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**ON A GENERAL SOLUTION OF FINITE ORDER
DIFFERENCE EQUATION WITH CONSTANT COEFFICIENTS**

MAREK PYCIA

ABSTRACT. In the present paper we give new formulas for a general solution of the linear difference equation of finite order with constant complex coefficients without necessity of solving the characteristic equation

Introduction. In this paper we deal with the following difference equation of order m :

$$(1) \quad x_{n+m} = \sum_{r=1}^m a_r x_{n+m-r}$$

with constant complex coefficients a_1, \dots, a_m . Our Theorem gives a simple formula for the general solution depending only on the coefficients a_1, \dots, a_m . We do not have to solve the characteristic equation as it is usually done (cf for instance [1], [2]) and, in general, it is often impossible to find the exact solutions of it.

To formulate our Theorem we adopt the following convention:

$$(-1)! \cdot 0 = 1 .$$

Theorem. Let x_0, \dots, x_{m-1} be arbitrary complex numbers, let h_1, \dots, h_m be nonnegative integers. The general solution of equation (1) is of the form (2):

$$x_n = \sum_{l=0}^{m-1} \left[\sum_{1h_1 + \dots + mh_m = n-l} \frac{(h_1 + \dots + h_m - 1)(h_{m-l} + \dots + h_m)}{h_1! \cdot \dots \cdot h_m!} \prod_{i=1}^m a_i^{h_i} \right] x_l$$

for $n = 0, 1, \dots$

Proof of the Theorem. The proof is by induction with respect to n .

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In the first part we show that formula (2) holds for $n = 0, \dots, m-1$. Let us take $l \in \{0, \dots, m-1\}$ and consider three cases depending on l is smaller, greater than n or equal to n .

In the case $l < n$ we have $1h_1 + \dots + mh_m = n-l > 0$, and therefore $h_1 + \dots + h_m - 1 \geq 0$. Let us note that $h_{m-l} + \dots + h_m = 0$. In fact, if $h_{m-l} + \dots + h_m > 0$ then would exist $i \in \{m-l, \dots, m\}$ such that $h_i > 0$. Consequently we would have $1h_1 + \dots + mh_m \geq ih_i \geq (m-l) > n-l$; which is a contradiction. Therefore we have:

$$\frac{(h_1 + \dots + h_m - 1)(h_{m-l} + \dots + h_m)}{h_1! \dots h_m!} = 0 .$$

In the case $l > n$ we have $1h_1 + \dots + mh_m = n-l < 0$. Since the set of all such sequences (h_1, \dots, h_m) is empty, the sum over this set of indices is 0.

For $l = n$ we have $1h_1 + \dots + mh_m = n-l = 0$. Consequently $h_1 = \dots = h_m = 0$ and, applying our convention, we get:

$$\frac{(h_1 + \dots + h_m - 1)(h_{m-l} + \dots + h_m)}{h_1! \dots h_m!} = \frac{(-1)!0}{1 \dots 1} = 1 .$$

Summing up this three cases we can observe that formula (2) holds for $n = 0, \dots, m-1$.

Now, for an inductive step, we assume that Theorem is true for m consecutive indices $n, \dots, n+m-1$.

Substituting the right hand side of formula (2) into equality (1) (we change simultaneously n for $n+m-r$ in formula (2)) and changing the order of summation we get:

$$\begin{aligned} (3) \quad x_{n+m} &= \sum_{r=1}^m a_r \sum_{l=0}^{m-1} \left[\sum_{1h_1 + \dots + mh_m = n-l} \right. \\ &\quad \left. \frac{(h_1 + \dots + h_m - 1)(h_{m-l} + \dots + h_m)}{h_1! \dots h_m!} \prod_{i=1}^m a_i^{h_i} \right] x_l = \\ &= \sum_{l=0}^{m-1} \left[\sum_{r=1}^m a_r \sum_{1h_1 + \dots + mh_m = n+m-r-l} \right. \\ &\quad \left. \frac{(h_1 + \dots + h_m - 1)(h_{m-l} + \dots + h_m)}{h_1! \dots h_m!} \prod_{i=1}^m a_i^{h_i} \right] x_l . \end{aligned}$$

Let us fix $l \in \{0, \dots, m-1\}$, and consider the coefficient standing before x_l . Performing the indicated operations we obtain that this coefficient is equal to:

$$(4) \quad \sum_{1g_1 + \dots + mg_m = n+m-l} c_{g_1, \dots, g_m} \prod_{i=1}^m a_i^{g_i} .$$

where every c_{g_1, \dots, g_m} is uniquely defined coefficient. We will determine the value of it depending on g_1, \dots, g_m . Let us fix g_1, \dots, g_m .

Since we get $\prod_{i=1}^m a_i^{g_i}$ as a product of admissible a_r (i.e., such that $g_r > 0$) and suitable uniquely defined $\prod_{i=1}^m a_i^{h_i}$:

$$(5) \quad g_i = \begin{cases} h_i & \text{for } i = 1, \dots, r-1, r+1, \dots, m, \\ h_i + 1 & \text{for } i = r. \end{cases}$$

Therefore to obtain the coefficient c_{g_1, \dots, g_m} it is enough to add all the coefficients of the form:

$$\frac{(h_1 + \dots + h_m - 1)(h_{m-l} + \dots + h_m)}{h_1! \cdot \dots \cdot h_m!}$$

standing before admissible $\prod_{i=1}^m a_i^{h_i}$.

Let us put:

$$P(i, j) := \{r : g_r > 0\} \cap \{i, \dots, j\} .$$

We have:

$$\begin{aligned} c_{g_1, \dots, g_m} &= \sum_{P(1, m)} \frac{(h_1 + \dots + h_m - 1)(h_{m-l} + \dots + h_m)}{h_1! \cdot \dots \cdot h_m!} = \\ &= \sum_{P(1, m-l-1)} \frac{(h_1 + \dots + h_m - 1)(h_{m-l} + \dots + h_m)}{h_1! \cdot \dots \cdot h_m!} + \\ &+ \sum_{P(m-l, m)} \frac{(h_1 + \dots + h_m - 1)(h_{m-l} + \dots + h_m)}{h_1! \cdot \dots \cdot h_m!} . \end{aligned}$$

Because if $r \in \{i, \dots, j\} - P(i, j)$ then $g_r = 0$, it follows that $\sum_{P(i, j)} g_r = \sum_{r=i}^j g_r$. Applying formula (5), hence we get:

$$\begin{aligned} c_{g_1, \dots, g_m} &= \sum_{P(1, m-l-1)} \frac{(g_1 + \dots + (g_r - 1) + \dots + g_m - 1)(g_{m-l} + \dots + g_m)}{g_1! \cdot \dots \cdot (g_r - 1)! \cdot \dots \cdot g_m!} + \\ + \sum_{P(m-l, m)} \frac{(g_1 + \dots + (g_r - 1) + \dots + g_m - 1)(g_{m-l} + \dots + (g_r - 1) + \dots + g_m)}{g_1! \cdot \dots \cdot (g_r - 1)! \cdot \dots \cdot g_m!} = \\ &= \sum_{P(1, m-l-1)} g_r \frac{(g_1 + \dots + g_m - 1)(g_{m-l} + \dots + g_m)}{(g_1 + \dots + g_m - 1) \cdot g_1! \cdot \dots \cdot g_m!} + \\ &+ \sum_{P(m-l, m)} g_r \frac{(g_1 + \dots + g_m - 1)(g_{m-l} + \dots + g_m - 1)}{(g_1 + \dots + g_m - 1) \cdot g_1! \cdot \dots \cdot g_m!} = \\ &= \left[\sum_{r=1}^{m-l-1} \frac{g_r}{(g_1 + \dots + g_m - 1)} + \sum_{r=m-l}^m \frac{g_r(g_{m-l} + \dots + g_m - 1)}{(g_1 + \dots + g_m - 1)(g_{m-l} + \dots + g_m)} \right] \cdot \\ &\quad \cdot \left[\frac{(g_1 + \dots + g_m - 1)(g_{m-l} + \dots + g_m)}{g_1! \cdot \dots \cdot g_m!} \right] = \end{aligned}$$

$$\begin{aligned}
&= \left[\frac{(g_1 + \dots + g_{m-l-1})}{(g_1 + \dots + g_m - 1)} + \frac{(g_{m-l} + \dots + g_m)(g_{m-l} + \dots + g_m - 1)}{(g_1 + \dots + g_m - 1)(g_{m-l} + \dots + g_m)} \right] \\
&\cdot \left[\frac{(g_1 + \dots + g_m - 1)!(g_{m-l} + \dots + g_m)}{g_1! \dots g_m!} \right] = \\
&= \left[\frac{(g_1 + \dots + g_{m-l-1}) + (g_{m-l} + \dots + g_m - 1)}{(g_1 + \dots + g_m - 1)} \right] \\
&\cdot \left[\frac{(g_1 + \dots + g_m - 1)!(g_{m-l} + \dots + g_m)}{g_1! \dots g_m!} \right] = \\
&= \frac{(g_1 + \dots + g_m - 1)!(g_{m-l} + \dots + g_m)}{g_1! \dots g_m!} .
\end{aligned}$$

Inserting this into (4) and then into (3) we obtain:

$$x_{n+m} = \sum_{l=0}^{m-1} \left[\sum_{1g_1+\dots+mg_m=n-l} \frac{(g_1 + \dots + g_m - 1)!(g_{m-l} + \dots + g_m)}{g_1! \dots g_m!} \prod_{i=1}^m a_i^{g_i} \right] x_l .$$

Now induction concludes the proof. \square

Remark 1. It may be interesting to note here that formula (2) can be written as follows:

$$x_n = \sum_{l=0}^{m-1} \left[\sum_{\{k_1, \dots, k_s \in \{1, \dots, m\} : k_1 + \dots + k_s = n-l, k_s \geq m-l, s \in \mathbb{N}\}} \prod_{i=1}^s a_{k_i} \right] x_l .$$

Remark 2. Theorem can be proved by some combinatorial reasoning.

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