Large vs. Continuum Assignment Economies: Efficiency and Envy-Freeness

Antonio Miralles
Marek Pycia

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Abstract

Continuum models are often used to study large finite assignment economies. However, some subtleties must be taken into account. We show that in the large finite random assignment problem without transfers, Competitive Equilibrium with vanishing income differences does not asymptotically characterize the set of efficient and envy-free random assignment profiles. This is in sharp contrast with the continuum model counterpart (Ashlagi and Shi, 2015). The problem is driven by the failure of local non-satiation inherent in no-transfer assignment.

Keywords: Random Assignments, Efficiency, Envy-freeness, Convergence Failure, Competitive Equilibrium from Equal Incomes.

1 Introduction

Since Aumann's (1964) pioneering work, it has been widely accepted that continuum economies constitute a valid approximation to big economies where no single agent can have an impact on general market conditions.[1] Following the work of Abdulkadiroglu, Che, and Yasuda (2015), Miralles (2008), Che and Kojima (2010), and Azevedo and Leshno (2013), continuum models became the workhorse models for analyzing large finite matching and assignment economies without transfers. Their usefulness goes beyond theory and extends to recent advances in empirical analysis of these markets (Agarwal and Somaini 2014, Calsamiglia, Fu and Güell, 2015).

We show that the use of the continuum models as approximations to large finite assignment economies without transfers requires care. We demonstrate it by examining a natural question—what assignment profiles[2] are efficient and fair?—in large finite economies, and by showing that the answer is substantially different

*Universitat Autonoma de Barcelona - Barcelona Graduate School of Economics (Miralles), and University of California Los Angeles (Pycia). We would like to thank Salvador Barberà, Jordi Massó, Moritz Meyer-ter-Vehn, William Zame, and seminar participants at UCLA and UAB for helpful comments. Antonio Miralles acknowledges financial support from the Ramón y Cajal contract (RYC2011-09665) of the Spanish Government, from the Spanish Plan Nacional I+D+I (ECO2014-53051-P), the Generalitat de Catalunya (SGR2014-505) and the Severo Ochoa program (SEV-2015-0563).

[1] Continuum approximations offer many methodological advantages. For instance, the standard abuse of the law of large numbers allows us to reduce complex stochastic problems to deterministic ones and the existence of pure-strategy Nash equilibria is assured when the strategy space remains finite (Mas-Colell 1984).

[2] An assignment profile gives a lottery over objects to each agent.

from the answer to the same question in the continuum economy limit. We study economies in which each agent evaluates the utility from a random assignment in line with the expected utility theory and we assume that each agent would like to receive at most one object as in Hylland and Zeckhauser (1979). In the continuum assignment economy limit, Ashlagi and Shi (2015) provided an elegant characterization of efficient and fair assignment profiles: in the full-support economies every such assignment profile can be implemented in a competitive equilibrium with equal incomes (CEEI); the CEEI mechanism has been first studied in Hylland and Zeckhauser's seminal work.[3]

We show that this elegant characterization fails to be true, even approximately, in large finite economies. To do so we study a sequence of growing finite assignment economies that converge to a full-support continuum economy. In these economies, we construct a sequence of efficient and fair assignment profiles that cannot be supported by competitive equilibria in which agents' budgets (or incomes) are asymptotically equal.

The key to our counterexample is the failure of local non-satiation that is inherent to Hylland and Zeckhauser's assignment economies. In fact, in economies in which agents are locally non-satiated Zhou (1992) shows that all fair and efficient allocations can be implemented via competitive equilibrium from equal incomes not only in the continuum limit but also in large finite economies.[4]

To understand the role of satiation, we first start at the limit continuum economy. From Ashlagi and Shi (2015) we know that any efficient and envy-free random assignment profile is the outcome of a CEEI. Now suppose that there is some object whose equilibrium price is not zero yet it lies below individual income. Due to a form of satiation inherent in random assignment economies, namely unit demands, all agents who prefer this object over the others would purchase a sure copy of it, hence not spending the whole budget. The gap between income and agents' expense is the basis of the failure of CEEI convergence in finite economies.

Going to finite economies, which are not dense in the preference space, we can almost satiate all individuals who prefer the aforementioned object, so that they do not envy other agents, even if there is a non-vanishing income gap between the latter agents and the former agents. We can do so for any arbitrarily, yet not fully, dense economy. That is why a converging sequence of growing finite economies,

[3]Thomson and Zhou (1993) have a similar characterization for assignment economies beyond unit demands. This result does not extend to our model because they only consider individual allocations belonging to the interior of the consumption space. In addition to Hylland and Zeckhauser, and Ashlagi and Shi, CEEI was studied by, among others, Azevedo and Budish (2013) who proved that CEEI becomes incentive compatible in large economies; Pycia (2011) who has shown that the CEEI assignment can be unboundedly more utilitarian-efficient than the best symmetric ordinal mechanism, and Hafalir and Miralles (2015) who have shown that under some conditions the CEEI assignment is utilitarian- and Rawlsian-optimal among all incentive-compatible assignment rules. Budish (2011) provided a deterministic approximation to CEEI, and Budish, Che, Kojima and Milgrom (2013) extended CEEI to multi-unit assignment problems. We study fairness in the standard sense of envy-free, see Foley (1967) and Kolm (1971).

[4]Zhou assumes that agents' utilities are monotone, quasiconcave and differentiable. His fairness concept is strict envy-freeness, which is equivalent to envy-freeness in our setting. In general, an allocation is strict envy-free if no agent envies the average bundle of a group of other agents. Zhou does not require agents to demand at most one object.

and a corresponding sequence of efficient and envy-free random assignment profiles can be constructed such that none of them are respectively supported by a sequence of competitive equilibria with income differences converging to zero.

This line of reasoning should convince the reader that the counterexample we find is not a knife-edge case. In fact, for any CEEI in a limit continuum economy with affordable, yet not free, sure copies of objects, one could construct such a sequence.

A natural follow-up question is whether there are other ways to restore the intuition, that CEEI approximately characterizes in some sense the set of efficient and envy-free assignment profiles in large finite economies. We provide an easy answer. For any converging sequence of growing economies with their corresponding efficient and envy-free assignment profiles, one can construct a parallel sequence of economies by only slightly modifying the supply vectors, converging to the same limit economy, such that a corresponding sequence of CEEI assignment profiles in the modified sequence yields asymptotically the same individual payoffs as the original sequence of assignment profiles. This is what we call payoff convergence under trembled supplies. As a corollary, we also show that the original sequence of assignments could be obtained through almost-optimal consumer decisions in a sort of quasi-CEEI.

Unfortunately, we cannot go beyond, in the sense of not needing a trembled supply vector. We show that CEEI prices and payoffs are not lower-hemicontinuous in the supply vector. Consequently, one cannot simply ignore trembled supplies and obtain a similar payoff convergence of CEEI in the original sequence.

Our note contributes to several strands of the literature. First, we provide a warning that solutions obtained in the continuum model are not necessarily indicative of solutions that would obtain in large finite economies. Second, by showing that CEEI is not the only assignment mechanism that is both efficient and fair, our paper poses the question what mechanisms are both efficient and fair (or, nearly equivalently in large markets, efficient and incentive compatible)?[5] The study of these mechanisms is a topic for further research.

Let us stress that in many problems the qualitative properties of continuum economies do parallel those of large finite economies we are interested in. For instance, the asymptotic equivalence of Random Priority and Probabilistic Serial mechanisms (Che and Kojima 2010) and the uniqueness of asymptotically ordinally efficient, symmetric, and strategy-proof mechanisms (Liu and Pycia 2013) obtain both in large finite and in continuum models.[6]

In more general economies, there is a rich literature shedding light on the benefits and shortcomings of working directly at limit economies. Roberts and

[5] We know from Miralles and Pycia (2014) that all such mechanisms can be described as competitive equilibria from some profile of incomes; the open question is how to assign the incomes in a way that is fair or incentive compatible.

[6] See also Miralles (2008) for an exploration of the continuum model. Beyond the no-transfer model we study, the convergence results have been obtained by many authors. See, for instance, Gretsky, Ostroy and Zame (1992) for a study of core convergence as the economy converges to the atomless housing assignment model with transfers.

Postlewaite (1976) find sequences of economies in which incentives to misreport preferences do not asymptotically vanish as the economy grows larger. Manelli (1991) constructs examples of sequences of increasingly large economies whose core cannot be decentralized via prices (a so-called core convergence failure), when preferences are not monotonic.[7] Serrano, Vohra and Volij (2001) show that, under asymmetric information, the core fails to converge to any sort of set of price equilibrium outcomes in replica economies.

2 Model

We study an economy with agents $i, j \in I \subset [0,1]$ and a finite, fixed set of indivisible objects $x, y \in X = \{1, 2, ..., |X|\}$. I is endowed with a measure λ, and the total mass of I is 1. We allow, yet not impose, I to be finite with $|I|$ individuals, considered as $|I|$ different atoms on the $[0,1]$ interval with mass $1/|I|$ each. In such a case we say that the economy is finite. When $I = [0,1]$, λ is the Lebesgue (uniform) measure. Each object x is represented by a mass of identical copies (or capacity) $s_x \in (0,1)$. By $S = (s_x)_{x \in X}$ we denote the total supply of object copies in the economy. If agents have outside options, we treat them as objects in X; in particular, this implies that $\sum_{x \in X} s_x \geq 1$.

We assume that agents demand at most one copy of an object. We allow random assignments, and denote by $q^x(i) \in [0,1]$ the probability that agent i obtains a copy of object x. Agent i's random *assignment* (or simply assignment hereafter) $q(i) = (q^1(i), ..., q^{|X|}(i)) \in \Delta^{|X|-1}$ is a probability distribution. The economy-wide assignment, or *assignment profile*, $q : I \to \Delta^{|X|-1}$ is feasible if $\int_I q(i) d\lambda \leq S$. Let $\mathcal{A}$ denote the set of all possible assignment profiles, and $\mathcal{F} \subset \mathcal{A}$ denote the set of feasible assignment profiles.

Agents are expected utility maximizers, and agent i's utility from assignment $q(i)$ equals the scalar product $u_i(q(i)) = v(i) \cdot q(i)$ where $v(i) = (v^x(i))_{x \in X}$ is a vector of agent i's von Neumann-Morgenstern valuations for objects $x \in X$. We assume that no agent is indifferent among all objects.[8] This allows us to normalize valuation vectors so that each agent's highest valuation is 1 and lowest is 0. Let $\mathcal{V}$ be the space of all such normalized valuation vectors:

$$\mathcal{V} = \{(v^1, ..., v^{|X|}) \in [0,1]^{|X|} \mid \min\{v^1, ..., v^{|X|}\} = 0, \max\{v^1, ..., v^{|X|}\} = 1\}.$$

A valuation profile is a function $v : [0,1] \to \mathcal{V}$ that is $\lambda-$measurable. An assignment *economy* is a tuple $E = (I, \lambda, X, S, v)$.

An economy is ε-*dense* if for any $\nu \in \mathcal{V}$ there is $i \in I$ with $\|v(i) - \nu\| \leq \varepsilon$.[9] An economy is *dense* if for every $\varepsilon > 0$ the economy is $\varepsilon-$dense. Clearly, no economy

[7] See also Anderson and Zame (1997) who study core convergence when the set of goods is a continuum. They show that Edgeworth's conjecture that the core can be asymptotically decentralized through prices as the number of agents increase is not always true.

[8] The assumption that no agent is indifferent among all objects is with no loss of generality. Such an agent would not ever envy other agents since she would be satiated by any probability bundle. Since she is satiated, it is easy to include her in any competitive equilibrium by endowing her with sufficient income.

[9] We use the Euclidean norm although results do not depend on this particular choice of norm.

with finite I can be dense. An economy has *full support* if for all $V \subset \mathcal{V}$ with $\dim(V) = \dim(\mathcal{V})$[10] we have $\lambda(V) > 0$. Obviously all fully supported economies are dense, yet the converse may not be true.

For an economy $E = (I, \lambda, X, S, v)$, a feasible random assignment $q^* \in \mathcal{F}$ is Pareto *efficient* (or, simply, efficient) if no other feasible assignment profile $q \in \mathcal{F}$ is weakly preferred by all agents and strictly preferred by a positive mass of agents. A feasible random assignment $q^* \in \mathcal{F}$ is *envy-free* if for every pair of agents $i, j \in I$ we have $u_i(q^*(i)) \geq u_i(q^*(j))$.

An assignment profile q^* and a price vector $p^* \in \mathbb{R}_+^{|X|}$ constitute a *competitive equilibrium* for a λ-measurable budget (or income) profile $w^* : I \to \mathbb{R}_+$ if q^* is feasible, $p^* \cdot q^*(i) \leq w^*(i)$ for any $i \in I$, and $u_i(q(i)) > u_i(q^*(i)) \implies p^* \cdot q(i) > w^*(i)$ for any $q \in \mathcal{A}$. In such a case we also say that q^* is supported by p^* and w^*. In a *competitive equilibrium with equal incomes* (CEEI) we additionally require w^* to be a constant function.

Notice that when all prices in p^* are equal then every agent obtains sure assignment of her most preferred object. This case is straightforward, and in the sequel we focus on the case in which not all prices are equal; this allows us to normalize prices and budgets so that the highest price is 1 and the lowest price is 0.[11] Let $\mathcal{P}$ denote the set of all such normalized price vectors.

Sequences of economies and convergence. Let $t = 1, 2, ...$ A sequence of finite economies $E_t = (I_t, \lambda_t, X\ , S_t, v_t)$ is a *growing sequence of economies* if $\forall t$, $I_t \subsetneq I_{t+1}$ and $\forall i \in I_t$, $v_{t+1}(i) = v_t(i)$. A growing sequence of finite economies is a *converging sequence of economies* with limit $E = (I, \lambda, X\ , S, v)$ if $\forall t$, $i \in I_t$ we have $v_t(i) = v(i)$, and both $I_t \to I$ and $S_t \to S$. [12]In such a growing sequence we use $t(i) = \min\{t : i \in I_t\}$ for the moment in which an agent is incorporated into the sequence of economies. An assignment profile q *corresponds* to an economy $E = (I, \lambda, X\ , S, v)$ if it gives a lottery to each element of I and it is feasible. In a converging sequence of economies $E_t = (I_t, \lambda_t, X\ , S_t, v_t)$ with limit $E = (I, \lambda, X\ , S, v)$, a sequence of corresponding assignment profiles q_t *payoff-converges* to q if for every $i \in I$ and for $t \geq t(i)$, $u(q_t(i)) \to u(q(i))$.[13]

3 The Failure of Convergence

Is it true that as the number of agents grow and the economy becomes denser, every efficient and envy-free assignment profile is supported by a competitive equilibrium with arbitrarily similar incomes? This is indeed the case in the limit continuum economy with full support of preferences. As implied by Ashlagi and Shi (2015),

[10] The dimension of V is the maximum dimension along all convex subsets of V.

[11] Let $\underline{p} = \min_{x \in X} p^x < \bar{p} = \max_{x \in X} p^x$. $\sum_{x \in X} p^x q^x \leq w$ implies that $\sum_{x \in X}(p^x - \underline{p})q^x \leq w - \underline{p}$ (since $\sum_{x \in X} q^x = 1$,) which in turn implies $\sum_{x \in X} \frac{p^x - \underline{p}}{\bar{p} - \underline{p}} q^x \leq \frac{w - \underline{p}}{\bar{p} - \underline{p}}$. That justifies the normalization $p'^x = \frac{p^x - \underline{p}}{\bar{p} - \underline{p}}$ and $w' = \min\left\{\frac{w - \underline{p}}{\bar{p} - \underline{p}}, 1\right\}$.

[12] I_t converges to I in the density sense: for any $i \in I$ and $\varepsilon > 0$ there is t and some $i_t \in I_t$ such that $|i - i_t| < \varepsilon$.

[13] We would also simply talk about *convergence* if $q_t(i) \to q(i)$. Obviously convergence implies pay-off convergence, while the converse may not be true.

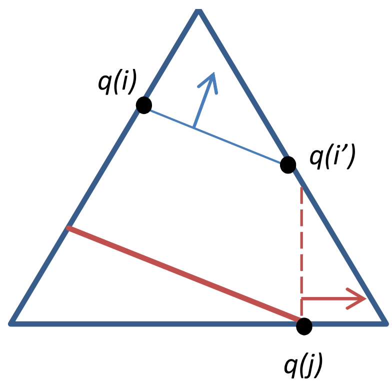

Figure 1: An efficient and envy-free assignment profile that cannot be supported by CEEI.

in any full-support economy $E = (I, \lambda, X\ , S, v)$ with a continuum of agents every efficient and envy-free assignment profile q^* can be supported by a CEEI.[14]

This natural result fails in finite economies. Not surprisingly, not every efficient and envy-free assignment profile can be supported by a CEEI in finite economies. Figure 1 illustrates an assignment profile for the set of agents $I = \{i, i', j\}$ in the simplex of all probabilistic assignments of objects $\{x, y, z\}$. The simplex is drawn so that the top corner is the sure assignment of object x, the right corner represents the sure assignment of object y, and the left corner represents the sure assignment of object z. Individual assignments are $q(i) = (3/4, 0, 1/4)$, $q(i') = (1/2, 1/2, 0)$ and $q(j) = (0, 3/4, 1/4)$. The supply vector such that this assignment profile is feasible: $S = [q(i)+q(i')+q(j)]/3 = (5/12, 5/12, 1/6)$. Agents i and i' are both indifferent between each other's assignment: $v(i) = v(i') = (1, 1/2, 0)$. Agent j's valuation vector is $v(j) = (1/2, 1, 0)$, and the dashed line represents her indifference curve. Arrows indicate the direction towards which agents would be better-off. Notice that no agent envies another agent's assignment. This assignment profile is efficient because the price vector $p^* = (1, 1/2, 0)$ supports this assignment profile in equilibrium. This price vector is in fact the unique equilibrium price vector supporting this assignment profile (according to our normalization); but at these prices agents i and i' need budget $\frac{3}{4}$ (thin blue line) to buy their bundles while agent j must have the budget of exactly $\frac{3}{8}$ (thick red line.) Hence this efficient and envy-free assignment profile cannot be implemented by a CEEI.

This example leaves open the possibility that the CEEI characterization becomes nearly true in large economies. Our main result addresses this possibility.

Theorem 1. *(CEEI Convergence Failure) There exists a limit economy* $E = (I, \lambda, X, S, v)$ *and a converging sequence of growing* ε_t*-dense finite economies* $E_t = (I_t, \lambda_t, X\ , S_t, v_t) \rightarrow$

[14] Ashlagi and Shi (2015)'s Theorem 1 says that in a full-support economy every efficient, symmetric, and incentive-compatible mechanism can be supported by a CEEI as long as changes of reports by a measure zero of agents have no impact on the assignment of other agents. The results are related as under the latter assumption, envy-freeness is equivalent to a conjunction of symmetry and incentive compatibility in any full support continuum economy.

$E = (I, \lambda, X, S, v)$, *with* $\varepsilon_t \to 0$ *and* E *having full support on the preference space, and a payoff-converging sequence of corresponding efficient and envy-free assignment profiles* $q_t^* \to q^*$ *(where* q^* *is an efficient and envy-free assignment profile corresponding to* E*) such that for every supporting corresponding sequence of equilibrium prices* $p_t^* \in P$ *and income functions* $w_t^* : I_t \to [0, 1]$ *there exist* $\alpha, \beta > 0$ *under which* $\lambda_t \left(\left\{ i \in I_t : \max_{j \in I_t} w_t^*(j) - w_t^*(i) \geq \alpha \right\} \right) \geq \beta$ *for all* t.

Proof: Let $X = \{x, y, z\}$. The limit economy $E = (I, \lambda, X, S, v)$ with $I = [0, 1]$ and λ being the Lebesgue uniform measure. We describe our valuation profile $v : I \to \mathcal{V}$ in Table 1:

$i \in ...$	$[0, 1/8]$	$[1/8, 1/4]$	$[1/4, 1/2]$	$[1/2, 5/8]$	$[5/8, 3/4]$	$[3/4, 7/8]$	$[7/8, 1]$
$v^x(i)$	1	1	1	$5 - 8i$	0	0	$8i - 7$
$v^y(i)$	0	$4i - 1/2$	$2i$	1	1	$7 - 8i$	0
$v^z(i)$	$1 - 8i$	0	0	0	$8i - 5$	1	1

Table 1: Limit distribution of preferences.

Notice that v is continuous in i, and that the economy E has full support (and hence it is dense) in $\mathcal{V}$. Agents in $[0, 1/4]$ prefer x the most and have a vNM valuation for y below $1/2$. Agent $i = 1/4$ (relevant in our example for the sake of equilibrium uniqueness) has valuation $v(i) = (1, 1/2, 0)$. Agents in $(1/4, 1/2)$ prefer object x and have valuation for y above 1/2. Agent $i = 1/2$ is indifferent between objects x and y (notice that no finite economy in the sequence below contains this agent.) Agents in $(1/2, 3/4)$ prefer y to the other objects. Agents in $[3/4, 1]$ have object z as favorite object.

Tables 2a and 2b summarize an efficient and envy-free assignment profile. This assignment profile can be supported solely by the following CEEI: $p^* = (p^{*x}, p^{*y}, p^{*z}) = (1, 1/2, 0)$ and $w^* = 3/4$. Since everyone enjoys the same income, x must be the most expensive object, since it is the only one that cannot be obtained with certainty, thus $p^{*x} = 1$. Now, agents in $[0, 1/4)$ have to buy less probabilities of the less preferred good (in this case z) than agents in $(1/4, 1/2]$ do (in this case object y). Recalling that everyone has the same income, it must be the case that z is cheaper than y. Hence $p^{*z} = 0$. This sets income, $w^* = 3/4$, which is the one that allows the latter agents to purchase the bundle $(3/4, 0, 1/4)$ at these prices. This in turn sets the price for object y, $p^{*y} = 1/2$, the only one that allows agents in $(1/4, 1/2]$ to purchase the exact bundle $(1/2, 1/2, 0)$.

$i \in ...$	Lebesgue measure	$q^{*x}(i)$	$q^{*y}(i)$	$q^{*z}(i)$	$p^* \cdot q^*(i)$	$w^*(i)$
$[0, 1/4)$	1/4	3/4	0	1/4	3/4	3/4
$\{1/4\}$	0	5/8	1/4	1/8	3/4	3/4
$(1/4, 1/2]$	1/4	1/2	1/2	0	3/4	3/4
$(1/2, 3/4)$	1/4	0	1	0	1/2	3/4
$[3/4, 1]$	1/4	0	0	1	0	3/4

Goods	p^*	S
x	1	5/16
y	1/2	3/8
z	0	5/16

Table 2: a) an efficient and envy-free assignment, with equilibrium expenses and budgets, and b) prices and supply.

Importantly, not every agent needs her full income in order to afford her assignment. This is the *key role of satiation*: the most preferred bundle is not necessarily on the hyperplane where expenses equal income. Thus, agents in $(1/2, 3/4)$ could have obtained their most-preferred assignment with income 1/2 instead of 3/4. This gap is the basis of our sequence of finite economies, where these agents have income slightly below 1/2.

We construct a sequence of growing finite economies $E_t = (I_t, \lambda_t, X, S_t, v_t)$ that converges to the limit economy E. For $t = 1, 2, ...$ set

$$I_t = \left\{ \frac{1}{4 \cdot 3^t}, \frac{3}{4 \cdot 3^t}, ..., \frac{4 \cdot 3^t - 3}{4 \cdot 3^t}, \frac{4 \cdot 3^t - 1}{4 \cdot 3^t} \right\}$$

implying $|I_t| = 2 \cdot 3^t$. Let $\hat{\imath}_t = \min\{i \in I_t : i \geq 1/2\} = 1/2 + \frac{1}{4 \cdot 3^t}$, such that $v(\hat{\imath}_t) = (1 - 2/3^t, 1, 0) = (1 - 4/|I_t|, 1, 0)$. This is the agent that has highest vNM valuation for object x among those who prefer y over all other objects. Note as well that $1/4 \in I_1$, and that $I_t \subset I_{t+1}$ for all t. For $\varepsilon_t = 2/3^t = 4/|I_t| \geq \frac{1}{2} \max_{i \in I_t \setminus \{\frac{1}{2|I_t|}\}} \left\| v(i) - v\left(i - \frac{1}{|I_t|}\right) \right\|$ the economy E_t is ε_t−dense in $\mathcal{V}$. We impose that for every t and $i \in I_t$, $v_t(i) = v(i)$. A payoff-convergent sequence of efficient and envy-free assignment profiles is summarized in Table 3, jointly with incomes and supply vectors. Only one vector of prices $p^* = (1, 1/2, 0)$ can sustain this assignment profile through a competitive equilibrium, as we show below.

$i \in ...$	Mass	$q_t^{*x}(i)$	$q_t^{*y}(i)$	$q_t^{*z}(i)$	$w_t^*(i)$	$\lim_{t\to\infty} w_t^*(i)$
$[0, 1/4)$	$1/4 - \frac{1}{2\|I_t\|}$	3/4	0	1/4	3/4	3/4
$\{1/4\}$	$1/\|I_t\|$	5/8	1/4	1/8	3/4	3/4
$(1/4, 1/2]$	$1/4 - \frac{1}{2\|I_t\|}$	1/2	1/2	0	3/4	3/4
$(1/2, 3/4)$	$1/4 - \frac{1}{2\|I_t\|}$	0	$1 - 2/\|I_t\|$	$2/\|I_t\|$	$1/2 - 1/\|I_t\|$	1/2
$[3/4, 1]$	$1/4 + \frac{1}{2\|I_t\|}$	0	0	1	3/4	3/4

Goods	$p_t^* = p^*$	S_t
x	1	5/16
y	1/2	$3/8 - \left(1 - \frac{1}{\|I_t\|}\right)\frac{1}{\|I_t\|}$
z	0	$5/16 + \left(1 - \frac{1}{\|I_t\|}\right)\frac{1}{\|I_t\|}$

Table 3: a) an efficient and envy-free assignment along the sequence, with incomes and limit incomes, and b) prices and supply along the sequence.

The assignment profiles q_t^* are efficient and envy-free. Efficiency is implied by the Hylland and Zeckhauser's (1979) First Welfare Theorem because there is a competitive equilibrium supporting it. The competitive equilibrium supporting q_t^* is constructed as follows. We set $p_t^* = v(1/4)$, that is, $p_t^{*x} = 1$, $p_t^{*y} = 1/2$ and $p_t^{*z} = 0$, $t = 1, 2, ...$ with incomes $w_t^*(i) = 1/2 - 1/|I_t| < 1/2$ if $i \in (1/2, 3/4)$, and $w_t(i) = 3/4$ otherwise.

The price vector is the unique (normalized) equilibrium price vector since agent $i = 1/4$'s assigned lottery is in the interior of the consumption space: linear utilities impose that $i = 1/4$ is indifferent among all the probability bundles that cost the same in any competitive equilibrium. Given the uniqueness of p_t^*, all agents $i \in [0, 1/2]$ must have income 3/4. Agents in $(1/2, 3/4)$ must have the assigned income $1/2 - 1/|I_t|$ to purchase their assigned bundles. Agents in $[3/4, 1]$ are satiated with any income, so without loss of generality we give them income 3/4.

To check envy-freeness note that the richest agents in each equilibrium, those with income 3/4, cannot envy any other agent's probability bundle (since it is also affordable with the highest income). Hence we just need to show that agents $i \in (1/2, 3/4)$, who obtain expected utility $1 - 2/|I_t|$, do not envy the richest agents. Bundle $(3/4, 0, 1/4)$ gives them payoff $3/4v^x(i) \leq 3/4v^x(\hat{\imath}_t) \leq 3/4(1-4/|I_t|) = 3/4-3/|I_t| < 1 - 2/|I_t|$ so that bundle is not envied. Bundle $(1/2, 1/2, 0)$ gives payoff $\frac{v^x(i)+1}{2} \leq \frac{v^x(\hat{\imath}_t)+1}{2} = 1 - 2/|I_t|$ thus that bundle is not envied either. Of course $(5/8, 1/4, 1/8)$ is a convex combination of the previous two bundles hence it is not envied either. Finally, the bundle $(0, 0, 1)$ is not envied either because these agents prefer y to z.

We complete the proof by setting the infimum income gap between poorer and richer agents, $\alpha = 1/4$, and the minimum mass of low-income agents along the sequence, $\beta = 1/4 - \frac{1}{2|I_1|} = 1/6$. **QED**

Notice what happens in the limit economy to which the sequence E_t converges in our counterexample. In the limit economy, the agents who prefer y over other objects obtain a sure copy of y, and hence we can increase their budget (or income) without affecting the equilibrium assignment; in particular, we can set their income to be equal to everyone else's. This is not possible at any finite economy of the sequence, no matter how close to the continuum limit it is. This observation reconciles our construction with the work of Ashlagi and Shi (2015): *The limit assignment profile of the sequence is a CEEI assignment profile; yet no element of the sequence is approximately CEEI.*

The role of satiation. To understand the idea behind this result, we pay attention to the unit demand constraint. In some sense, it is a satiation constraint. It could be taken as if individuals were not willing to purchase higher quantities. Observe that, with the same linear preferences yet ignoring the unit demand constraint, the counterexample proving Theorem 1 would have failed.[15] Agents $i \in (1/4, 1/2)$ in the counterexample above would now optimally buy the bundle (0,3/2,0) instead of (1/2,1/2,0). Agents who prefer y, with income below 1/2, would envy the former agents. With satiation (unit demands), two agents with just slightly different cardinal preferences yet different ordinal preferences could optimally choose very different bundles even with equal incomes. Hence envy may not arise when income differences between these agents stay away from zero. Without satiation, and unless both objects have the same price, differences in ordinal preferences induce no differences in the optimal choice when incomes are equal. Non-vanishing income differences would unavoidably induce envy eventually as the economy gets denser.

[15] Obviously and to start with, $p^z = 0$ would not be allowed in equilibrium. But even if we consider an economy with goods x and y only (and ignoring the agents who preferred object z), the counterexample fails, as explained in the main text.

4 Discussion: Other Convergence Criteria

A natural question then is whether there is a different convergence notion that restores the validity of CEEI in large markets. In this section we explore additional convergence criteria under which CEEI could asymptotically characterize the set of efficient and envy-free assignment profiles. Particularly we list the following alternative approaches:

- **Payoff convergence:** there exists a sequence of CEEI assignment profiles for the same sequence of economies that payoff-converges to the same limit assignment profile. CEEI would constitute a good approximation to the set of payoffs obtained under efficient and envy-free assignment profiles. A weak version of payoff convergence allows the sequence of CEEI assignment profiles to correspond to a sequence of slightly modified economies with trembled supplies, with this tremble converging to zero.
- **Nearly-optimal-choice convergence:** each element q_t^* of the sequence of efficient and envy-free assignment profiles is obtained through δ_t-optimal choices for a vector of prices p_t^* and the same income w_t^* for every individual, with $\delta_t \geq 0$ converging to 0. A choice $q_t^*(i)$ is δ_t-optimal given prices p_t^* and income w_t^* if $u_i(q) - u_i(q_t^*(i)) > \delta_t$ implies $p_t^* \cdot q > w_t^*$.

We start with a weak version of payoff convergence. A sequence of CEEI assignment profiles payoff-approaches any converging sequence of efficient and envy-free assignment profiles if the supply vector is slightly trembled (and this tremble converges to zero as the economy grows large). This in turns proves that nearly-optimal-choice convergence restores the validity of CEEI in characterizing efficient and envy-free assignment profiles in large economies.

Proposition 1 (CEEI Payoff Convergence under Trembling Supplies) Let $E_t = (I_t, \lambda_t, X\ , S_t, v_t)$, $t = 1, 2, ...$ be a converging sequence of ε_t-dense economies with full-support dense limit economy $E = (I, \lambda, X\ , S, v)$, and let us have a sequence of corresponding efficient and envy-free assignment profiles q_t^* that payoff-converges to q^*, an efficient and envy-free assignment profile corresponding to E. Then there is a parallel converging sequence of economies $\mathcal{E}_t = (I_t, \lambda_t, X\ , \Sigma_t, v_t)$ identical to E_t in all except for the supply vector, and with $\Sigma_t - S_t \to 0$, such that there is a corresponding sequence of CEEI assignment profiles q_t^{ceei} for each $\mathcal{E}_t$ in which for every i and $t \geq t(i)$, $u(q_t^*(i)) - u(q_t^{ceei}(i)) \to 0$.

Corollary 1 (Nearly-optimal-choice CEEI Convergence) Let $E_t = (I_t, \lambda_t, X\ , S_t, v_t)$, $t = 1, 2, ...$ be a converging sequence of ε_t-dense economies with full-support dense limit economy $E = (I, \lambda, X\ , S, v)$, and let us have a sequence of corresponding efficient and envy-free assignment profiles q_t^* that payoff-converges to q^*, an efficient and envy-free assignment profile corresponding to E. Then there is a sequence $\delta_t \to 0$ such that every q_t^* is a δ_t-optimal-choice CEEI outcome for its corresponding economy E_t.

Proof: The assignment profile q^* is an efficient and envy-free assignment profile corresponding to E, a full-support economy. By Ashlagi and Shi (2015), it can be supported by a CEEI with prices p^* and the same budget w^* for every agent $i \in I$. Now we use a consistency property of competitive equilibria. For each t, let $q_t^{ceei} : I_t \to \Delta^{|X|-1}$ be equal to q^* with domain I_t: $q_t^{ceei}(i) = q^*(i)$ for every $i \in I_t$. Let $\Sigma_t = \int_{I_t} q_t^{ceei}(i) d\lambda_t + [S - \int_I q^*(i) d\lambda]$. (The latter component of the sum is the possible excess supply in the limit economy). It is clear that q_t^{ceei} is a CEEI assignment profile corresponding to $\mathcal{E}_t = (I_t, \lambda_t, X\ , \Sigma_t, v_t)$ with the same prices p^* and the same budget w^* for every agent $i \in I_t$. Moreover $\mathcal{E}_t \to E$, thus $\Sigma_t - S_t \to 0$. Since q_t^* payoff-converges to q^* and q_t^{ceei} is equal to q^* with domain I_t, we conclude that $u(q_t^*(i)) - u(q_t^{ceei}(i)) \to 0$. This last statement directly proves the corollary, since q_t^* is a nearly-optimal choice profile under prices p^* and identical budget w^* for every agent and for every economy E_t. **QED**

A proper payoff convergence without trembling supplies needs that the set of payoff vectors obtained under CEEI be lower hemicontinuous in Σ_t, for every t. In that case, since $\Sigma_t - S_t \to 0$, for each sufficiently high value of t a CEEI under S_t would exist with payoffs close to those under Σ_t, which in turn approach those of the converging sequence of efficient and envy-free assignment profiles. However, this condition does not always hold.

Remark 1 (Discontinuity of Pseudomarket Equilibria) There is a finite economy $E = (I, \lambda,\ X\ , S, v)$ such that the set of CEEI price and income vectors is not singleton nor lower hemicontinuous in S. Moreover, the corresponding set of payoff profiles does not have any of these properties either.

Proof: Let $X = \{x, y, z\}$, and let $I = \{i_1, i_2, i_3\}$ (each element with mass 1/3). Supply is $S = (5/9, 2/9, 2/9)$. Valuations are $v(i_1) = (1, 0, 1 - \varepsilon)$, $v(i_2) = (1, 1 - \varepsilon, 0)$, $v(i_3) = (1, 1/2, 0)$, where ε is a small positive number.

For the easiness of exposition and only for this proof, we normalize all incomes to 1 instead of normalizing the highest price, which may now be higher than one. The lowest price is still normalized to zero. Uniqueness and lower hemicontinuity (as well as the lack of them) are preserved when we return to the original normalization.

Non-uniqueness.- The following are equilibrium price vectors: $p^* = (3/2, 3/4, 0)$, and $p' = (3/2, 0, 3/4)$. Corresponding assignments are $q^*(i_1) = q^*(i_3) = (2/3, 0, 1/3)$ and $q^*(i_2) = (1/3, 2/3, 0)$, under p^*, and $q'(i_1) = (1/3, 0, 2/3)$ and $q'(i_2) = q'(i_3) = (2/3, 1/3, 0)$, under p'. It is clear that both assignment profiles are feasible. It is also easy to check that agents obtain different payoffs depending on the equilibrium price vector. There are no other equilibria.

No lower hemicontinuity.- Consider the vector p^* and notice that i_3 is indifferent between her assignment and $q^*(i_2)$. Now let us marginally increase supply for object z, and both the other objects marginally decrease their supplies in order to make overall supply add up to 1. Any valid marginal variation around p^* involves $p^z = 0$ due to our normalization. In order to increase demand for z, a marginal increase of p^x (say ε^x) is necessary (for agents who spend their budgets on objects x and z only, higher price of x forces them to buy a bundle with a higher share of z). Since supply for object y has been reduced and equilibrium requires lower demand of it, a marginal decrease of p^y (which we call ε^y) is necessary (for agents who spend their budgets on objects x and y only, lower price of y allows to buy a bundle with a lower share of y). But then, the increase of p^x jointly with the decrease of p^y make i_3's optimal choice jump to a point close to $q^*(i_2)$. Indeed, i_3's payoff from spending her budget on objects x and y is

$$\frac{1-(p^{*y}-\varepsilon^y)}{(p^{*x}+\varepsilon^x)-(p^{*y}-\varepsilon^y)}+\frac{(p^{*x}+\varepsilon^x)-1}{(p^{*x}+\varepsilon^x)-(p^{*y}-\varepsilon^y)}\cdot 1/2=\frac{1/2+\varepsilon^x/2+\varepsilon^y}{3/4+\varepsilon^x+\varepsilon^y}$$

while her payoff from spending her budget on objects x and z is

$$\frac{1}{p^{*x}+\varepsilon^x}=\frac{1}{3/2+\varepsilon^x}$$

It is mechanical to check that for any $\varepsilon^x, \varepsilon^y > 0$ the former expression is higher than the latter expression. Therefore, i_3's demand for object z decreases from 1/3 to zero. This breaks any possible equilibrium with prices around p^* when S is marginally modified in the way mentioned before. Indeed the equilibrium is now unique with some price vector close to p', thus payoffs also non-marginally differ from those under S and p^*. **QED**

A pseudomarket equilibrium could just vanish, not only vary, with a small variation of supplies. This happens because an affordable good (one of which a sure copy can be bought) could behave as a Giffen good, an issue first pointed out by Hylland and Zeckhauser (1979). In our counterexample we observe that, in order to decrease the demand for good y, we actually have to *decrease* its price. But this might backfire because it makes a bundle containing positive probabilities for y more attractive (as indeed happens in the counterexample.) Hence the tremble of supplies we use in Proposition 1 cannot be left aside. *Payoff convergence* cannot be proved using this strategy (that is, trembling supplies and then invoking continuity of CEEI with respect to the supply vector).

5 Conclusions

We show that the relation between large no-transfer assignment markets and their continuum-agent models has to be taken with care. The problems are driven by the inherent failure of local non-satiation in such markets. By showing that the class of mechanisms that are fair and efficient (or symmetric, incentive-compatible, and efficient) in large finite markets is larger than CEEI, our note also poses the question what non-CEEI mechanisms satisfy these standard requirements.

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