^*)m^\alpha(S) + \lambda^* m(S)$$

for each $S \in \Sigma_0$, where λ^* is the solution of the problem

$$\max\{\lambda \in [0,1] \mid (1-\lambda)m^\alpha(S) + \lambda m(S) \in S\}.$$

The negotiated aspirations solution[6] is a natural counterpart to and generalization of the Kalai-Smorodinsky solution. The latter prescribes the largest feasible point in the ray connecting the disagreement point and the utopia point. In contrast, the negotiated aspirations solution determines the line passing through the utopia point which respects the pre-specified bargaining weights. Intuitively, the solution then minimizes the losses with respect to the (typically unattainable) utopia point, distributing them according to bargaining power. Hence, an alternative, more descriptive but uncomfortably long name would be "the *weighted proportional losses solution*."

The Kalai-Smorodinsky solution corresponds to the particular case of the negotiated aspirations solution where all agents have identical bargaining power, i.e. $\alpha^{KS} = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$. This follows because the line determined by the utopia point and $m^{\alpha^{KS}}(S)$ passes through the disagreement point. Hence, the negotiated aspirations solution can be seen as a generalization of the Kalai-Smorodinsky solution taking into account bargaining power.

[6]Strictly speaking, this solution is actually a family of solutions, indexed by α. For simplicity, we drop the dependence on α whenever it does not lead to confusion.

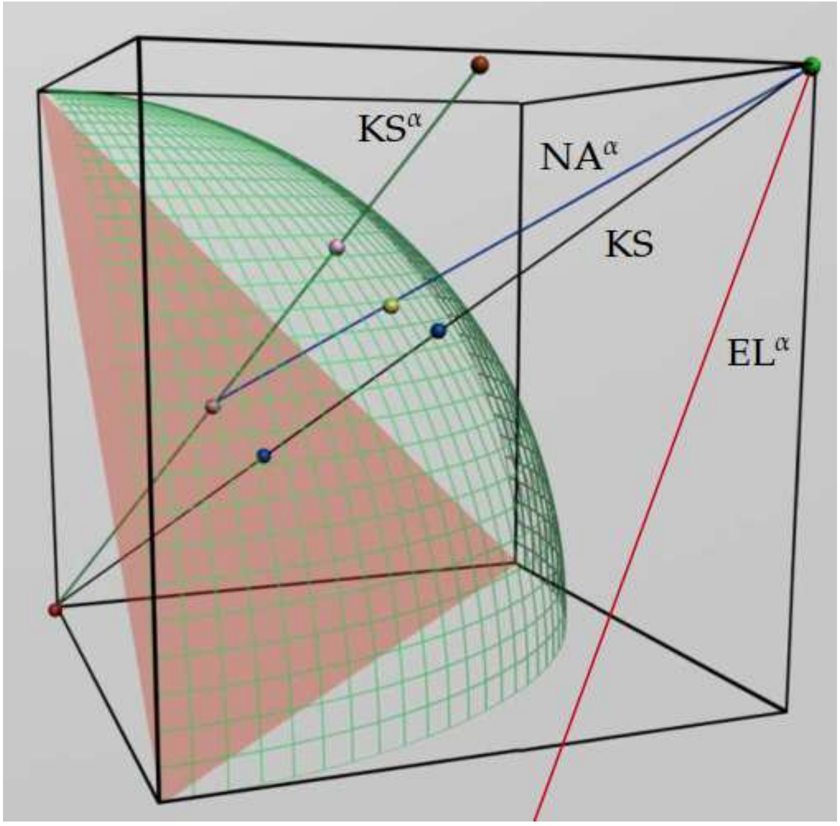

Figure 2: Illustration of the negotiated aspirations solution and other solutions for 3 agents.

To clarify the intuition, Figure 2 illustrates the negotiated aspirations solution for the case of $n = 3$ agents, compared to the Kalai-Smorodinsky solution and other prominent solutions. For example, the equal losses solution (Chun, 1988) prescribes to reduce aspirations proportionally to bargaining power starting from the utopia point, but can fail individual rationality for $n \geq 3$ (Chun and Peters, 1991; Herrero and Marco, 1993). In contrast, NA^{α}, although also based on reducing aspirations from utopia, always satisfies individual rationality. Figure 2 illustrates this point. The triangular surface is simply the simplex of a bargaining set S, i.e. the surface connecting the points $\hat{z}^i$ with $\hat{z}^i_i = m_i(S)$, $\hat{z}^i_j = 0$ for all $j \neq i$. The point $NA^{\alpha}(S)$ is simply the intersection of the surface $WPO(S)$ and the line from the utopia point to an *endogenous reference point* which lies in the simplex, hence in S (and typically in its interior) . Hence individual rationality is guaranteed.

A conceptual relation to the weighted Kalai-Smorodinsky solution (KS^{α}) introduced by Thomson (1994) can also be readily observed in Figure 2. This solution can be viewed as prescribing an allocation of gains proportional to a given vector of bargaining weights starting at the disagreement point. Hence, the constructions of KS^{α} and NA^{α} share the same reference point in the simplex, but, while KS^{α} anchors on disagreement, the negotiated aspirations solution anchors on utopia. These two solutions can be considered dual in the sense that they allocate loses or gains (respectively) proportionally to the maximum aspirations of agents (adjusted by bargaining power).[7]

A further, interesting conceptual relation is worth mentioning. Thomson (1981) introduced the family of reference-function solutions which includes those resulting from the intersection of the convex and comprehensive hull (the "upper" boundary of the bargaining set) with all possible lines starting at the disagreement point. One could define the family of *dual-reference-function solutions* by considering the intersections of all possible lines starting at the utopia point with the convex and comprehensive hull. This family includes the negotiated aspirations solution(s) and the Kalai-Smorodinsky solution.

3 Axioms

We now briefly present the properties which will be part of our characterizations. The first is Pareto efficiency. It states that all gains from cooperation should be exhausted, in the sense that the solution should not prescribe a Pareto-dominated alternative. The *strong Pareto frontier* of $S \in \Sigma_0$ is the set $\text{SPO}(S) = \{x \in S \mid \text{if } y > x \text{ then } y \notin S\}$; that is, the set of alternatives which are not Pareto-dominated by other alternatives in S.

Strong Pareto optimality: For every $S \in \Sigma_0$, then $f(S) \in SPO(S)$.

This property is one of the axioms introduced by Nash (1950) as one of four axioms characterizing the Nash Bargaining Solution. The adjective "strong" is added to distinguish it from weakenings as the following. Let the *weak Pareto frontier* be the set $\text{WPO}(S) = \{x \in S \mid \text{if } y_i > x_i \ \forall i = 1, \ldots, n \text{ then } y \notin S\}$.

Weak Pareto optimality: For every $S \in \Sigma_0$, then $f(S) \in WPO(S)$.

The second property is *scale invariance*, also introduced by Nash (1950). Intuitively, this property means that the scale of units in which each agent is measuring its utility does not matter.

Scale invariance: For every $S \in \Sigma_0$ and $p \in \mathbb{R}^n_{++}$,

$$f(pS) = (p_1 f_1(S), \ldots, p_n f_n(S))$$

where $pS = \{(p_1 s_1, \ldots, p_n s_n) \mid s \in S\}$.

If a solution satisfies scale invariance, by non-degeneracy every bargaining set S can be normalized by transforming it into a new bargaining set with utopia point $m(S) = (1, \ldots, 1)$. That is, the solution is completely determined by its restriction to the set of normalized bargaining problems in Σ_0.

The third property concerns monotonicity. The simplest and strongest monotonicity property simply states that, if the feasible set is expanded, no agent should lose utility. However, it is well-known that, even for 2-agent problems, there is no solution satisfying

[7] This is conceptually similar to the notion of duality introduced by Thomson (2015a) for bankruptcy problems, namely that the dual of a rule should allocate losses the same way the original rule allocates gains.

that version of monotonicity and strong Pareto optimality. Consequently, Rosenthal (1976) introduced the *restricted monotonicity axiom*, which states that whenever the expansion in the bargaining set does not alter the utopia point, no agent should lose utility.[8]

Restricted monotonicity: For each pair $S, T \in \Sigma_0$, if $S \subseteq T$ and $m(S) = m(T)$ then $f(S) \leq f(T)$.

This property, used e.g. by Peters and Tijs (1985), is compatible with strong Pareto optimality for $n = 2$ (but not for $n \geq 3$; see García-Segarra and Ginés-Vilar, 2015).

The last property we require is the axiom of *concavity* (Myerson, 1981), which is related to the interpretation of bargaining problems under conditions of probabilistic uncertainty. When this property is satisfied, agents are willing to reach an agreement before the uncertainty is resolved because each and every agent benefits from this early agreement (see Thomson, 1994, 2010).[9]

Concavity: For each $S, T \in \Sigma_0$ and each $\lambda \in [0, 1]$, $f(\lambda S + (1 - \lambda)T) \geq \lambda f(S) + (1 - \lambda)f(T)$.

We will actually rely on a weaker concavity property, which was first introduced in García-Segarra (2012). Suppose again that bargaining takes place now but the feasible set will be known only later. However, agents additionally know that the utopia points of all possible sets are identical, and hence there is no uncertainty in the reference point represented by utopian anchors. The following property requires that, under these conditions, agents prefer an early compromise to waiting for the resolution of uncertainty.

Restricted concavity: For each $S, T \in \Sigma_0$ and each $\lambda \in [0, 1]$, if $m(S) = m(T)$ then $f(\lambda S + (1 - \lambda)T) \geq \lambda f(S) + (1 - \lambda)f(T)$.

The following result enumerates the properties satisfied by our solution.

Proposition 1. *Let $\alpha \in \Delta^{n-1}$. The negotiated aspirations solution with weights α satisfies weak Pareto optimality, scale invariance, restricted monotonicity, and restricted concavity.*

Proof. Weak Pareto optimality, scale invariance, and restricted monotonicity are obviously fulfilled. To see restricted concavity, let $S, T \in \Sigma_0$ be bargaining sets with $m(S) = m(T)$, and let $\lambda \in [0, 1]$. For convenience, denote

$$g(\mu) = (1 - \mu)m^{\alpha}(S) + \mu m(S)$$

for each $\mu \in [0, 1]$. Note that $m(\lambda S + (1 - \lambda)T) = m(S)$. By construction, there exist $r, s, t \in [0, 1]$ such that $NA^{\alpha}(\lambda S + (1 - \lambda)T) = g(r)$, $NA^{\alpha}(S) = g(s)$, and $NA^{\alpha}(T) = g(t)$. Note that $\lambda g(s) + (1 - \lambda)g(t) = g(\lambda s + (1 - \lambda)t)$. Since $g(s) \in S$ and $g(t) \in T$, we have that $g(\lambda s + (1 - \lambda)t) \in \lambda S + (1 - \lambda)T$ and, by definition of $NA^{\alpha}(\lambda S + (1 - \lambda)T)$, it follows that $NA^{\alpha}(\lambda S + (1 - \lambda)T) = g(r) \geq g(\lambda s + (1 - \lambda)t) = \lambda g(s) + (1 - \lambda)g(t) = \lambda NA^{\alpha}(S) + (1 - \lambda)NA^{\alpha}(T)$. This completes the proof of restricted concavity. □

[8] This axiom is also well-known in the domain of claims problems (see, e.g., Thomson, 2003, 2015a,b; Harless, 2017).

[9] Alós-Ferrer et al (2017) show that, under strong Pareto optimality, concavity is equivalent to the axiom of super-additivity (Perles and Maschler, 1981; Peters, 1985).

4 The Two-Agents Case

In bargaining theory, the two-agent case is of fundamental importance. Many seminal contributions have focused exclusively on this case, ranging from Nash (1950) to Peters and Tijs (1985) and Dubra (2001). Often, extensions of results to the general case remain elusive, and the $n = 2$ case remains a useful playground to test the compatibility of desirable properties. We hence devote our attention to the two-agent setting before tackling the general n-agent case.

Surprisingly, even though the solution we propose is new to the literature, for two agents we obtain a characterization which relies exclusively on already-known, standard axioms (or natural restrictions thereof). This observation is of independent interest, since ex post our result provides an implementation for a combination of known, normatively desirable properties.

For the case of two agents, we can give a simpler definition of the negotiated aspirations solution as follows. First, abusing notation, write the vector of bargaining weights as $(\alpha, 1-\alpha)$ with $\alpha \in [0,1]$. Then, given $S \in \Sigma_0$ with $m_1(S) > 0$ and $m_2(S) > 0$, $NA^\alpha(S)$ is the largest (with respect to any coordinate with strictly positive bargaining weight) point in S fulfilling the equation

$$\frac{\alpha(m_1(S) - x_1)}{m_1(S)} = \frac{(1-\alpha)(m_2(S) - x_2)}{m_2(S)},$$

which describes the line passing through $m(S)$ and $m^\alpha(S)$.

Before we state our characterization, we need a preliminary result. Peters and Tijs (1985) introduce a family of bargaining solutions as follows. A *monotonic curve* is a function $\gamma : [1,2] \mapsto \nabla := \text{conv}\{(1,0),(0,1),(1,1)\}$ (where conv denotes the convex hull of a set) fulfilling the following property.

(C) For all $s, t \in [1,2]$ with $s \leq t$, $\gamma(s) \leq \gamma(t)$ and $\gamma_1(s) + \gamma_2(s) = s$.

Given a monotonic curve γ, the associated bargaining solution f^γ is defined as follows. For every two-agent problem $S \in \Sigma_0$ with $m(S) = (1,1)$ (i.e., a normalized bargaining problem), $f^\gamma(S)$ is the unique point of $SPO(S)$ which lies on $\{\gamma(t) \mid t \in [1,2]\}$. Extend f^γ to all $S \in \Sigma_0$ by assuming scale invariance.

Given any $\alpha \in [0,1]$, define the function $\gamma^\alpha : [1,2] \mapsto \nabla$ by $\gamma^\alpha(s) = (2-s)(\alpha, 1-\alpha) + (s-1)(1,1)$ for all $s \in [1,2]$, that is, an affine function tracing a line from the point $(\alpha, 1-\alpha)$ to $(1,1)$. Note that γ^α is a monotonic curve and $f^{\gamma^\alpha} = NA^\alpha$. That is, the negotiated aspirations solution belongs to the family of solutions f^γ studied by Peters and Tijs (1985). These authors show that the set of bargaining solutions f^γ is characterized by the combination of strong Pareto optimality, scale invariance, and restricted monotonicity (Peters and Tijs, 1985, Proposition 1 and Theorems 2 and 3). The proof of our characterization below relies on this result.

Theorem 1. *Consider bargaining problems with two agents. The negotiated aspirations solution is the only solution satisfying strong Pareto optimality, scale invariance, restricted monotonicity, and restricted concavity.*

Proof. By Peters and Tijs (1985, Proposition 1 and Theorem 2), $NA^\alpha = f^{\gamma^\alpha}$ fulfills strong Pareto optimality. The other properties hold by Proposition 1. Conversely, suppose f satisfies all listed properties. By Peters and Tijs (1985, Proposition 1 and Theorem 3), there exists a monotonic curve γ such that $f(S) = f^\gamma(S)$ for each bargaining set S. Let $\alpha = \gamma_1(1) \in [0,1]$ and let

$$V_t = \left\{x \in \mathbb{R}^2 \mid x_1 + x_2 \leq t\right\} \cap [0,1]^2$$

for each $t \in [1,2]$. Notice that $m(V_t) = (1,1)$ for all $t \in [1,2]$. By property (C) and Lemma 1(b,d) in the Appendix, $f^\gamma(V_t) = \gamma(t)$ and $V_t = (2-t)V_1 + (t-1)V_2$. Let $y = (2-t)\gamma(1) + (t-1)\gamma(2)$. Applying restricted concavity shows that $\gamma(t) = f^\gamma(V_t) \geq y$. Since $y_1 + y_2 = t = \gamma_1(t) + \gamma_2(t)$ by property (C) and $\gamma(t) \geq y$, we conclude that $\gamma(t) = y$. Since $\gamma(2) = (1,1)$, $\gamma(t) = (2-t)(\alpha, 1-\alpha) + (t-1)(1,1) = \gamma^\alpha(t)$ for all $t \in [1,2]$. □

The axioms used in Theorem 1 are logically independent. The weighted Kalai-Smorodinsky solution KS^α (Thomson, 1994) fulfills all axioms except strong Pareto optimality. The equal losses solution EL^α for $n = 2$ (Chun, 1988) fulfills all axioms except scale invariance. The asymmetric Kalai-Smorodinsky solution lKS^α (Dubra, 2001) fulfills all axioms except restricted concavity.[10] It can be shown that Example 3 below for $n = 2$ satisfies all axioms except restricted monotonicity.

5 The General Case

We now turn our attention to the general case of n-agent bargaining problems. For this case, it is well-known that strong Pareto optimality is too strong. For instance, it is incompatible with monotonicity (even for $n = 2$; Rosenthal, 1976) and with restricted monotonicity (for $n \geq 3$; García-Segarra and Ginés-Vilar, 2015). For the case of n agents, the negotiated aspirations solution has a characterization where strong Pareto optimality is replaced by its weaker version. The price to pay is adding two additional axioms. The first is a weak, straightforward version of the contraction independence axiom introduced by Nash (1950) (see also Chun, 2005).

Restricted contraction independence: For every $S, T \in \Sigma_0$ with $m(S) = m(T)$, if $S \subseteq T$ and $f(T) \in S$, then $f(S) = f(T)$.

This property (see Roth, 1977) is compatible with restricted monotonicity. The second axiom is exceedingly weak. It merely states the minimal requirement that, in the absence of any bargaining conflict, every agent is granted his or her wishes. That is, in the extreme situation where the utopia point is feasible, the solution must prescribe it.

Utopia fulfillment: For every $S \in \Sigma_0$, if $m(S) \in S$, then $f(S) = m(S)$.

The following result presents the characterization for the general case. Unlike Theorem 1, the proof does not follow from Peters and Tijs (1985), since the result in that work is restricted to 2-agent problems.

[10] The solution of Dubra (2001), which is only defined for $n = 2$, prescribes the (unique) Pareto optimal point which dominates $KS^\alpha(S)$, and hence it can be considered a lexicographic extension of the solution by Thomson (1994) in the case of two agents.

Theorem 2. *Consider n-agent bargaining problems. A bargaining solution f satisfies weak Pareto optimality, scale invariance, restricted monotonicity, restricted concavity, restricted contraction independence, and utopia fulfillment if and only if there exists $\alpha \in \Delta^{n-1}$ such that f is the negotiated aspirations solution with weights α.*

Proof. The negotiated aspirations solution clearly satisfies utopia fulfillment and restricted contraction independence. The remaining properties hold by Proposition 1.

Let f be a bargaining solution fulfilling all properties in the statement. As in the proof of Theorem 1, define the instrumental sets

$$V_t = \left\{ x \in \mathbb{R}^n_+ \;\middle|\; \sum_{i=1}^{n} x_i \leq t \right\} \cap [0,1]^n \tag{1}$$

for each $t \in [1, n]$. Let $\alpha = f(V_1)$. By weak Pareto optimality and Lemma 1(c) in the Appendix, $\alpha \in \Delta^{n-1}$. We want to show that $f(S) = NA^\alpha(S)$ for all $S \in \Sigma_0$. We first establish this for the sets V_t.

Claim. $f(V_t) = NA^\alpha(V_t)$ for all $t \in [1, n]$.

By utopia fulfillment, $f(V_n) = (1, \ldots, 1)$. By Lemma 2(a) in the Appendix, $f(V_t) = \left(\frac{n-t}{n-1}\right)\alpha + \left(1 - \frac{n-t}{n-1}\right)(1, \ldots, 1) = \left(\frac{n-t}{n-1}\right) m^\alpha(V_t) + \left(1 - \frac{n-t}{n-1}\right) m(V_t)$. By Lemma 2(b), $f(V_t) \in SPO(V_t)$. The latter implies that $f(V_t) = NA^\alpha(V_t)$ by definition of NA^α and Lemma 1(b).

Let $S \in \Sigma_0$. By scale invariance, without loss of generality we can assume that $m(S) = (1, \ldots, 1)$. By $\mathbf{0}$-comprehensiveness it follows that all unit vectors are in S and hence, by convexity of S and weak Pareto optimality of f, $t = \sum_{i=1}^n f_i(S) \geq 1$. Hence $f(S) \in SPO(V_t)$ (Lemma 1(b)).

Note that $m(S \cap V_t) = (1, \ldots, 1) = m(S)$. By restricted monotonicity, $f(S \cap V_t) \leq f(S)$. Since $f(S) \in S \cap SPO(V_t) \subseteq S \cap V_t$, restricted contraction independence implies that $f(S) = f(S \cap V_t)$. Again by restricted monotonicity, $f(S) = f(S \cap V_t) \leq f(V_t)$ and since $f(S) \in SPO(V_t)$, it follows that $f(S) = f(V_t) = NA^\alpha(V_t)$, where the last equality follows from the claim above. In particular, $NA^\alpha(V_t) \in S$ and by construction of NA^α it follows that $NA^\alpha(S) \geq NA^\alpha(V_t) = f(S)$.

Applying an analogous argument to the bargaining solution NA^α yields $NA^\alpha(S) = NA^\alpha(V_k) = f(V_k)$ where $k = \sum_{i=1}^n NA^\alpha_i(S)$. Hence $f(V_k) \in S \cap V_k$ and restricted contraction independence yields $f(S \cap V_k) = f(V_k)$. Applying restricted monotonicity, $NA^\alpha(S) = f(V_k) = f(S \cap V_k) \leq f(S)$. Together with $NA^\alpha(S) \geq f(S)$, this yields $f(S) = NA^\alpha(S)$. □

We now show that the axioms used in Theorem 2 are logically independent. First, the weighted Kalai-Smorodinsky solution KS^α (Thomson, 1994) fails utopia fulfillment but satisfies all other axioms. Second, following an idea by Peters and Tijs (1984, 1985), consider monotone paths connecting the disagreement point to the utopia point. For each such path, define a bargaining solution which picks the maximum point of the path belonging to the bargaining set. Such a solution fails restricted concavity, but fulfills all other axioms. For each of the remaining axioms, one of the following examples exhibits a bargaining solution failing it but fulfilling the rest. For the sake of concreteness, in all examples we omit the (straightforward) axiom checks.

Example 1. (Violation of weak Pareto optimality) For each $S \in \Sigma_0$, define $G(S) = \mathbf{0}$ if $\sum_{i=1}^{n} \frac{s_i}{m_i(S)} \leq 1$ for all $s \in S$, and $G(S) = KS(S)$ otherwise. This solution agrees with the Kalai-Smorodinsky solution except for the sets which coincide with the convex hull of $(0, \ldots, 0)$ and the points $(0, \ldots, 0, m_i(S), 0, \ldots)$, $i = 1, \ldots, n$. For those sets, $G(S) = \mathbf{0}$, in violation of weak Pareto optimality by non-degeneracy.

Example 2. (Violation of scale invariance) Fix $\alpha \in \Delta^{n-1}$ with $\alpha \neq \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$, hence $NA^{\alpha}(S) \neq KS(S)$. For each $S \in \Sigma_0$, define $H(S) = NA^{\alpha}(S)$ if $m(S) = \mathbf{1}$ and $H(S) = KS(S)$ otherwise. Let $S \in \Sigma_0$ be such that $m(S) \neq \mathbf{1}$. Taking $p = \left(\frac{1}{m_1(S)}, \ldots, \frac{1}{m_n(S)}\right)$, we obtain that $m(pS) = \mathbf{1}$. Hence, $H(pS) = NA^{\alpha}(pS) \neq pKS(S) = pH(S)$ and H fails scale invariance.

Example 3. (Violation of restricted monotonicity) Given $S \in \Sigma_0$, define

$$U(S) = \left\{ x \in S \;\middle|\; \sum_{i=1}^{n} \frac{x_i}{m_i(S)} \geq \sum_{i=1}^{n} \frac{y_i}{m_i(S)} \text{ for all } y \in S \right\}.$$

We adopt a tie-breaking rule to obtain a single-valued solution. Note that $U(S)$ is a non-empty, compact, and convex set. Denote by $\succ_L$ the (strict) lexicographic order on $U(S)$. That is, given $x, y \in U(S)$, we say that $x \succ_L y$ if and only if either $x_1 > y_1$, or $x_1 = y_1$ and $x_2 > y_2$, or $x_1 = y_1$, $x_2 = y_2$, and $x_3 > y_3$, and so on. We say that $x \succeq_L y$ if and only if either $x = y$ or $x \succ_L y$. Define

$$f_L(S) = \{x \in U(S) \mid x \succeq_L y \text{ for all } y \in U(S)\}.$$

To see that $f_L(S)$ exists and is single-valued, let $U^0(S) = U(S)$ and $U^k(S) = \left\{x \in U^{k-1}(S) \mid x_k \geq y_k \text{ for all } y \in U^{k-1}(S)\right\}$, for $k = 1, \ldots, n$. The sets $U^k(S)$ are nonempty, convex, and compact with $U^1(S) \supseteq U^2(S) \supseteq \ldots \supseteq U^n(S)$. Suppose $x, y \in U^n(S)$. By construction, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$. Hence, $f_L(S) = U^n(s)$ is a singleton. This solution selects one of the outcomes recommended by a normalized version of the utilitarian solution. To see that this solution fails restricted monotonicity, let $S, T \in \Sigma_0$ be given by

$$S = \left\{ x \in [0,1]^n \;\middle|\; \sum_{i=1}^{n} x_i \leq 1 \right\} \quad \text{and} \quad T = \left\{ x \in [0,1]^n \;\middle|\; \sum_{i=1}^{n} x_i^2 \leq 1 \right\}.$$

We have that $S \subseteq T$, $m(S) = m(T) = (1, \ldots, 1)$, $f_L(S) = (1, 0, \ldots, 0)$, and $f_L(T) = \left(\frac{\sqrt{n}}{n}, \ldots, \frac{\sqrt{n}}{n}\right)$, in contradiction with restricted monotonicity.

Example 4. (Violation of restricted contraction independence) Consider the sets V_t given by (1). For each $t \in [1, n]$, define $L(V_t) = NA^{\alpha^1}(V_t)$ with $\alpha^1 = (1, 0, \ldots, 0) \in \Delta^{n-1}$. For any $S \in \Sigma_0$ with $m(S) = \mathbf{1}$, let $L(S) = L(V_t)$ where $t = \max\{t' \in [1, n] \mid V_{t'} \subseteq S\}$. Since $V_1 \subseteq S$ and S is compact, $L(S)$ is well defined. Finally, whenever $m(S) \neq \mathbf{1}$, let

$p = \left(\frac{1}{m_1(S)}, \ldots, \frac{1}{m_n(S)}\right)$ and define $L_i(S) = m_i(S)L_i(pS)$ for each $i = 1, \ldots, n$ (note that $m(pS) = \mathbf{1}$). By Lemma 1(d) in the Appendix, the convex combination of two sets of the form V_t is also of the same form. Therefore L satisfies restricted concavity.

Note that $L(V_2) = NA^\alpha(V_2) = \left(1, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right)$, hence $L(V_2) \neq L(V_1) = (1, 0, \ldots, 0)$. Define a set $S^1 \subseteq V_2$ by

$$S^1 = \left\{ x \in [0,1]^n \;\middle|\; x_1 \leq 1, \sum_{i=2}^{n} x_i \leq 1 \right\}.$$

For each $t > 1$ we have that $(0, t, 0, \ldots, 0) \in V_t$ and $(0, t, 0, \ldots, 0) \notin S^1$. Therefore, $L(S^1) = L(V_1)$, in contradiction with restricted contraction independence since $S^1 \subseteq V_2$, $L(V_2) \in S^1$, and $L(S^1) \neq L(V_2)$.

6 Conclusion

Bargaining solutions have either anchored on the worst-case scenario disagreement point or the highly desirable but typically unattainable utopia point. The solution of Kalai and Smorodinsky (1975) does both, but since a line is fully defined by any two different points it crosses, any generalization taking bargaining power into account must neglect one. The proportional solutions of Kalai (1977) anchor on disagreement, and can be seen as the allocation of potential gains from the worst-case scenario, proportionally to bargaining power. The weighted Kalai-Smorodinsky solutions of Thomson (1994) also anchor on disagreement, but potential gains are allocated proportionally taking into account both bargaining power and the utopian claims. The equal-losses solution (Chun, 1988) anchors on utopia and allocates necessary losses (proportionally to bargaining power), but runs into feasibility problems (Chun and Peters, 1991). Here we propose and characterize a solution which anchors on utopia, adjusts the utopian aspirations according to bargaining power to create a new, feasible reference point (hence "negotiated aspirations solution"), and can then be seen either as allocating gains towards utopia from this new reference point, or allocating losses from utopia towards the reference point.

The negotiated aspirations solutions has a natural procedural description in terms of intuitive concepts: bargaining power and maximal aspirations. Further, its axiomatic characterization requires no conceptual departures from the well-established axioms previously considered in the literature, which are universally accepted as identifying attractive characteristics of solutions. For the case of two agents, the negotiated aspirations solution is the only solution fulfilling strong Pareto optimality, scale invariance, the axiom of monotonicity restricted to comparisons among sets with the same utopia point (restricted monotonicity), and the axiom of concavity restricted in the same sense. For n agents, the characterization replaces strong with weak Pareto optimality and adds restricted contraction independence plus the minimal requirement that if utopia is actually attainable (hence there is no conflict), it should be chosen.

Appendix G Properties of the sets V_t

For each $t \in [1, n]$ let V_t be given by (1). The following lemma summarizes the geometric properties of these sets as required in the proofs in the main text.

Lemma 1. *For every* $t \in [1, n]$,

(a) $m(V_t) = (1, \ldots, 1)$;

(b) $SPO(V_t) = \{x \in V_t \mid \sum_{i=1}^n x_i = t\}$;

(c) $WPO(V_t) = SPO(V_t) \cup \{x \in V_t \mid x_j = 1$ *for some* $j \in \{1, \ldots, n\}\}$; *and*

(d) $V_t = \left(\frac{n-t}{n-1}\right) V_1 + \left(1 - \frac{n-t}{n-1}\right) V_n$.

Proof. (a), (b), and (c) are straightforward. To see (d), we proceed by double inclusion. Let $x \in V_1$, $y \in V_n$, and $z = \left(\frac{n-t}{n-1}\right) x + \left(1 - \frac{n-t}{n-1}\right) y$. Then $z \in [0,1]^n$ and $\sum_{i=1}^n z_i = \left(\frac{n-t}{n-1}\right) \sum_{i=1}^n x_i + \left(1 - \frac{n-t}{n-1}\right) \sum_{i=1}^n y_i \leq \left(\frac{n-t}{n-1}\right)(1) + \left(1 - \frac{n-t}{n-1}\right)(n) = t$. Hence, $y \in V_t$.

Let $z \in V_t$. Note that $y = (1, \ldots, 1) \in V_n$ and define x by $x_i = \frac{(n-1)z_i - t + 1}{n-t}$. Then, $\sum_{i=1}^n x_i = \frac{1}{n-t}\left((n-1)\sum_{i=1}^n z_i + n(1-t)\right) \leq \frac{1}{n-t}\left((n-1)t + n(1-t)\right) = 1$ and $x \in V_1$. Note that $\left(\frac{n-t}{n-1}\right) x_i + \left(1 - \frac{n-t}{n-1}\right) y_i = \frac{n-t}{n-1}\left(\frac{(n-1)z_i - t + 1}{n-t}\right) + \left(1 - \frac{n-t}{n-1}\right) = \frac{1}{n-1}((n-1)z_i) = z_i$. Hence, $z \in \left(\frac{n-t}{n-1}\right) V_1 + \left(1 - \frac{n-t}{n-1}\right) V_n$. □

The next result identifies two useful properties of bargaining solutions satisfying restricted concavity, when applied to the sets V_t.

Lemma 2. *Let f be a bargaining solution for n-agent problems satisfying weak Pareto optimality, restricted concavity, and utopia fulfillment. Then, for every* $t \in [1, n]$,

(a) $f(V_t) = \left(\frac{n-t}{n-1}\right) f(V_1) + \left(1 - \frac{n-t}{n-1}\right) f(V_n)$, *and*

(b) $f(V_t) \in SPO(V_t)$.

Proof. (a) By Lemma 1(a), $m(V_1) = m(V_n)$. Let $z = \left(\frac{n-t}{n-1}\right) f(V_1) + \left(1 - \frac{n-t}{n-1}\right) f(V_n)$. By weak Pareto optimality, $f(V_1) \in WPO(V_1)$ and by Lemma 1(b,c), $SPO(V1) = WPO(V1)$. Hence, by Lemma 1(b), $\sum_{i=1}^n f_i(V_1) = 1$. By utopia fulfillment, $f(V_n) = (1, \ldots, 1)$. We obtain that $\sum_{i=1}^n z_i = t$. By restricted concavity, $f(V_t) \geq z$. Since $\sum_{i=1}^n f_i(V_t) \leq t$, it follows that $f(V_t) = z$.

(b) Follows from part (a) and Lemma 1(b). □

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