**University of
Zurich**^{UZH}

**Zurich Open Repository and
Archive**

University of Zurich
University Library
Strickhofstrasse 39
CH-8057 Zurich
www.zora.uzh.ch

Year: 2018

A general framework to construct schemes satisfying additional conservation relations. Application to entropy conservative and entropy dissipative schemes

Abgrall, Rémi

Abstract: We are interested in the approximation of a steady hyperbolic problem. In some cases, the solution can satisfy an additional conservation relation, at least when it is smooth. This is the case of an entropy. In this paper, we show, starting from the discretization of the original PDE, how to construct a scheme that is consistent with the original PDE and the additional conservation relation. Since one interesting example is given by the systems endowed by an entropy, we provide one explicit solution, and show that the accuracy of the new scheme is at most degraded by one order. In the case of a discontinuous Galerkin scheme and a Residual distribution scheme, we show how not to degrade the accuracy. This improves the recent results obtained in [1], [2], [3], [4] in the sense that no particular constraints are set on quadrature formula and that a priori maximum accuracy can still be achieved. We study the behavior of the method on a non linear scalar problem. However, the method is not restricted to scalar problems.

DOI: <https://doi.org/10.1016/j.jcp.2018.06.031>

Posted at the Zurich Open Repository and Archive, University of Zurich

ZORA URL: <https://doi.org/10.5167/uzh-157549>

Journal Article

Accepted Version

Originally published at:

Abgrall, Rémi (2018). A general framework to construct schemes satisfying additional conservation relations. Application to entropy conservative and entropy dissipative schemes. *Journal of Computational Physics*, 372:640-666.

DOI: <https://doi.org/10.1016/j.jcp.2018.06.031>

A general framework to construct schemes satisfying additional conservation relations.

Application to entropy conservative and entropy dissipative schemes

R. Abgrall

Institute of Mathematics, University of Zurich, Switzerland

October 3, 2018

Abstract

We are interested in the approximation of a steady hyperbolic problem. In some cases, the solution can satisfy an additional conservation relation, at least when it is smooth. This is the case of an entropy. In this paper, we show, starting from the discretisation of the original PDE, how to construct a scheme that is consistent with the original PDE and the additional conservation relation. Since one interesting example is given by the systems endowed by an entropy, we provide one explicit solution, and show that the accuracy of the new scheme is at most degraded by one order. In the case of a discontinuous Galerkin scheme and a Residual distribution scheme, we show how not to degrade the accuracy. This improves the recent results obtained in [1, 2, 3, 4] in the sense that no particular constraints are set on quadrature formula and that a priori maximum accuracy can still be achieved. We study the behaviour of the method on a non linear scalar problem. However, the method is not restricted to scalar problems.

1 Introduction

In this paper, we are interested in the approximation of non-linear hyperbolic problems. To make things more precise, our target are the Euler equations in the compressible regime, other examples are the MHD equations. The case of parabolic problems in which the elliptic terms play an important role only in some area of the computational domain, such as the Navier-Stokes equations in the compressible regime, or the resistive MHD equations, can be dealt with in a similar way.

Let $\mathcal{D}$ an open subset of $\mathbb{R}^p$ and d C^1 functions $\mathbf{f}_j, j = 1, \dots, d$ defined on $\mathcal{D}$. We consider the hyperbolic problem, with $\mathbf{u} : \Omega \subset \mathbb{R}^d \rightarrow \mathcal{D} \subset \mathbb{R}^p$, defined by

$$\operatorname{div} \mathbf{f}(\mathbf{u}) = \sum_{j=1}^d \frac{\partial \mathbf{f}_j}{\partial x_j}(\mathbf{u}) = 0 \quad (1a)$$

subjected to

$$\left(\nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}) \cdot \mathbf{n}(\mathbf{x}), 0 \right)^- (\mathbf{u} - \mathbf{u}_b) = 0 \text{ on } \partial\Omega. \quad (1b)$$

In (1b), $\mathbf{n}(\mathbf{x})$ is the outward unit vector at $\mathbf{x} \in \partial\Omega$ and $\mathbf{u}_b$ is a regular enough function. We have also used standard notations: if A is a $p \times p$ diagonalisable matrix in $\mathbb{R}$, $A = L\Lambda R$ where L and R are two $p \times p$ matrices with $LR = \operatorname{Id}_{p \times p}$ and $\Lambda = \operatorname{diag}(\lambda_i)_{i=1, \dots, p}$ diagonal, then

$$A^{\pm} = L\Lambda^{\pm}R \text{ where } \Lambda^{\pm} = \operatorname{diag}(\lambda_i^{\pm})_{i=1, \dots, p}$$

and, for $x \in \mathbb{R}$, $x^+ = \max(x, 0)$ and $x^- = \min(x, 0)$. The weak formulation of (1) is: $\mathbf{u} \in L^2(\Omega)^d \cap L^\infty(\Omega)^d$ is a weak solution of (1) if $\mathbf{u} \in \mathcal{D}$ and for any $\varphi \in C_0^1(\Omega)^d$,

$$-\int_{\Omega} \nabla \varphi \cdot \mathbf{f}(\mathbf{u}) \, d\mathbf{x} + \int_{\partial\Omega} \varphi (\mathcal{F}_{\mathbf{n}}(\mathbf{u}, \mathbf{u}_b) - \mathbf{f}(\mathbf{u}) \cdot \mathbf{n}) \, d\gamma = 0 \quad (2)$$

where $\mathcal{F}_{\mathbf{n}}$ is a flux that is almost everywhere the upwind flux:

$$\mathcal{F}_{\mathbf{n}}(\mathbf{u}, \mathbf{u}_g) = (\nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}) \cdot \mathbf{n}(\mathbf{x}))^+ \mathbf{u} + (\nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}) \cdot \mathbf{n}(\mathbf{x}))^- \mathbf{u}_g$$

In general, schemes are constructed starting from the conservation relation. In some cases, such as when the solution is smooth enough, it appears that the solution of (1) also satisfies an additional conservation relation. There is no reason that the initial discretisation will also be consistent with the new conservation relation. A typical example is the entropy. If U , defined on $\mathcal{D}$, is a convex and C^1 function and if there exists g , a C^1 function, $g = (g_1, \dots, g_d) : \mathcal{D} \rightarrow \mathbb{R}^d$, such that: for all $j = 1, \dots, d$,

$$\nabla_{\mathbf{u}} U \cdot \nabla_{\mathbf{u}} \mathbf{f}_j = \nabla_{\mathbf{u}} g_j, \quad (3a)$$

then when $\mathbf{u}$ is smooth we get the additional conservation relation

$$\operatorname{div} \mathbf{g}(\mathbf{u}) = \sum_{j=1}^d \frac{\partial g_j}{\partial x_j}(\mathbf{u}) = 0. \quad (3b)$$

An entropy solution of (1) will satisfy $\operatorname{div} \mathbf{g}(\mathbf{u}) \leq 0$ in the sense of distributions.

Hence a rather natural question is: given a scheme for (1), how can we modify it so that the modified scheme is consistent with an additional conservation relation, i.e. in the case of an entropy, how to modify it so that it is also compatible with an entropy inequality? This is not a new question: there has been lots of work on that particular question, see [5, 6, 7, 8, 9, 10], and more recently see [11, 12, 4, 3, 2, 13, 1, 14, 15, 16, 17, 18, 19, 20, 21, 22] and references therein for an un-complete list of contributions. One of the salient aspects of research on this problem is a focus on the discrete version of the schemes: for example, when some integral appear in the formulation, as in finite element method, quadrature formula are needed, and the integrals are never computed exactly. The central question is then how to translate, from a discrete point of view, the 'continuous' differential relations (3a) and the algebra associated to them at the discrete level. Up to our knowledge, entropy stable schemes can be rigorously designed only for special quadrature formula. The objective of this paper is to provide a different point of view within the Residual Distribution framework. In particular, we show that we can consider general quadrature formula.

The outline of the paper is as follows. In a first part, we recall the class of schemes (nicknamed as Residual distribution schemes or RD or RDS for short) we are interested in, and show their link with more classical methods such as finite element ones. Then we recall a condition that guaranties that the numerical solution will converge to a weak solution of the problem, and a second one about entropy condition, and recall a systematic way to check the formal accuracy of the scheme. In the third part, we give a general recipe that show how to modify a scheme in order to fulfill an additional conservation law. The fourth part is devoted to a discussion on the entropy conservation: we show explicitly how to modify a given scheme. The next section is devoted to the construction of entropy dissipative schemes, and we provide numerical examples. A conclusion follows.

2 Notations

From now on, we assume that Ω has a polyhedral boundary in $\mathbb{R}^d$, $d = 1, 2, 3$. This simplification is by no mean essential. We denote by $\mathcal{E}_h$ the set of internal edges/faces of a triangulation $\mathcal{T}_h$, and by $\mathcal{F}_h$ those contained in $\partial\Omega$. $\mathcal{K}$ stands either for an element K or a face/edge $e \in \mathcal{E}_h \cup \mathcal{F}_h$. The boundary faces/edges are denoted by Γ . The mesh is assumed to be shape regular, h_K represents the diameter of the element K .

Similarly, if $e \in \mathcal{E}_h \cup \mathcal{F}_h$, h_e represents its diameter. Throughout the text, we will restrict ourselves to the case where the elements K are simplices, mostly for simplicity reasons.

Throughout this paper, we follow Ciarlet's definition [23, 24] of a finite element approximation: for any element K , we have a set of linear forms Σ_K acting on the set $\mathbb{P}^k$ of polynomials of degree k such that the linear mapping

$$q \in \mathbb{P}^k \mapsto (\sigma_1(q), \dots, \sigma_{|\Sigma_K|}(q))$$

is one-to-one. The space $\mathbb{P}^k$ is spanned by the basis function $\{\varphi_\sigma\}_{\sigma \in \Sigma_K}$ defined by

$$\forall \sigma, \sigma', \sigma(\varphi_{\sigma'}) = \delta_{\sigma}^{\sigma'}.$$

We have in mind either the Lagrange interpolation where the degrees of freedom are associated to the Lagrange points in K , or other type of polynomials approximation such as using Bézier polynomials where we will also do the same geometrical identification. More specifically, if $\{\mathbf{A}_1, \dots, \mathbf{A}_{d+1}\}$ are the vertices of K (i.e K is the convex hull of the vertices $\{\mathbf{A}_1, \dots, \mathbf{A}_{d+1}\}$), the Lagrange points of degree k are defined by their barycentric coordinates

$$L_\alpha = \sum_{l=1}^{d+1} \frac{i_l}{d+1} \mathbf{A}_l, \quad \alpha = (i_1, \dots, i_{d+1}) \quad (4)$$

with i_l positive integers such that $i_1 + \dots + i_{d+1} = k$

Any point $\mathbf{M}$ of $\mathbb{R}^d$ can be uniquely represented by its barycentric coordinates $\{\lambda_1(\mathbf{M}), \dots, \lambda_{d+1}(\mathbf{M})\}$ with

$$\mathbf{M} = \sum_{l=1}^{d+1} \lambda_l(\mathbf{M}) \mathbf{A}_l, \quad \sum_{l=1}^{d+1} \lambda_l(\mathbf{M}) = 1.$$

Note that $\lambda_l(\mathbf{M}) \geq 0$ if and only if $\mathbf{M} \in K$. The Bézier polynomials of degree k , associated to the multi-index $\alpha = (i_1, \dots, i_{d+1})$, i_l positive integer, $i_1 + \dots + i_{d+1} = k$ as in (4), are the polynomials

$$B_\alpha = \frac{k!}{i_1! \dots i_{d+1}!} \lambda_1^{i_1} \dots \lambda_{d+1}^{i_{d+1}}.$$

They sum up to unity, are positive on K , are interpolatory on the vertices only, span $\mathbb{P}^k$ as well as the Lagrange polynomials of degree k . For the Lagrange interpolation of degree k , the set of degrees of freedom is the set of Lagrange points of degree k , i.e. the L_α where α is as in (4). For the Bézier approximation, we make the geometrical identification between the multi-index α that defines the polynomial B_α and the Lagrange point L_α , though the Bézier polynomials are not interpolatory in general. However, the Bézier polynomials of degree k span $\mathbb{P}_k$, as well as the Lagrange polynomials of degree k .

Considering all the elements covering Ω , the set of degrees of freedom is denoted by $\mathcal{S}$ and a generic degree of freedom by σ . We note that for any K ,

$$\forall \mathbf{x} \in K, \quad \sum_{\sigma \in K} \varphi_\sigma(\mathbf{x}) = 1.$$

For any element K , $\#K$ is the number of degrees of freedom in K . If Γ is a face or a boundary element, $\#\Gamma$ is also the number of degrees of freedom in Γ .

The solution is sought for in the space V_h^k (or V_h for short since the degree k will be assumed to be constant) defined by, setting

$$\mathcal{V}_h = \bigoplus_K \{\mathbf{v} \in L^2(K), \mathbf{v}|_K \in \mathbb{P}^k\}.$$

We will consider two cases $V_h = \mathcal{V}_h \cap C^0(\Omega)$ in which case the triangulation needs to be conformal, and $V_h = \mathcal{V}_h$ where the continuity requirement is released: the triangulation does not need to be conformal anymore.

Throughout the text, we need to integrate functions. This is done via quadrature formula, and the symbol $\oint$ used in volume integrals

$$\oint_K v(\mathbf{x}) d\mathbf{x} := |K| \sum_q \omega_q^K v(\mathbf{x}_q^K) \quad (5)$$

or boundary/face integrals

$$\oint_{\partial K} v(\mathbf{x}) d\gamma := |f| \sum_q \omega_q^f v(\mathbf{x}_q^f) \quad (6)$$

means that these integrals are done via user defined numerical quadratures.

If $e \in \mathcal{E}_h$, represents any *internal* edge, i.e. $e \subset K \cap K^+$ for two elements K and K^+ , we define for any function ψ the jump $[\nabla \psi] = \nabla \psi|_K - \nabla \psi|_{K^+}$. Similarly, $\{\mathbf{v}\} = \frac{1}{2}(\mathbf{v}|_K + \mathbf{v}|_{K^+})$.

If $\mathbf{x}$ and $\mathbf{y}$ are two vectors of $\mathbb{R}^q$, for q integer, $\langle \mathbf{x}, \mathbf{y} \rangle$ is their scalar product. In some occasions, it can also be denoted as $\mathbf{x} \cdot \mathbf{y}$ or $\mathbf{x}^T \mathbf{y}$. Similarly, if $\mathbf{x} = (x_1, \dots, x_d)$ is a vector and $\mathbf{f} = (f_1, \dots, f_d)$ with $f_j \in \mathbb{R}^m$, then we denote

$$\mathbf{x} \cdot \mathbf{f} = \mathbf{x}^T \mathbf{f} = \sum_{i=1}^d x_i f_i.$$

Last, $v^h \in V^h$ will denote any test function, $\mathbf{u}$ will be the conserved quantities defined in (1), $\mathcal{V}$ will be the entropy variable and $\mathcal{V}^h \in V^h$ will be its interpolant in $\mathbb{P}^k$. Depending on the context, $\mathbf{u}^h \in V^h$ will denote either an approximation of $\mathbf{u}$ or $\mathbf{u}(\mathcal{V}^h)$. Similarly, $\pi_h(\mathbf{u}) \in V^h$ is the interpolant of $\mathbf{u}$ or $\mathbf{u}(\pi_h(\mathcal{V}))$, depending on the context.

3 Schemes, conservation, entropy dissipation

3.1 Schemes

In order to integrate the steady version of (1) on a domain $\Omega \subset \mathbb{R}^d$ with the boundary conditions (1b), on each element K and any degree of freedom $\sigma \in \mathcal{S}$ belonging to K , we define residuals $\Phi_\sigma^K(\mathbf{u}^h)$. Following [25, 26], they are assumed to satisfy the following conservation relations: For any element K , and any $\mathbf{u}^h \in V^h$,

$$\sum_{\sigma \in K} \Phi_\sigma^K(\mathbf{u}^h) = \oint_{\partial K} \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}|_K, \mathbf{u}|_{K^-}) d\gamma. \quad (7a)$$

where $u|_K$ is the restriction of $\mathbf{u}^h$ in the element K while $\mathbf{u}|_{K^-}$ is the restriction of $\mathbf{u}^h$ on the other side of the local edge/face of K . In addition, $\hat{\mathbf{f}}_{\mathbf{n}}$ is a constant numerical flux, i.e. $\hat{\mathbf{f}}(\mathbf{u}, \mathbf{u}) = \mathbf{f}(\mathbf{u}) \cdot \mathbf{n}$. Similarly, we consider residuals on the boundary elements Γ . On any such Γ , for any degree of freedom $\sigma \in \mathcal{S} \cap \Gamma$, we consider boundary residuals $\Phi_\sigma^\Gamma(\mathbf{u}^h)$ that will satisfy the conservation relation

$$\sum_{\sigma \in \Gamma} \Phi_\sigma^\Gamma(\mathbf{u}^h) = \oint_\Gamma (\hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}|_\Gamma, \mathbf{u}_b) - \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n}) d\gamma \quad (7b)$$

where, for $\mathbf{x} \in \Gamma$, $\mathbf{u}|_\Gamma(\mathbf{x}) = \lim_{\mathbf{y} \rightarrow \mathbf{x}, \mathbf{y} \in \Omega} \mathbf{u}^h(\mathbf{y})$. Once this is done, the discretisation of (1) is achieved via: for any $\sigma \in \mathcal{S}$,

$$\sum_{K \subset \Omega, \sigma \in K} \Phi_\sigma^K(\mathbf{u}^h) + \sum_{\Gamma \subset \partial \Omega, \sigma \in \Gamma} \Phi_\sigma^\Gamma(\mathbf{u}^h) = 0. \quad (8)$$

In (8), the first term represents the contribution of the internal elements. The second exists if $\sigma \in \Gamma$ and it represents the contribution of the boundary conditions.

In fact, the formulation (8) is very natural. Consider a variational formulation of the steady version of (1):

$$\text{find } \mathbf{u}^h \in V^h \text{ such that for any } \mathbf{v}^h \in V^h, a(\mathbf{u}^h, \mathbf{v}^h) = 0.$$

Let us show on three examples that this variational formulation leads to (8). They are

- The SUPG [27] variational formulation, with $\mathbf{u}^h, \mathbf{v}^h \in V^h = \mathcal{V}^h \cap C^0(\Omega)$:

$$\begin{aligned} a(\mathbf{u}^h, \mathbf{v}^h) := & - \oint_{\Omega} \nabla \mathbf{v}^h \cdot \mathbf{f}(\mathbf{u}^h) d\mathbf{x} + \sum_{K \subset \Omega} h_K \oint_K [\nabla \mathbf{f}(\mathbf{u}^h) \cdot \nabla \mathbf{v}^h] \tau_K [\nabla \mathbf{f}(\mathbf{u}^h) \cdot \nabla \mathbf{u}^h] d\mathbf{x} \\ & + \oint_{\partial\Omega} \mathbf{v}^h (\mathcal{F}_{\mathbf{n}}(\mathbf{u}^h, \mathbf{u}_b) - \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n}) d\gamma. \end{aligned} \quad (9)$$

and $\tau_K > 0$.

- The Galerkin scheme with jump stabilization, see [28] for details. We have

$$\begin{aligned} a(\mathbf{u}^h, \mathbf{v}^h) := & - \oint_{\Omega} \nabla \mathbf{v}^h \cdot \mathbf{f}(\mathbf{u}^h) d\mathbf{x} + \sum_{e \in \Omega} \theta_e h_K^2 \oint_e [\nabla \mathbf{v}^h] \cdot [\nabla \mathbf{u}^h] d\gamma \\ & + \oint_{\partial\Omega} \mathbf{v}^h (\mathcal{F}_{\mathbf{n}}(\mathbf{u}^h, \mathbf{u}_b) - \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n}) d\gamma. \end{aligned} \quad (10)$$

Here, $\mathbf{u}^h, \mathbf{v}^h \in V^h = \mathcal{V}^h \cap C^0(\Omega)$, and θ_e is a positive parameter.

- The discontinuous Galerkin formulation: we look for $\mathbf{u}^h, \mathbf{v}^h \in V^h = \mathcal{V}^h$ such that

$$a(\mathbf{u}^h, \mathbf{v}^h) := \sum_{K \subset \Omega} \left(- \oint_K \nabla \mathbf{v}^h \cdot \mathbf{f}(\mathbf{u}^h) d\mathbf{x} + \oint_{\partial K} \mathbf{v}^h \cdot \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_{|K}^h, \mathbf{u}_{|K-}^h) d\gamma \right). \quad (11)$$

In (11), the boundary integral is a sum of integrals on the faces of K , and here for any face of K $\mathbf{u}_{|K-}^h$ represents the approximation of $\mathbf{u}$ on the other side of that face in the case of internal elements, and $\mathbf{u}_b$ when that face is on $\partial\Omega$. In (11), $\hat{\mathbf{f}}$ is a consistent numerical flux. Note that to fully comply with (8), we should have defined for boundary faces $\mathbf{u}_{|K-}^h = \mathbf{u}_{|K}^h$, and then (11) is rewritten as

$$\begin{aligned} a(\mathbf{u}^h, \mathbf{v}^h) := & \sum_{K \subset \Omega} \left(- \oint_K \nabla \mathbf{v}^h \cdot \mathbf{f}(\mathbf{u}^h) d\mathbf{x} + \oint_{\partial K} \mathbf{v}^h \cdot \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_{|K}^h, \mathbf{u}_{|K}^h) d\gamma \right) \\ & + \sum_{\Gamma \subset \partial\Omega} \oint_{\Gamma} \mathbf{v}^h \cdot \left(\hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_{\Gamma}^h, \mathbf{u}_b) - \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n} \right) d\gamma. \end{aligned} \quad (12)$$

with by abuse of language, $\hat{\mathbf{f}} = \mathcal{F}$ on the boundary edges.

In the SUPG, Galerkin scheme with jump stabilisation or the DG scheme, the boundary flux can be chosen different from $\mathcal{F}$. This can lead to boundary layers if these flux are not "enough" upwind, but we are not interested in these issues here.

Using the fact that the basis functions that span V_h have a *compact* support, then each scheme can be rewritten in the form(8) with the following expression for the residuals:

- For the SUPG scheme (9), the residual are defined by

$$\Phi_{\sigma}^K(\mathbf{u}^h) = \oint_{\partial K} \varphi_{\sigma} \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n} d\gamma - \oint_K \nabla \varphi_{\sigma} \cdot \mathbf{f}(\mathbf{u}^h) d\mathbf{x} + h_K \oint_K \left(\nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}^h) \cdot \nabla \varphi_{\sigma} \right) \tau_K \left(\nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}^h) \cdot \nabla \mathbf{u}^h \right) d\mathbf{x} \quad (13)$$

with $\tau_K > 0$.

- For the Galerkin scheme with jump stabilization (10), the residuals are defined by:

$$\Phi_{\sigma}^K(\mathbf{u}^h) = \oint_{\partial K} \varphi_{\sigma} \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n} d\gamma - \oint_K \nabla \varphi_{\sigma} \cdot \mathbf{f}(\mathbf{u}^h) d\mathbf{x} + \sum_{e \text{ faces of } K} \theta_e h_e^2 \oint_{\partial K} [\nabla \mathbf{u}] \cdot [\nabla \varphi_{\sigma}] d\gamma \quad (14)$$

with $\theta_e > 0$. Here, since the mesh is conformal, any internal edge e (or face in 3D) is the intersection of the element K and an other element denoted by K^+ .

- For the discontinuous Galerkin scheme,

$$\Phi_\sigma^K(\mathbf{u}^h) = - \oint_K \nabla \varphi_\sigma \cdot \mathbf{f}(\mathbf{u}^h) d\mathbf{x} + \oint_{\partial K} \varphi_\sigma \cdot \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_{|K}^h, \mathbf{u}_{|K^-}^h) d\gamma \quad (15)$$

using the second definition of $\mathbf{u}_{|K^-}^h$.

- The boundary residuals are

$$\Phi_\sigma^\Gamma(\mathbf{u}^h) = \oint_\Gamma \varphi_\sigma (\hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_\Gamma^h, \mathbf{u}_b) - \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n}) d\gamma \quad (16)$$

All these residuals satisfy the relevant conservation relations, namely (7a) or (7b), depending if we are dealing with element residuals or boundary residuals.

For now, we are just rephrasing classical finite element schemes into a purely numerical framework. However, considering the pure numerical point of view and forgetting the variational framework, we can go further and define schemes that have no clear variational formulation. These are the limited Residual Distribution Schemes, see [25, 26], namely

$$\Phi_\sigma^K(\mathbf{u}^h) = \beta_\sigma \oint_{\partial K} \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n} d\gamma \quad (17)$$

or

$$\Phi_\sigma^K(\mathbf{u}^h) = \beta_\sigma \oint_{\partial K} \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n} d\gamma + h_K \theta_S \oint_K \left(\nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}^h) \cdot \nabla \varphi_\sigma \right) \tau_K \left(\nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}^h) \cdot \nabla \mathbf{u}^h \right) d\mathbf{x} \quad (18)$$

or

$$\Phi_\sigma^K(\mathbf{u}^h) = \beta_\sigma \oint_{\partial K} \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n} d\gamma + \theta_J h_K^2 \oint_{\partial K} [\nabla \mathbf{u}^h] \cdot [\nabla \varphi_\sigma] d\gamma. \quad (19)$$

or, combining the two kind of stabilisations

$$\Phi_\sigma^K(\mathbf{u}^h) = \beta_\sigma \oint_{\partial K} \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n} d\gamma + \theta_J h_K^2 \oint_{\partial K} [\nabla \mathbf{u}^h] \cdot [\nabla \varphi_\sigma] d\gamma + h_K \theta_S \oint_K \left(\nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}^h) \cdot \nabla \varphi_\sigma \right) \tau \left(\nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}^h) \cdot \nabla \mathbf{u}^h \right) d\mathbf{x} \quad (20)$$

where the parameters β_σ are defined to guarantee conservation,

$$\sum_{\sigma \in K} \beta_\sigma = 1$$

and such that (18) without the streamline term and (19) without the jump terms satisfy a discrete maximum principle. The parameters τ_K , θ_J and θ_S in (18), (19) and (20) are positive real numbers. The streamline term and jump term are introduced because one can easily see that spurious modes may exist, but their role is very different compared to (13) and (14) where they are introduced to stabilize the Galerkin scheme: if formally the maximum principle is violated, experimentally the violation is extremely small, if not existant. See [29, 25] for more details.

A similar construction can be done starting from a discontinuous Galerkin scheme without non-linear stabilisation such as limiting, has been applied.

The non-linear stability is provided by the coefficient β_σ which is a non-linear function of $\mathbf{u}^h$. Possible values of β_σ are described in the appendix A.

3.2 Conservation

It is time to state the geometrical assumptions we make on the quadrature formula:

Assumption 3.1 (Assumption on the quadrature formula). In (7b) and (7a), numerical integrations are made on the faces of the elements of $\mathcal{T}_h$, boundary of Ω included. Hence when we write, for an element K $\oint_{\partial K} \psi \cdot \mathbf{n} \, d\gamma$, or for a face on $\partial\Omega$ $\oint_{\Gamma} \psi \, d\gamma$, we are adding contributions 'living' on the skeleton of $\mathcal{T}_h$. We request that the quadrature formula depend only on the skeleton elements, so that if $f = K_1 \cap K_2$ then

$$\oint_{f \cap \partial K_1} \psi \cdot \mathbf{n} \, d\gamma + \oint_{f \cap \partial K_2} \psi \cdot \mathbf{n} \, d\gamma = 0. \quad (21)$$

In the previous relation, the first integral is a numerical formula on f seen from K_1 , and the second term is an integral seen from K_2 . Since the normal are opposite, this means that the geometrical location of the quadrature points are the same on f seen from K_1 and f seen from K_2 . If f is on the boundary of K , this is also the intersection of an element K and the boundary of Ω , and again, the quadrature points have the same geometrical locations whatever be the way we consider f .

In practice, this is not a restriction at all since quadrature points are defined by their barycentric coordinates and symmetric with respect to permutations of the vertices that respect orientation.

From (8), using the conservation relations (7b) and (7a), we obtain for any $\mathbf{v}^h \in V^h$,

$$\mathbf{v}_h = \sum_{\sigma \in S} \mathbf{v}_\sigma \varphi_\sigma,$$

the following relation:

$$\begin{aligned} 0 = & - \oint_{\Omega} \nabla \mathbf{v}^h \cdot \mathbf{f}(\mathbf{u}^h) \, d\mathbf{x} + \oint_{\partial\Omega} \mathbf{v}^h (\hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}^h, \mathbf{u}_b) - \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n}) \, d\gamma \\ & + \sum_{e \in \mathcal{E}_h} \oint_e [\mathbf{v}^h] \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_{|K}^h, \mathbf{u}_{|K-}^h) \, d\gamma + \sum_{K \subset \Omega} \frac{1}{\#K} \left(\sum_{\sigma, \sigma' \in K} (\mathbf{v}_\sigma - \mathbf{v}_{\sigma'}) \left(\Phi_\sigma^K(\mathbf{u}^h) - \Phi_\sigma^{K, Gal}(\mathbf{u}^h) \right) \right) \\ & + \sum_{\Gamma \subset \partial\Omega} \frac{1}{\#\Gamma} \left(\sum_{\sigma, \sigma' \in \Gamma} (\mathbf{v}_\sigma - \mathbf{v}_{\sigma'}) (\Phi_\sigma^\Gamma(\mathbf{u}^h, \mathbf{u}_b) - \Phi_\sigma^{Gal, \Gamma}(\mathbf{u}^h, \mathbf{u}_b)) \right) \end{aligned} \quad (22a)$$

where

$$\begin{aligned} \Phi_\sigma^{K, Gal}(\mathbf{u}^h) &= - \oint_K \nabla \varphi_\sigma \cdot \mathbf{f}(\mathbf{u}^h) \, d\mathbf{x} + \oint_{\partial K} \varphi_\sigma \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_{|K}^h, \mathbf{u}_{|K-}^h) \, d\gamma, \\ \Phi_\sigma^{\Gamma, Gal}(\mathbf{u}^h, \mathbf{u}_b) &= \oint_\Gamma \varphi_\sigma (\hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}^h, \mathbf{u}_b) - \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n}) \, d\gamma. \end{aligned} \quad (22b)$$

Note that

$$\sum_{\sigma \in K} \Phi_\sigma^{K, Gal}(\mathbf{u}^h) = \oint_{\partial K} \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_{|K}^h, \mathbf{u}_{|K-}^h) \, d\gamma$$

and

$$\sum_{\sigma \in \Gamma} \Phi_\sigma^{\Gamma, Gal}(\mathbf{u}^h, \mathbf{u}_b) = \oint_\Gamma (\hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}^h, \mathbf{u}_b) - \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n}) \, d\gamma.$$

The proof of (22a) is given in appendix C. The relation (22a) is instrumental in proving the following results. The first one is proved in [30], and is a generalisation of the classical Lax-Wendroff theorem.

Theorem 3.2. Assume the family of meshes $\mathcal{T} = (\mathcal{T}_h)$ is shape regular. We assume that the residuals $\{\Phi_\sigma^K\}_{\sigma \in \mathcal{K}}$, for $\mathcal{K}$ an element or a boundary element of $\mathcal{T}_h$, satisfy:

- For any $M \in \mathbb{R}^+$, there exists a constant C which depends only on the family of meshes $\mathcal{T}_h$ and M such that for any $\mathbf{u}^h \in V^h$ with $\|\mathbf{u}^h\|_\infty \leq M$, then

$$|\Phi_\sigma^K(\mathbf{u}^h)| \leq C \sum_{\sigma, \sigma' \in \mathcal{K}} |\mathbf{u}_\sigma^h - \mathbf{u}_{\sigma'}^h|$$

- The conservation relations (7a) and (7b) with quadrature formulas satisfying assumption 3.1.

Then if there exists a constant C_{max} such that the solutions of the scheme (8) satisfy $\|\mathbf{u}^h\|_\infty \leq C_{max}$ and a function $\mathbf{v} \in L^2(\Omega)$ such that $(\mathbf{u}^h)_h$ or at least a sub-sequence converges to $\mathbf{v}$ in $L^2(\Omega)$, then $\mathbf{v}$ is a weak solution of (1)

Proof. The proof can be found in [30], it uses (22a) and some adaptation of the ideas of [31]. One of the key arguments comes from the consistency of the flux $\mathbf{f}$ as well as (21) applied to $\hat{\mathbf{f}}$ $\square$

Another consequence of (22a) is the following result on entropy inequalities:

Proposition 3.3. *Let $(U, \mathbf{g})$ be a entropy-flux pair for (1) and $\hat{\mathbf{g}}_{\mathbf{n}}$ be a numerical entropy flux consistent with $\mathbf{g} \cdot \mathbf{n}$. Assume that the residuals satisfy: for any element K ,*

$$\sum_{\sigma \in K} \langle \mathcal{V}_\sigma, \Phi_\sigma^K \rangle \geq \oint_{\partial K} \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}|_K^h, \mathcal{V}|_{K-}^h) d\gamma \quad (23a)$$

and for any boundary edge e ,

$$\sum_{\sigma \in e} \langle \mathcal{V}_\sigma, \Phi_\sigma^e \rangle \geq \oint_e (\hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_\Gamma^h, \mathcal{V}_b) - \mathbf{g}(\mathcal{V}^h) \cdot \mathbf{n}) d\gamma. \quad (23b)$$

Then, under the assumptions of theorem 3.2, the limit weak solution also satisfies the following entropy inequality: for any $\varphi \in C^1(\bar{\Omega})$, $\varphi \geq 0$,

$$-\oint_{\Omega} \nabla \varphi \cdot \mathbf{g}(\mathcal{V}) d\mathbf{x} + \oint_{\partial\Omega^-} \varphi \mathbf{g}(\mathcal{V}_b) \cdot \mathbf{n} d\gamma \leq 0.$$

Proof. The proof is similar to that of theorem 3.2 once we are writing the conservative variables in term of the entropy variable $\square$

Another consequence of (22a) is the following condition under which one can guarantee to have a $k+1$ -th order accurate scheme. We first introduce the (weak) truncation error

$$\mathcal{E}(\mathbf{u}^h, \varphi) = \sum_{\sigma \in \mathcal{S}_h} \varphi_\sigma \left[\sum_{K \subset \Omega, \sigma \in K} \Phi_\sigma^K + \sum_{\Gamma \subset \partial\Omega, \sigma \in \Gamma} \Phi_\sigma^\Gamma \right]. \quad (24)$$

Proposition 3.4. *Assuming that:*

1. The solution of the steady problem $\mathbf{u}$ is smooth enough,
2. The approximation $\pi_h(\mathbf{u})$ of $\mathbf{u}$ is accurate with order $k+1$, and $\mathbf{f}$ Lipschitz continuous,
3. The quadrature formula (5) and (6) are of order $k+1$: for any ψ of class at least C^{k+1} , for any element K and face f ,

$$\int_K \psi(\mathbf{x}) d\mathbf{x} = \oint_K \psi(\mathbf{x}) d\mathbf{x} + O(h^{d+k+1}), \quad \int_f \psi d\gamma = \oint_f \psi d\gamma + O(h^{d+k}),$$

4. The residuals, computed with the interpolant $\pi_h(\mathbf{u})$ of the solution, are such that for any element K and boundary element Γ

$$\Phi_\sigma^K(\pi_h(\mathbf{u})) = \mathcal{O}(h^{k+d}), \quad \Phi_\sigma^\Gamma(\pi_h(\mathbf{u})) = \mathcal{O}(h^{k+d-1}) \quad (25)$$

then the truncation error satisfies the following relation

$$|\mathcal{E}(\pi_h(\mathbf{u}), \varphi)| \leq C(\mathbf{f}, \mathbf{u}) \|\varphi\|_{H^1(\Omega)} h^{k+1},$$

with C a constant which depends only on $\mathbf{f}$, and $\|\mathbf{u}\|_\infty$.

The proof can be found in appendix C.

Local conservation. In [32], we show that schemes satisfying the conservation requirements (7a) and (7b) are indeed locally conservative. More precisely, for each degree of freedom, one can define a control volume C_σ of polygonal type, and numerical flux $\hat{f}_{\mathbf{n}_{\sigma\sigma'}}(\mathbf{u}^h)$ that depend on the values $\mathbf{u}_{\sigma'}$ in a neighborhood of C_σ and the local normal of $C_\sigma \cup C_{\sigma'}$ which is planar. This set of connection defines a dual graph. By construction, we have

$$\hat{f}_{\mathbf{n}_{\sigma\sigma'}}(\mathbf{u}^n) = -\hat{f}_{\mathbf{n}_{\sigma'\sigma}}(\mathbf{u}^h),$$

which ensure local conservation. Details can be found in [32], but we give two examples in appendix B.

This means that (8) can be rewritten as

$$\sum_{\sigma' \in \mathcal{N}_\sigma} \hat{f}_{\mathbf{n}_{\sigma\sigma'}}(\mathbf{u}^h) = 0$$

where $\mathcal{N}_\sigma$ is the set of degrees of freedom connected to σ by the dual graph defined from the cells.

3.3 Construction of entropy conservative schemes

In the rest of the paper, $\mathcal{V}^h \in V^h$ represents the approximation of the entropy variable $\mathcal{V} = \nabla_{\mathbf{u}} U(\mathbf{u})$ in V^h , and since, the mapping $\mathbf{u} \rightarrow \mathcal{V}(\mathbf{u})$ is one-to-one, $\mathbf{u}^h := \mathbf{u}(\mathcal{V}^h)$ here. Hence in general it is not a polynomial.

Starting from a scheme $\{\Phi_\sigma\}$, we show in this section how to construct a new scheme, with residuals $\{\Phi'_\sigma\}$ such that

$$\sum_{\sigma \in K} \langle \mathcal{V}_\sigma, \Phi'_\sigma \rangle = \oint_{\partial K} \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) d\gamma. \quad (26)$$

This relation is motivated by Proposition 3.3 where the inequality is replaced by an equality, so that combined with Theorem 3.2, we know that if the scheme converges, the limit solution will satisfy two conservation relations: one on $\mathbf{u}$ and one on the entropy. *The equality, together with the remark of the appendix B shows that we get indeed local conservation of the entropy, too.*

Following [33, 34], the idea is to write:

$$\Phi'_\sigma = \Phi_\sigma + \mathbf{r}_\sigma \quad (27)$$

such that conservation is still guaranteed

$$\sum_{\sigma \in K} (\Phi_\sigma^K + \mathbf{r}_\sigma) = \int_{\partial K} \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) d\gamma,$$

i.e.

$$\sum_{\sigma \in K} \mathbf{r}_\sigma = 0, \quad (28a)$$

and (26) holds true, i.e.

$$\sum_{\sigma \in K} \langle \mathcal{V}_\sigma, \mathbf{r}_\sigma \rangle = \oint_{\partial K} \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) d\gamma - \sum_{\sigma \in K} \langle \mathcal{V}_\sigma, \Phi_\sigma \rangle := \mathcal{E} \quad (28b)$$

The two relations (28) defines a linear system with always at least two unknowns: the solution which is always valid is

$$\mathbf{r}_\sigma = \alpha(\mathcal{V}_\sigma - \bar{\mathcal{V}}) \text{ with } \bar{\mathcal{V}} = \frac{1}{\#K} \sum_{\sigma \in K} \mathcal{V}_\sigma. \quad (29a)$$

Since

$$\begin{aligned} \sum_{\sigma \in K} \langle \mathcal{V}_\sigma, \mathcal{V}_\sigma - \bar{\mathcal{V}} \rangle &= \sum_{\sigma \in K} \langle \mathcal{V}_\sigma - \bar{\mathcal{V}}, \mathcal{V}_\sigma - \bar{\mathcal{V}} \rangle + \langle \bar{\mathcal{V}}, \sum_{\sigma \in K} (\mathcal{V}_\sigma - \bar{\mathcal{V}}) \rangle = \sum_{\sigma \in K} \langle \mathcal{V}_\sigma - \bar{\mathcal{V}}, \mathcal{V}_\sigma - \bar{\mathcal{V}} \rangle \\ &= \sum_{\sigma \in K} (\mathcal{V}_\sigma - \bar{\mathcal{V}})^2 \end{aligned}$$

since $\sum_{\sigma \in K} (\mathcal{V}'_{\sigma} - \bar{\mathcal{V}}) = 0$ by construction. Thus we get

$$\alpha = \frac{\mathcal{E}}{\sum_{\sigma \in K} (\mathcal{V}'_{\sigma} - \bar{\mathcal{V}})^2}. \quad (29b)$$

Remark 3.5. 1. Since (28b) means that the new set of residuals satisfy a conservation relation for the entropy on each element, from [32] the new scheme will be locally conservative for the variable $\mathbf{u}$ and the entropy.

2. Relation (29b) becomes singular for a constant state. In practice, we use

$$\alpha = \frac{\mathcal{E}}{\sum_{\sigma \in K} (\mathcal{V}'_{\sigma} - \bar{\mathcal{V}})^2 + \varepsilon} \quad (30)$$

and we have taken $\varepsilon = 10^{-20}$.

3. Interpretation of (29). In fact adding $\alpha(\mathcal{V}_{\sigma} - \bar{\mathcal{V}}) = \frac{\alpha}{\#K} \sum_{\sigma' \in K} (\mathcal{V}_{\sigma} - \mathcal{V}_{\sigma'})$ to each residual is like adding diffusion. If we had a Cartesian mesh, this amounts to adding an approximation of

$$\Delta x \frac{\partial}{\partial x} \left(\alpha \frac{\partial \mathcal{V}}{\partial x} \right) + \Delta y \frac{\partial}{\partial y} \left(\alpha \frac{\partial \mathcal{V}}{\partial y} \right).$$

Since α has a priori no sign, depending on the local entropy production of the original scheme, one adds or removes the right amount to be entropy conservative.

Now, let us have a look at the accuracy of such a scheme. Let us remind we are looking at *steady* problems. According to the accuracy conditions (25), we assume $\Phi_{\sigma} = O(h^{k+d})$. Similarly, since the numerical flux $\hat{\mathbf{g}}$ is Lipschitz continuous and consistent, we also have point-wise, for a smooth steady exact solution $\mathcal{V}$, $\text{div } \mathbf{g}(\mathcal{V}) = 0$, so that

$$0 = \int_K \text{div } \mathbf{g}(\mathcal{V}) \, d\mathbf{x} = \int_{\partial K} \mathbf{g}(\mathcal{V}) \cdot \mathbf{n} \, d\gamma$$

and then

$$\begin{aligned} \oint_{\partial K} \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}|_K^h, \mathcal{V}|_{K-}^h) \, d\gamma &= \oint_{\partial K} \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}|_K^h, \mathcal{V}|_{K-}^h) \, d\gamma - \int_{\partial K} \mathbf{g}(\mathcal{V}) \cdot \mathbf{n} \, d\gamma \\ &= \oint_{\partial K} (\hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}|_K^h, \mathcal{V}|_{K-}^h) - \mathbf{g}(\mathcal{V}) \cdot \mathbf{n}) \, d\gamma + \oint_{\partial K} \mathbf{g}(\mathcal{V}) \cdot \mathbf{n} \, d\gamma - \int_{\partial K} \mathbf{g}(\mathcal{V}) \cdot \mathbf{n} \, d\gamma \\ &= O(h^{k+d}) \end{aligned} \quad (31)$$

provided the quadrature formula on the boundary is at least of order k . This is the case by assumption. In general, we cannot have more than $O(h^{k+d})$ because even if the integration is exact, we cannot have

$$\oint_{\partial K} (\hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}|_K^h, \mathcal{V}|_{K-}^h) - \mathbf{g}(\mathcal{V})) \, d\gamma = O(h^{k+d+1}).$$

Hence, $\mathcal{E} = O(h^{k+d})$. However, for smooth problems, we have

$$\alpha = O(h^{k-2+d}),$$

so that the correction $\mathbf{r}_{\sigma}$ is only $O(h^{k-1+d})$ a priori: we loose one order. Thus we get

Proposition 3.6. *The scheme defined by (27), (29) is locally entropy conservative and at least of order k at steady state.*

Up to now, we have shown, starting from a general additional conservation constraint, how to modify a scheme so that the modified scheme will satisfy one additional conservation relation, and what we see is that a priori we lose one order of accuracy. This is a general statement that assumes there is no particular connection between the 'old' conservation relation and the new one. However, there is a connection for the entropy flux, it is given by the following relation, see [7]: the entropy flux g is related to the flux $\mathbf{f}$ by the relation

$$\mathbf{g}(\mathcal{V}) = \langle \mathcal{V}, \mathbf{f}(\mathcal{V}) \rangle - \theta(\mathcal{V})$$

In addition, the potential satisfies

$$\nabla_{\mathcal{V}} \theta(\mathcal{V}) = \mathbf{f}(\mathcal{V}).$$

Using this, and a proper definition of the entropy numerical flux, it is possible to do better. We explore this question for the discontinuous Galerkin method, and the non linear RD scheme. The discussion on the discontinuous Galerkin method will also embed the continuous Galerkin approximation.

3.3.1 Discontinuous Galerkin methods

In the case of the discontinuous Galerkin methods, we get

$$\mathcal{E} = - \oint \nabla \mathcal{V}^h \cdot \mathbf{f}(\mathcal{V}^h) d\mathbf{x} + \oint_{\partial K} \langle \mathcal{V}^h, \mathbf{f}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) \rangle d\gamma - \oint_{\partial K} \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) d\gamma.$$

The precise definition of $\hat{\mathbf{g}}_{\mathbf{n}}$ will be done later in this section. Since point-wise¹, we have

$$\nabla \mathcal{V}^h \cdot \mathbf{f}(\mathcal{V}^h) = \operatorname{div} (\langle \mathcal{V}^h, \mathbf{f}(\mathcal{V}^h) \rangle) - \mathcal{V}^h \cdot \nabla \mathbf{f}(\mathcal{V}^h)$$

and $\mathcal{V}^h \cdot \operatorname{div} \mathbf{f}(\mathcal{V}^h) = \operatorname{div} \mathbf{g}(\mathcal{V}^h)$, we can write:

$$\begin{aligned} \mathcal{E} &= - \oint_K \nabla \mathcal{V}^h \cdot \mathbf{f}(\mathcal{V}^h) d\mathbf{x} + \oint_{\partial K} \langle \mathcal{V}^h, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) \rangle d\gamma - \oint_{\partial K} \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) d\gamma \\ &= - \oint_K \operatorname{div} (\langle \mathcal{V}^h, \mathbf{f}(\mathcal{V}^h) \rangle) d\mathbf{x} + \oint_K \mathcal{V}^h \cdot \operatorname{div} \mathbf{f}(\mathcal{V}^h) d\mathbf{x} + \oint_{\partial K} \langle \mathcal{V}^h, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) \rangle d\gamma - \oint_{\partial K} \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) d\gamma \\ &= - \oint_K \operatorname{div} (\langle \mathcal{V}^h, \mathbf{f}(\mathcal{V}^h) \rangle) d\mathbf{x} + \oint_K \operatorname{div} \mathbf{g}(\mathcal{V}^h) d\mathbf{x} + \oint_{\partial K} \langle \mathcal{V}^h, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) \rangle d\gamma - \oint_{\partial K} \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) d\gamma \\ &= - \underbrace{\left(\oint_K \operatorname{div} (\langle \mathcal{V}^h, \mathbf{f}(\mathcal{V}^h) \rangle) d\mathbf{x} + \int_K \operatorname{div} (\langle \mathcal{V}^h, \mathbf{f}(\mathcal{V}^h) \rangle) d\mathbf{x} \right)}_{(I)} + \underbrace{\left(\oint_K \operatorname{div} \mathbf{g}(\mathcal{V}^h) d\mathbf{x} - \int_K \operatorname{div} \mathbf{g}(\mathcal{V}^h) d\mathbf{x} \right)}_{(II)} \\ &\quad - \int_K (\operatorname{div} (\langle \mathcal{V}^h, \mathbf{f}(\mathcal{V}^h) \rangle) - \mathbf{g}(\mathcal{V}^h)) d\mathbf{x} + \oint_{\partial K} (\langle \mathcal{V}^h, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) \rangle d\gamma - \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) d\gamma) \\ &= (I) + (II) + \underbrace{\left(\int_{\partial K} (\mathcal{V}^h \mathbf{f}_{\mathbf{n}}(\mathcal{V}^h) - \mathbf{g}_{\mathbf{n}}(\mathcal{V}^h)) d\gamma - \oint_{\partial K} (\mathcal{V}^h \cdot \mathbf{f}_{\mathbf{n}}(\mathcal{V}^h) - \mathbf{g}_{\mathbf{n}}(\mathcal{V}^h)) d\gamma \right)}_{III} \\ &\quad + \oint_{\partial K} \left(\langle \mathcal{V}^h \cdot (\hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) - \mathbf{f}_{\mathbf{n}}(\mathcal{V}^h)) \rangle - (\hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) - \mathbf{g}_{\mathbf{n}}(\mathcal{V}^h)) \right) d\gamma \end{aligned} \tag{32}$$

Since $\mathcal{V}^h$ is a polynomial, if the quadrature formulas are of order k , we get that I , II and III in (32) are

$$I = O(h^{d+k+1}), \quad II = O(h^{d+k}), \quad III = O(h^{d+k}).$$

Let us have a look at the last term. If we define the entropy flux to be

$$\hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}^h, \mathcal{V}^{h,-}) = \langle \{\mathcal{V}^h\}, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) - \theta(\{\mathcal{V}^h\}) \rangle, \tag{33}$$

¹Note that we are computing the divergence (in $\mathbf{x}$) of the function $\langle \mathcal{V}^h, \mathbf{f}(\mathcal{V}^h) \rangle$

we have

$$\begin{aligned}
& -\langle \mathcal{V}^h, \mathbf{f}(\mathcal{V}^h) \rangle + \langle \mathcal{V}^h, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) + \mathbf{g}(\mathcal{V}^h) \cdot \mathbf{n} - \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) \rangle \\
& = \langle \mathcal{V}^h - \{\mathcal{V}^h\}, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K-}^h, \mathcal{V}_{|K-}^h) \rangle + (\theta(\{\mathcal{V}^h\}) - \theta(\mathcal{V}^h)) \cdot \mathbf{n} \\
& = \langle \mathcal{V}^h - \{\mathcal{V}^h\}, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) \rangle - \langle \nabla_{\mathbf{v}} \theta(\bar{\mathcal{V}}), \mathcal{V}^h - \{\mathcal{V}^h\} \rangle \\
& = \langle \mathcal{V}^h - \{\mathcal{V}^h\}, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) - \nabla_{\mathcal{V}} \theta(\bar{\mathcal{V}}) \rangle
\end{aligned}$$

where $\bar{\mathcal{V}} = t\{\mathcal{V}^h\} + (1-t)\mathcal{V}$ by abuse of language ². Since $\nabla_{\mathcal{V}} \theta = \mathbf{f}$, we have indeed

$$\begin{aligned}
& -\langle \mathcal{V}^h, \mathbf{f}(\mathcal{V}^h) \rangle + \langle \mathcal{V}^h, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) + \mathbf{g}(\mathcal{V}^h) \cdot \mathbf{n} - \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) \rangle \\
& = \langle \mathcal{V}^h - \{\mathcal{V}^h\}, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) - \mathbf{f}(\bar{\mathcal{V}}) \rangle = O(h^{2(k+1)})
\end{aligned}$$

because the numerical flux is Lipschitz continuous. Hence

$$\oint_{\partial K} \left(-\langle \mathcal{V}^h, \mathbf{f}(\mathcal{V}^h) \rangle + \langle \mathcal{V}^h, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) + \mathbf{g}(\mathcal{V}^h) \cdot \mathbf{n} - \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) \right) d\gamma = O(h^{d+2k+1})$$

provided suitable quadrature formulas. In the end we get

Proposition 3.7. *The scheme defined by (27), (29) is locally entropy conservative. In addition $\mathcal{E} = \sum_{\sigma \in K} \langle \mathcal{V}_{\sigma}, \Phi_{\sigma} \rangle - \oint_{\partial K} \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) d\gamma$, with*

$$\hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) = \langle \{\mathcal{V}\}, \hat{\mathbf{f}} \rangle - \theta(\{\mathcal{V}^h\})$$

satisfies

$$\mathcal{E} = O(h^{d+k+1}) \tag{34}$$

if the quadrature formula are of order k for the surface/volume integrals and $k+1$ for the boundary, so that the scheme (27)-eqrefent is stil of order $k+1$ at steady state.

Remark 3.8. 1. We do not use the fact we are solving a steady problem: the same would be true for the semi-discrete unsteady problem.

2. We have only used approximation properties: the same would be true when non linear limiting is applied, provided the formal order is kept for sommth solutions

3.3.2 Non linear RD schemes

Let us recall that for the non linear RD schemes defined in appendix A, the accuracy condition is that the boundary integrals are of order $k-1$, since then

$$\oint_{\partial K} \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) d\gamma = O(h^{k+d}).$$

²We write $\theta(\{\mathcal{V}^h\}) - \theta(\mathcal{V}^h)$ component by component. For the component $\#l$, we get

$$\theta_l(\{\mathcal{V}^h\}) - \theta_l(\mathcal{V}^h) = \left\langle \frac{\partial \theta_l}{\partial \mathcal{V}}(t_l \{\mathcal{V}^h\} + (1-t_l)\mathcal{V}), \{\mathcal{V}^h\} - \mathcal{V} \right\rangle,$$

$t_l \in [0, 1]$ Hence

$$\langle \mathcal{V}^h - \{\mathcal{V}^h\}, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) - \nabla_{\mathcal{V}} \theta(\bar{\mathcal{V}}) \rangle$$

means, by abuse of language

$$\sum_{l=1}^p (\mathcal{V}^h - \{\mathcal{V}^h\})_l \left((\hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h))_l - \frac{\partial \theta_l}{\partial \mathcal{V}}(t_l \{\mathcal{V}^h\} + (1-t_l)\mathcal{V}) \right).$$

If $\hat{\mathbf{g}}_{\mathbf{n}}$ is a Lipschitz continuous flux that is consistent with the entropy flux, we also have

$$\oint_{\partial K} \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) d\gamma = O(h^{k+d}).$$

Let us look at $\mathcal{E}$ in more detail when $\Phi_{\sigma} = \beta_{\sigma} \Phi$, $\sum_{\sigma \in K} \beta_{\sigma} = \text{Id}$. We have, defining again $\hat{\mathbf{g}} = \langle \{\mathcal{V}\}, \hat{\mathbf{f}} \rangle - \theta(\{\mathcal{V}^h\})$ and $\underline{\mathbf{V}} = \sum_{\sigma \in K} \beta_{\sigma} \mathcal{V}_{\sigma}$, and taking into account we assume a smooth steady solution

$$\begin{aligned} \mathcal{E} &= \oint_{\partial K} \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) d\gamma - \oint_{\partial K} \langle \underline{\mathbf{V}}, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K-}^h, \mathcal{V}_{|K-}^h) \rangle d\gamma \\ &= \oint_{\partial K} \left(\hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) - \mathbf{g}(\mathcal{V}) \right) d\gamma - \oint_{\partial K} \left(\langle \underline{\mathbf{V}}, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K-}^h, \mathcal{V}_{|K-}^h) \rangle - \langle \underline{\mathbf{V}}, \mathbf{f}(\mathcal{V}) \rangle \right) d\gamma. \end{aligned}$$

We look at each term. First, using Taylor formula, and the fact that $\nabla_{\mathcal{V}} \theta(\mathcal{V}) = \mathbf{f}(\mathcal{V})$,

$$\begin{aligned} \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) - \mathbf{g}(\mathcal{V}) &= \langle \{\mathcal{V}^h\}, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) \rangle - \theta(\{\mathcal{V}^h\}) - \langle \mathcal{V}, \mathbf{f}(\mathcal{V}) \rangle + \theta(\mathcal{V}) \\ &= \langle \{\mathcal{V}^h\} - \mathcal{V}, \mathbf{f}(\mathcal{V}) \rangle + \langle \{\mathcal{V}^h\}, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) - \mathbf{f}(\mathcal{V}) \rangle \\ &\quad + \langle \nabla_{\mathcal{V}} \theta(\mathcal{V}), \mathcal{V} - \{\mathcal{V}^h\} \rangle + O((\mathcal{V} - \{\mathcal{V}^h\})^2) \\ &= \langle \{\mathcal{V}^h\}, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) - \mathbf{f}(\mathcal{V}) \rangle + O((\mathcal{V} - \{\mathcal{V}^h\})^2) \end{aligned}$$

so that

$$\begin{aligned} \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) - \mathbf{g}(\mathcal{V}) - \langle \mathcal{V}, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K-}^h, \mathcal{V}_{|K-}^h) - \mathbf{f}(\mathcal{V}) \rangle &= \langle \{\mathcal{V}^h\} - \mathcal{V}, \hat{\mathbf{f}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) - \mathbf{f}(\mathcal{V}) \rangle + O((\mathcal{V} - \{\mathcal{V}^h\})^2) \\ &= O(h) \times O(h^{k+1}) \end{aligned}$$

if the boundary quadrature formula is of order k . We have

Proposition 3.9. *The scheme defined by (27), (29) is locally entropy conservative. In addition, $\mathcal{E} = \sum_{\sigma \in K} \langle \mathcal{V}_{\sigma}, \Phi_{\sigma} \rangle - \oint_{\partial K} \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) d\gamma$, with*

$$\hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) = \langle \{\mathcal{V}\}, \hat{\mathbf{f}} \rangle - \theta(\{\mathcal{V}^h\})$$

satisfies

$$\mathcal{E} = O(h^{k+d+1}) \quad (35)$$

if the boundary quadrature formula is of order k .

Remark 3.10. *Here we must use the fact we are solving a steady problem.*

3.4 Construction of entropy stable schemes

Starting from the residuals $\{\Phi_{\sigma}^K\}_{K \in \mathcal{T}_h, \sigma \in K}$, we want to construct a scheme $\{\Phi_{\sigma}^{'K}\}_{K \in \mathcal{T}_h, \sigma \in K}$, that is entropy stable, i.e., for any K ,

$$\sum_{\sigma \in K} \langle \mathcal{V}_{\sigma}, \Phi_{\sigma}^{'K}(\mathcal{V}^h) \rangle \geq \oint_{\partial K} \hat{\mathbf{g}}_{\mathbf{n}}(\mathcal{V}_{|K}^h, \mathcal{V}_{|K-}^h) d\gamma.$$

Starting from the previous construction, and dropping the superscript K because we will consider only quantities in K , the idea is to write the new residuals as:

$$\Phi_{\sigma}'(\mathcal{V}^h) = \Phi_{\sigma} + \mathbf{r}_{\sigma} + \Psi_{\sigma} \quad (36)$$

where the $\mathbf{r}_\sigma$ are defined by (29) and the Ψ_σ satisfy

$$\sum_{\sigma \in K} \Psi_\sigma = 0 \text{ and } \sum_{\sigma \in K} \langle \mathcal{V}_\sigma, \Psi_\sigma(\mathcal{V}^h) \rangle \geq 0$$

without violating the accuracy condition (25).

Let us consider these two expressions for $\Psi_\sigma(\mathcal{V}^h)$ which obviously satisfy the conservation requirement

$$\sum_{\sigma \in K} \Psi_\sigma(\mathcal{V}^h) = 0$$

since $\sum_{\sigma \in K} \varphi_\sigma = 1$:

1. With jumps: We define

$$\Psi_\sigma = \theta h_K^2 \oint_{\partial K} [\nabla_{\mathbf{x}} \varphi_\sigma] \cdot [\nabla_{\mathbf{x}} \mathcal{V}^h] d\gamma \quad (37)$$

We have

$$\sum_{\sigma \in K} \langle \mathcal{V}_\sigma, \Psi_\sigma(\mathcal{V}^h) \rangle = \theta h_K^2 \sum_{e \text{ edge of } K} \oint_K [\nabla_{\mathbf{x}} \mathcal{V}^h]^2 d\gamma \geq 0$$

for any $\theta > 0$. It is also easy to check that if $\mathcal{V}^h$ is an approximation of order k of a smooth $\mathcal{V}$, we have, *provided the quadrature formula is of order k* that

$$\Psi_\sigma = O(h^{k+d})$$

so that the accuracy requirement are met.

2. With streamline: we define

$$\Psi_\sigma = \theta h_K \oint_K (\nabla_{\mathcal{V}} \mathbf{f}(\mathcal{V}^h) \cdot \nabla_{\mathbf{x}} \varphi_\sigma) \tau_K (\nabla_{\mathcal{V}} \mathbf{f}(\mathcal{V}^h) \cdot \nabla_{\mathbf{x}} \mathcal{V}^h) d\mathbf{x}. \quad (38)$$

Clearly,

$$\sum_{\sigma \in K} \langle \mathcal{V}_\sigma, \Psi_\sigma(\mathcal{V}^h) \rangle = \theta h_K \oint_K (\nabla_{\mathcal{V}} \mathbf{f}(\mathcal{V}^h) \cdot \nabla_{\mathbf{x}} \mathcal{V}_\sigma) \tau_K (\nabla_{\mathcal{V}} \mathbf{f}(\mathcal{V}^h) \cdot \nabla_{\mathbf{x}} \mathcal{V}^h) d\mathbf{x} \geq 0$$

for any $\theta > 0$ and any $\tau > 0$. In practice, we take τ_K as the scalar/matrix defined by

$$\tau^{-1} = \sum_{\sigma' \in K} K_{\sigma'}^-, K_\sigma = \frac{1}{|K|} \int_K \overline{\nabla_{\mathbf{u}} \mathbf{f}} \cdot \nabla_{\mathbf{x}} \varphi_\sigma d\mathbf{x}, \quad K_\sigma^+ = \max(K_\sigma, 0), K_\sigma^- = \min(K_\sigma, 0).$$

It is also easy to check that if $\mathcal{V}^h$ is an approximation of order k of a smooth $\mathcal{V}$, we have, *provided the quadrature formula is of order k* that

$$\Psi_\sigma = O(h^{k+d})$$

so that the accuracy requirement are met.

In practice, the quadrature formula involved in the edge integrals and the surface integrals need to be such that:

- For $\oint_e [\nabla_{\mathbf{x}} \mathcal{V}^h]^2 d\gamma$: if $\mathcal{V}^h$ is not constant,

$$\oint_e [\nabla \mathcal{V}^h]^2 d\gamma > 0.$$

- For $\oint_K (\nabla_{\mathcal{V}} \mathbf{f}(\mathcal{V}^h) \cdot \nabla_{\mathbf{x}} \mathcal{V}^h) \tau_K (\nabla_{\mathcal{V}} \mathbf{f}(\mathcal{V}^h) \cdot \nabla_{\mathbf{x}} \mathcal{V}^h) d\mathbf{x}$: if $\nabla_{\mathcal{V}} \mathbf{f}(\mathcal{V}^h) \cdot \nabla_{\mathbf{x}} \mathcal{V}^h$ is not identically 0, then

$$\oint_K (\nabla_{\mathcal{V}} \mathbf{f}(\mathcal{V}^h) \cdot \nabla_{\mathbf{x}} \mathcal{V}_\sigma) \tau_K (\nabla_{\mathcal{V}} \mathbf{f}(\mathcal{V}^h) \cdot \nabla_{\mathbf{x}} \mathcal{V}^h) d\mathbf{x} > 0.$$

In [25], we have provided minimal conditions on these quadrature formula, such that this conditions are met. Note that these quadrature formulas may be non-consistent: the only things we need is to keep the accuracy, and have a strictly positive entropy production.

4 Numerical illustrations

In this section, we illustrate the behaviour of the methods on non linear examples. We first study the conservation issues, for mass and entropy, and then show the behavior of the scheme on a Burger-like problem.

4.1 Conservation

In $\Omega = [-20, 20] \times [-20, 20]$ we look at the problem

$$\frac{\partial u}{\partial t} + \frac{\partial \sqrt{u}}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad (39a)$$

with

$$u_0(x, y) = \begin{cases} 1 & \text{if } r = \sqrt{(x+5)^2 + (y+5)^2} > 0.5 \\ 1 + 2 \sin\left(\frac{\pi}{4}(1-2r)\right) & \text{else.} \end{cases} \quad (39b)$$

$u(x, y, t) = u_0(x, y)$ for $t \geq 0$, and $\mathbf{x} \in \partial\Omega$.

We integrate for $t \in [0, 5]$: at time $t = 5$, the non constant part of the solution has not reached the boundary, this is why we have taken such a large domain. We want to avoid any interference with the boundary conditions. The flux is non polynomial, so that any standard quadrature rule will *never* be exact.

Following the notations of (8), we integrate in time with an Euler forward method

$$\mathbf{u}_\sigma^{n+1} = \mathbf{u}_\sigma^n - \frac{\Delta t}{C_\sigma} \left(\sum_{K \subset \Omega, \sigma \in K} \Phi_\sigma^K(\mathbf{u}^{n,h}) + \sum_{\Gamma \subset \partial\Omega, \sigma \in \Gamma} \Phi_\sigma^\Gamma(\mathbf{u}^{n,h}) \right), \quad (40)$$

where C_σ is a dual control volume. This method is not accurate in time, but we can check numerically the conservation of mass,

$$M = \sum_{\sigma \in \Omega} |C_\sigma| \mathbf{u}_\sigma.$$

An entropy for (39a) is $U(\mathbf{u}) = \frac{\mathbf{u}^2}{2}$, the entropy flux is $\mathbf{g}(\mathbf{u}) = (\frac{\mathbf{u}^2}{2}, \frac{1}{3}\mathbf{u}\sqrt{\mathbf{u}})$. Since

$$\langle \mathcal{V}(\mathbf{u}^n), \mathbf{u}^{n+1} - \mathbf{u}^n \rangle \leq U(\mathbf{u}^{n+1}) - U(\mathbf{u}^n)$$

we will not get conservation of entropy, but we will check that, the entropy residuals defined by (27) Φ'_σ are such that

$$\sum_{\sigma \in \Omega} \left(\sum_{K \subset \Omega, \sigma \in K} \langle \mathcal{V}_\sigma, \Phi_\sigma^K(\mathbf{u}^{n,h}) \rangle + \sum_{\Gamma \subset \partial\Omega, \sigma \in \Gamma} \langle \mathcal{V}_\sigma, \Phi_\sigma^\Gamma(\mathbf{u}^{n,h}) \rangle \right) = 0. \quad (41)$$

Up to our knowledge, any of the so-called entropy conservative schemes show rigorous conservation of the spatial discretisation terms only. To get also entropy conservation at the time-discrete level, it seems that the scheme must be implicit in time, see [17]. This is the reason why we are not showing

$$E = \int_{\Omega} U(\mathbf{u}^h) d\mathbf{x}$$

but how well (41) is fulfilled in figure 1. We see that the mass is always conserved up to machine accuracy, while the correction has a clear effect on the spatial conservation of entropy. This was shown for the pure Galerkin scheme.

Figure 1: Plot of (41) for the pure galerkin scheme, with and without the correction

4.2 Accuracy

In this section we want to show that the correction does not spoil the accuracy. This is done on the following test case: in $\Omega = [0, 1] \times [0, 1]$,

$$\frac{\partial \sinh u}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad (42a)$$

with the boundary condition, for $y = 0$,

$$u(x, 0) = x - \frac{1}{2} \quad (42b)$$

The solution is regular and computed by the method of characteristics. We have chosen the exponential flux to make sure that none of the standard quadrature formula is exact. The quadrature formula are:

- Linear approximation. the quadrature point in the triangle K is its centroid, and the weight is 1. On the boundary, we have chosen the Gaussian formula associated to the Legendre polynomials of degree 2: there are two points of weight $\frac{1}{2}$ that are the image of $\pm \frac{1}{\sqrt{3}}$ in the mappings from $[-1, 1]$ to any of the edges of K .
- Quadratic approximation. The surface integrals are computed with 7 quadrature points, the formula is exact for degree 5

- The centroid with weight 0.225,
- The points defined by the barycentric coordinates (a, b, b) ,

$$a = 0.797426985353087, \quad a + 2b = 1$$

and cyclic permutations of the coordinates. The weights are $\omega = 0.125939180544827$.

- The points defined by the barycentric coordinates (a, b, b) ,

$$a = 0.059715871789770, \quad a + 2b = 1$$

and cyclic permutations of the coordinates. The weights are $\omega = 0.132394152788506$.

The integral on the edges are evaluated by the same Gaussian formula as before.

h	n_{dofs}	L^1	slope	L^2	slope	L^∞	slope
$7.06 \cdot 10^{-2}$	525	$9.16 \cdot 10^{-5}$	—	$1.39 \cdot 10^{-4}$	—	$1.14 \cdot 10^{-3}$	—
$3.53 \cdot 10^{-2}$	2017	$2.12 \cdot 10^{-5}$	2.11	$3.30 \cdot 10^{-5}$	2.07	$3.23 \cdot 10^{-4}$	1.81
$1.76 \cdot 10^{-2}$	7905	$5.04 \cdot 10^{-6}$	2.06	$7.96 \cdot 10^{-6}$	2.04	$8.75 \cdot 10^{-5}$	1.87
$8.82 \cdot 10^{-3}$	31297	$1.51 \cdot 10^{-6}$	1.74	$2.76 \cdot 10^{-6}$	1.53	$3.13 \cdot 10^{-5}$	1.5

Table 1: Error for problem (42), linear approximation.

h	n_{dofs}	L^1	slope	L^2	slope	L^∞	slope
$1.41 \cdot 10^{-1}$	525	$1.35 \cdot 10^{-5}$	—	$2.46 \cdot 10^{-5}$	—	$2.15 \cdot 10^{-4}$	—
$7.06 \cdot 10^{-2}$	2017	$1.88 \cdot 10^{-6}$	2.84	$3.62 \cdot 10^{-6}$	2.76	$4.08 \cdot 10^{-5}$	2.39
$3.53 \cdot 10^{-2}$	7905	$2.33 \cdot 10^{-7}$	3.01	$4.77 \cdot 10^{-7}$	2.92	$6.28 \cdot 10^{-6}$	2.69
$1.76 \cdot 10^{-2}$	31297	$3.12 \cdot 10^{-8}$	2.88	$6.35 \cdot 10^{-8}$	2.90	$8.88 \cdot 10^{-7}$	2.81

Table 2: Error for problem (42), quadratic approximation.

We have used the scheme 36 where Φ_σ is obtained by the Galerkin formulation using the above quadrature and Ψ_σ is defined by (37) with $\theta = 0.01$. The boundary conditions are implemented with a local Lax Friedrich numerical scheme. The errors are given in table 1 and 2.

As expected the optimal orders are reached. We can also observe the improved accuracy for a given number of freedom, between second and third order.

4.3 Behavior on discontinuous solutions

The non linear example is an adaptation of the Burgers equation:

$$\begin{aligned}
\frac{\partial u}{\partial y} + \frac{\partial \sinh u}{\partial x} &= 0 & \text{if } x \in [0, 1]^2 \\
u(x, y) &= \frac{1}{2} - x & \text{on the inflow boundary.}
\end{aligned} \tag{43}$$

If the y -coordinate corresponds to time, we are back to a more standard formulation.

The exact solution consists in a fan that merges into a shock. We have performed the simulations using a quadratic approximation. The mesh (i.e. all the degrees of freedom) is displayed in figure 2.

Figure 2: Mesh for the numerical experiments. All the Lagrange degrees of freedom are represented. From the representation, we evaluate the solution at the $\mathbb{P}^2$ degrees of freedom and then plot the solution using a local $\mathbb{P}^1$ representation in all the triangles represented on the figure.

(a) SUPG- $\theta = 0.1$

(b) SUPG- $\theta = 0.01$

(c) SUPG- $\theta = 0.02$

(d) GALJ- $\theta = 0.01$

Figure 3: Results without entropy correction, linear schemes

The numerical solutions are obtained by several methods:

- The SUPG scheme with different values of θ in (38).
- The Galerkin scheme with jumps and different values of θ in (37)
- The non-linear RD scheme where the first order scheme is the Rusanov one (see appendix A)
- The non-linear RD scheme with jump filtering, (19), the jumps being tuned by θ
- The non-linear RD scheme with streamline filtering, (18), the jumps being tuned by θ

These schemes are nicknamed respectively as SUPG- θ , GalJ- θ , RD, RD-J- θ and RD-S- θ . We also have made a second set of simulations where the schemes are modified by adding the entropy correction. More precisely, this is done after the evaluation of the Galerkin term for the SUPG- θ and GALJ- θ schemes, and after the nonlinear RD step of the RD-J- θ and RD-S- θ steps, i.e. before the filtering step in all cases, as described in section 3.4. These schemes are nicknamed as SUPG-E- θ , GalJ-E- θ , RD, RD-J-E- θ and RD-S-E- θ respectively. Since the numerical flux is not polynomial, the Galerkin step is only approximate: the entropy correction is a priori acting. We have used the same quadrature points as in section 4.2.

Our goal is to see if we can lower the parameter θ in the filtering term because we have a fine tuning of the local entropy production. All the simulations are done with the Euler forward time stepping, CFL=0.3. The plots use the same isolines between 2 and -2, 100 isolines: we can see the numerical oscillations, if any.

(a) RD scheme

(b) RDJ-0.1

(c) RDJ-0.02

(d) RDJS-0.02-0.02

Figure 4: Results without entropy correction, non linear schemes

From figure 3, we see that the SUPG scheme without correction and $\theta_S = 0.1$ provides very good results in the fan but is almost unstable in the shock (the calculation has been stopped since it is meaningless). When we lower the parameter θ to 0.01, the shock does not improve, and some wiggles exist in the fan, so we are not filtering enough. A relatively good compromise is reached for $\theta = 0.02$. This is in contrast with the GalJ-0.01 result that are comparable to the SUPG-0.02 ones (with smaller oscillations at the shock, by the way).

From figure 4, we see that $\theta = 0$ leads to wiggles in the fan. They are due to spurious modes, and not a stability problem, see [35, 36]. We have tried several values of θ for the streamline and the jump term, it seems that the best results are obtained for $\theta = 0.1$, both for the fan and the discontinuity (remember that we have 100 isolines between -2 and 2 , so 25 between -0.5 and 0.5).

From figure 5, we see that Gal-E-0.01 behaves much better than SUPG-E-0.01. This is another indication (comparing to figure 3-(b) that the streamline term is not sufficient). The solution is much better with $\theta = 0.02$, see figure 5-(c), but it is similar to 3-(c). We have also experienced similar disappointments with the unsteady version of these schemes in [37] where the jump term seems to filter out more efficiently. Comparing figures 4-(c) and 5-(d), we see that the quality of the fan has improved with the entropy correction, while, of course, the shock becomes wiggly. If we decrease the value of θ , then the solution becomes similar to 5-(b).

(a) GALJ-E- $\theta = 0.01$

(b) SUPG-E- $\theta = 0.01$

(c) SUPG-E- $\theta = 0.02$

(d) RDJ-E-0.02

Figure 5: Results with with the entropy term

5 Conclusion

We have developed a general technique to guaranty that the discretisation of a conservative hyperbolic problem is consistent with an additional conservation relation. This is a generalisation of the work [34] and a sequel of [32]. This construction uses in depth a reinterpretation of conservative schemes as Residual distribution schemes.

Of course, the modifications will depend on the problem. Aiming at illustrating this idea, this is why we have focussed on an entropy equality and have constructed explicit modifications of the original discretisation that guaranty a consistency with the original PDE and the additional entropy inequality. We also have shown that independently of the choice of the entropy flux, one lose one order of accuracy. However, in the case of discontinuous Galerkin schemes, and stabilized residual distribution schemes, we can provide an explicit form of the entropy flux that does not lead to a degradation of the accuracy. To achieve this goal, we do not need any particular structure of the quadrature formula, in particular they do not need to be exact nor to achieve the summation-by-part property.

We have tested the method on scalar problem, the formula we have given also work for systems. We have shown, numerically, at least for finite element like methods, that the optimal accuracy is kept. We have also checked that the method is globally conservative for the original conserved variable and the entropy. Using the results of [32], it is easy to see that, at least for the steady case, that these schemes can be reformulated as finite volume schemes. In the case of entropy stable scheme, we can get an *explicit* form of the numerical flux for the conserved variable and the entropy.

We have also shown that adding this entropy correction can help to lower the amount of artificial diffusion that is needed to filter spurious modes for the Galerkin method and the non linear residual distribution ones. Our study also confirm that the stabilisation by jump is more efficient than the streamline diffusion one. One reason is probaly that the jump stabilisation allow inter-element exchange of informations, contrarily to the streamline one.

The emphasis of this paper is put on the steady case, but the unsteady state is similar, see [38] and [37, 39]. However, if the unsteady case can be considered as well, this is to the price of having implicit formulations, in the spirit of [34]. This issue will be considered in future work. Future work will also deal with the question of how to have an entropy conservative scheme in smooth regions, and how to guaranty non oscillatory behaviour in discontinuous ones

We conclude this work in noticing that one of the ingredients of the method is that any element has at least 2 vertices: the conservation relation translates by a linear relations on the residuals, and the modification by another one. Having more than 2 degrees of freedom, one solution is a priori possible. As a consequence, it could be possible, using the same kind of technique, to guaranty more that one additional conservation relation, this will also be considered for future work.

Acknowledgements

The author has been funded in part by the SNSF project 200021_153604 "High fidelity simulation for compressible materials". I would also like to thanks Anne Burbeau (CEA-DEN) for her critical reading of the first draft of this paper. Her input has hopefully helped to improve the readability of this paper. I also warmly thanks the three anonymous referees whose comments and criticisms have, I hope, improved the original version of the paper. The remaining mistakes are mine.

A personal note to end: though the author is curently editor in chief of this Journal, the EES system has been parametrised in such a way that I, of course, have absolutely no access to the identity of the referees, nor to the identity of the editor in charge of the paper. This is, of course, the only paper of this type. Hence I would like to thanks the Elsevier team for making this singular situation possible.

References

- [1] T. Chen and C.-W. Shu. Entropy stable high order discontinuous Galerkin methods with suitable quadrature rules for hyperbolic conservation laws. *J. Comput. Phys.*, 345:427 – 461, 2017.
- [2] A. Hildebrand and S. Mishra. Entropy stable shock capturing spacetime discontinuous Galerkin schemes for systems of conservation laws. *Numer. Math.*, 126:103–151, 2014.
- [3] G.J. Gassner. A skew-symmetric discontinuous Galerkin spectral element discretisation and its relation to SBP-SAT finite difference methods. *SIAM J. Sci. Comput.*, 35:A1233–A1253, 2013.
- [4] J.E. Hicke, D.C. Del Rey, and D.W. Zingg. Multidimensional summation-by-part operators: general theory and application to simplex elements. *SIAM J. Sci. Comput.*, 38:A1935–A1958, 2016.
- [5] E. Tadmor. *Handbook of Numerical Methods for Hyperbolic Problems, Basic and fundamental issues*, volume 17 of *Handbook of Numerical Analysis*, chapter Entropy Stable Schemes, pages 467–490. North-Holland, 2016.
- [6] E. Tadmor. The numerical viscosity of entropy stable schemes for systems of conservation laws, I. *Math. Comp.*, 49:91–103, 1987.
- [7] E. Tadmor. Entropy stability theory for difference approximations of nonlinear conservation laws and related time-dependent problems. *Acta Numerica*, 13:451–512, 2003.
- [8] G.S. Jiang and C.-W. Shu. On a cell entropy inequality for discontinuous Galerkin methods. *Math. Comput.*, 62:531–53, 1994.
- [9] F. Bouchut, C. Bourdarias, and B. Perthame. A MUSCL method satisfying all the numerical entropies. *Math. Comp.*, 65(1439-1461), 1996.
- [10] P.G. Lefloch, J.-M. Mercier, and C. Rohde. Fully discrete, entropy conservative schemes of arbitrary order. *SIAM J. Numer. Anal.*, 40:1968–1992, 2002.
- [11] S. Hou and X.-D. Liu. Solutions of multi-dimensional hyperbolic systems of conservation laws by square entropy condition satisfying discontinuous Galerkin method. *J. Sci. Comput.*, 31:127–151, 2007.
- [12] F. Ismail and P.L. Roe. Affordable, entropy-consistent Euler flux functions II: entropy production at shocks. *J. Comput. Phys.*, 228:5410–5436, 2009.
- [13] J.-L. Guermond, R. Pasquetti, and B. Popov. Entropy viscosity method for nonlinear conservation laws. *J. Comput. Phys.*, 230:4248–4267, 2011.
- [14] H. Ranocha, Ph. Öffner, and Th. Sonar. Summation-by-parts operators for correction procedure via reconstruction. *J. Comput. Phys.*, 311:299–328, 2016.
- [15] M. Parsani, M. H. Carpenter, T. C. Fisher, and Eric J. Nielsen. Entropy stable staggered grid discontinuous spectral collocation methods of any order for the compressible Navier-Stokes equations. *SIAM J. Sci. Comput.*, 38(5):a3129–a3162, 2016.
- [16] M. Svärd and J. Nordström. Review of summation-by-parts schemes for initial-boundary-value problems. *J. Comput. Phys.*, 268:17–38, 2014.
- [17] A. Hildebrand and S. Mishra. Entropy stable shock capturing spacetime discontinuous Galerkin schemes for systems of conservation laws. *Numer. Math.*, 126(1):103–151, 2014.
- [18] M. Svärd and H. Özcan. Entropy-stable schemes for the Euler equations with far-field and wall boundary conditions. *J. Sci. Comput.*, 58:61–89, 2014.

- [19] Deep Ray and Praveen Chandrashekar. Entropy stable schemes for compressible Euler equations. *Int. J. Numer. Anal. Model., Ser. B*, 4(4):335–352, 2013.
- [20] Deep Ray, Praveen Chandrashekar, Ulrik S. Fjordholm, and Siddhartha Mishra. Entropy stable scheme on two-dimensional unstructured grids for Euler equations. *Commun. Comput. Phys.*, 19(5):1111–1140, 2016.
- [21] Gregor J. Gassner, Andrew R. Winters, and David A. Kopriva. A well balanced and entropy conservative discontinuous Galerkin spectral element method for the shallow water equations. *Applied Mathematics and Computation*, 272:291 – 308, 2016. Recent Advances in Numerical Methods for Hyperbolic Partial Differential Equations.
- [22] P. Chandrasekar. Kinetic energy preserving and entropy stable finite volume scheme for compressible Euler and Navier-Stokes equations. *Commun. Comput. Phys.*, 14:1252–1286, 2013.
- [23] P. Ciarlet. *The finite element method for elliptic problems*. North-Holland, Amsterdam, 1978.
- [24] A. Ern and J.L. Guermond. *Theory and practice of finite elements*, volume 159 of *Applied Mathematical Sciences*. Springer verlag, 2004.
- [25] R. Abgrall, A. Larat, and M. Ricchiuto. Construction of very high order residual distribution schemes for steady inviscid flow problems on hybrid unstructured meshes. *J. Comput. Phys.*, 230(11):4103–4136, 2011.
- [26] R. Abgrall and D. de Santis. High-order preserving residual distribution schemes for advection-diffusion scalar problems on arbitrary grids. *SIAM J. Sci. Comput.*, 36(3):A955–A983, 2014. also <http://hal.inria.fr/docs/00/76/11/59/PDF/8157.pdf>.
- [27] T.J.R. Hughes, L.P. Franca, and M. Mallet. A new finite element formulation for CFD: I. symmetric forms of the compressible Euler and Navier-Stokes equations and the second law of thermodynamics. *Comput. Methods. Appl. Mech. Engrg.*, 54:223–234, 1986.
- [28] E. Burman and P. Hansbo. Edge stabilization for Galerkin approximation of convection-diffusion-reaction problems. *Comput. Methods Appl. Mech. Engrg.*, 193:1437–1453, 2004.
- [29] R. Abgrall. Essentially non-oscillatory residual distribution schemes for hyperbolic problems. *J. Comput. Phys.*, 214(2):773–808, 2006.
- [30] R. Abgrall and P. L. Roe. High-order fluctuation schemes on triangular meshes. *J. Sci. Comput.*, 19(1-3):3–36, 2003.
- [31] D. Kröner, M. Rokyta, and M. Wierse. A Lax-Wendroff type theorem for upwind finite volume schemes in 2-d. *East-West J. Numer. Math.*, 4(4):279–292, 1996.
- [32] R. Abgrall. Some remarks about conservation for residual distribution schemes. *Computational Methods in Applied Mathematics*, 2017. published online 2017-12-06, doi:<https://doi.org/10.1515/cmam-2017-0056>.
- [33] R. Abgrall and S. Tokareva. Staggered grid residual distribution scheme for lagrangian hydrodynamics. *SIAM J. Sci. Comput.*, 39(5):A2317–A2344, 2017. see also <https://hal.inria.fr/hal-01327473>.
- [34] R. Abgrall, P. Baccigalupi, and S. Tokareva. A high-order nonconservative approach for hyperbolic equations in fluid dynamics. *Computers and Fluids*, doi:<https://doi.org/10.1016/j.compfluid.2017.08.019>, 2018. see also <https://hal.archives-ouvertes.fr/hal-01476636v1>.
- [35] R. Abgrall. Essentially non oscillatory residual distribution schemes for hyperbolic problems. *J. Comput. Phys.*, 214(2):773–808, 2006.

- [36] R. Abgrall. About non linear stabilization for scalar hyperbolic problems. <https://hal.archives-ouvertes.fr/hal-01572473>, July 2016.
- [37] R. Abgrall. High order schemes for hyperbolic problems using globally continuous approximation and avoiding mass matrices. *J. Sci. Comput.*, 73, 2017. <https://hal.archives-ouvertes.fr/hal-01445543v2>.
- [38] M. Ricchiuto and R. Abgrall. Explicit Runge-Kutta residual distribution schemes for time dependent problems: second order case. *J. Comput. Phys.*, 229(16):5653–5691, 2010.
- [39] R. Abgrall, P. Bacigaluppi, and S. Tokareva. High-order residual distribution scheme for the time-dependent Euler equations of fluid dynamics. *Computer & Mathematics with Applications*, 2018. doi: <https://doi.org/10.1016/j.camwa.2018.05.009>.
- [40] R. Abgrall. On a class of high order schemes for hyperbolic problems. In *Proceedings of the International Conference of Mathematicians, volume IV*, pages 699–726, Seoul, 2014.

A Construction of the non linear stabilisation for RD schemes

Here we consider a globally continuous approximation: $\mathbf{u}^h \in \mathcal{V}_h \cap C^0(\Omega)$.

Consider one element K . Since there is no ambiguity, the drop, for the residuals, any reference to K in the following. The total residual is defined by

$$\Phi(\mathbf{u}^h) = \int_{\partial K} \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n} \, d\gamma,$$

and we assume to have monotone residuals $\{\Phi_\sigma^L\}_{\sigma \in K}$. By this we mean

$$\Phi_\sigma^L(\mathbf{u}^h) = \sum_{\sigma' \in K} c_{\sigma\sigma'}^L (\mathbf{u}_\sigma - \mathbf{u}_{\sigma'})$$

with $c_{\sigma\sigma'} \geq 0$ that also satisfies

$$\sum_{\sigma \in K} \Phi_\sigma^L = \Phi.$$

It can easily be shown that the condition $c_{\sigma\sigma'} \geq 0$ guaranties that the scheme is monotone under a CFL like condition. One example is given by the Rusanov residuals:

$$\Phi_\sigma^{Rus}(\mathbf{u}^h) = - \int_K \nabla \varphi_\sigma \cdot \mathbf{f}(\mathbf{u}^h) \, d\mathbf{x} + \int_{\partial K} \varphi_\sigma \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n} \, d\gamma + \alpha(\mathbf{u}_\sigma - \bar{\mathbf{u}}),$$

where $\bar{\mathbf{u}}$ is the arithmetic average of of the $\mathbf{u}'_\sigma$ s on K and α satisfies:

$$\alpha \geq \#K \max_{\sigma, \sigma' \in K} \left| \int_K \varphi_\sigma \nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}^h) \, d\mathbf{x} \right|.$$

Here $\#K$ is the number of degrees of freedom in K . Indeed, this residual can be rewritten as

$$\Phi_\sigma^{Rus}(\mathbf{u}^h) = \sum_{\sigma' \in K} c_{\sigma\sigma'} (\mathbf{u}_\sigma - \mathbf{u}_{\sigma'})$$

with

$$c_{\sigma\sigma'} = \int_K \varphi_\sigma \cdot \nabla_{\mathbf{u}} \mathbf{f}(\mathbf{u}^h) \, d\mathbf{x} + \frac{\alpha}{\#K}.$$

Under the condition above, $c_{\sigma\sigma'} \geq 0$ and hence we have a maximum principle.

The coefficients β_σ introduced in the relations (18) and (19) are defined by:

$$\beta_\sigma = \frac{\max(0, \frac{\Phi_\sigma^L}{\Phi})}{\sum_{\sigma' \in K} \max(0, \frac{\Phi_{\sigma'}^L}{\Phi})}.$$

and can be shown to be always defined, to guaranty a local maximum principle for (18) and (19), see [25].

Remark A.1 (About the coefficients $c_{\sigma\sigma'}^L$). *All the examples of monotone residual we are aware of are such that for linear problems, the $c_{\sigma\sigma'}^L$ are independant of $\mathbf{u}^h$. Then one can show that for any $\sigma \in K$,*

$$\sum_{\sigma' \in K} (c_{\sigma\sigma'}^L - c_{\sigma'\sigma}^L) = \int_K \nabla \mathbf{f}(\mathbf{u}^h) \cdot \nabla \varphi_\sigma d\mathbf{x}.$$

This relation implies the conservation relation (7a).

B Examples of finite volume writing of Residual distribution schemes

This part is taken from [32]. As a motivation, we first show that any finite volume scheme can be written as a residual distribution scheme. Then we give two example of the converse. The general statement can be found in [32], it applies also to discontinuous representations, see the same reference

B.1 Finite volume as Residual distribution schemes

The notations are defined in Figure 7. We specialize ourselves to the case of triangular elements, but *exactly*

Figure 6: Notations for the finite volume schemes. On the left: definition of the control volume for the degree of freedom σ . The vertex σ plays the role of the vertex 1 on the left picture for the triangle K. The control volume C_σ associated to $\sigma = 1$ is green on the right and corresponds to $1PGR$ on the left. The vectors $\mathbf{n}_{ij}$ are normal to the internal edges scaled by the corresponding edge length

the same arguments can be given for more general elements, provided a conformal approximation space can be constructed. This is the case for triangle elements, and we can take $k = 1$.

The control volumes in this case are defined as the median cell, see figure 7. We concentrate on the approximation of $\text{div } \mathbf{f}$, see equation (1). Since the boundary of C_σ is a closed polygon, the scaled outward normals $\mathbf{n}_\gamma$ to ∂C_σ sum up to 0:

$$\sum_{\gamma \subset \partial C_\sigma} \mathbf{n}_\gamma = 0$$

where γ is any of the segment included in ∂C_σ , such as PG on Figure 7. Hence

$$\begin{aligned} \sum_{\gamma \subset \partial C_\sigma} \hat{\mathbf{f}}_{\mathbf{n}_\gamma}(\mathbf{u}_\sigma, \mathbf{u}_-) &= \sum_{\gamma \subset \partial C_\sigma} \hat{\mathbf{f}}_{\mathbf{n}_\gamma}(\mathbf{u}_\sigma, \mathbf{u}_-) - \left(\sum_{\gamma \subset \partial C_\sigma} \mathbf{n}_\gamma \right) \cdot \mathbf{f}(\mathbf{u}_\sigma) \\ &= \sum_{K, \sigma \in K} \sum_{\gamma \subset \partial C_\sigma \cap K} (\hat{\mathbf{f}}_{\mathbf{n}_\gamma}(\mathbf{u}_\sigma, \mathbf{u}_-) - \mathbf{f}(\mathbf{u}_\sigma) \cdot \mathbf{n}_\gamma) \end{aligned}$$

To make things explicit, in K , the internal boundaries are PG , QG and RG , and those around $\sigma \equiv 1$ are PG and RG . We set

$$\begin{aligned} \Phi_\sigma^K(\mathbf{u}^h) &= \sum_{\gamma \subset \partial C_\sigma \cap K} (\hat{\mathbf{f}}_{\mathbf{n}_\gamma}(\mathbf{u}_\sigma, \mathbf{u}_-) - \mathbf{f}(\mathbf{u}_\sigma) \cdot \mathbf{n}_\gamma) \\ &= \sum_{\gamma \subset \partial(C_\sigma \cap K)} \hat{\mathbf{f}}_{\mathbf{n}_\gamma}(\mathbf{u}_\sigma, \mathbf{u}_-). \end{aligned} \tag{44}$$

The last relation uses the consistency of the flux and the fact that $C_\sigma \cap K$ is a closed polygon. The quantity $\Phi_\sigma^K(\mathbf{u}^h)$ is the normal flux on $C_\sigma \cap K$. If now we sum up these three quantities and get:

$$\begin{aligned} \sum_{\sigma \in K} \Phi_\sigma^K(\mathbf{u}^h) &= \left(\hat{\mathbf{f}}_{\mathbf{n}_{12}}(\mathbf{u}_1, \mathbf{u}_2) - \hat{\mathbf{f}}_{\mathbf{n}_{13}}(\mathbf{u}_1, \mathbf{u}_3) - \mathbf{f}(\mathbf{u}_1) \cdot \mathbf{n}_{12} + \mathbf{f}(\mathbf{u}_1) \cdot \mathbf{n}_{31} \right) \\ &\quad + \left(\hat{\mathbf{f}}_{\mathbf{n}_{23}}(\mathbf{u}_2, \mathbf{u}_3) - \hat{\mathbf{f}}_{\mathbf{n}_{12}}(\mathbf{u}_2, \mathbf{u}_1) + \mathbf{f}(\mathbf{u}_2) \cdot \mathbf{n}_{12} - \mathbf{f}(\mathbf{u}_2) \cdot \mathbf{n}_{23} \right) \\ &\quad + \left(-\hat{\mathbf{f}}_{\mathbf{n}_{23}}(\mathbf{u}_3, \mathbf{u}_2) + \hat{\mathbf{f}}_{\mathbf{n}_{31}}(\mathbf{u}_3, \mathbf{u}_1) - \mathbf{f}(\mathbf{u}_3) \cdot \mathbf{n}_{23} + \mathbf{f}(\mathbf{u}_3) \cdot \mathbf{n}_{31} \right) \\ &= \mathbf{f}(\mathbf{u}_1) \cdot (\mathbf{n}_{12} - \mathbf{n}_{31}) + \mathbf{f}(\mathbf{u}_2) \cdot (-\mathbf{n}_{23} + \mathbf{n}_{31}) + \mathbf{f}(\mathbf{u}_3) \cdot (\mathbf{n}_{31} - \mathbf{n}_{23}) \\ &= \mathbf{f}(\mathbf{u}_1) \cdot \frac{\mathbf{n}_1}{2} + \mathbf{f}(\mathbf{u}_2) \cdot \frac{\mathbf{n}_2}{2} + \mathbf{f}(\mathbf{u}_3) \cdot \frac{\mathbf{n}_3}{2} \end{aligned}$$

where $\mathbf{n}_j$ is the scaled inward normal of the edge opposite to vertex σ_j , i.e. twice the gradient of the $\mathbb{P}^1$ basis function φ_{σ_j} associated to this degree of freedom. Thus, we can reinterpret the sum as the boundary integral of the Lagrange interpolant of the flux. The finite volume scheme is then a residual distribution scheme with residual defined by (44) and a total residual defined by

$$\Phi^K := \int_{\partial K} \mathbf{f}^h \cdot \mathbf{n}, \quad \mathbf{f}^h = \sum_{\sigma \in K} \mathbf{f}(\mathbf{u}_\sigma) \varphi_\sigma. \tag{45}$$

Let K be a fixed triangle. We are given a set of residues $\{\Phi_\sigma^K\}_{\sigma \in K}$, our aim here is to define a flux function such that relations similar to (44) hold true. We explicitly give the formula for $\mathbb{P}^1$ and $\mathbb{P}^2$ interpolant.

B.2 General case

One can deal with the general case, i.e when K is a polytope contained in $\mathbb{R}^d$ with degrees of freedoms on the boundary of K . The set $\mathcal{S}$ is the set of degrees of freedom. We consider a triangulation $\mathcal{T}_K$ of K whose vertices are exactly the elements of $\mathcal{S}$. Choosing an orientation of K , it is propagated on $\mathcal{T}_K$: the edges are oriented.

Figure 7: Notations for the finite volume schemes. On the left: definition of the control volume for the degree of freedom σ . The vertex σ plays the role of the vertex 1 on the left picture for the triangle K. The control volume C_σ associated to $\sigma = 1$ is green on the right and corresponds to $1PGR$ on the left. The vectors $\mathbf{n}_{ij}$ are normal to the internal edges scaled by the corresponding edge length

The problem is to find quantities $\hat{\mathbf{f}}_{\mathbf{n}_{\sigma,\sigma'}}$ for any edge $[\sigma, \sigma']$ of $\mathcal{T}_K$ such that:

$$\Phi_\sigma = \sum_{\text{edges } [\sigma, \sigma']} \hat{\mathbf{f}}_{\mathbf{n}_{\sigma,\sigma'}} + \hat{\mathbf{f}}_\sigma^b \quad (46a)$$

with

$$\hat{\mathbf{f}}_{\mathbf{n}_{\sigma,\sigma'}} = -\hat{\mathbf{f}}_{\mathbf{n}_{\sigma',\sigma}} \quad (46b)$$

and

$$\hat{\mathbf{f}}_\sigma^b = \oint_{\partial K} \varphi_\sigma \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}^h, \mathbf{u}_-^h) d\gamma. \quad (46c)$$

The control volumes will be defined by their normals so that we get consistency. The discontinuous nature of $\mathbf{u}^h$, if any, is handled through the numerical flux $\hat{\mathbf{f}}_{\mathbf{n}}$.

Note that (46b) implies the conservation relation

$$\sum_{\sigma \in K} \Phi_\sigma = \sum_{\sigma \in K} \hat{\mathbf{f}}_\sigma^b, \quad (46d)$$

so that we rewrite (46a), introducing $\Psi_\sigma = \Phi_\sigma - \hat{\mathbf{f}}_\sigma^b$ as:

$$\Psi_\sigma = \sum_{\text{edges } [\sigma, \sigma']} \hat{\mathbf{f}}_{\mathbf{n}_{\sigma,\sigma'}} \quad (46e)$$

The general $\mathbb{P}^1$ case. A solution is

$$\begin{aligned}\hat{\mathbf{f}}_{\mathbf{n}_{13}} &= \frac{\Psi_1 - \Psi_3}{3} \\ \hat{\mathbf{f}}_{\mathbf{n}_{23}} &= \frac{\Psi_2 - \Psi_3}{3} \\ \hat{\mathbf{f}}_{\mathbf{n}_{32}} &= \frac{\Psi_3 - \Psi_2}{3}\end{aligned}$$

In order to describe the control volumes, we first have to make precise the normals $\mathbf{n}_\sigma$ in that case. It is easy to see that in all the cases described above, we have

$$\mathbf{N}_\sigma = -\frac{\mathbf{n}_\sigma}{2}.$$

Then a short calculation shows that

$$\begin{pmatrix} \mathbf{n}_{12} \\ \mathbf{n}_{23} \\ \mathbf{n}_{31} \end{pmatrix} = \frac{1}{6} \begin{pmatrix} \mathbf{n}_1 - \mathbf{n}_2 \\ \mathbf{n}_2 - \mathbf{n}_3 \\ \mathbf{n}_3 - \mathbf{n}_1 \end{pmatrix}.$$

Using elementary geometry of the triangle, we see that these are the normals of the elements of the dual mesh. For example, the normal $\mathbf{n}_{12}$ is the normal of PG , see figure 7.

Relying more on the geometrical interpretation (once we know the control volumes), we can recover the same formula by elementary calculations, see [40].

The general example of the $\mathbb{P}^2$ approximation. Using a similar method, we get (see figure 8 for some notations):

$$\begin{aligned}\hat{\mathbf{f}}_{14} &= \frac{1}{12}(\Psi_1 - \Psi_4) + \frac{1}{36}(\Psi_6 - \Psi_5) + \frac{7}{36}(\Psi_1 - \Psi_2) + \frac{5}{36}(\Psi_3 - \Psi_1) \\ \hat{\mathbf{f}}_{16} &= \frac{1}{12}(\Psi_4 - \Psi_1) + \frac{5}{36}(\Psi_5 - \Psi_1) + \frac{7}{36}(\Psi_6 - \Psi_1) + \frac{1}{36}(\Psi_3 - \Psi_2) \\ \hat{\mathbf{f}}_{46} &= \frac{2}{9}(\Psi_2 - \Psi_6) + \frac{1}{9}(\Psi_3 - \Psi_5) \\ \hat{\mathbf{f}}_{54} &= \frac{2}{9}(\Psi_5 - \Psi_2) + \frac{1}{9}(\Psi_5 - \Psi_1) \\ \hat{\mathbf{f}}_{42} &= \frac{7}{36}(\Psi_2 - \Psi_3) + \frac{5}{36}(\Psi_1 - \Psi_3) + \frac{1}{12}(\Psi_6 - \Psi_3) + \frac{1}{36}(\Psi_5 - \Psi_4) \\ \hat{\mathbf{f}}_{25} &= \frac{1}{36}(\Psi_2 - \Psi_1) + \frac{5}{36}(\Psi_3 - \Psi_5) + \frac{7}{36}(\Psi_3 - \Psi_5) + \frac{1}{12}(\Psi_3 - \Psi_6) \\ \hat{\mathbf{f}}_{53} &= \frac{1}{36}(\Psi_1 - \Psi_6) + \frac{5}{36}(\Psi_3 - \Psi_5) + \frac{7}{36}(\Psi_4 - \Psi_5) + \frac{1}{12}(\Psi_2 - \Psi_5) \\ \hat{\mathbf{f}}_{63} &= \frac{1}{36}(\Psi_4 - \Psi_3) + \frac{5}{36}(\Psi_5 - \Psi_1) + \frac{7}{36}(\Psi_5 - \Psi_6) + \frac{1}{12}(\Psi_5 - \Psi_2) \\ \hat{\mathbf{f}}_{65} &= \frac{1}{9}(\Psi_1 - \Psi_3) + \frac{2}{9}(\Psi_6 - \Psi_4)\end{aligned}$$

Then we choose the boundary flux:

$$\hat{\mathbf{f}}_\sigma^b = \int_{\partial K} \varphi_\sigma \mathbf{n} \, d\gamma$$

and get:

$$\begin{aligned}\mathbf{N}_l &= -\frac{\mathbf{n}_l}{6} \quad \text{if } l = 1, 2, 3 \\ \mathbf{N}_4 &= \frac{\mathbf{n}_3}{3} \quad \mathbf{N}_5 = \frac{\mathbf{n}_1}{3} \quad \mathbf{N}_6 = \frac{\mathbf{n}_2}{3}\end{aligned}$$

The normals are given by:

$$\begin{aligned}\mathbf{n}_{14} &= \frac{1}{12}(\mathbf{N}_1 - \mathbf{N}_4) + \frac{1}{36}(\mathbf{N}_6 - \mathbf{N}_5) + \frac{7}{36}(\mathbf{N}_1 - \mathbf{N}_2) + \frac{5}{36}(\mathbf{N}_3 - \mathbf{N}_1) \\ \mathbf{n}_{16} &= \frac{1}{12}(\mathbf{N}_4 - \mathbf{N}_1) + \frac{5}{36}(\mathbf{N}_5 - \mathbf{N}_1) + \frac{7}{36}(\mathbf{N}_6 - \mathbf{N}_1) + \frac{1}{36}(\mathbf{N}_3 - \mathbf{N}_2) \\ \mathbf{n}_{46} &= \frac{2}{9}(\mathbf{N}_2 - \mathbf{N}_6) + \frac{1}{9}(\mathbf{N}_3 - \mathbf{N}_5) \\ \mathbf{n}_{54} &= \frac{2}{9}(\mathbf{N}_5 - \mathbf{N}_2) + \frac{1}{9}(\mathbf{N}_5 - \mathbf{N}_1) \\ \mathbf{n}_{42} &= \frac{7}{36}(\mathbf{N}_2 - \mathbf{N}_3) + \frac{5}{36}(\mathbf{N}_1 - \mathbf{N}_3) + \frac{1}{12}(\mathbf{N}_6 - \mathbf{N}_3) + \frac{1}{36}(\mathbf{N}_5 - \mathbf{N}_4) \\ \mathbf{n}_{25} &= \frac{1}{36}(\mathbf{N}_2 - \mathbf{N}_1) + \frac{5}{36}(\mathbf{N}_3 - \mathbf{N}_5) + \frac{7}{36}(\mathbf{N}_3 - \mathbf{N}_5) + \frac{1}{12}(\mathbf{N}_3 - \mathbf{N}_6) \\ \mathbf{n}_{53} &= \frac{1}{36}(\mathbf{N}_1 - \mathbf{N}_6) + \frac{5}{36}(\mathbf{N}_3 - \mathbf{N}_5) + \frac{7}{36}(\mathbf{N}_4 - \mathbf{N}_5) + \frac{1}{12}(\mathbf{N}_2 - \mathbf{N}_5) \\ \mathbf{n}_{63} &= \frac{1}{36}(\mathbf{N}_4 - \mathbf{N}_3) + \frac{5}{36}(\mathbf{N}_5 - \mathbf{N}_1) + \frac{7}{36}(\mathbf{N}_5 - \mathbf{N}_6) + \frac{1}{12}(\mathbf{N}_5 - \mathbf{N}_2) \\ \mathbf{n}_{65} &= \frac{1}{9}(\mathbf{N}_1 - \mathbf{N}_3) + \frac{2}{9}(\mathbf{N}_6 - \mathbf{N}_4)\end{aligned}$$

There is not uniqueness, and it is possible to construct different solutions to the problem. In what follows, we show another possible construction. We consider the set-up defined by Figure 8.

C Proof of (22a).

Proof of (22a). We start from (8) which is multiplied by $\mathbf{v}_\sigma$, and these relations are added for each $\sigma \in \mathcal{S}$. We get:

$$0 = \sum_{\sigma \in \mathcal{S}} \mathbf{v}_\sigma \left(\sum_{K \subset \Omega, \sigma \in K} \Phi_\sigma^K(\mathbf{u}^h) + \sum_{\Gamma \subset \partial\Omega, \sigma \in \Gamma} \Phi_\sigma^\Gamma(\mathbf{u}^h, \mathbf{u}_b) \right).$$

Permuting the sums on σ and K , then on σ and Γ , we get:

$$0 = \sum_{K \subset \Omega} \left(\sum_{\sigma \in K} \mathbf{v}_\sigma \Phi_\sigma^K(\mathbf{u}^h) \right) + \sum_{\Gamma \subset \partial\Omega} \left(\sum_{\sigma \in \Gamma} \mathbf{v}_\sigma \Phi_\sigma^\Gamma(\mathbf{u}^h, \mathbf{u}_b) \right).$$

Figure 8: Geometrical elements for the $\mathbb{P}^2$ case. I_{ij} is the mid-point between the vertices i and j . The intersections of the dotted lines are the centroids of the sub-elements.

We look at the first term, the second is done similarly. We have, introducing $\Phi_\sigma^{K,Gal}$ and $\#K$ the number of degrees of freedom in K ,

$$\begin{aligned}
\sum_{\sigma \in K} \mathbf{v}_\sigma \Phi_\sigma^K(\mathbf{u}^h) &= \sum_{\sigma \in K} \mathbf{v}_\sigma \Phi_\sigma^{K,Gal}(\mathbf{u}^h) + \sum_{\sigma \in K} \mathbf{v}_\sigma \left(\Phi_\sigma^K(\mathbf{u}^h) - \Phi_\sigma^{K,Gal}(\mathbf{u}^h) \right) \\
&= - \oint_K \nabla \mathbf{v}_h \cdot \mathbf{f}(\mathbf{u}^h) d\mathbf{x} + \oint_{\partial K} \mathbf{v}^h \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_{|K}^h, \mathbf{u}_{|K-}^h) d\gamma + \sum_{\sigma \in K} \mathbf{v}_\sigma \left(\Phi_\sigma^K(\mathbf{u}^h) - \Phi_\sigma^{K,Gal}(\mathbf{u}^h) \right) \\
&= - \oint_K \nabla \mathbf{v}_h \cdot \mathbf{f}(\mathbf{u}^h) d\mathbf{x} + \oint_{\partial K} \mathbf{v}^h \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_{|K}^h, \mathbf{u}_{|K-}^h) d\gamma \\
&\quad + \frac{1}{\#K} \sum_{\sigma, \sigma' \in K} (\mathbf{v}_\sigma - \mathbf{v}_{\sigma'}) \left(\Phi_\sigma^K(\mathbf{u}^h) - \Phi_\sigma^{K,Gal}(\mathbf{u}^h) \right)
\end{aligned}$$

because from (7a),

$$\sum_{\sigma \in K} (\Phi_\sigma^K(\mathbf{u}^h) - \Phi_\sigma^{K,Gal}(\mathbf{u}^h)) = 0.$$

Similarly, we have

$$\sum_{\sigma \in \Gamma} \mathbf{v}_\sigma \Phi_\sigma^\Gamma(\mathbf{u}^h) = \oint_\Gamma \mathbf{v}^h (\hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}^h, \mathbf{u}_b) - \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n}) d\gamma + \sum \frac{1}{\#\Gamma} \sum_{\sigma, \sigma' \in \Gamma} (\mathbf{v}_\sigma - \mathbf{v}_{\sigma'}) (\Phi_\sigma^\Gamma(\mathbf{u}^h, \mathbf{u}_b) - \Phi_\sigma^{Gal, \Gamma}(\mathbf{u}^h, \mathbf{u}_b))$$

Adding all the relations and using the assumption 3.1 on the quadrature formulas, we get:

$$\begin{aligned}
0 &= \sum_{K \subset \Omega} \left(- \oint_K \nabla \mathbf{v}_h \cdot \mathbf{f}(\mathbf{u}^h) d\mathbf{x} + \oint_{\partial K} \mathbf{v}^h \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}_{|K}^h, \mathbf{u}_{|K-}^h) d\gamma \right) + \sum_{\Gamma \subset \partial \Omega} \oint_\Gamma \mathbf{v}^h (\hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}^h, \mathbf{u}_b) - \mathbf{f}(\mathbf{u}^h) \cdot \mathbf{n}) d\gamma \\
&\quad + \sum_{K \subset \Omega} \frac{1}{\#K} \left(\sum_{\sigma, \sigma' \in K} (\mathbf{v}_\sigma - \mathbf{v}_{\sigma'}) \left(\Phi_\sigma^K(\mathbf{u}^h) - \Phi_\sigma^{K,Gal}(\mathbf{u}^h) \right) \right) \\
&\quad + \sum_{\Gamma \subset \partial \Omega} \frac{1}{\#\Gamma} \left(\sum_{\sigma, \sigma' \in \Gamma} (\mathbf{v}_\sigma - \mathbf{v}_{\sigma'}) (\Phi_\sigma^\Gamma(\mathbf{u}^h, \mathbf{u}_b) - \Phi_\sigma^{Gal, \Gamma}(\mathbf{u}^h, \mathbf{u}_b)) \right)
\end{aligned}$$

i.e. after having defined $[\mathbf{v}^h] = \mathbf{v}|_K^h - \mathbf{v}|_{K^-}^h$ and chosen one orientation of the internal edges $e \in \mathcal{E}_h$, we get (22a). $\square$

Proof of Proposition 3.4. We first show that $\Phi_\sigma^{K,Gal}(\pi_h(\mathbf{u})) = \mathcal{O}(h^{k+d})$. Since $\mathbf{u}$ is regular enough, we have pointwise $\operatorname{div} \mathbf{f}(\mathbf{u}) = 0$ on K , so that, by consistency of the flux,

$$0 = - \int_K \nabla \varphi \cdot \mathbf{f}(\mathbf{u}) \, d\mathbf{x} + \int_{\partial K} \varphi \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}, \mathbf{u}) \, d\gamma.$$

This shows that

$$- \oint_K \nabla \varphi \cdot \mathbf{f}(\mathbf{u}) \, d\mathbf{x} + \oint_{\partial K} \varphi \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}, \mathbf{u}) \, d\gamma = \mathcal{O}(h^{k+d+1}).$$

Then,

$$\begin{aligned} \Phi_\sigma^{K,Gal}(\pi_h(\mathbf{u})) &= - \oint_K \nabla \varphi_\sigma \cdot (\mathbf{f}(\pi_h(\mathbf{u})) - \mathbf{f}(\mathbf{u})) \, d\mathbf{x} + \oint_{\partial K} \varphi_\sigma (\hat{\mathbf{f}}_{\mathbf{n}}(\pi_h(\mathbf{u}), \pi_h(\mathbf{u})) - \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}, \mathbf{u})) \, d\gamma \\ &\quad + \oint_K \nabla \varphi \cdot \mathbf{f}(\mathbf{u}) \, d\mathbf{x} - \oint_{\partial K} \varphi \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}, \mathbf{u}) \, d\gamma \\ &= |K| \times \mathcal{O}(h^{-1}) \times \mathcal{O}(h^{k+1}) + |\partial K| \times \mathcal{O}(1) \mathcal{O}(h^{k+1}) \\ &\quad + \mathcal{O}(h^{k+d+1}) \\ &= \mathcal{O}(h^d) \times \mathcal{O}(h^{-1}) \times \mathcal{O}(h^{k+1}) + \mathcal{O}(h^{d-1}) \times \mathcal{O}(1) \times \mathcal{O}(h^{k+1}) \\ &\quad + \mathcal{O}(h^{k+d+1}) \\ &= \mathcal{O}(h^{d+k}) \end{aligned}$$

because the flux is Lipschitz continuous and the mesh is regular.

The result on the boundary term is similar since the boundary numerical flux is upwind and the boundary of Ω is not characteristic: only two types of boundary faces exists, the upwind and downwind ones. On the downwind faces, the boundary flux vanishes. On the upwind ones, we get the estimate for the Galerkin boundary residuals thanks to the same approximation argument.

The mesh is assumed to be regular: the number of elements (resp. edges) is $\mathcal{O}(h^{-d})$ (resp. $\mathcal{O}(h^{-d+1})$). Let us assume (25). Let $\mathbf{v} \in C_0^1(\bar{\Omega})$. Using (22a) for $\pi_h(\mathbf{u})$,

$$\begin{aligned} \mathcal{E}(\mathbf{u}^h, \varphi) &= - \oint_\Omega \nabla \pi_h(v) \cdot \mathbf{f}(\pi_h(\mathbf{u})) \, d\mathbf{x} + \oint_{\partial\Omega} \pi_h(v) (\hat{\mathbf{f}}_{\mathbf{n}}(\pi_h(\mathbf{u}), \mathbf{u}_b) - \mathbf{f}(\pi_h(\mathbf{u})) \cdot \mathbf{n}) \, d\gamma \\ &\quad + \sum_{e \in \mathcal{E}_h} \oint_e [\pi_h(v)] \hat{\mathbf{f}}_{\mathbf{n}}(\pi_h(\mathbf{u}), \pi_h(\mathbf{u})_-) \, d\gamma + \sum_{K \subset \Omega} \frac{1}{\#K} \left(\sum_{\sigma, \sigma' \in K} (\mathbf{v}_\sigma - \mathbf{v}_{\sigma'}) \left(\Phi_\sigma^K(\pi_h(\mathbf{u})) - \Phi_\sigma^{K,Gal}(\pi_h(\mathbf{u})) \right) \right) \\ &\quad + \sum_{\Gamma \subset \partial\Omega} \frac{1}{\#\Gamma} \left(\sum_{\sigma, \sigma' \in \Gamma} (\mathbf{v}_\sigma - \mathbf{v}_{\sigma'}) (\Phi_\sigma^\Gamma(\pi_h(\mathbf{u}), \mathbf{u}_b) - \Phi_\sigma^{Gal,\Gamma}(\pi_h(\mathbf{u}), \mathbf{u}_b)) \right). \end{aligned}$$

where $\pi_h(\mathbf{u})_-$ represents the interpolant of $\mathbf{u}$ on K^- .

We have, using

$$\begin{aligned} &- \oint_\Omega \nabla \pi_h(v) \cdot \mathbf{f}(\mathbf{u}) \, d\mathbf{x} + \oint_{\partial\Omega} \pi_h(v) (\mathcal{F}_{\mathbf{n}}(\mathbf{u}, \mathbf{u}_b) - \mathbf{f}(\mathbf{u}) \cdot \mathbf{n}) \, d\gamma = 0, \\ &- \oint_\Omega \nabla \pi_h(v) \cdot \mathbf{f}(\pi_h(\mathbf{u})) \, d\mathbf{x} + \oint_{\partial\Omega} \pi_h(v) (\hat{\mathbf{f}}_{\mathbf{n}}(\pi_h(\mathbf{u}), \mathbf{u}_b) - \mathbf{f}(\pi_h(\mathbf{u})) \cdot \mathbf{n}) \, d\gamma \\ &= - \oint_\Omega \nabla \pi_h(v) \cdot (\mathbf{f}(\pi_h(\mathbf{u})) - \mathbf{f}(\mathbf{u})) \, d\mathbf{x} + \oint_{\partial\Omega} \pi_h(v) (\hat{\mathbf{f}}_{\mathbf{n}}(\pi_h(\mathbf{u}), \mathbf{u}_b) - \hat{\mathbf{f}}_{\mathbf{n}}(\mathbf{u}, \mathbf{u}_b)) \, d\gamma \\ &\quad - \oint_{\partial\Omega} \pi_h(v) (\mathbf{f}(\pi_h(\mathbf{u})) - \mathbf{f}(\mathbf{u})) \cdot \mathbf{n} \, d\gamma \\ &= \mathcal{O}(h^{k+1}) \end{aligned}$$

since the flux on the boundary is the upwind flux $\mathcal{F}_{\mathbf{n}}$, and using the approximation properties of $\pi_h(\mathbf{u})$.
Then

$$\sum_{e \in \mathcal{E}_h} \oint_e [\pi_h(v)] \hat{\mathbf{f}}_{\mathbf{n}}(\pi_h(\mathbf{u}), \pi_h(\mathbf{u})^-) d\gamma = \mathcal{O}(h^{-d+1}) \times \mathcal{O}(h^{d-1}) \times \mathcal{O}(h^{k+1}) \times \mathcal{O}(1) = \mathcal{O}(h^{k+1}),$$

$$\sum_{K \subset \Omega} \frac{1}{\#K} \left(\sum_{\sigma, \sigma' \in K} (\mathbf{v}_{\sigma} - \mathbf{v}_{\sigma'}) \left(\Phi_{\sigma}^K(\pi_h(\mathbf{u})) - \Phi_{\sigma}^{K, Gal}(\pi_h(\mathbf{u})) \right) \right) = \mathcal{O}(h^{-d}) \times \mathcal{O}(h) \times \mathcal{O}(h^{k+d}) = \mathcal{O}(h^{k+1}),$$

and similarly

$$\sum_{\Gamma \subset \partial\Omega} \frac{1}{\#\Gamma} \left(\sum_{\sigma, \sigma' \in \Gamma} (\mathbf{v}_{\sigma} - \mathbf{v}_{\sigma'}) (\Phi_{\sigma}^{\Gamma}(\pi_h(\mathbf{u}), \mathbf{u}_b) - \Phi_{\sigma}^{Gal, \Gamma}(\pi_h(\mathbf{u}), \mathbf{u}_b)) \right) = \mathcal{O}(h^{-d+1}) \times \mathcal{O}(h) \times \mathcal{O}(h^{k+d-1}) = \mathcal{O}(h^{k+1})$$

thanks to the regularity of the mesh, that $\pi_h(v)$ is the interpolant of a C^1 function and the previous estimates. $\square$