Quantum gravitational Bremsstrahlung: massless versus massive case

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Quantum Gravitational Bremsstrahlung
Massless Versus Massive Case

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Zusammenfassung

In der vorliegenden Arbeit wird masselose und massive Gravitation als Quanteneichtheorie untersucht. Es wird gezeigt, dass die sechs physikalischen Freiheitsgrade des massiven Gravitons in die zwei des masselosen Gravitons plus vier entkoppelte Vektor Ghost Freiheitsgrade im Limes von verschwindender Gravitonmasse übergehen. Weiter wird im Formalismus der kausalen Störungstheorie der differentielle Wirkungsquerschnitt und die total abgestrahlte Energie der gravitationellen Bremsstrahlung berechnet. Wir betrachten den Fall von ultra relativistischer Teilchenenergie und kleiner Frequenz. Zusätzlich betrachten wir das Teilchen in einer ungebundenen Bahn, gestreut vom gravitationellen Zentrum unter einem kleinen Winkel. Falls das abgestrahlte Bremsstrahlgraviton als ein asymptotisch freies, masseloses Spin-2 Teilchen betrachtet wird, stimmt das Resultat im betrachteten Limes mit der klassischen Theorie überein. Im Falle des massiven Bremsstrahlgravitons geht das dazugehörige Matrixelement im Limes von verschwindender Gravitonmasse in dasjenige der masselosen Theorie über.

Abstract

In the present work, massless and massive gravity as a quantum gauge theory is investigated. It is shown, that the six physical degrees of freedom of the massive graviton go over smoothly into the two massless ones and four decoupled vector ghost degrees of freedom in the limit of zero graviton mass. Furthermore, the differential cross section and the total radiated energy for quantum gravitational Bremsstrahlung are calculated in the framework of causal perturbation theory on the full tensorial level. We consider the case of ultra relativistic particle energy and low frequency. In addition, the particle is considered to follow an unbound trajectory and be scattered by the gravitational center through a small angle. In the case of treating the emerging Bremsstrahl graviton to be the free asymptotic spin-2 particle with mass zero and helicity $\pm$ 2, the result agrees in the considered limit with the classical theory. In the case of a massive Bremsstrahl graviton, the corresponding matrix element goes over smoothly into the one of the massless theory in the limit of zero graviton mass.

Contents

1 Introduction

The problem of gravitational Bremsstrahlung is a wide subject and has been discussed by many authors in a variety of different approaches. The classical picture represents mainly the process of two massive bodies (particles), flying past each other, while deflecting their trajectories and thus emitting gravitational waves. In an unbound orbit, a particle follows an asymptotic trajectory through a gravitational field, induced by a central mass. Under the influence of the central potential, the particle is scattered and looses energy, which in the case of high energies, is lost mainly by radiation. In the case of quantum field theory, in lowest order, four tree-graph Feynman diagrams are to be considered, where the interaction with central potential is treated according to the external field approximation. The scattered particle is described by a scalar field, carrying an energy-momentum tensor to which the spin-2 field, representing the graviton, couples. Under the influence of the central potential, the particle emits a Bremsstrahl graviton.

The classical problem was first treated by *Einstein* culminating in the well known quadrupole formula. It is a non-relativistic formula which provides good agreement with experimental observations. In order to widen the range of its validity, different approaches [17, 19, 34, 39] have been carried out. They provide a good range of validity for astrophysical sources of gravitational radiation. The classical phenomenon of gravitational Bremsstrahlung and its energy loss is completely understood.

In the framework of quantum theory, different approaches [14, 24, 26, 49, 52] have been carried out, too. However, the full tensorial problem was never attacked in the limit of comparing it with the classical theory. Worth mentioning in this context is the work by *Galtsov* et al. [26]. They treated a toy model of scalar gravitational Bremsstrahlung and recovered the result for the energy loss by a independent classical calculation. Most of these approaches start with a Lagrangian description of the interaction.

The theory of massive gravity has also been discussed by several authors [16, 18, 25, 33, 43, 45, 53]. Again, all these approaches start with a classical Lagrangian, enriched with a mass term. Notions of quantum theory, such as spin-2 field and propagator, are sprinkled into the subject. However, the dividing line between treating the fields classically or quantum is in most cases not clear.

In recent years, *Scharf* and collaborators [8, 32, 40] have treated gravity as the spin-2 quantum gauge theory in the sense of the causal approach. They enriched the structure of causal perturbation theory with the concept of perturbative quantum gauge invariance and have shown, that in the massless case, the full Einstein gravity is recovered. Recent works [28, 29] have expanded the theory to describe the massive graviton where no satisfactory classical theory exists.

We will postpone the discussion whether it makes sense to even speak of massive gravitons. In the course of the present work, it will become clear how the situation behaves. In the causal framework of massive spin-2 quantum gauge theory, all objects are well defined elements of a quantum field algebra and the extension from the massless to the massive theory goes quite naturally. Therefore, we will treat this two consistent, nearby theories with equally legitimation and discuss their experimental consequences at the end.

The key ingredient of treating gravitational processes, where real, free, massless or massive, gravitons appear, is the derivation of the physical degrees of freedom they carry. In the massless case, the well known ± 2 helicity is found whereas in the massive case, six physical degrees of freedom are found. The concept of quantum gauge invariance imposes a one parameter gauge transformation on the field operators, generated by a nilpotent operator Q. In the massless case, the classical analogon of the gauge transformation generated by the gauge charge Q leaves the transverse- and traceless gauge invariant and represents the infinitesimal version of the diffeomorphic coordinate transformations of general relativity. In the massive case, the definition of Q is smoothly adapted in order to keep it nilpotent. It turns out, that the main difference between the massless and the massive theory lies in the gauge structure generated by Q. The concept of spin-2 quantum gauge theory in the causal approach allows us to fully develop the theory in a Lorentz invariant way without referring to classical concepts. We discuss those only for reasons of comparison.

It is important to note that the structure of massive gravity and especially the physical degrees of freedom of the massive graviton has been discussed by many authors [16, 18, 25, 33, 44, 45, 53]. They basically found a key difference to massive quantum electrodynamics where the additional massive transversal degree of freedom completely decouple in the limit of zero photon mass allowing for the smooth limit between the massive and the massless theory. In contrast, they argue in the case of spin-2, that not all additional massive degrees of freedom decouple in the limit of zero graviton mass, lying the basis for the discontinuity between the massive and the massless theory. It was extensively discussed how the massive degrees of freedom arise, what symmetries the theory carries, how massive spin-2 propagators and the polarization sum have to be derived and what observational implications are to be expected. The presented theory suffered from the lack of consistency, namely gauge invariance, Lorentz invariance, positivity of energy and coupling to a conserved source could not have been satisfied simultaneously, not to mention the smooth limit between the two theories. These results are based mainly on arguments from the structure of the Einstein-Hilbert action leading to the inconsistencies mentioned above and it is not clear, if they hold in quantum field theory, too. One may guess that this approach to massive gravity where concepts of classical and quantum theory are mixed, completely fails.

From the point of view of the gauge structure induced by perturbative gauge invariance, there exists indeed a smooth limit between the massive and the massless theory, as it was already shown in [28, 29]. Furthermore, we will show by explicitly constructing the six massive physical states how they decouple in the limit of zero graviton mass finding the two massless and four completely decoupled bosonic vector ghost degrees of freedom. Moreover, in the massive case, up to second order perturbation theory, no additional Higgs field has to be introduced as it is the case for Yang-Mills fields [28, 29] to save gauge invariance. One is tempted to think that the massive spin-2 theory cannot be found by the conventional Higgs mechanism and that the Lagrangian approach completely fails in the case of massive gravity. One further aspect to care of is the presumed smallness of the graviton mass. Therefore, the definition of the polarization sum needs special care.

In the course of the present work, we will treat the full tensorial gravitational Bremsstrahlung in the quantum framework of causal perturbation theory on Minkowski background of space-time. In a first step, we discuss the massless versus the massive free graviton as the asymptotic spin-2 particle. It is shown, how the different gauge structures arise and how they induce the difference in the physical degrees of freedom.

The present work is organized as follows: In section 2, an overview over causal perturbation theory and perturbative quantum gauge invariance is given. In Section 3, we define what we mean by a graviton. We introduce the basic objects of the theory and give the derivation of the physical degrees of freedom in both, the massless and the massive case. It is outlined, how the different couplings are derived from principles of perturbative gauge invariance. In section 4, we treat gravitational Bremsstrahlung in the causal framework. We calculate the S-matrix element, the differential cross section and the total energy radiated of the entire Bremsstrahl process, starting with the massless case, followed by the treatment of the massive case. Section 5 is devoted to the conclusion, applications of the results and an outlook. In the appendix, the steps of the calculation are outlined which are less important for the general understanding of the work.

2 Causal Approach to the Theory of Quantized Fields

The causal approach to quantum field theory goes back to the work by *Stückelberg* [42], *Bogoliubov* and *Shirkov* [1] and *Epstein* and *Glaser* [23]. *Scharf* [9] elaborated the theory successfully for gauge theories. It is an S-matrix theory which is constructed as a perturbation series in terms of asymptotic free fields and the interaction is considered to be adiabatically switched. One would like to formulate gauge invariance without referring to classical concepts. This requires two basic ingredients to be fulfilled. First, since the S-matrix is constructed as a formal power series, a perturbative definition of gauge invariance is needed. Second, it is clear, that in this definition, we have to introduce free ghost fields from the beginning in order to achieve the quantum formulation of gauge invariance. This implies that we have to deal with an enlarged Hilbert space $\mathcal{H}^{gh}$, containing physical, unphysical and also ghost states and to elaborate a mechanism, how to sort the physical states out of $\mathcal{H}^{gh}$.

2.1 Causal Perturbation Theory

In causal perturbation theory, all objects are manifest elements of a quantum field algebra. There is no classical Lagrangian description of the interaction, no interacting fields and no classical gauge conditions. One may have an eye on the Wightman axioms [5, 10] to be inspired to the notion of a Hilbert space, a quantum field, Lorentz invariance, stability of the vacuum, locality and completeness of the theory. In the causal approach, the basic object, the S-matrix, is constructed as a perturbation series. It maps the asymptotically incoming free fields on the outgoing scattered ones. The idea is to fix the first order by giving an interaction and construct higher orders of the scattering matrix S by induction using free fields only. Since the free quantum fields are operator-valued distributions, the S-matrix $S(g)$ will be constructed as an operator-valued functional on the space of test functions where g is an element of it. In the present context, it is appropriate to take this space as the Schwartz space of tempered distributions with rapid decrease since additionally, it is invariant under the Fourier transform. As the word causality says, the basic requirement for the inductive construction of the S-matrix is the causality requirement. This basically means, that the scattering-matrix for a sum of two test functions factorizes if the support of the test functions can be separated in time. By this construction, all objects are well defined tempered distributions. The interaction is switched off at large distances in space-time through the test function g. As an unphysical operation, this has to be removed at the end. Therefore, we refer to $S = S(1)$ as the scattering matrix in the limit of $S(g)$ when the region in which $g(x) = 1$ is expanded infinitely to include the whole space-time. This is the so called adiabatic limit. In this way, we investigate the long range behavior of the theory. Since only few models of quantum field theories have been solved exactly, it is necessary to set up the S-matrix as a formal power series where all inductive steps are under control.

The inductive construction starts with the definition of the formal power series in the coupling constant κ of the theory

$$S(g) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{\kappa^n}{n!} \int d^4x_1 \cdots d^4x_n \; T_n(x_1, \ldots, x_n) \; g(x_1) \cdots g(x_n) \tag{2.1}$$

where the g is a test function in Schwartz space $\mathcal{S}(\mathbb{R}^4)$. In general, the coupling constant κ will be absorbed in the time ordered products of free field operators $T_n(x_1, \ldots, x_n)$. Its first order, T_1, defines the interaction to first order of perturbation theory. The test function $g(x)$ may be viewed as a certain external classical field, an interaction which takes place at each vertex of a Feynman diagram. In this way, $g(x)$ violates the law of energy-momentum conservation at each vertex. Therefore, in order to restore energy-momentum conservation at each vertex, it is necessary to perform the adiabatic limit

$$\lim_{g \to 1} S(g)$$

at the end. A key point in the inductive construction of the formal power series 2.1 is the decomposition of distributions with causal support into retarded and advanced parts. If this distribution splitting is carried out without care by multiplication with step functions, the usual ultraviolet divergencies appear. If for example, the first order interaction T_1 is given, the naive construction of the time ordered higher order term T_n is given by

$$\begin{aligned} T_n(x_1, \ldots, x_n) &= T\{T_1(x_1) \cdot \ldots \cdot T_1(x_n)\} \\ &:= \sum_{\pi} \Theta(x^0_{\pi_1} - x^0_{\pi_2}) \cdot \ldots \cdot \Theta(x^0_{\pi_{n-1}} - x^0_{\pi_n}) \cdot T_1(x_{\pi_1}) \cdot \ldots \cdot T_1(x_{\pi_n}) \end{aligned} \tag{2.2}$$

which leads to a ultraviolet divergent expression, because it involves the ill-defined products of distributions. The work due to *Epstein* and *Glaser* [23] avoids this defect by carefully constructing the S-matrix order by order under the requirement of causality. This means that for two test functions g_1 and g_2, for which we can find a separation of the supports by a spacelike surface, i.e. $supp(g_1) > supp(g_2)$, the S-matrix should factorize as

$$S(g_1 + g_2) = S(g_1) \cdot S(g_2) \tag{2.3}$$

For arbitrary sets X, Y of points $\{x_1, \ldots, x_s\}$ in Minkowski space, the relation $X > Y$ means that all points $x \in X$ are later than all $y \in Y$ in some Lorentz frame. The causality condition 2.3 can be translated into a condition on the time ordered products as

$$T_n(x_1, \ldots, x_n) = T_m(x_1, \ldots, x_m) \cdot T_{n-m}(x_{m+1}, \ldots, x_n) \tag{2.4}$$

if $\{x_1, \ldots, x_m\} > \{x_{m+1}, \ldots, x_n\}$. Now, the inductive construction of the time ordered distribution T_n goes as follows: Suppose all $T_m(x_1, \ldots, x_m)$ for $1 < m \leq n-1$ are known. Then it is possible to construct the auxiliary distributions $R'_n(x_1, \ldots, x_n)$ and $A'_n(x_1, \ldots, x_n)$ from the T_m

$$
\begin{aligned}
R'_n(x_1,\dots,x_n) &= \sum_P T_{n-n_1}(Y,x_n)\tilde{T}_{n_1}(X) \\
A'_n(x_1,\dots,x_n) &= \sum_P \tilde{T}_{n_1}(X)T_{n-n_1}(Y,x_n)
\end{aligned}
\tag{2.5}
$$

by carrying out all possible contractions between the field operators and using Wick's theorem to obtain normally ordered expressions. The n-point distribution $\tilde{T}_m$, given by the formal power series for $S(g)^{-1}$, follows from 2.1 by inversion. The sum runs over all partitions of the set of points $\{x_1,\dots,x_m\}$ into two subsets X and Y whereas X is not empty. These distributions R'_n and A'_n have no causal support but its difference $R'-A'$ has, forming the distribution D according to

$$D_n(x_1,\dots,x_n) = R'_n(x_1,\dots,x_n) - A'_n(x_1,\dots,x_n)$$

Thus, $D_n(x_1,\dots,x_n)$ may be splitted leaving two distributions A and R which have proper support in the advanced and the retarded light-cone as

$$D_n(x_1,\dots,x_n) \stackrel{\text{splitting}}{=} R_n(x_1,\dots,x_n) - A_n(x_1,\dots,x_n)$$

Then, the time ordered distribution $T_n(x_1,\dots,x_n)$ is given by

$$
\begin{aligned}
T_n(x_1,\dots,x_n) &= R_n(x_1,\dots,x_n) - R'_n(x_1,\dots,x_n) \\
&= A_n(x_1,\dots,x_n) - A'_n(x_1,\dots,x_n)
\end{aligned}
$$

The distribution splitting of a tree graph is always trivial, since it is proportional to the Jordan-Pauli distribution[1] D which has causal support. We emphasize that the causal construction of the S-matrix leave us with well defined objects.

2.2 Perturbative Quantum Gauge Invariance

Since we want to describe a gauge theory in the framework of causal perturbation theory, the concept of formal power series 2.1 is useful only if gauge invariance of the S-matrix $S(g)$ can be defined for power series [8], leading to the concept of perturbative gauge invariance. Subject to this, we then are able to deduce the couplings of the theory. Since we only deal with free asymptotic quantum fields, it is clear that a gauge transformation must be manifest quantum. This means that free ghost fields have to be introduced from the beginning on. This quantum gauge transformation contains no classical functions nor vector fields but quantum fields of the theory. The one parameter gauge transformation is infinitesimally generated by a gauge charge Q, first introduced by *Kugo* and *Ojima* [35] and is given by

$$F' = e^{-i\lambda Q} F e^{i\lambda Q} \tag{2.6}$$

where F is a free quantum field, an operator valued distribution on an enlarged Hilbert space $\mathcal{H}^{\text{gh}}$, carrying also the unphysical states[2]. The form of 2.6 is dictated by

[1] The Jordan-Pauli distribution [9] for mass m is given by

$$D_m(x-y) \;:=\; (2\pi)^{-3} \int d^4p\, \delta(p^2-m^2)\, sign(p^0)\, e^{-ip(x-y)}$$

[2] The Q generated gauge transformation is similar to a BRST-transformation with the difference, that the BRST-transformation acts on interacting fields whereas the gauge transformation, generated by Q, acts only on asymptotic free fields. For the connection between them, see [21].

the requirement that the transformed field F' satisfies the same commutation relations as the original field F. The gauge charge Q is defined as a nilpotent operator on $\mathcal{H}^{\text{gh}}$. The gauge variation d_Q is obtained with the help of the Lie series expansion of 2.6 and is given by

$$d_Q F := [Q,\ F] \tag{2.7}$$

if F contains only Bose fields and an even number of ghost fields and

$$d_Q F := \{Q,\ F\} \tag{2.8}$$

if F contains an odd number of ghost fields[3]. The nature of the operator Q and the gauge variation d_Q will become clear when defining it explicitly in the next section. In order to reduce the enlarged Hilbert space $\mathcal{H}^{\text{gh}}$ to the Hilbert space $\mathcal{H}_{\text{phys}}$ containing only the physical states, we need a mechanism which does the job. If Q is now defined as a nilpotent operator on $\mathcal{H}^{\text{gh}}$, we achieve unitarity of the S-matrix and are able to define the physical subspace by [8, 31, 29]

$$\mathcal{H}_{\text{phys}} = \frac{Ker(Q)}{Ran(Q)}$$

or equivalently

$$\mathcal{H}_{\text{phys}} = Ker\{Q, Q^\dagger\}$$

where $\mathcal{H}$ is the Hilbert space of asymptotic free fields and $Ker(Q)$, $Ran(Q)$ the kernel and the range of the operator Q, respectively. It is worth mentioning that these concepts hold for massless as well as for massive theories quite naturally. Then the condition of gauge invariance of the S-matrix can be stated as

$$\lim_{g\to 1} d_Q S(g) = 0 \tag{2.9}$$

and by transforming it to the perturbation series of the S-matrix it takes the form

$$d_Q T_n := [Q,\ T_n] = i\sum_{l=1}^{n} \frac{\partial}{\partial x_l^\mu} T^\mu_{n/l}(x_1, \ldots x_l \ldots x_n) \tag{2.10}$$

becoming a condition on the causal distributions T_n. In other words, gauge invariance is fulfilled if $d_Q T_n$ is a divergence where n means the order in perturbation theory and l the gauge variation on the vertex x_l.

One further fundamental ingredient of quantum field theory has to be mentioned [8]. In contrast to the classical theory, a gauge condition[4] like $\partial_\mu A^\mu = 0$ cannot hold on the entire Fock space as an operator equation[5]. In quantum field theory, this means

[3] $[.,.]$ means the commutator whereas $\{.,.\}$ means the anticommutator.

[4] which has necessarily to be imposed in order to reduce the superfluous degrees of freedom.

[5] Consider for example the Lorentz condition as an operator equation [8]

$$\partial_\mu A^\mu(x)\Omega = \partial_\mu A^{\mu+}(x)\Omega \neq 0$$

where $+$ $(-)$ means the emission (absorbtion) part of $A^\mu(x)$ and Ω the vacuum. It holds only as an expectation value

$$(\Omega, \partial_\mu A^\mu(x)\Omega) = \big(\Omega, \partial_\mu A^{\mu+}(x)\Omega\big) = \big(\mp\partial_\mu A^{\mu-}(x)\Omega, \Omega\big) = 0$$

where the upper $-$ sign corresponds to $\mu = 0$.

that a gauge transformation and a space-time transformation like Lorentz- or Poincaré transformations must be formulated on states in $\mathcal{H}^{\text{gh}}$. Moreover, the unitary transformation from a physical state $\Phi \in \mathcal{H}_{\text{phys}}$ to another physical state $\tilde{\Phi} \in \mathcal{H}_{\text{phys}}$ turns out to consists of a Lorentz transformation, which in general transforms Φ out of H_{phys} *and* a gauge transformation, generated by Q which transforms the state back into $\mathcal{H}_{\text{phys}}$. Thus in quantum field theory, we can generally not speak of a gauge transformation which leaves a gauge invariant. Indeed, we will see in section 3.1.2 that the gauge conditions do not hold on the field operators but on the test functions. This will be a crucial fact in the extension of the massless to the massive spin-2 quantum gauge theory.

There is one further remark to be mentioned. So far, the framework of causal perturbation theory and perturbative gauge invariance has recovered all main features of conventional gauge theories but with the important difference that there are no ultraviolet divergencies and other ill-defined operations appearing. In the case of supersymmetry [27], first inconsistencies between the conventional approach and the causal framework have shown up. But in the case of massless and massive gravity as two nearby theories, the spin-2 quantum gauge theory is the only mechanism to fully develop the theories in a consistent way, without giving up basic physical principles like positivity of the energy, conserved energy-momentum and Lorentz invariance.

3 The graviton as the asymptotic spin-2 particle

Since gravity couples to the energy-momentum tensor of physical matter, the graviton[6] has to be represented by a symmetric tensor field[7] of rank two. It transforms under the proper Lorentz group $\mathcal{L}_+^\uparrow$ according to the tensor product of two spinor representations $D^{(1/2,\ 1/2)}$ which can be decomposed into irreducible representations as

$$D^{(1/2,\ 1/2)} \otimes D^{(1/2,\ 1/2)} = D^{(1,\ 1)} \oplus D^{(1,\ 0)} \oplus D^{(0,\ 1)} \oplus D^{(0,\ 0)} \tag{3.1}$$

A symmetric tensor field with arbitrary trace transforms under the irreduzible representations as $D^{(1,\ 1)} \oplus D^{(0,\ 0)}$ where $D^{(1,\ 1)}$ is nine-dimensional and given by symmetric, traceless tensors whereas $D^{(0,\ 0)}$ is one dimensional and given by a scalar field, summing up to ten independent degrees of freedom. Therefore, the one particle Hilbert space $\mathcal{H}^{(1)}$ carries the direct sum of the two unitary, irreduzible representations of the proper Lorentz group as $D^{(1,\ 1)} \oplus D^{(0,\ 0)}$ corresponding to two particles of mass zero or m_0 and spins 2 and 0, respectively. In consequence of this, we will understand by a free graviton, the spin-2 particle, which is represented on the physical one particle Hilbert space $\mathcal{H}^{(1)}_{\text{phys}}$, corresponding to $\mathcal{H}^{(1)}_{\text{phys}} = Ker^{(1)}(Q)/Ran^{(1)}(Q)$ or $\mathcal{H}^{(1)}_{\text{phys}} = Ker^{(1)}\{Q, Q^\dagger\}$. This becomes clear when identifying the matter- and selfcouplings, derived in the classical case by expanding the linearized Einstein-Hilbert Lagrangian and in the quantum case from perturbative gauge invariance. $\mathcal{H}^{(1)}_{\text{phys}}$ is isomorphic to a certain L^2-space over the upper hyperboloid $X^+_{m_0}$ of mass $m_0 \geq 0$ which is by definition $X^+_{m_0} \equiv \{k \in \mathbb{R}^4 |\ ||k||^2 = m_0^2\}$.

We do not refer with our notation of the symmetric tensor field $h^{\mu\nu}$ to the classical linearized theory of general relativity. Only when treating Bremsstrahlung in the external field mechanism, we will make contact to the Newtonian limit of it, where the metric

[6] A highly recommended comment on the particle or field point of view is *J. Schwinger* in Theoretical Physics, International Atomic Energy Agency, Vienna, 1963,

[7] We will therefore take the quantum field as the fundamental object and not the particle since all properties of the latter can be assigned to the former leaving the one-to-one correspondence particle $\leftrightarrow$ quantum field $\leftrightarrow$ local quantum field operator.

tensor is linearized. From the beginning on, we will treat the fields on Minkowski background of space-time where its metric is chosen according to $\eta^{\mu\nu} = diag(+,-,-,-)$. The indices are raised and lowered in the usual way, comma means ordinary partial derivative and the covariant summation convention is used. The vacuum Ω is normalized according to $(\Omega, \Omega) = 1$. Furthermore, c and $\hbar$ are set to one. In general, it should be clear without subscript for the graviton mass (m_0) what case we are investigating.

3.1 Massless case

In describing the massless graviton as the asymptotic spin-2 particle by a symmetric tensor field $h^{\mu\nu}$, we will move along a path close to the classical concepts, but will stress the difference when ever needed. In the massless case [8, 40], the one particle Hilbert space $\mathcal{H}^{(1)}$ carrying the unitary, irreducible representation of the proper Lorentz group as $D^{(1,\,1)} \oplus D^{(0,\,0)}$, is $\mathcal{H}^{(1)} = \mathcal{H}^{(0,2)} \oplus \mathcal{H}^{(0,0)}$ i.e. describes two particles of mass zero and spin 2 and one of spin 0, respectively. We will see later, that for the free, physical massless graviton, the spin-0 part is absent.

3.1.1 Massless tensor field, operator gauge transformation and gauge charge Q

In the following, most of the presented material is elaborated in [8]. The fundamental free asymptotic fields in the massless case are the symmetric tensor field of rank two with arbitrary trace $h^{\mu\nu}(x)$, a ghost $u^{\mu}(x)$ and a anti-ghost $\tilde{u}^{\mu}(x)$. They all satisfy the massless Klein-Gordon equation

$$\begin{aligned} \Box h^{\mu\nu}(x) &= 0 \\ \Box u^{\mu}(x) &= 0 \\ \Box \tilde{u}^{\mu}(x) &= 0 \end{aligned} \tag{3.2}$$

Since a symmetric tensor field carries ten degrees of freedom, five more than the five independent components of a spin-2 field, they must be reduced by an additional condition. In the classical case, where $h^{\mu\nu}$ represents the metric perturbation, the five degrees of freedom are reduced with the two gauge conditions

$$\begin{aligned} \bar{h}^{\mu\nu}_{,\mu}(x) &= 0 \\ \bar{h}^{\mu}_{\mu}(x) &= 0 \end{aligned} \tag{3.3}$$

where $\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h$ and $h = \eta_{\mu\nu}h^{\mu\nu}$. They are referred to as the Hilbert-gauge and the trace condition, summarized as the transverse- traceless (tt) condition. Since we want to treat our tensor field $h^{\mu\nu}(x)$ as a gauge field in the sense of perturbative gauge theory, we must impose a quantum gauge transformation[8] on it. Its classical analogon

$$\bar{h}'^{\mu\nu}(x) = \bar{h}^{\mu\nu}(x) + \xi^{\mu,\nu}(x) + \xi^{\nu,\mu}(x) - \eta^{\mu\nu}\xi^{\gamma}_{,\gamma}(x) \tag{3.4}$$

leave the tt-condition 3.3 invariant where ξ^{μ} is a vector field.

[8]The main ingredient of a quantum gauge transformation is the fact, that there appear no ordinary functions or vector fields but quantum fields, which have to be treated as such.

Therefore, the quantum gauge transformation of the tensor field $h^{\mu\nu}(x)$ takes the form

$$\begin{aligned} h'^{\mu\nu}(x) &= h^{\mu\nu}(x) + \lambda\left(u^{\mu,\nu} + u^{\nu,\mu} - \eta^{\mu\nu} u^{\gamma}_{,\gamma}\right)(x) + O(\lambda^2) \\ &= h^{\mu\nu}(x) + 2\lambda\, b^{\mu\nu\alpha\beta}\, u_{\alpha,\beta}(x) + O(\lambda^2) \end{aligned} \tag{3.5}$$

where $u^{\mu}(x)$ is the vector ghost field and $b^{\mu\nu\alpha\beta}$ is the b-tensor given by

$$b^{\mu\nu\alpha\beta} = \frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}\right)$$

The vector ghost $u^{\mu}(x)$ has to satisfy the massless Klein-Gordon equation, in order to leave the transformed field $h'^{\mu\nu}$ satisfying the Klein-Gordon equation, too. In the classical case, the transformation 3.4 leaves the gauge condition 3.3 invariant and reduces three further degrees of freedom leaving two of them. But in our context, the tensor field $h^{\mu\nu}(x)$ as an operator valued functional, cannot fulfill a classical gauge condition. Moreover, we know from section 2.2, that a gauge condition cannot hold as an operator equation on the entire Fock space. The main ingredient is now to consider 3.5 as an operator gauge transformation, generated by a infinitesimal generator Q as

$$h'^{\mu\nu}(x) = e^{-i\lambda Q}\; h^{\mu\nu}(x)\; e^{i\lambda Q} \tag{3.6}$$

where according to 3.1, $h^{\mu\nu}(x)$ is considered to represent the spin-2 tensor field with arbitrary trace. The explicit form of Q is given by

$$Q = \int\limits_{x^0=t} d^3x\, h^{\mu\nu}_{,\nu} \overleftrightarrow{\partial}{}^{x}_{0}\, u_{\mu} \tag{3.7}$$

We will see later that a gauge condition induced by the gauge generator Q will hold on the corresponding test function of the smeared field operator $h^{\mu\nu}(x)$. In order to achieve the required nilpotency of Q, the vector ghost field $u^{\mu}(x)$ has to be quantized with anticommutators

$$\{u^{\mu}(x),\ \tilde{u}^{\nu}(y)\} = i\ \eta^{\mu\nu}\ D_0(x-y) \tag{3.8}$$

becoming a fermionic vector ghost. $D_0(x-y)$ is the massless Jordan-Pauli distribution [9]. According to 3.6, the operator Q as given in 3.7, must generate the infinitesimal operator gauge transformation 3.5. Therefore, the gauge variation on the tensor field $h^{\mu\nu}(x)$ must take the form

$$\begin{aligned} d_Q\, h^{\mu\nu}(x) = \left[Q, h^{\mu\nu}(x)\right] &= -\frac{i}{2}\left(u^{\mu,\nu} + u^{\nu,\mu} - \eta^{\mu\nu} u^{\sigma}_{,\sigma}\right)(x) \\ &:= -i\, b^{\mu\nu\alpha\beta} u_{\alpha,\beta}(x) \end{aligned} \tag{3.9}$$

where the factor $i/2$ is convention. Subject to 3.9, the massless tensor field $h^{\mu\nu}(x)$ must be quantized [8, 40] as

$$\left[h^{\mu\nu}(x), h^{\alpha\beta}(y)\right] = -\, i\, b^{\mu\nu\alpha\beta}\, D_0(x-y) \tag{3.10}$$

For the explicit representation in Fock space $\mathcal{F}$, the space of asymptotic free fields, $h^{\mu\nu}(x)$ is decomposed according to $D^{(1,1)} \oplus D^{(0,0)}$, the unitary, irreducible representations of the Lorentz group as

$$h^{\mu\nu}(x) = H^{\mu\nu}(x) + \frac{1}{4}\eta^{\mu\nu}h(x) \tag{3.11}$$

where $H^{\mu\nu}(x)$ represents the traceless part of the spin-2 field i.e. $\eta_{\mu\nu}H^{\mu\nu} = 0$ and $h(x)$ its trace part i.e. $\eta_{\mu\nu}h^{\mu\nu} = h$. According to [8], the fields are Fourier represented by

$$H^{\mu\nu}(x) = (2\pi)^{-\frac{3}{2}} \int \frac{d^3k}{\sqrt{2E_k}} \left(A_{\mu\nu}(\vec{k})\; e^{-ikx} + \eta^{\mu\mu}\eta^{\nu\nu} A^{\dagger}_{\mu\nu}(\vec{k})\; e^{ikx} \right) \tag{3.12}$$

where the $A_{\mu\nu}(\vec{k})$, $A^{\dagger}_{\mu\nu}(\vec{k})$ are annihilation and creation operators on a bosonic Fock space. From 3.10 we find the commutation relation

$$\eta^{\alpha\alpha}\eta^{\beta\beta} \left[A_{\mu\nu}(\vec{k}), A^{\dagger}_{\alpha\beta}(\vec{k'}) \right] \;=\; t^{\mu\nu\alpha\beta}\, \delta(\vec{k} - \vec{k'}) \tag{3.13}$$

where $t^{\mu\nu\alpha\beta}$ is given by

$$t^{\mu\nu\alpha\beta} = \frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta} \right)$$

Note, that our notation differs slightly from the one in [8] because in the present work, we have to distinguish between operators which create and annihilate traceless modes and some which create and annihilate modes including trace degrees of freedom. The scalar part $h(x)$ is represented in momentum space as

$$h(x) = (2\pi)^{-\frac{3}{2}} \int \frac{d^3k}{\sqrt{2E_k}} \left(d(\vec{k})\; e^{-ikx} - d^{\dagger}(\vec{k})\; e^{ikx} \right) \tag{3.14}$$

where the creation and annihilation operators $d^{\dagger}(\vec{k})$ and $d(\vec{k})$ on a bosnic Fock space satisfy, according to 3.10, the commutation relation

$$\left[d(\vec{k}),\; d^{\dagger}(\vec{k'}) \right] = 4\delta(\vec{k} - \vec{k'}) \tag{3.15}$$

In 3.12 and 3.14, we could have also used the Krein operator [32] as another conjugation. But they are equivalent and their definition does not matter for physical quantities anyway. In consequence of 3.10, the gauge variation for the fundamental fields take, according to 3.7 and 3.9, the following form

$$\begin{aligned} d_Q\, h^{\mu\nu} &= -\frac{i}{2}\left(u^{\mu,\nu} + u^{\nu,\mu} - \eta^{\mu\nu} u^{\sigma}_{,\sigma} \right) \\ d_Q\, h &= i u^{\sigma}_{,\sigma} \\ d_Q\, \tilde{u}^{\mu} &= i h^{\mu\nu}_{,\nu} \\ d_Q\, u^{\mu} &= 0 \end{aligned} \tag{3.16}$$

Note, that $d_Q\; h^{\mu\nu}_{,\nu} = 0$ thus the definition of Q in the massless case corresponds to the classical gauge condition 3.3, recovering the well known classical framework of linearized gravity.

It is worth mentioning, in consequence of 2.7, 2.8 and the gauge variations defined above, that the general form of the gauge variation may be cast into the following form

$$d_Q F := QF - (-1)^{n_g(F)} FQ$$

where F is a free quantum field on $\mathcal{H}^{\text{gh}}$. n_g is the ghost number, the number of ghost fields minus the number of anti-ghost fields in F. Then, the operator d_Q has all properties of an antiderivation, obeying the Leibniz rule

$$d_Q(AB) = (d_Q A)B + (-)^{n_g(A)} A d_Q B$$

with $A,\ B$ being arbitrary operators on $\mathcal{H}^{\text{gh}}$.

3.1.2 The physical degrees of freedom of the free massless graviton

In the derivation of what physical degrees of freedom are carried by the free massless graviton, we have two options. One way is to consider the gauge charge Q in momentum space and calculate explicitly the physical subspace, defined as the kernel of $\{Q, Q^+\}$. For the massless case, this is done in [8, 32]. We will outline here only the main ingredients. The other way is to consider the physical space defined as $\mathcal{H}_{\text{phys}} = Ker(Q)/Ran(Q)$ and to derive the physical degrees of freedom in x-space.

p-space derivation

In the massless case, the gauge charge Q is given, according to 3.7, by

$$Q = \int\limits_{x^0=t} d^3x\, h^{\mu\nu}_{,\nu} \overleftrightarrow{\partial}{}^x_0\, u_\mu$$

and we consider the definition of the physical sub-space as $Ker\{Q, Q^+\}$. The task is now to write down Q in momentum space and to calculate explicitly the anticommutator $\{Q, Q^+\}$. It contains the particle number operators of all unphysical particles of the theory. The physical states ψ are then identified as the particle number operators with vanishing eigenvalues acting on the vacuum Ω. This means that we have to insert into the definition of Q the momentum space representations for the fields involved. An important point lies in the fact that the Fock space $\mathcal{F}$ generated by the creation and annihilation operators $A_{\mu\nu}(\vec{k})$, $A^\dagger_{\mu\nu}(\vec{k})$ and their commutation relation 3.13 is not diagonal due to $t^{\mu\nu\alpha\beta}$. For remedy this defect, we have to apply a linear transformation on the diagonal operators $A_{\mu\mu}$ and $A^\dagger_{\mu\mu}$ which takes the form

$$\begin{aligned}
A_{00} &= \frac{1}{2}\Big(\tilde{A}_{11} + \tilde{A}_{22} + \tilde{A}_{33}\Big) \\
A_{11} &= \frac{1}{2}\Big(-\tilde{A}_{11} + \tilde{A}_{22} + \tilde{A}_{33}\Big) \\
A_{22} &= \frac{1}{2}\Big(\tilde{A}_{11} - \tilde{A}_{22} + \tilde{A}_{33}\Big) \\
A_{33} &= \frac{1}{2}\Big(\tilde{A}_{11} + \tilde{A}_{22} - \tilde{A}_{33}\Big)
\end{aligned} \tag{3.17}$$

Using the notation of these operators as $\frac{1}{2}A_{00} = B_{00}$, $\tilde{A}_{ii} = B_{ii}$ and $A_{\alpha\beta} = B_{\alpha\beta}$ for $\alpha \neq \beta$, they satisfy the commutation relations

$$\left[B_{\mu\nu}(\vec{k}), B^\dagger_{\alpha\beta}(\vec{k'}) \right] \;=\; \frac{1}{2}\left(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}\right)\delta(\vec{k} - \vec{k'}) \tag{3.18}$$

and thus construct the Fock representation in the usual way. This means that we have to substitute in the calculation of $\{Q, Q^+\}$ the diagonal operators $A_{\mu\mu}$ and $A^\dagger_{\mu\mu}$ by the $\tilde{A}_{\mu\mu}$ and $\tilde{A}^\dagger_{\mu\mu}$. Choosing an explicit frame as $k^\mu = (\omega, 0, 0, \omega)$ with $k^\mu k_\mu = 0$ and diagonalizing the emerging quadratic form, we then may identify the massless physical states as

$$\begin{aligned} \psi_1 \;&=\; A^\dagger_{12}\,\Omega \\ \psi_2 \;&=\; \left(\tilde{A}_{11} - \tilde{A}_{22}\right)^\dagger \Omega \end{aligned}$$

In order to prepare the calculation for the massless Bremsstrahl matrix element, we have to normalize these two physical states due to 3.13, 3.15, 3.18 and transform the $\tilde{A}_{\mu\nu}$'s back to the $A_{\mu\nu}$'s because the commutation relations 3.13 will be used there. In the following, we will write the massless physical state as a two-tuple as

$$\vec{\psi} \;=\; \begin{pmatrix} \psi_{\hat{1}} \\ \\ \psi_{\hat{2}} \end{pmatrix} \;=\; \begin{pmatrix} \sqrt{2}\,A^\dagger_{12} \\ \\ \frac{1}{\sqrt{2}}\left(A_{22} - A_{11}\right)^\dagger \end{pmatrix}(\vec{k})\,\Omega \tag{3.19}$$

Transforming 3.19 into helicity formalism, we find the well known ± 2 helicity states. It is important to note that the trace degree of freedom, created by $d^\dagger$, is not physical.

x-space derivation

In the derivation of the massless physical degrees of freedom in x-space, we will have to use some arguments from the former p-space derivation, in order to establish the way presented here. It is based on the work [28, 31]. For simplicity, we will consider only one-particle states.

The physical one-particle space corresponding to $\mathcal{H}^{(1)}_{\text{phys}} = Ker^{(1)}(Q)/Ran^{(1)}(Q)$ where $Ker^{(1)}(Q) \equiv Ker(Q) \cap \mathcal{H}^{(1)}$ and $Ran^{(1)}(Q) \equiv Ran(Q) \cap \mathcal{H}^{(1)}$ is $H^{[0,2]} \oplus H^{[0,0]}$ i.e. describing two massless particles with spin 2 and one of spin 0, respectively. A general one-particle state is

$$\Phi \;=\; \int d^4x \left[f^{\mu\nu} h_{\mu\nu} + f_1^\mu u_\mu + f_2^\mu \tilde{u}_\mu \right] \Omega \tag{3.20}$$

where the wave functions $f^{\mu\nu}(x)$[9] have Fourier transforms with support on the mass shell, they are symmetric in μ, ν and traceless[10].

[9] i.e. test functions $f^{\mu\nu} \in \mathcal{S}(\mathbb{R}^4)$, the Schwartz space of tempered distributions.

[10] In this approach, we do not have an independent argument for the $f^{\mu\nu}$, $h^{\mu\nu}$ to be traceless, but we make use of this argument from the derivations in [8, 32].

$Ker(Q)$:

In order to derive $Ker^{(1)}Q$, we have to act with the gauge charge Q on the general one-particle state Φ using $Q\ \Omega = 0$, hence

$$\begin{aligned} Q\,\Phi &= \int d^4x \left[f^{\mu\nu} \left[Q, h_{\mu\nu} \right] + f_1^{\mu} \{Q, u_\mu\} + f_2^{\mu} \{Q, \tilde{u}_\mu\} \right] \Omega \\ &= \int d^4x \left[-\frac{i}{2} f^{\mu\nu} \left(\partial_\mu u_\nu + \partial_\nu u_\mu - \eta_{\mu\nu} \partial^\gamma u_\gamma \right) + i f_2^{\mu} \partial^\nu h_{\mu\nu} \right] \Omega \end{aligned} \tag{3.21}$$

The condition $\Phi \in Ker^{(1)}(Q)$ takes the form

$$\begin{aligned} 0 &\stackrel{!}{=} \int d^4x \left[\frac{i}{2} \left(\partial_\mu f^{\mu\nu} u_\nu + \partial_\nu f^{\mu\nu} u_\mu - \partial^\gamma f\, u_\gamma \right) - i \partial^\nu f_2^{\mu} h_{\mu\nu} \right] \Omega \\ &= \frac{i}{2} \int d^4x \left[\left(2 \partial_\nu f^{\mu\nu} - \partial^\mu f \right) u_\mu - \left(\partial^\nu f_2^{\mu} + \partial^\mu f_2^{\nu} \right) h_{\mu\nu} \right] \Omega \end{aligned} \tag{3.22}$$

where $f = \eta_{\mu\nu} f^{\mu\nu}$ leaving two conditions on the wave functions as

$$\partial^\nu f_2^{\mu} + \partial^\mu f_2^{\nu} = 0 \tag{3.23}$$

$$2 \partial_\nu f^{\mu\nu} - \partial^\mu f = 0 \tag{3.24}$$

It is important to note, that the classical Hilbert gauge condition $\partial_\mu \bar{h}^{\mu\nu} = 0$ acts on the classical tensor field $\bar{h}^{\mu\nu}$. In quantum field theory, $h^{\mu\nu}$ becomes an operator valued distribution and the gauge condition 3.24 holds on its corresponding wave function $f^{\mu\nu}(x)$[11]. This is a crucial difference to the classical case. If we define $\bar{f}^{\mu\nu} = f^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} f$, the condition 3.24 becomes a Hilbert gauge condition on the wave function i.e.

$$\partial_\mu \bar{f}^{\mu\nu} = 0 \tag{3.25}$$

Therefore, a state $\Phi \in Ker^{(1)}(Q)$ takes the form 3.20 subject to 3.23 and 3.25.

$Ran(Q)$**:**

Not all states $\Phi \in Ker^{(1)}(Q)$ describe distinct physical situations. We must factor out the states $\Phi' \in Ran^{(1)}(Q)$ i.e. states from $Ker^{(1)}(Q)$ verifying

$$\Phi' = Q \Phi_0 \tag{3.26}$$

for some state Φ_0. Taking 3.25 as well as restrictions on ghost numbers into account, Φ_0 then must be of the form

$$\Phi_0 = \int d^4x \, (g_1^{\mu} u_\mu + g_2^{\mu} \tilde{u}_\mu) \, \Omega \tag{3.27}$$

[11]For a symmetric array, there is a TT-decomposition according to $f_{ij} = f_{ij}^{TT} + f_{ij}^{T} + (f_{i\ ,j} + f_{ij})$ where each of the quantities on the right-hand side can be expressed uniquely as a linear functional of f_{ij}. The quantities f_{ij}^{TT} are the two transverse traceless components of f_{ij}. The transverse part f_{ij}^{T} is uniquely defined through the trace f^{T}. See Arnowitt, Deser, Misner: The dynamics of General Relativity in L- Witten: Gravitation: an introduction to current research, Wiley, 1962.

since u_ν and $\tilde{u}_\mu$ are the only ghost-fields in the massless case. Inserting this into 3.26 and commuting Q to the right, we get

$$
\begin{aligned}
Q\Phi_0 &= -\frac{i}{2}\int d^4x \left[(\partial^\nu g_2^\mu + \partial^\mu g_2^\nu)\, H_{\mu\nu} + \frac{1}{2}\partial_\gamma g_2^\gamma\, h\right]\Omega \\
&\stackrel{!}{=} \int d^4x f^{\mu\nu} H_{\mu\nu}\Omega
\end{aligned}
\tag{3.28}
$$

where 3.28 must hold in order to satisfy 3.26 leaving

$$
\partial_\mu g_2^\mu = 0 \tag{3.29}
$$

$$
f^{\mu\nu} = -\frac{i}{2}(\partial^\nu g_2^\mu + \partial^\mu g_2^\nu) \tag{3.30}
$$

This is the general form of a state from $Ran^{(1)}(Q)$. It follows now that the physical states in the massless case are given by wave functions with Fourier transforms on the mass shell, verifying 3.25, modulo wave functions of the type 3.30 where the ghost wave functions are subject to 3.29.

In order to fix the number of physical degrees of freedom, we have to count from the various conditions the reduction of degrees of freedom they imply. A symmetric tensor field $f^{\mu\nu}$ has 10 degrees of freedom. Therefore, the general one-particle state 3.20 carries 18 degrees of freedom. The conditions 3.23 and 3.24 reduce 14 degrees of freedom plus the additionally traceless condition $f_\mu^\mu = 0$, leaving three degrees of freedom. From $Ran^{(1)}Q$ there is the condition 3.29 to be satisfied where 3.30 sets the form of the wave functions $f^{\mu\nu}$. If we now count the degrees of freedom from $Ker^{(1)}(Q)$ and from $Ran^{(1)}(Q)$, we are left with two physical degrees of freedom as it should be for the massless free graviton.

3.1.3 Polarization sum

Since we are interested in quantum gravitational Bremsstrahlung where the massless graviton carries two physical polarizations $\pm$ (helicities), we must sum over them in the final cross section. The aim of the following discussion is to elaborate the nature of the polarization "tensor" and to determine its polarization sum, the sum over the two physical degrees of freedom.

A general graviton state, representing a physical massless graviton, is given by

$$
h_f = \int d^3k\ \hat{f}_{\mu\nu}(\vec{k})\ \varepsilon^{\mu\nu}_{\lambda\lambda'}(\vec{k})\ A^\dagger_{\lambda\lambda'}(\vec{k})\ \Omega \tag{3.31}
$$

where the object $\varepsilon^{\mu\nu}_{\lambda\lambda'}(\vec{k})$ is the polarization "tensor", defined below, referring to the propagation direction $\vec{k}$ of the Bremsstrahl graviton with polarization indices $\lambda\lambda'$ and $\hat{f}_{\mu\nu}(\vec{k})$ is the Fourier transform of a test function in Schwartz space. In the massless case, $\lambda\lambda'$ parametrizes the ± 2 helicity, writing the polarization "tensor" as $\varepsilon^{\mu\nu}_{\pm}(\vec{k})$. The sum in 3.31 over the $\lambda\lambda'$-indices is not a Minkowski scalar product since $\lambda\lambda'$ are not true covariant or contravariant indices with respect to the Minkowski metric $\eta^{\mu\nu}$ whereas the sum over the indices $\mu\nu$ is. We may extract the polarization "tensor" $\varepsilon^{\mu\nu}_{\pm}$ from the matrix element $\mathcal{M}$ as

$$\mathcal{M}_\pm = \varepsilon_\pm^{\mu\nu *}\mathcal{M}_{\mu\nu} \tag{3.32}$$

where $\mathcal{M}_{\mu\nu}$ represents the remaining tensorial part of it and $*$ means complex conjugation. In the calculation of the scattering cross section, we have to square 3.32 and sum over the two physical polarizations $\pm$. The cross section is proportional to

$$\begin{aligned}\sum_\pm \left|\mathcal{M}_\pm\right|^2 &= \sum_\pm \left|\varepsilon_\pm^{\mu\nu *}\,\mathcal{M}_{\mu\nu}\right|^2 \\ &= \sum_\pm \left|\varepsilon_\pm^{\mu\nu *}\right|^2 \left|\mathcal{M}_{\mu\nu}\right|^2 \\ &= \sum_\pm \varepsilon_\pm^{\mu\nu}\,\varepsilon_\pm^{\mu'\nu' *}\,\mathcal{M}_{\mu\nu}\,\mathcal{M}^*_{\mu'\nu'}\end{aligned} \tag{3.33}$$

The expression $\sum_\pm \varepsilon_\pm^{\mu\nu}\varepsilon_\pm^{\mu'\nu' *}$ has now to be investigated. According to the pioneering work by Weinberg [46, 47, 48, 49], we are able to write the polarization "tensor" $\varepsilon_\pm^{\mu\nu *}$ as the product of two polarization "vectors" $\varepsilon_\pm^{\mu *}$

$$\varepsilon_\pm^{\mu\nu *} = \varepsilon_\pm^{\mu *}\varepsilon_\pm^{\nu *} \tag{3.34}$$

A polarization "vector" $\varepsilon_\pm^\mu(\hat{k})$ for the propagation direction $\vec{k}$ of the Bremsstrahl graviton is defined by $\varepsilon_\pm^\mu(\hat{k}) \equiv \mathcal{R}(\hat{k})^\mu_\nu\ \varepsilon_\pm^\nu$ where $\mathcal{R}(\hat{k})^\mu_\nu$ is a standard rotation that carries a fixed coordinate axis into the direction of $\vec{k}$. According to our convention of the metric tensor $\eta_{\mu\nu} = \eta^{\mu\nu} = diag(+,-,-,-)$, the explicit form of the two polarization "vectors" for the momentum in the z-direction takes the form

$$\varepsilon_\pm^\mu = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$$

and the following properties hold

$$\begin{aligned}\varepsilon^*_{\pm\mu}(\hat{k})\,\varepsilon_\pm^\mu(\hat{k}) &= -1 && (3.35)\\ \varepsilon_{\pm\mu}(\hat{k})\,\varepsilon_\pm^\mu(\hat{k}) &= 0 && (3.36)\\ \varepsilon_\pm^{\mu *}(\hat{k}) &= \varepsilon_\mp^\mu(\hat{k}) && (3.37)\\ \varepsilon_\pm^0(\hat{k}) &= 0 && (3.38)\\ k_\mu\,\varepsilon_\pm^\mu(\hat{k}) &= 0 && (3.39)\end{aligned}$$

where the summation convention over repeated covariant indices is used. The two conditions 3.36 and 3.39 represent 3.25 holding on the polarization "vector".

The transformation properties of the polarizations "vectors" $\varepsilon_\pm^\mu(\hat{k})$ are based on the work by Wigner [11, 13]. They are described by the little group for massless particles, the group consisting of all Lorentz transformations which leave a standard light-like four vector invariant. Wigner showed that this group is isomorphic to the group of rotations by an angle $\Theta[\vec{k}, \Lambda]$ and a translation in the Euclidian plane. The transformation property takes the form [47]

$$\left(\frac{\Lambda^\mu_\nu - k^\mu \Lambda^0_\nu}{|\vec{k}|}\right)\varepsilon_\pm^\nu(\Lambda\hat{k}) = \exp\{\pm i\Theta[\vec{k}, \Lambda]\}\varepsilon_\pm^\mu(\hat{k}) \tag{3.40}$$

The term k^μ in 3.40 prevent the polarization vector $\varepsilon^\mu_\pm(\hat{k})$ to be a true four-vector, hence $\varepsilon^{\mu\nu}_\pm$ is not a true tensor according to 3.34. The transformation property 3.40 implies the S-matrix to be invariant under the classical gauge transformation

$$\varepsilon^\mu_\pm \rightarrow \varepsilon^\mu_\pm + \lambda_\pm k^\mu$$

with $\lambda_\pm$ arbitrary. Since the polarization tensor carries the two physical polarizations, it is worth to pause and summarize the difference between the classical and the quantum case. In the classical framework, the superfluous degrees of freedom of the tensor field are reduced by imposing the tt-condition 3.3 on the classical tensor field $h^{\mu\nu}$ and its invariance under the coordinate transformation 3.4. In the case of gravity as a quantum gauge theory, the superfluous degrees of freedom are reduced to the two physical ones by the gauge charge Q, acting on the field operator $h^{\mu\nu}(x)$ and filtering out the physical states by $\mathcal{H}_{\text{phys}} = Ker(Q)/Ran(Q)$. This difference in reducing the superfluous degrees of freedom is crucial in the discussion for the massive case. So far the properties of the polarization "tensor".

To come back to the polarization sum, we find, using 3.34

$$\begin{aligned}\sum_\pm \varepsilon^{\mu\nu}_\pm(\hat{k})\,\varepsilon^{\mu'\nu'*}_\pm(\hat{k}) &= \sum_\pm \varepsilon^{\mu}_\pm(\hat{k})\,\varepsilon^{\nu}_\pm(\hat{k})\,\varepsilon^{\mu'*}_\pm(\hat{k})\,\varepsilon^{\nu'*}_\pm(\hat{k}) \\ &= \sum_\pm \varepsilon^{\mu}_\pm(\hat{k})\,\varepsilon^{\mu'*}_\pm(\hat{k})\,\varepsilon^{\nu}_\pm(\hat{k})\,\varepsilon^{\nu'*}_\pm(\hat{k}) \qquad (3.41)\end{aligned}$$

which is valid in any coordinate system. The expression 3.41 can be factorized in the form

$$\sum_\pm \varepsilon^{\mu}_\pm(\hat{k})\,\varepsilon^{\mu'*}_\pm(\hat{k})\,\varepsilon^{\nu}_\pm(\hat{k})\,\varepsilon^{\nu'*}_\pm(\hat{k}) = P\left(\Pi^{\mu\mu'}\cdot\Pi^{\nu\nu'}\right) \qquad (3.42)$$

where P is an arbitrary function and

$$\Pi^{\mu\mu'} := \sum_\pm \varepsilon^{\mu}_\pm(\hat{k})\ \varepsilon^{\mu'*}_\pm(\hat{k})$$

We find

$$\sum_\pm \varepsilon^{\mu}_\pm(\hat{k})\,\varepsilon^{\nu}_\pm(\hat{k})\,\varepsilon^{\mu'*}_\pm(\hat{k})\,\varepsilon^{\nu'*}_\pm(\hat{k}) = \frac{1}{2}\left\{\Pi^{\mu\mu'}\Pi^{\nu\nu'} + \Pi^{\mu\nu'}\Pi^{\nu\mu'} - \Pi^{\mu\nu}\Pi^{\mu'\nu'}\right\} \qquad (3.43)$$

which is proven by using 3.37, thus

$$\begin{aligned}P\left(\Pi^{\mu\mu'}\cdot\Pi^{\nu\nu'}\right) &= \frac{1}{2}\left\{\Pi^{\mu\mu'}\Pi^{\nu\nu'} + \Pi^{\mu\nu'}\Pi^{\nu\mu'} - \Pi^{\mu\nu}\Pi^{\mu'\nu'}\right\} \qquad (3.44)\\ &=: P^{\mu\nu\mu'\nu'}_{(0)} \qquad (3.45)\end{aligned}$$

where we will call $P^{\mu\nu\mu'\nu'}_{(0)}$ the polarization sum function in the massless case. In Minkowski space, the "tensor" $\Pi^{\mu\nu}$ for a massless particle has to be constructed out of k_μ, $\tilde{k}_\mu$ and $\eta_{\mu\nu}$ and is given according to [47] by

$$\sum_{\pm} \varepsilon^{\mu}_{\pm}(\hat{k})\ \varepsilon^{\mu'*}_{\pm}(\hat{k}) = \Pi^{\mu\mu'} = \frac{\tilde{k}^{\mu}k^{\mu'} + k^{\mu}\tilde{k}^{\mu'}}{k\tilde{k}} - \eta^{\mu\mu'} \tag{3.46}$$

where $\tilde{k}^{\mu} = (\omega,\ -\vec{k})$ is obtained from k^{μ} by a space reflection. It is important to note, that the form 3.46 of the Π-"tensor" depends uniquely on the conditions 3.35 - 3.39 and they themselves on condition 3.25 and on completeness arguments. In 3.41, we were forced to do the rearrangement in this way because of 3.36. Then, the polarization sum for the massless graviton in any coordinate system is given by

$$\sum_{\pm} \varepsilon^{\mu\nu}_{\pm}(\hat{k})\, \varepsilon^{\mu'\nu'*}_{\pm}(\hat{k}) \ = \ \sum_{\pm} \varepsilon^{\mu}_{\pm}(\hat{k})\, \varepsilon^{\nu}_{\pm}(\hat{k})\, \varepsilon^{\mu'*}_{\pm}(\hat{k})\, \varepsilon^{\nu'*}_{\pm}(\hat{k}) \ = \ P^{\mu\nu\mu'\nu'}_{(0)} \tag{3.47}$$

Squaring the matrix element $\mathcal{M}_{\pm}$ and summing over the two physical polarizations $\pm$ becomes then, according to 3.33 and 3.47,

$$\sum_{\pm} |\mathcal{M}_{\pm}|^2 \ = \ P^{\mu\nu\mu'\nu'}_{(0)} \mathcal{M}_{\mu\nu}\mathcal{M}^*_{\mu'\nu'} \tag{3.48}$$

We would like to emphasize that, subject to 3.40, the polarization sum function $P^{\mu\nu\mu'\nu'}_{(0)}$ is not covariant hence not a true four rank tensor but can be written in terms of k_{μ}, $\tilde{k}_{\mu}$ and $\eta_{\mu\nu}$.

3.2 Massive case

In the massive case [28, 29], the one particle Hilbert space, carrying the unitary, irreducible representation of the proper Lorentz group as $D^{(1,\ 1)} \oplus D^{(0,\ 0)}$, is $\mathcal{H}^{(1)} = \mathcal{H}^{(m_0,2)} \oplus \mathcal{H}^{(m_0,0)}$ i.e. describes two particles of mass m_0 and spin 2 and one of spin 0, respectively. It will be shown that in the massive case, the spin-0 part will become physical.

3.2.1 Massive tensor field, operator gauge transformation and gauge charge Q

One may ask how to extend the massless theory to the massive one [28, 29]. Clearly, all fields become massive, satisfying the massive Klein-Gordon equation and the Jordan-Pauli distribution of mass zero is extended to the one of mass m_0. But the most important ingredient in the context of causal perturbation theory and perturbative quantum gauge invariance is to keep the gauge charge Q nilpotent, in order to use it for defining $\mathcal{H}_{\rm phys}$. The nilpotency of Q requires a new, massive bosonic vector ghost field $v_{\mu}(x)$. This implies, by the derivation of the physical subspace according to $\mathcal{H}_{\rm phys} = Ker(Q)/Ran(Q)$, that in the massive case, the spin-0 part becomes physical. One could have defined the gauge charge Q to generate a gauge transformation which itself leaves a gauge condition invariant. But this is a classical concept and not relevant for our discussion. The fundamental principle lies on the level of the physical Hilbert space, where a state is transformed into another by a quantum gauge transformation, generated by Q *and* a Lorentz transformation.

Therefore, the fundamental free asymptotic fields in the massive case are the symmetric tensor field of rank two $h^{\mu\nu}(x)$ with arbitrary trace, the fermionic vector ghost $u^{\mu}(x)$ and anti-vector ghost $\tilde{u}^{\mu}(x)$, respectively and a bosonic ghost $v^{\mu}(x)$. They all satisfy the massive Klein-Gordon equation

$$
\begin{aligned}
(\Box + m_0^2)\, h^{\mu\nu}(x) &= 0 \\
(\Box + m_0^2)\, u^{\mu}(x) &= 0 \\
(\Box + m_0^2)\, \tilde{u}^{\mu}(x) &= 0 \\
(\Box + m_0^2)\, v^{\mu}(x) &= 0
\end{aligned}
\tag{3.49}
$$

The operator gauge transformation in the massive case is dictated by the form of the massive gauge charge Q. It takes the form

$$
Q = \int\limits_{x^0=t} d^3x \left[h^{\mu\nu}_{,\nu}(x) + m_0\, v^{\mu}(x) \right] \overleftrightarrow{\partial_0^x}\, u_\mu(x) \tag{3.50}
$$

where the fermionic vector ghost field $u_\mu(x)$ must be quantized again according to 3.8 with anticommutators and the bosonic vector ghost $v^\mu(x)$ satisfies the commutation relation

$$
[v^\mu(x),\ v^\nu(y)] = \frac{i}{2}\, \eta^{\mu\nu}\ D_{m_0}(x-y)
$$

in order to render the massive gauge charge Q nilpotent. $D_{m_0}(x-y)$ is the Jordan-Pauli distribution for mass m_0. As in the massless case, the gauge variation on the tensor field takes the form

$$
\begin{aligned}
d_Q\, h^{\mu\nu}(x) = \Big[Q, h^{\mu\nu}(x) \Big] &= -\frac{i}{2}\Big(u^{\mu,\nu} + u^{\nu,\mu} - \eta^{\mu\nu} u^{\sigma}_{,\sigma} \Big)(x) \\
&:= -i\, b^{\mu\nu\alpha\beta} u_{\alpha,\beta}
\end{aligned}
\tag{3.51}
$$

provided the massive tensor field $h^{\mu\nu}(x)$ is quantized as

$$
\Big[h^{\mu\nu}(x), h^{\alpha\beta}(y) \Big] = -\, i\, b^{\mu\nu\alpha\beta}\, D_\mu(x-y) \tag{3.52}
$$

where the b-tensor is given by

$$
b^{\mu\nu\alpha\beta} = \frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta} \right)
$$

Now, one has to be careful with the decomposition of the massive spin-2 field and its representation in momentum space. First, since we do need a representation where the creation and annihilation operators include the trace degrees of freedom, we must first show that such a representation exists. This representation then could be related to the field decomposition 3.11. Furthermore, we need to derive two representations for the massive tensor field $h^{\mu\nu}(x)$, one in which all massive degrees of freedom are included (for the general contraction in the matrix element) and one, which parameterizes the massive physical degrees of freedom. The latter is necessary to define the final state of the massive Bremsstrahl graviton.

The representation of the massive field operator $h^{\mu\nu}(x)$ which create and annihilate both, the traceless and the trace degrees of freedom, is given by

$$
h^{\mu\nu}(x) = (2\pi)^{-\frac{3}{2}} \int \frac{d^3k}{\sqrt{2E_k}} \left(a_{\mu\nu}(\vec{k})\ e^{-ikx} + \eta^{\mu\mu}\eta^{\nu\nu} a^{\dagger}_{\mu\nu}(\vec{k})\ e^{ikx} \right) \tag{3.53}
$$

where the $a_{\mu\nu}(\vec{k})$, $a^\dagger_{\mu\nu}(\vec{k})$ satisfy the commutation relation

$$\eta^{\mu\mu}\eta^{\nu\nu}\left[a_{\mu\nu}(\vec{k}), a^\dagger_{\alpha\beta}(\vec{k'})\right] = b^{\mu\nu\alpha\beta}\,\delta(\vec{k}-\vec{k'}) \tag{3.54}$$

This representation is consistent with 3.52 and its field decomposition in Fock space $\mathcal{F}$ according to $D^{(1,1)} \oplus D^{(0,0)}$, the unitary, irreducible representations of the Lorentz group carries the same structure as 3.11, hence

$$h^{\mu\nu}(x) = H^{\mu\nu}(x) + \frac{1}{4}\eta^{\mu\nu}h(x) \tag{3.55}$$

with the only difference that the fields in question are all massive. Therefore we may summarize as follows: The field operator $h^{\mu\nu}(x)$ include the creation and annihilation operators for both, the traceless and the trace degrees of freedom whereas the field operator $H^{\mu\nu}(x)$ and $h(x)$ create and annihilate only the traceless and the trace degrees of freedom, respectively. Therefore, the field decomposition 3.55 can be used in both, the massless and the massive case.

In order to define the final state, we have to derive a representation for $h^{\mu\nu}(x)$ which carries an explicit object, representing the polarization of the massive graviton. For this, consider the operator

$$\Psi = \int d^4x\; f^{\mu\nu}(x)\; h_{\mu\nu}(x) \tag{3.56}$$

where the sum over the indices $\mu\nu$ is a true Minkowski scalar product with respect to the metric $\eta^{\mu\nu}$ and $f^{\mu\nu}(x)$[12] is a wave function, symmetric in the indices $\mu\nu$ and not necessarily traceless anymore. Its Fourier transform with support on the mass shell is given by

$$\begin{aligned} \hat{f}^{\mu\nu}(-k) &= (2\pi)^{-2}\int d^4x\, f^{\mu\nu}(x)\, e^{-ikx} \\ \hat{f}^{\mu\nu}(k) &= (2\pi)^{-2}\int d^4x\, f^{\mu\nu}(x)\, e^{ikx} \end{aligned} \tag{3.57}$$

Then using 3.53, the operator becomes

$$\begin{aligned} \Psi &= \int d^4x\, f^{\mu\nu}(x)\, h_{\mu\nu}(x) \\ &= (2\pi)^{-\frac{3}{2}}\int d^4x \int \frac{d^3k}{\sqrt{2E_k}}\, f_{\mu\nu}(x)\left[a_{\mu\nu}(\vec{k})\, e^{-ikx} + \eta^{\mu\mu}\eta^{\nu\nu}\; a^\dagger_{\mu\nu}(\vec{k})\, e^{ikx}\right] \\ &= \sqrt{2\pi}\int \frac{d^3k}{\sqrt{2E_k}}\left[\hat{f}_{\mu\nu}(-k)\, a_{\mu\nu}(\vec{k}) + \eta^{\mu\mu}\eta^{\nu\nu}\; \hat{f}_{\mu\nu}(k)\, a^\dagger_{\mu\nu}(\vec{k})\right] \\ &= \sqrt{2\pi}\int \frac{d^3k}{\sqrt{2E_k}}\left[\hat{f}_{\lambda\lambda'}(-k)\, a_{\lambda\lambda'}(\vec{k}) + \eta^{\lambda\lambda}\eta^{\lambda'\lambda'}\; \hat{f}_{\lambda\lambda'}(k)\, a^\dagger_{\lambda\lambda'}(\vec{k})\right] \end{aligned} \tag{3.58}$$

where we write the indices downstairs indicating that the sum is not a Minkowski scalar product since they are not true covariant or contravariant indices. The indices $\lambda\lambda'$

[12]i.e. test functions $f^{\mu\nu} \in \mathcal{S}(\mathbb{R}^4)$, the Schwartz space of tempered distributions.

are introduced here, they will become the indices parameterizing the physical degrees of freedom of the massive graviton. Now define [30]

$$\begin{aligned} a'_{\lambda\lambda'}(\vec{k}) &:= a_{\mu\nu}(\vec{k})\, \hat{f}_{\mu\nu\,\lambda\lambda'}(-k) \\ \hat{f}^{\mu\nu}(-k) &:= \hat{f}^{\mu\nu}_{\lambda\lambda'}(-k)\, \hat{f}^{\lambda\lambda'}(-k) \end{aligned} \tag{3.59}$$

where $\hat{f}^{\mu\nu}_{\lambda\lambda'}$ will play the analogon of the polarization "tensor" $\varepsilon^{\mu\nu}_{\pm}$ in the massless case. Then it follows that

$$\hat{f}^{\lambda\lambda'}(-k) = \hat{f}^{\lambda\lambda'}_{\mu\nu}(-k)\, \hat{f}^{\mu\nu}(-k) \tag{3.60}$$

and the operator becomes

$$\Psi = \sqrt{2\pi}\int \frac{d^3k}{\sqrt{2E_k}} \left[\hat{f}_{\mu\nu}(-k)\, \hat{f}^{\mu\nu}_{\lambda\lambda'}(-k)\, a_{\lambda\lambda'}(\vec{k}) + \eta^{\lambda\lambda}\eta^{\lambda'\lambda'}\, \hat{f}_{\mu\nu}(k)\, \hat{f}^{\mu\nu}_{\lambda\lambda'}(k)\, a^{\dagger}_{\lambda\lambda'}(\vec{k})\right] \tag{3.61}$$

Substituting the Fourier transform back, we find for the field operator $h^{\mu\nu}(x)$

$$h^{\mu\nu}(x) = (2\pi)^{-\frac{3}{2}}\int \frac{d^3k}{\sqrt{2E_k}} \left[\hat{f}^{\mu\nu}_{\lambda\lambda'}(-k)\, a_{\lambda\lambda'}(\vec{k})\, e^{-ikx} + \eta^{\lambda\lambda}\eta^{\lambda'\lambda'}\, \hat{f}^{\mu\nu}_{\lambda\lambda'}(k)\, a^{\dagger}_{\lambda\lambda'}(\vec{k})\, e^{ikx}\right] \tag{3.62}$$

This momentum space representation is consistent with the commutation relation 3.52. In addition to 3.62, we need a representation of $h^{\mu\nu}(x)$ which is manifest in the polarization indices $\lambda\lambda'$. To do this, we define instead of 3.59

$$\begin{aligned} a'_{\mu\nu}(\vec{k}) &:= \hat{f}_{\mu\nu\,\lambda\lambda'}(-k)\, a_{\lambda\lambda'}(\vec{k}) \\ \hat{f}^{\lambda\lambda'}(-k) &:= \hat{f}^{\lambda\lambda'}_{\mu\nu}(-k)\, \hat{f}^{\mu\nu}(-k) \end{aligned} \tag{3.63}$$

and find for the field operator

$$h_{\lambda\lambda'}(x) = (2\pi)^{-\frac{3}{2}}\int \frac{d^3k}{\sqrt{2E_k}} \left[\hat{f}_{\mu\nu\,\lambda\lambda'}(-k)\, a_{\mu\nu}(\vec{k})\, e^{-ikx} + \eta^{\mu\mu}\eta^{\nu\nu}\, \hat{f}_{\mu\nu\,\lambda\lambda'}(k)\, a^{\dagger}_{\mu\nu}(\vec{k})\, e^{ikx}\right] \tag{3.64}$$

where again 3.52 is satisfied. For the sake of completeness, the representation of the vector ghost fields are

$$\begin{aligned} u^{\mu}(x) &= (2\pi)^{-\frac{3}{2}}\int \frac{d^3k}{\sqrt{2E_k}} \Big(c_2^{\mu}(\vec{k})\, e^{-ikx} - \eta^{\mu\mu} c_1^{\mu\dagger}(\vec{k})\, e^{ikx}\Big) \\ \tilde{u}^{\mu}(x) &= (2\pi)^{-\frac{3}{2}}\int \frac{d^3k}{\sqrt{2E_k}} \Big(- c_1^{\mu}(\vec{k})\, e^{-ikx} - \eta^{\mu\mu} c_2^{\mu\dagger}(\vec{k})\, e^{ikx}\Big) \\ v^{\mu}(x) &= (2\pi)^{-\frac{3}{2}}\int \frac{d^3k}{2\sqrt{E_k}} \Big(b^{\mu}(\vec{k})\, e^{-ikx} - \eta^{\mu\mu} b^{\mu\dagger}(\vec{k})\, e^{ikx}\Big) \end{aligned} \tag{3.65}$$

where the creation and annihilation operators satisfy the following commutation relations

$$\begin{aligned}\left\{c_j^{\mu}(\vec{k}),\, c_l^{\nu\dagger}(\vec{k'})\right\} &= \delta_{jl}\,\delta^{\mu}_{\nu}\,\delta^3(\vec{k}-\vec{k'})\\ \left[b^{\mu}(\vec{k}),\, b^{\nu\dagger}(\vec{k'})\right] &= \delta^{\mu}_{\nu}\,\delta^3(\vec{k}-\vec{k'})\end{aligned}\tag{3.66}$$

The gauge variation for the fundamental fields in the massive case take, according to 3.50 and 3.51, the form

$$\begin{aligned}d_Q\, h^{\mu\nu} &= -\frac{i}{2}\Big(u^{\mu,\nu}+u^{\nu,\mu}-\eta^{\mu\nu}u^{\sigma}_{,\sigma}\Big)\\ d_Q\, h &= iu^{\sigma}_{,\sigma}\\ d_Q\,\tilde{u}^{\mu} &= i\left(h^{\mu\nu}_{,\nu}-m_0\,v^{\mu}\right)\\ d_Q\, u^{\mu} &= 0\\ d_Q\, v^{\mu} &= -\frac{im_0}{2}\,u^{\mu}\end{aligned}\tag{3.67}$$

where again $h=\eta_{\mu\nu}h^{\mu\nu}$.

3.2.2 The physical degrees of freedom of the free massive graviton

The physical degrees of freedom of the free, massive graviton are again derived either in momentum-space with the definition of the physical Hilbert space by the kernel of $\{Q,Q^{\dagger}\}$ or in x-space, where the definition of the physical Hilbert space by $\mathcal{H}_{\rm phys}=Ker(Q)/Ran(Q)$ is used.

p-space derivation

We know from the previous section that the fields in the massive case can be represented in momentum space in the same way as in the massless case. The only difference lies in the fact that they satisfy the massive Klein-Gordon equation instead of the massless one and the fields are quantized with the massive Pauli-Jordan distribution. Since we want to compare the result of the following derivation with the massless case of section 3.1.2, we will use the decomposition of $h^{\mu\nu}(x)$ according to 3.55 as $h^{\mu\nu}(x)=H^{\mu\nu}(x)+\frac{1}{4}\eta^{\mu\nu}h(x)$ treating the traceless and the trace part of $h^{\mu\nu}(x)$ separately. One crucial advantage of the derivation in p-space is the clear control over the expressions proportional to the graviton mass m_0, since up to now, massive gravity as the spin-2 quantum gauge theory arise smoothly out of the massless one or vice versa.

We start with the definition of the physical Hilbert space by $\mathcal{F}_{\rm phys}=Ker\{Q,\,Q^{\dagger}\}$ and calculate the commutator $\{Q,\,Q^{\dagger}\}$ in momentum space. According to 3.50, the massive gauge charge is given by

$$Q=\int\limits_{x^0=t} d^3x\left[h^{\mu\nu}_{,\nu}(x)+m_0\,v^{\mu}(x)\right]\overleftrightarrow{\partial_0^x}\,u_{\mu}(x)$$

When substituting the representation of the fields according to 3.65, 3.12 and 3.14 into the definition of Q, we must be careful in writing the Lorentz indices of k^{μ} correctly.

These are true covariant or contravariant indices with respect to the Minkowski metric $\eta^{\mu\nu}$ while the indices of the creation and annihilation operators are not. We will write the latter always upstairs in what follows. Then, the gauge charge Q can be written in momentum space as follows

$$Q \;=\; \int d^3k \Big(A^{\alpha\,\dagger}(\vec{k})\, c_2^{\gamma}(\vec{k}) - B^{\alpha}(\vec{k})\, c_1^{\gamma\,\dagger}(\vec{k}) \Big) \eta_{\gamma\alpha} \tag{3.68}$$

where

$$\begin{aligned} A^{\alpha} &= \eta^{\alpha\alpha} A^{\alpha\beta} k^{\beta} - \frac{1}{4}\, k^{\alpha}\, d - i m_1\, \eta^{\alpha\alpha} b^{\alpha} \\ B^{\alpha} &= \Big(A^{\alpha\beta} k_{\beta} + \frac{1}{4}\, k^{\alpha}\, d + i m_1\, b^{\alpha} \Big) \eta^{\alpha\alpha} \end{aligned} \tag{3.69}$$

The adjoint is given by

$$Q^{\dagger} \;=\; \int d^3k \Big(c_2^{\delta\,\dagger}(\vec{k})\, A^{\beta}(\vec{k}) - c_1^{\delta}(\vec{k})\, B^{\beta\,\dagger}(\vec{k}) \Big) \eta_{\delta\beta} \tag{3.70}$$

where

$$m_1 = \frac{m_0}{\sqrt{2}} \tag{3.71}$$

With the help of

$$\big\{ ab, c \big\} \;=\; a\big\{ b, c \big\} - \big[a, c \big] b$$

and

$$\big\{ b_1^{\dagger} f_1, f_2^{\dagger} b_2 \big\} \;=\; b_1^{\dagger} b_2 \big\{ f_1, f_2^{\dagger} \big\} + f_2^{\dagger} f_1 \big[b_2, b_1^{\dagger} \big]$$

where b, f is a bose- fermi-operator, respectively, the anticommutator $\{Q,\ Q^{\dagger}\}$ becomes

$$\begin{aligned} \Big\{ Q, Q^{\dagger} \Big\} \;=\; \int d^3k\, d^3k' \Big\{ & A^{\alpha\,\dagger}(\vec{k}) A^{\beta}(\vec{k}') \Big\{ c_2^{\gamma}(\vec{k}), c_2^{\delta\,\dagger}(\vec{k}') \Big\} \\ & + c_2^{\delta\,\dagger}(\vec{k}') c_2^{\gamma}(\vec{k}) \Big[A^{\beta}(\vec{k}'), A^{\alpha\,\dagger}(\vec{k}) \Big] + B^{\beta\,\dagger}(\vec{k}') B^{\alpha}(\vec{k}) \Big\{ c_1^{\delta}(\vec{k}'), c_1^{\gamma\,\dagger}(\vec{k}) \Big\} \\ & + c_1^{\gamma\,\dagger}(\vec{k}) c_1^{\delta}(\vec{k}') \Big[B^{\alpha}(\vec{k}), B^{\beta\,\dagger}(\vec{k}') \Big] \Big\} \eta_{\gamma\alpha} \eta_{\delta\beta} \end{aligned} \tag{3.72}$$

Evaluating the anticommutation relations for the fermionic vector ghost creation and annihilation operators and splitting off the relevant graviton sector, we obtain

$$\Big\{ Q, Q^{\dagger} \Big\} \Big|_{\text{graviton}} \;=\; \int d^3k \sum_{\alpha=0}^{3} \Big(A^{\alpha\,\dagger} A^{\alpha} + B^{\alpha\,\dagger} B^{\alpha} \Big) \tag{3.73}$$

In order to perform the sum in 3.73, we are forced to divide the expression for A^α and B^α, subject to 3.69, into timelike-(00) and spacelike-(ij) indices. They are

$$
\begin{aligned}
A^0 &= k_0 \Big(A^{00} - A^0_{||} - \frac{1}{4}\, d - \frac{im_1}{k_0}\, b^0 \Big) \\
A^j &= k_0 \Big(- A^{j0} + A^j_{||} - \frac{k^j}{4k_0}\, d + \frac{im_1}{k_0}\, b^j \Big) \\
B^0 &= k_0 \Big(A^{00} + A^0_{||} + \frac{1}{4}\, d + \frac{im_1}{k_0}\, b^0 \Big) \\
B^j &= k_0 \Big(- A^{j0} - A^j_{||} - \frac{k^j}{4k_0}\, d - \frac{im_1}{k_0}\, b^j \Big)
\end{aligned}
\tag{3.74}
$$

where the abbreviation for the longitudinal modes $A^\mu_{||} = \frac{k_i}{k_0} A^{\mu i}$ is used. As a side remark, note, as we started with an expression which is smooth in the limit of zero graviton mass m_0, we may realize in the following expressions the same smooth limit. Since the unphysical states obviously depend on k, let us choose, in analogy to the massless case, a system in which the propagation direction is along the 3-axis i.e. $k_\mu = (\omega, 0, 0, k_3)$. Then, substituting 3.74 back into 3.73, we find

$$
\begin{aligned}
\sum_{\alpha=0}^{3} \Big(A^{\alpha\,\dagger} A^\alpha + B^{\alpha\,\dagger} B^\alpha \Big) &= 2\omega^2 \Big\{ A^{00\,\dagger} A^{00} + \frac{k_3^2}{\omega^2}\, A^{03\,\dagger} A^{03} \\
&\quad + A^{01\,\dagger} A^{01} + A^{02\,\dagger} A^{02} + A^{03\,\dagger} A^{03} \\
&\quad + \frac{k_3^2}{\omega^2} \Big[A^{13\,\dagger} A^{13} + A^{23\,\dagger} A^{23} + A^{33\,\dagger} A^{33} \Big] \\
&\quad + \frac{1}{16} (1 + \frac{k_3^2}{\omega^2})\, d^\dagger d + \frac{im_1}{4\omega}\, d^\dagger b^0 - \frac{im_1}{4\omega}\, b^{0\,\dagger} d \\
&\quad + \frac{im_1 k_3}{\omega^2}\, A^{03\,\dagger} b^0 - \frac{im_1 k_3}{\omega^2}\, b^{0\,\dagger} A^{03} \\
&\quad + \frac{im_1 k_3}{\omega^2} \Big[A^{13\,\dagger} b^1 + A^{23\,\dagger} b^2 + A^{33\,\dagger} b^3 \\
&\quad - b^{1\,\dagger} A^{13} - b^{2\,\dagger} A^{23} - b^{3\,\dagger} A^{33} \Big] \\
&\quad + \frac{m_1^2}{\omega^2} \Big[b^{0\,\dagger} b^0 + \sum_{j=1}^{3} b^{j\,\dagger} b^j \Big] \Big\}
\end{aligned}
\tag{3.75}
$$

The particle number operators $A^{01\;\dagger} A^{01}$ and $A^{02\;\dagger} A^{02}$ are diagonal with eigenvalues $+1$, representing two unphysical states $A^{01\;\dagger}\Omega$ and $A^{02\;\dagger}\Omega$. Already at this stage of the derivation, we may recognize that the particle number operator $A^{12\;\dagger} A^{12} \in Ker\{Q,\, Q^\dagger\}$ is physical since its eigenvalue vanishes. In 3.75, there are 14 free parameters that will be divided into physical and unphysical ones. Since we used the creation and annihilation operators for the traceless tensor field $H^{\mu\nu}(x)$, we have to transform, according to 3.17, the diagonal parts of $A^{\mu\nu}$ to $\tilde{A}^{\mu\nu}$, since only the $\tilde{A}^{\mu\nu}$ operators create the diagonal Fock space [8]. Then, since 3.75 is a quadratic form in the particle number operators, we can write it in matrix form as

$$\sum_{\alpha=0}^{3}\left(A^{\alpha\,\dagger}A^{\alpha}+B^{\alpha\,\dagger}B^{\alpha}\right) \;=\; \vec{A}^{\dagger}\,X\,\vec{A} \tag{3.76}$$

with

$$\vec{A}^{\dagger} \;=\; \left(\tilde{A}_{11}^{\dagger},\tilde{A}_{22}^{\dagger},\tilde{A}_{33}^{\dagger},\sqrt{2}A_{01}^{\dagger},\sqrt{2}A_{02}^{\dagger},\sqrt{2}A_{03}^{\dagger},\sqrt{2}A_{13}^{\dagger},\sqrt{2}A_{23}^{\dagger},1/2d^{\dagger},b_0^{\dagger},b_1^{\dagger},b_2^{\dagger},b_3^{\dagger}\right) \tag{3.77}$$

referred to standard basis $e_i = (0,\ldots,\underbrace{1}_{\text{i'th slot}},\ldots,0)$. According to 3.75, X is a 13x13 hermitian matrix of the form

$$X \;=\; \left(\begin{array}{ccccccc}
z & z & m_1^2 & 0 & 0 & 0 & 0\\
z & z & m_1^2 & 0 & 0 & 0 & 0\\
m_1^2 & m_1^2 & z & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \omega^2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \omega^2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 2z & 0\\
0 & 0 & 0 & 0 & 0 & 0 & k_3^2\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -\sqrt{2}im_1k_3 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}im_1k_3\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
-im_1k_3 & -im_1k_3 & im_1k_3 & 0 & 0 & 0 & 0
\end{array}\right.$$

$$\left.\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & im_1k_3\\
0 & 0 & 0 & 0 & 0 & im_1k_3\\
0 & 0 & 0 & 0 & 0 & -im_1k_3\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \sqrt{2}im_1k_3 & 0 & 0 & 0\\
0 & 0 & 0 & \sqrt{2}im_1k_3 & 0 & 0\\
k_3^2 & 0 & 0 & 0 & \sqrt{2}im_1k_3 & 0\\
0 & z & im_1\omega & 0 & 0 & 0\\
0 & -im_1\omega & 2m_1^2 & 0 & 0 & 0\\
0 & 0 & 0 & 2m_1^2 & 0 & 0\\
-\sqrt{2}im_1k_3 & 0 & 0 & 0 & 2m_1^2 & 0\\
0 & 0 & 0 & 0 & 0 & 2m_1^2
\end{array}\right) \tag{3.78}$$

where

$$z = \frac{1}{2}(\omega^2 + k_3^2) \tag{3.79}$$

Since the different creation and annihilation operators in 3.75 mix among each other, we have to diagonalize the hermitian, quadratic form $\vec{A}^{\dagger}X\vec{A}$ in order to identify the pure particle number operators with vanishing eigenvalues. This means, that we are looking for a transformation of the standard basis into the eigenbasis in which the matrix X is diagonal i.e. $C^{-1}XC = D = diag(\lambda_1, \ldots, \lambda_{13})$ where λ_j are the eigenvalues of X and the quadratic form becomes $\vec{d}^\dagger D\vec{d}$. One finds the transformation properties

$$\begin{aligned} \vec{A} &= C\,\vec{d} \\ \vec{A}^\dagger &= \left(C^{-1\dagger}\,\vec{d}\right)^\dagger \\ \vec{d} &= C^{-1}\,\vec{A} \end{aligned} \tag{3.80}$$

whereas the eigenvalues take the form

$$\begin{aligned} \lambda_1 &= \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0 \\ \lambda_6 &= \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{11} = \omega^2 = 2m_1^2 + k_3^2 \\ \lambda_{12} &= \lambda_{13} = 3m_1^2 + 2k_3^2 \end{aligned} \tag{3.81}$$

One recognizes that there are five eigenvalues equals to zero, plus the one from the particle number operator $A_{12}^\dagger A_{12} \in Ker\{Q, Q^\dagger\}$ in 3.75 summing up to six, which means that there are six particle number operators $d_a^\dagger d_a \in Ker\{Q, Q^\dagger\}$, $a = 1, \ldots, 6$ inducing the six physical states. Since the quadratic form $\vec{A}^\dagger X \vec{A}$ is hermitian, the basis transformation from the standard basis into the eigenbasis is unitary assuring that the creation operators in the eigenbasis $d_a^\dagger$ construct the Fock space in the usual way.

As a side remark, the features of the anticommutator $\{Q, Q^\dagger\}$ allow for the identification as a contracting homotopy[13] Λ with respect to Q, since the nilpotent operator Q acts as a odd derivation i.e. $d_Q\ .\ = \{Q,\ .\ \}$.

physical states

By diagonalizing the matrix X in the frame $k_\mu = (\omega, 0, 0, k_3)$, we may identify out of the six particle number operators $d_a^\dagger d_a \in Ker\{Q, Q^\dagger\}$ the six massive physical states. They are outlined as functions of (m_1, k_3) whereas the limit of zero graviton mass and zero three-momentum is indicated. The hat on the subscribt indicates that the states are normalized to one.

$\psi_{\hat{1}}$: The first physical state $\psi_{\hat{1}}$ originates out of the quadratic form 3.75 where the particle number operator $A_{12}^\dagger A_{12} \in Ker\{Q, Q^\dagger\}$ carries the eigenvalue zero, thus

$$\begin{aligned} \Rightarrow\ \psi_{\hat{1}} &= c_1\, A_{12}^\dagger(\vec{k})\,\Omega \\ c_1 &= \sqrt{2} \end{aligned} \tag{3.82}$$

$$\begin{aligned} \lim_{m_1 \searrow 0} \psi_{\hat{1}} &= \sqrt{2}\, A_{12}^\dagger(\vec{k})\,\Omega \\ \lim_{k_3 \searrow 0} \psi_{\hat{1}} &= \sqrt{2}\, A_{12}^\dagger(\vec{k})\,\Omega \end{aligned}$$

[13] M. Henneaux, C. Teitelboim: *Quantization of gauge systems.* Princeton University Press, Princeton 1992. I thank G. Scharf for drawing my attention to this point.

$\psi_{\hat{2}}$: The second physical state $\psi_{\hat{2}}$ originates from the particle number operator $\lambda_1 d_1^\dagger d_1 \in Ker\{Q, Q^\dagger\}$ with eigenvalue $\lambda_1 = 0$ and takes the form

$$\begin{aligned} \Rightarrow\ \psi_{\hat{2}}(m_1, k_3) &= c_2 \left(\frac{im_1}{k_3} (\tilde{A}_{11} - \tilde{A}_{33}) + b_3 \right)^\dagger (\vec{k})\, \Omega \\ c_2 &= \left(1 + \frac{2m_1^2}{k_3^2} \right)^{-\frac{1}{2}} \end{aligned} \tag{3.83}$$

$$\begin{aligned} \lim_{m_1 \searrow 0} \psi_{\hat{2}}(m_1, k_3) &= b_3^\dagger(\vec{k})\, \Omega \\ \lim_{k_3 \searrow 0} \psi_{\hat{2}}(m_1, k_3) &= \frac{1}{\sqrt{2}} \left(\tilde{A}_{11} - \tilde{A}_{33} \right)^\dagger (\vec{k})\, \Omega \end{aligned}$$

$\psi_{\hat{3}}$: The third physical state $\psi_{\hat{3}}$ originates from the particle number operator $\lambda_2 d_2^\dagger d_2 \in Ker\{Q, Q^\dagger\}$ with eigenvalue $\lambda_2 = 0$ and takes the form

$$\begin{aligned} \Rightarrow\ \psi_{\hat{3}}(m_1, k_3) &= c_3 \left(\frac{2im_1}{k_3} A_{23} + b_2 \right)^\dagger (\vec{k})\, \Omega \\ c_3 &= \left(1 + \frac{2m_1^2}{k_3^2} \right)^{-\frac{1}{2}} \end{aligned} \tag{3.84}$$

$$\begin{aligned} \lim_{m_1 \searrow 0} \psi_{\hat{3}}(m_1, k_3) &= b_2^\dagger(\vec{k})\, \Omega \\ \lim_{k_3 \searrow 0} \psi_{\hat{3}}(m_1, k_3) &= \sqrt{2}\, A_{23}^\dagger(\vec{k})\, \Omega \end{aligned}$$

$\psi_{\hat{4}}$: The fourth physical state $\psi_{\hat{4}}$ originates from the particle number operator $\lambda_3 d_3^\dagger d_3 \in Ker\{Q, Q^\dagger\}$ with eigenvalue $\lambda_3 = 0$ and takes the form

$$\begin{aligned} \Rightarrow\ \psi_{\hat{4}}(m_1, k_3) &= c_4 \left(\frac{2im_1}{k_3} A_{13} + b_1 \right)^\dagger (\vec{k})\, \Omega \\ c_4 &= \left(1 + \frac{2m_1^2}{k_3^2} \right)^{-\frac{1}{2}} \end{aligned} \tag{3.85}$$

$$\begin{aligned} \lim_{m_1 \searrow 0} \psi_{\hat{4}}(m_1, k_3) &= b_1^\dagger(\vec{k})\, \Omega \\ \lim_{k_3 \searrow 0} \psi_{\hat{4}}(m_1, k_3) &= \sqrt{2}\, A_{13}^\dagger(\vec{k})\, \Omega \end{aligned}$$

$\psi_{\hat{5}}$: The fifth physical state $\psi_{\hat{5}}$ originates from the particle number operator $\lambda_4 d_4^\dagger d_4 \in Ker\{Q, Q^\dagger\}$ with eigenvalue $\lambda_4 = 0$ and takes the form

$$
\begin{aligned}
\Rightarrow\ \psi_{\hat{5}}(m_1,k_3) &= c_5\left(\frac{im_1}{2(m_1^2+k_3^2)}\left(2k_3\,A_{03}+\sqrt{2m_1^2+k_3^2}\,d\right)+b_0\right)^{\dagger}(\vec{k})\,\Omega \\
c_5 &= \left(1+m_1^2\,\frac{3k_3^2+4m_1^2}{2(m_1^2+k_3^2)^2}\right)^{-\frac{1}{2}}
\end{aligned}
\tag{3.86}
$$

$$
\begin{aligned}
\lim_{m_1\searrow 0}\psi_{\hat{5}}(m_1,k_3) &= b_0^{\dagger}(\vec{k})\,\Omega \\
\lim_{k_3\searrow 0}\psi_{\hat{5}}(m_1,k_3) &= \frac{1}{\sqrt{3}}\left(\frac{i}{\sqrt{2}}\,d+b_0\right)^{\dagger}(\vec{k})\,\Omega
\end{aligned}
$$

$\psi_{\hat{6}}$: The sixth physical state $\psi_{\hat{6}}$ originates from the particle number operator $\lambda_5 d_5^{\dagger} d_5 \in Ker\{Q, Q^{\dagger}\}$ with eigenvalue $\lambda_5 = 0$ and takes the form

$$
\begin{aligned}
\Rightarrow\ \psi_{\hat{6}}(m_1,k_3) &= c_6\left(\tilde{A}_{11}-\tilde{A}_{22}\right)^{\dagger}(\vec{k})\,\Omega \\
c_6 &= \frac{1}{\sqrt{2}}
\end{aligned}
\tag{3.87}
$$

$$
\begin{aligned}
\lim_{m_1\searrow 0}\psi_{\hat{6}}(m_1,k_3) &= \frac{1}{\sqrt{2}}\left(\tilde{A}_{11}-\tilde{A}_{22}\right)^{\dagger}(\vec{k})\,\Omega \\
\lim_{k_3\searrow 0}\psi_{\hat{6}}(m_1,k_3) &= \frac{1}{\sqrt{2}}\left(\tilde{A}_{11}-\tilde{A}_{22}\right)^{\dagger}(\vec{k})\,\Omega
\end{aligned}
$$

In some sense, the limit k_3 to zero can be compared with the x-space derivation where we will explicitly construct the basis polarization tensor in the rest frame $k_\mu = (m_0, 0, 0, 0)$. However one must be aware of the difference in the definition of the physical subspace. It is worth mentioning one more important point. In the physical states ψ_i derived above, there is always an admixture of ghost creation operators $b_\delta^{\dagger}$. It has to be shown in the following sections that either they fail to couple to the fields involved in the Bremsstrahl process or the contributions of diagrams where the bosonic vector ghost $v^\mu(x)$ couples to the massive spin-2 field $h^{\mu\nu}(x)$ can be neglected.

In analogy to the massless case, we may write the six massive physical states as a six-tuple, representing the six massive degrees of freedom where the diagonal annihilation operators $\tilde{A}_{\mu\nu}$ are already transformed back to the regular ones $A_{\mu\nu}$ i.e.

$$\vec{\psi}\,(m_1,k_3) \;=\; \begin{pmatrix} \psi_{\hat{1}} \\ \psi_{\hat{2}} \\ \psi_{\hat{3}} \\ \psi_{\hat{4}} \\ \psi_{\hat{5}} \\ \psi_{\hat{6}} \end{pmatrix} (m_1,k_3) \;=\; \begin{pmatrix} c_1\, A_{12}^\dagger \\ c_2 \left(\frac{im_1}{k_3}\,(A_{33}-A_{11}) + b_3\right)^\dagger \\ c_3 \left(\frac{2im_1}{k_3}\,A_{23} + b_2\right)^\dagger \\ c_4 \left(\frac{2im_1}{k_3}\,A_{13} + b_1\right)^\dagger \\ c_5 \left(\frac{im_1}{2(m_1^2+k_3^2)}\,\left(2k_3\,A_{03} + \sqrt{2m_1^2+k_3^2}\; d\right) + b_0\right)^\dagger \\ c_6 \left(A_{22}-A_{11}\right)^\dagger \end{pmatrix} (\vec{k})\,\Omega \tag{3.88}$$

It is interesting to outline the zero mass limit of 3.88. It takes the form

$$\lim_{m_1 \searrow 0}\; \vec{\psi}\,(m_1,k_3) \;=\; \lim_{m_1 \searrow 0} \begin{pmatrix} \psi_{\hat{1}} \\ \psi_{\hat{2}} \\ \psi_{\hat{3}} \\ \psi_{\hat{4}} \\ \psi_{\hat{5}} \\ \psi_{\hat{6}} \end{pmatrix} (m_1,k_3) \;=\; \begin{pmatrix} \sqrt{2}\, A_{12}^\dagger \\ b_3^\dagger \\ b_2^\dagger \\ b_1^\dagger \\ b_0^\dagger \\ \frac{1}{\sqrt{2}} \left(A_{22}-A_{11}\right)^\dagger \end{pmatrix} (\vec{k})\,\Omega \tag{3.89}$$

The crucial point in this derivation is that in the limit of zero graviton mass, the massive physical degrees of freedom go over smoothly into the two massless ones and four completely decoupled vector ghost degrees of freedom. We may recognize already at this stage, that, if the bosonic vector ghost does not couple to matter, there will be a smooth limit between the massive and the massless Bremsstrahl matrix element.

x-space derivation

The aim of the following discussion is to prepare the formalism of massive gravity as the spin-2 quantum gauge theory in order to find the massive polarization sum analogous to 3.47 in the massless case.

Consider a physical state Φ_{phys}. According to the discussion in section 3.2.1, one may ask where to put the information about the physical degrees of freedom of the massive graviton. It could be cast into the test function $f_0^{\mu\nu}$ or into the field operator $h_{\mu\nu}^{\text{phys}}$ i.e.

$$\Phi_{\text{phys}} = \int d^4x\, f_0^{\mu\nu}(x)\, h_{\mu\nu}(x)\, \Omega \tag{3.90}$$

$$= \int d^4x\, f^{\mu\nu}(x)\, h_{\mu\nu}^{\text{phys}}(x)\, \Omega \tag{3.91}$$

It is not hard to show that 3.91 can only be derived from 3.90 but the derivation itself is rather complicated. However, $h_{\mu\nu}^{\text{phys}}(x)$ is a rather academic object and it is useful only when discussing the structure of the theory, and it is not too useful for the derivation of the massive polarization sum. Indeed, the commutator $\left[h_{\mu\nu}^{\text{phys}}(x), h_{\alpha\beta}^{\text{phys}}(y)\right]$ is an interesting object. But it is fare more adequate to follow the path to absorb the physical degrees of freedom in the test function $f_0^{\mu\nu}$ and to treat the field operator $h^{\mu\nu}(x)$ in the regular way as it was done in the massless case, too.

Again, the one-particle Hilbert space corresponding to $Ker^{(1)}(Q)/Ran^{(1)}(Q)$ is $H^{[m_0,2]} \oplus H^{[m_0,0]}$ i.e. describing two particles of mass m_0 and spin 2 and one of spin 0, respectively. A general one-particle state is [30]

$$\Phi = \int d^4x\ \left[f^{\mu\nu} h_{\mu\nu} + f_1^\mu u_\mu + f_2^\mu \tilde{u}_\mu + f_3^\mu v_\mu\right] \Omega \tag{3.92}$$

where the test function $f^{\mu\nu}$ is symmetric in the indices μ, ν and not necessarily traceless anymore. As mentioned above, we will derive in the following the physical content of the massive graviton absorbed into the test function $f_0^{\mu\nu}$.

$Ker(Q)$:

According to [30] and using $Q\ \Omega = 0$ we have

$$\begin{aligned} Q\,\Phi &= \int d^4x \Big[f^{\mu\nu}\left[Q, h_{\mu\nu}\right] + f_1^\mu \{Q, u_\mu\} + f_2^\mu \{Q, \tilde{u}_\mu\} + f_3^\mu \left[Q, v_\mu\right]\Big]\,\Omega \\ &= \int d^4x \Big[-\frac{i}{2} f^{\mu\nu}\left(\partial_\mu u_\nu + \partial_\nu u_\mu - \eta_{\mu\nu}\partial^\gamma u_\gamma\right) \\ &\quad + i f_2^\mu \left(\partial^\nu h_{\mu\nu} + m_0\, v_\mu\right) - \frac{i m_0}{2} f_3^\mu\, u_\mu\Big]\,\Omega \end{aligned} \tag{3.93}$$

In order to have $\Phi \in Ker(Q)$, the test function of the fermionic vector ghost has to vanish i.e. $f_2^\mu = 0$, and

$$\begin{aligned} 0 &\stackrel{!}{=} \int d^4x \Big[\frac{i}{2}\left(\partial_\mu f^{\mu\nu} u_\nu + \partial_\nu f^{\mu\nu} u_\mu - \partial^\gamma f\, u_\gamma\right) - \frac{i m_0}{2} f_3^\mu u_\mu\Big]\,\Omega \\ &= \frac{i}{2}\int d^4x \Big[2\partial_\nu f^{\mu\nu} u_\mu - \partial^\mu f\, u_\mu - m_0\, f_3^\mu u_\mu\Big]\,\Omega \\ &= \frac{i}{2}\int d^4x \left(2\partial_\nu f^{\mu\nu} - \partial^\mu f - m_0\, f_3^\mu\right) u_\mu\, \Omega \end{aligned} \tag{3.94}$$

where $f = \eta_{\mu\nu} f^{\mu\nu}$ is the trace of the test function. The condition 3.94 requires that

$$f_3^\mu = \frac{1}{m_0}\left(2\partial_\nu f^{\mu\nu} - \partial^\mu f\right)$$

Then it follows that $\Phi \in Ker(Q)$ takes the form

$$\Phi = \int d^4x \left[f^{\mu\nu} h_{\mu\nu} + f_1^\mu u_\mu + \frac{2}{m_0} (\partial_\nu f^{\mu\nu} - \frac{1}{2} \partial^\mu f) \, v_\mu \right] \Omega \tag{3.95}$$

One may recognize that the fraction with the graviton mass m_0 appearing in the denominator will lead to difficulties subject to the presumed smallness of m_0, unless this kind of fraction disappears in the polarization sum.

Ran(Q):

Not all this states $\Phi \in Ker^{(1)}(Q)$ describe distinct physical situations. We must again factor out the states $\Phi' \in Ran^{(1)}(Q)$ i.e. states from $Ker^{(1)}(Q)$ verifying

$$\Phi' = Q\Phi_0$$

for some arbitrary Φ_0. According to 3.92, we have

$$\Phi_0 = \int d^4x \; [g^{\mu\nu} h_{\mu\nu} + g_1^\mu u_\mu + g_2^\mu \tilde{u}_\mu + g_3^\mu v_\mu] \, \Omega$$

and

$$\begin{aligned} Q\,\Phi_0 &= \int d^4x \left[g^{\mu\nu} \left[Q, h_{\mu\nu} \right] + g_1^\mu \{ Q, u_\mu \} + g_2^\mu \{ Q, \tilde{u}_\mu \} + g_3^\mu \left[Q, v_\mu \right] \right] \Omega \\ &= \int d^4x \left[\frac{i}{2} \left(\partial_\mu g^{\mu\nu} u_\nu + \partial_\nu g^{\mu\nu} u_\mu - \partial^\gamma g \, u_\gamma \right) - \frac{i}{2} \left(\partial^\mu g_2^\nu + \partial^\nu g_2^\mu \right) h_{\mu\nu} \right. \\ &\quad \left. + \, i m_0 \, g_2^\mu v_\mu - \frac{i m_0}{2} \, g_3^\mu u_\mu \right] \Omega \end{aligned} \tag{3.96}$$

with $g = \eta_{\mu\nu} g^{\mu\nu}$. Then $\Phi' \in Ran^{(1)}(Q)$ and

$$\Phi' = \int d^4x \left[-\frac{i}{2} (\partial^\mu g_2^\nu + \partial^\nu g_2^\mu) \, h_{\mu\nu} + i \big(\partial_\nu g^{\mu\nu} - \frac{1}{2} \, \partial^\mu g - \frac{1}{2} \, m_0 \, g_3^\mu \big) \, u_\mu + i m_0 \, g_2^\mu v_\mu \right] \Omega \tag{3.97}$$

A representative in $Ker^{(1)}(Q)/Ran^{(1)}(Q)$ is of the form $\Phi + \Phi'$ i.e.

$$\begin{aligned} \Phi + \Phi' &= \int d^4x \left[\Big(f^{\mu\nu} - \frac{i}{2} (\partial^\mu g_2^\nu + \partial^\nu g_2^\mu) \Big) \, h_{\mu\nu} \right. \\ &\quad + \Big(f_1^\mu + i \, (\partial_\nu g^{\mu\nu} - \frac{1}{2} \, \partial^\mu g - \frac{1}{2} \, m_0 \, g_3^\mu) \Big) \, u_\mu \\ &\quad \left. + \Big(\frac{1}{m_0} \, (2 \partial_\nu f^{\mu\nu} - \partial^\mu f) + i m_0 \, g_2^\mu \Big) \, v_\mu \right] \Omega \\ &:= \int d^4x \, f_0^{\mu\nu} h_{\mu\nu} \, \Omega \end{aligned} \tag{3.98}$$

where the definition of $f_0^{\mu\nu}$ allows us to absorb all massive physical degrees of freedom in $f_0^{\mu\nu}$. This is achieved by choosing the coefficient of g_3^μ and f_1^μ conveniently, and making null the coefficient of v^μ by imposing

$$g_2^\mu \stackrel{!}{=} \frac{i}{m_0^2}\left(2\partial_\nu f^{\mu\nu} - \partial^\mu f\right)$$

Then, the physical test function $f_0^{\mu\nu}$ takes the form

$$\begin{aligned} f_0^{\mu\nu} &= f^{\mu\nu} - \frac{i}{2}\left[\partial^\mu\left(\frac{i}{m_0^2}(2\partial_\rho f^{\nu\rho} - \partial^\nu f)\right) + \partial^\nu\left(\frac{i}{m_0^2}(2\partial_\rho f^{\mu\rho} - \partial^\mu f)\right)\right] \\ &= f^{\mu\nu} + \frac{1}{m_0^2}\left(\partial^\mu\partial_\rho f^{\nu\rho} + \partial^\nu\partial_\rho f^{\mu\rho} - \partial^\mu\partial^\nu f\right) \end{aligned} \tag{3.99}$$

In 3.98, we could have also chosen the physical test function to be splitted between the pure graviton part and a bosonic vector ghost part i.e. $\Phi + \Phi' = \int d^4x (f_0^{\mu\nu} h_{\mu\nu} + f_0^\mu v_\mu)\Omega$. But this does not lead to a convincing result.

The gauge condition which in the classical case holds on the classical tensor field $h^{\mu\nu}$ i.e. $h^{\mu\nu}_{,\nu} = 0$, turns out to hold on the physical test function $f_0^{\mu\nu}$, known as the De Donder gauge condition

$$\partial_\nu f_0^{\mu\nu} = \frac{1}{2}\partial^\mu f_0 \tag{3.100}$$

with $f_0 = \eta_{\mu\nu} f_0^{\mu\nu}$. The gauge condition 3.100 holds without prior conditions on the symmetric test function $f^{\mu\nu}$. This shows the important fact, already mentioned in section 2.2, that in quantum field theory, the gauge condition holds on the physical test function and is not realized as an operator equation in Fock space. In order to construct explicit physical polarization tensors that represents the six physical degrees of freedom of the massive graviton, it is useful to consider $f_0^{\mu\nu}$ in momentum space with Fourier transform, according to 3.57, as

$$\hat{f}_0^{\mu\nu}(k) = \hat{f}^{\mu\nu}(k) - \frac{1}{m_0^2}\left(k^\mu k_\rho \hat{f}^{\nu\rho}(k) + k^\nu k_\rho \hat{f}^{\mu\rho}(k) - k^\mu k^\nu \hat{f}(k)\right) \tag{3.101}$$

and to introduce real polarization vectors $\varepsilon_\lambda^\mu(k)$, $\lambda = 0, 1, 2, 3$, forming an orthonormal basis in Minkowski space and satisfying the following conditions

$$\varepsilon_0^\mu(k) = \frac{k^\mu}{m_0} \tag{3.102}$$

$$k_\mu \varepsilon_j^\mu(k) = 0 \tag{3.103}$$

$$\sum_{j=1}^{3} \varepsilon_j^\mu(k)\,\varepsilon_j^\nu(k) = \frac{k^\mu k^\nu}{m_0^2} - \eta^{\mu\nu} \tag{3.104}$$

They are inspired by massive vector particles [8]. In the following, we will use Greek indices for Lorentz objects and Latin indices for space indices. The last condition 3.104 is the completeness relation in Minkowski space for two polarization vectors subject to 3.102 and 3.103. Note that in order to determine the explicit form of $\varepsilon_\lambda^\mu(k)$, there is still one condition to be addressed. As we will see later, a completeness relation for four polarization vectors in Minkowski space would be appropriate. The transformation properties of the polarization vectors are dictated by the little group for massive particles. According to [11, 13], this group is isomorphic to the three dimensional rotation group $SO(3)$. We then define

$$\hat{f}^{\mu\nu}_{\lambda\lambda'} := \frac{1}{2}\left(\varepsilon^{\mu}_{\lambda}\varepsilon^{\nu}_{\lambda'} + \varepsilon^{\nu}_{\lambda}\varepsilon^{\mu}_{\lambda'}\right) \tag{3.105}$$

where the trace with respect to the Minkowski metric $\eta_{\mu\nu}$ is $\eta_{\mu\nu}\hat{f}^{\mu\nu}_{\lambda\lambda'} = \hat{f}_{\lambda\lambda'} = \eta_{\lambda\lambda'}$. In section 3.2.1, we have already defined how $\hat{f}^{\mu\nu}_{\lambda\lambda'}$ behaves by representing the fields in momentum space. In consequence of 3.102 - 3.104, the polarization vectors $\varepsilon^{\mu}_{\lambda}$ may be rendered in such a way that $\hat{f}^{\mu\nu}_{\lambda\lambda'}(-k) = \hat{f}^{\mu\nu}_{\lambda\lambda'}(k)$ holds. If we insert 3.105 into 3.101, one finds the physical polarization function $\hat{f}^{\mu\nu}_{0\lambda\lambda'}$ as

$$\hat{f}^{\mu\nu}_{0\lambda\lambda'} = \hat{f}^{\mu\nu}_{\lambda\lambda'} - \frac{1}{m_0^2}\left(k^{\mu}k_{\rho}\hat{f}^{\nu\rho}_{\lambda\lambda'} + k^{\nu}k_{\rho}\hat{f}^{\mu\rho}_{\lambda\lambda'} - k^{\mu}k^{\nu}\hat{f}_{\lambda\lambda'}\right) \tag{3.106}$$

which is the analogon of the polarization "tensor" $\varepsilon^{\mu\nu}_{\pm}$ in the massless case. It satisfies the de Donder gauge condition

$$k_{\nu}\hat{f}^{\mu\nu}_{0\lambda\lambda'} - \frac{1}{2}k^{\mu}\hat{f}_{0\lambda\lambda'} = 0$$

without prior conditions on $\varepsilon^{\mu}_{\lambda}(k)$ and $\hat{f}^{\mu\nu}_{\lambda\lambda'}(k)$. As we know from the derivation of the massive physical degrees of freedom in momentum space, the polarization function $\hat{f}^{\mu\nu}_{0\lambda\lambda'}$ must parameterize six physical degrees of freedom with the two indices $\lambda\lambda'$. If we analyze with the help of 3.102 - 3.104, 3.105 and 3.106 the time like and spatial components of it, we find

$$\begin{aligned}
\hat{f}^{\mu\nu}_{0\,00} &= 0\\
\hat{f}^{\mu\nu}_{0\,0j} &= 0\\
\hat{f}^{\mu\nu}_{0\,ij} &= \frac{1}{2}\left(\varepsilon^{\mu}_{i}\varepsilon^{\nu}_{j} + \varepsilon^{\nu}_{i}\varepsilon^{\mu}_{j}\right) + \frac{k^{\mu}k^{\nu}}{m_0^2}\eta_{ij}\\
\hat{f}^{\mu\nu}_{0\,ii} &= \varepsilon^{\mu}_{i}\varepsilon^{\nu}_{i} + \frac{k^{\mu}k^{\nu}}{m_0^2}\eta_{ii}\\
\hat{f}^{\mu\nu}_{0\,kl} &= \frac{1}{2}\left(\varepsilon^{\mu}_{k}\varepsilon^{\nu}_{l} + \varepsilon^{\nu}_{k}\varepsilon^{\mu}_{l}\right) \quad k \neq l
\end{aligned} \tag{3.107}$$

It then follows that $\hat{f}^{\mu\nu}_{0\lambda\lambda'}$ parameterizes six independent degrees of freedom, which are orthogonal to each other. This allows us to construct explicit polarization tensors which are for convenience defined in the rest frame, where $k^{\mu} = (m_0,\ 0,\ 0,\ 0)$. There are three orthogonal basis polarization tensors originating out of $\hat{f}^{\mu\nu}_{0ii}$ with trace $\eta_{\mu\nu}\hat{f}^{\mu\nu}_{0ii} = 2\eta_{ii} = -2$ with respect to the Minkowski metric and three orthogonal basis polarization tensors originating out of $\hat{f}^{\mu\nu}_{0kl}$ with $k \neq l$ and with trace $\eta_{\mu\nu}\hat{f}^{\mu\nu}_{0kl} = 2\eta_{kl} = 0$. The physical polarization function $\hat{f}^{\mu\nu}_{0\lambda\lambda'}$, restricted to the rest frame, will be indicated by $\tilde{f}^{\mu\nu}_{0\lambda\lambda'}$. Therefore, we find

$$\tilde{f}_1^{\mu\nu} = \tilde{f}_{0\,11}^{\mu\nu} = \varepsilon_1^\mu \varepsilon_1^\nu + \varepsilon_0^\mu \varepsilon_0^\nu \,\eta_{11} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{f}_2^{\mu\nu} = \tilde{f}_{0\,22}^{\mu\nu} = \varepsilon_2^\mu \varepsilon_2^\nu + \varepsilon_0^\mu \varepsilon_0^\nu \,\eta_{22} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (3.108)$$

$$\tilde{f}_3^{\mu\nu} = \tilde{f}_{0\,33}^{\mu\nu} = \varepsilon_3^\mu \varepsilon_3^\nu + \varepsilon_0^\mu \varepsilon_0^\nu \,\eta_{33} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The basis polarization tensors from the off-diagonal part are

$$\tilde{f}_4^{\mu\nu} = \tilde{f}_{0\,12}^{\mu\nu} = \frac{1}{2}\Big(\varepsilon_1^\mu \varepsilon_2^\nu + \varepsilon_2^\mu \varepsilon_1^\nu\Big) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{f}_5^{\mu\nu} = \tilde{f}_{0\,13}^{\mu\nu} = \frac{1}{2}\Big(\varepsilon_1^\mu \varepsilon_3^\nu + \varepsilon_3^\mu \varepsilon_1^\nu\Big) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad (3.109)$$

$$\tilde{f}_6^{\mu\nu} = \tilde{f}_{0\,23}^{\mu\nu} = \frac{1}{2}\Big(\varepsilon_2^\mu \varepsilon_3^\nu + \varepsilon_3^\mu \varepsilon_2^\nu\Big) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

By normalizing the basis tensors, we find the ortho-normal basis for the six massive physical degrees of freedom as

$$\tilde{f}^{\mu\nu}_{\hat{1}} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad \tilde{f}^{\mu\nu}_{\hat{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{f}^{\mu\nu}_{\hat{3}} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad \tilde{f}^{\mu\nu}_{\hat{4}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.110)$$

$$\tilde{f}^{\mu\nu}_{\hat{5}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} , \quad \tilde{f}^{\mu\nu}_{\hat{6}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

3.2.3 Polarization sum

When considering quantum gravitational Bremsstrahlung with an emitted massive graviton, we have to sum in the final cross section over the six massive degrees of freedom. According to 3.62, a general graviton state, representing a massive physical graviton with fixed polarization $\lambda\lambda'$, is given by

$$h_f = \int d^3k \, \hat{f}_{\mu\nu}(\vec{k}) \, \hat{f}^{\mu\nu}_{\lambda\lambda'}(\vec{k}) \, a^{\dagger}_{\lambda\lambda'}(\vec{k}) \, \Omega \quad (3.111)$$

It was already indicated in the definition of the basis polarization tensors $\tilde{f}^{\mu\nu}_{a}$ that there is a linear dependence between the indices $\lambda\lambda'$, parameterizing the massive physical degrees of freedom and the summation index a. It may be cast into a linear function $a(\lambda\lambda')$ defined as

$$\begin{aligned} a(11) &= 1 \\ a(22) &= 2 \\ a(33) &= 3 \\ a(12) &= 4 \\ a(13) &= 5 \\ a(23) &= 6 \end{aligned} \quad (3.112)$$

As in the massless case, we may extract the polarization function $\hat{f}^{\mu\nu}_{0\lambda\lambda'}$ from the matrix element $\mathcal{M}$ as

$$\mathcal{M}_{a(\lambda\lambda')} = \hat{f}^{\mu\nu}_{0\lambda\lambda'} \mathcal{M}_{\mu\nu} \quad (3.113)$$

where again, $\mathcal{M}_{\mu\nu}$ represents the remaining tensorial part of it. As a side remark, $\mathcal{M}_{\mu\nu}$ is derived by using the representation of the h-field according to 3.53. In calculating the scattering cross section, we have to square 3.113 and sum over the six massive degrees of freedom $a = 1, \dots, 6$. The cross section is proportional to

$$
\begin{aligned}
\sum_{a(\lambda\lambda')=1}^{6} \left|\mathcal{M}_{a(\lambda\lambda')}\right|^2 &= \sum_{a(\lambda\lambda')=1}^{6} \left|\hat{f}^{\mu\nu}_{0\lambda\lambda'}\, \mathcal{M}_{\mu\nu}\right|^2 \\
&= \sum_{a(\lambda\lambda')=1}^{6} \hat{f}^{\mu\nu}_{0\lambda\lambda'}\, \hat{f}^{\mu'\nu'}_{0\lambda\lambda'}\, \mathcal{M}_{\mu\nu}\mathcal{M}^{*}_{\mu'\nu'} \qquad (3.114)\\
&= \sum_{a(ij)=1}^{6} \hat{f}^{\mu\nu}_{0\,ij}\, \hat{f}^{\mu'\nu'}_{0\,ij}\, \mathcal{M}_{\mu\nu}\mathcal{M}^{*}_{\mu'\nu'}
\end{aligned}
$$

where the last line is due to $\hat{f}^{\rho\sigma}_{0\;00} = \hat{f}^{\rho\sigma}_{0\;0j} = 0$. In consequence of 3.107, the explicit sum over the six massive polarizations $a = 1, \dots, 6$ is

$$
\sum_{a(ij)=1}^{6} \hat{f}^{\mu\nu}_{0\,ij}\, \hat{f}^{\mu'\nu'}_{0\,ij} = \frac{1}{2}\Big[\underbrace{\sum_{i,j} \hat{f}^{\mu\nu}_{0\,ij}\, \hat{f}^{\mu'\nu'}_{0\,ij}}_{\text{sum over i,j=1,2,3}} + \underbrace{\sum_{i} \hat{f}^{\mu\nu}_{0\,ii}\, \hat{f}^{\mu'\nu'}_{0\,ii}}_{\text{sum over three-trace}} \Big] := P^{\mu\nu\mu'\nu'}_{(m_0)} \qquad (3.115)
$$

Unfortunately, by performing the sum over the massive degrees of freedom using 3.107 and 3.104, one is led to a expression which is not Lorentz covariant but worse, may not be written in terms of k_μ, $\tilde{k}_\mu$ and $\eta_{\mu\nu}$.

One naively may restrict the sum to the rest frame $k^\mu = (m_0, 0, 0, 0)$, perform it with the explicit basis tensors 3.110 and make a most general Ansatz in terms of k_μ, $\tilde{k}_\mu$ and $\eta_{\mu\nu}$ as

$$
\begin{aligned}
P^{\mu\nu\mu'\nu'}_{(m_0)} &= \alpha\left(\eta^{\mu\mu'}\eta^{\nu\nu'} + \eta^{\mu\nu'}\eta^{\nu\mu'}\right) + \beta\left(\eta^{\mu\nu}\eta^{\mu'\nu'}\right) \\
&+ \frac{\gamma}{m_0^2}\left(k^\mu k^{\mu'}\eta^{\nu\nu'} + k^\nu k^{\nu'}\eta^{\mu\mu'} + k^\mu k^{\nu'}\eta^{\nu\mu'} + k^\nu k^{\mu'}\eta^{\mu\nu'}\right) \qquad (3.116)\\
&+ \frac{\delta}{m_0^2}\left(k^\mu k^\nu \eta^{\mu'\nu'} + k^{\mu'}k^{\nu'}\eta^{\mu\nu}\right) + \frac{\tau}{m_0^4}\, k^\mu k^\nu k^{\mu'} k^{\nu'}
\end{aligned}
$$

In order to determine the coefficients α, β, γ, δ, τ, one would have to restrict $P^{\mu\nu\mu'\nu'}_{(m_0)}$ to rest frame

$$
\begin{aligned}
P^{\mu\nu\mu'\nu'}_{(m_0)}\Big|_{\text{rf.}} &= \alpha\left(\eta^{\mu\mu'}\eta^{\nu\nu'} + \eta^{\mu\nu'}\eta^{\nu\mu'}\right) + \beta\left(\eta^{\mu\nu}\eta^{\mu'\nu'}\right) \\
&+ \gamma\left(\eta^{\nu\nu'}\Big|_{\mu\mu'=00} + \eta^{\mu\mu'}\Big|_{\nu\nu'=00} + \eta^{\nu\mu'}\Big|_{\mu\nu'=00} + \eta^{\mu\nu'}\Big|_{\nu\mu'=00}\right)\Big]_{=0,\,\text{else}} \\
&+ \delta\left(\eta^{\mu'\nu'}\Big|_{\mu\nu=00} + \eta^{\mu\nu}\Big|_{\mu'\nu'=00}\right)\Big]_{=0,\,\text{else}} + \tau\Big]_{\mu\nu\mu'\nu'=0000}
\end{aligned}
\qquad (3.117)
$$

and compare the coefficients for each $\mu\nu\mu'\nu'$ with $\sum_a \hat{f}^{\mu\nu}_a \hat{f}^{\mu'\nu'}_a$. One finds then

$$
\begin{aligned}
P^{\mu\nu\mu'\nu'}_{(m_0)} &= \frac{1}{2}\left(\eta^{\mu\mu'}\eta^{\nu\nu'} + \eta^{\mu\nu'}\eta^{\nu\mu'} - \eta^{\mu\nu}\eta^{\mu'\nu'}\right) \\
&\quad - \frac{1}{2m_0^2}\left(k^\mu k^{\mu'}\eta^{\nu\nu'} + k^\nu k^{\nu'}\eta^{\mu\mu'} + k^\mu k^{\nu'}\eta^{\nu\mu'} + k^\nu k^{\mu'}\eta^{\mu\nu'}\right) \\
&\quad + \frac{1}{m_0^2}\left(k^\mu k^\nu \eta^{\mu'\nu'} + k^{\mu'}k^{\nu'}\eta^{\mu\nu}\right) + \frac{1}{m_0^4} k^\mu k^\nu k^{\mu'} k^{\nu'}
\end{aligned}
\tag{3.118}
$$

But this object, written in terms of k_μ, $\tilde{k}_\mu$ and $\eta_{\mu\nu}$, does not transform in the same way as the polarization sum over the six massive degrees of freedom $\sum_a \hat{f}_a^{\mu\nu}\hat{f}_a^{\mu'\nu'}$ does. It is not clear if the massive polarization sum $\sum_a \hat{f}_a^{\mu\nu}\hat{f}_a^{\mu'\nu'}$ could be even written in terms of k_μ, $\tilde{k}_\mu$ and $\eta_{\mu\nu}$.

Therefore, we are left with explicitly calculating the matrix element for each massive degree of freedom according to the definition in p-space and at the end to square and sum them up for deriving the cross section.

3.3 Quantum gauge invariance and couplings

The only underlying principle to derive the couplings is perturbative quantum gauge invariance. In the massless case, the couplings are derived in [8, 40], where the trilinear, pure graviton couplings coincide with the ones derived from expanding the classical Einstein-Hilbert Lagrangian. For the massive case, the couplings are discussed in [29].

Matter coupling

We first consider the massless case: $T^m(x_1)$ stands for the coupling of a graviton to matter, in our case represented by a charged scalar field φ with mass m. In consequence of first order gauge invariance, the graviton couples to the energy-momentum tensor of φ. It reads

$$T^m(x_1) = \frac{i}{2}\kappa\ h^{\alpha\beta}(x_1)\ b_{\alpha\beta\mu\nu}\ T_m^{\mu\nu}$$

In classical field theory, the Lagrangian for φ is

$$\mathcal{L} = \partial_\mu\varphi^\dagger\partial^\mu\varphi - m^2\varphi^\dagger\varphi$$

then $T_m^{\mu\nu}$ is given by

$$
\begin{aligned}
T_m^{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial\varphi^\dagger_{,\nu}}\varphi^{\dagger,\mu} + \frac{\partial\mathcal{L}}{\partial\varphi_{,\mu}}\varphi^{,\nu} - \eta^{\mu\nu}\mathcal{L} \\
&= \varphi^\dagger_{,\nu}\varphi_{,\mu} + \varphi^\dagger_{,\mu}\varphi_{,\nu} - \eta_{\mu\nu}\left(\varphi^\dagger_{,\mu}\varphi^{,\mu} - m^2\varphi^\dagger\varphi\right)
\end{aligned}
$$

leading to the classical expression for $T^m(x_1)$

$$T^m(x_1) = i\kappa\ h^{\alpha\beta}(x_1)\left(\varphi^\dagger_{,\alpha}(x_1)\varphi_{,\beta}(x_1)\ -\ \frac{m^2}{2}\ \eta_{\alpha\beta}\ \varphi^\dagger(x_1)\varphi(x_1)\right)$$

In the passage to quantum field theory, products of φ get normally ordered. Since $h^\alpha_\alpha = h$, we arrive at

$$T^m(x_1) = i\kappa\left(h^{\alpha\beta}(x_1) \ :\varphi^\dagger_{,\alpha}(x_1)\varphi_{,\beta}(x_1): \ - \ \frac{m^2}{2}h(x_1) \ :\varphi^\dagger(x_1)\varphi(x_1):\right) \tag{3.119}$$

The general Ansatz in the case of an external field $h^{\alpha\beta}_{\mathrm{ext}}(x)$ is

$$\begin{aligned} T^m(x_1) \ &= \ i\kappa\Big(\left(h^{\alpha\beta}(x_1) + h^{\alpha\beta}_{\mathrm{ext}}(x_1)\right) :\varphi^\dagger_{,\alpha}(x_1)\varphi_{,\beta}(x_1): \\ &- \frac{m^2}{2}\left(h(x_1) + h_{\mathrm{ext}}(x_1)\right) :\varphi^\dagger(x_1)\varphi(x_1):\Big) \end{aligned}$$

This is reduced to the pure external field coupling as

$$T^m_{ext}(x_1) = i\kappa\left(h^{\alpha\beta}_{\mathrm{ext}}(x_1) \ :\varphi^\dagger_{,\alpha}(x_1)\varphi_{,\beta}(x_1): \ - \ \frac{m^2}{2}h_{\mathrm{ext}}(x_1) \ :\varphi^\dagger(x_1)\varphi(x_1):\right) \tag{3.120}$$

The coupling of a massive graviton to matter takes the same structure as in the massless case because the gauge variations according to 3.67 do not alter

$$d_Q h^{\mu\nu} = d_Q h^{\mu u}_{(m_0)} = -ib^{\mu\nu\alpha\beta}u_{\alpha,\beta}$$

and it is again coupled to a conserved source i.e. $\partial_\mu T^{\mu\nu} = 0$ as well as the gauge variation $d_Q T^{\mu\nu} = 0$ vanishes for the matter-tensor. Thus we can use the same matter coupling as in the massless case i.e.

$$T^m_{(m_0)\ \mathrm{ext}}(x_1) = i\kappa\left(h^{\alpha\beta}_{\mathrm{ext}}(x_1) \ :\varphi^\dagger_{,\alpha}(x_1)\varphi_{,\beta}(x_1): \ - \ \frac{m^2}{2}h_{\mathrm{ext}}(x_1) \ :\varphi^\dagger(x_1)\varphi(x_1):\right) \tag{3.121}$$

Trilinear coupling

The trilinear self-coupling $T^h_1(x)$ is derived with a general Ansatz of all possible couplings between $h^{\alpha\beta}$ and ghosts. The terms have to fulfill first order gauge invariance, which in this case means $d_Q(T^h_1 + T^u_1) = i\partial_\mu T^\mu_{1/1}$, where T^h_1 represents the graviton sector whereas the T^u represents the ghost sector. It is shown that one recovers the same graviton couplings as by expanding the classical Einstein-Hilbert theory where the coefficients are set in order to agree. The trilinear coupling of the h-field without ghosts is

$$T^h(x_1) = i\kappa h^{\alpha\beta}(x_1)\left(-\frac{1}{4}h_{,\alpha}(x_1)h_{,\beta}(x_1) + \frac{1}{2}h_{\lambda\eta,\alpha}(x_1)h^{\lambda\eta}_{,\beta}(x_1) + h^\lambda_{\beta,\eta}(x_1)h^\eta_{\alpha,\lambda}(x_1)\right) \tag{3.122}$$

In the case of the external field, the trilinear self-coupling becomes

$$T^h_{\mathrm{ext}}(x_1) = i\kappa h^{\alpha\beta}_{\mathrm{ext}}(x_1)\left(-\frac{1}{4}h_{,\alpha}(x_1)h_{,\beta}(x_1) + \frac{1}{2}h_{\lambda\eta,\alpha}(x_1)h^{\lambda\eta}_{,\beta}(x_1) + h^\lambda_{\beta,\eta}(x_1)h^\eta_{\alpha,\lambda}(x_1)\right) \tag{3.123}$$

In the massive case, the situation looks not too different. The trilinear self-coupling for the massive graviton is derived in [29]. Without ghosts, the coupling for the massive field $h^{\mu\nu}(x)$ reads

$$T^h_{(m_0)} = h^{\alpha\beta}\Big(h_{,\alpha}h_{,\beta} - 2h_{\lambda\eta,\alpha}h^{\lambda\eta}_{,\beta} - 4h^{\eta}_{\alpha,\lambda}h^{\lambda}_{\beta,\eta}\Big) \tag{3.124}$$

$$- h^{\alpha\beta}\Big(2h_{\alpha\beta,\lambda}h^{,\lambda} - 4h_{\alpha\lambda,\eta}h^{\lambda,\eta}_{\beta}\Big) \tag{3.125}$$

$$+ m_0^2\Big(- \frac{4}{3}h^{\alpha\beta}h_{\alpha\lambda}h^{\lambda}{}_{\beta} + h^{\alpha\beta}h_{\alpha\beta}h - \frac{1}{6}h^3\Big) \tag{3.126}$$

where 3.124 coincides with the massless case. The terms in 3.125 are of divergence type and therefore physically not relevant. The terms in 3.126 are additional terms in the massive case. But since the graviton mass m_0 is assumed to be very small compared to the mass-scale in question, they can be neglected, leaving the same trilinear self-coupling as in the massless case.

Quartic coupling

The quartic coupling in the massless case is a direct consequence of quantum gauge invariance when discussing anomalies for second-order tree graphs i.e they must hold in order to restore second order gauge invariance. In the case of an external field, it is given by

$$\begin{aligned} T_2^h(x_1, x_2) = \frac{i\kappa^2 m^2}{4}\Big(& h^{\alpha\beta}_{\text{ext}}(x_1)h_{\alpha\beta}(x_2) : \varphi^\dagger(x_1)\varphi(x_2) : \\ & - \frac{1}{2}h_{\text{ext}}(x_1)h(x_2) : \varphi^\dagger(x_1)\varphi(x_2) : \Big)\,\delta(x_1 - x_2) \end{aligned} \tag{3.127}$$

For the massive case there are as well quartic couplings derived in [29] but again, the additional terms in the massive case are quadratic in the graviton mass m_0 and therefore can be neglected, leaving the quartic coupling in the massive case to coincide with the massless case.

Bosonic vector ghost coupling

In the context of gravitational Bremsstrahlung, it is most important to know whether the bosonic vector ghost $v^\mu(x)$ couples to matter. In order to investigate this, we consider the most general coupling of v^μ to a charged scalar field φ. It takes the form [30]

$$T_1 = a_1\partial_\mu v^\mu \varphi^\dagger\varphi + a_2 v^\mu\partial_\mu\varphi^\dagger\varphi + a_3 v^\mu\varphi^\dagger\partial_\mu\varphi \tag{3.128}$$

where a_i are arbitrary constants. Note, that 3.128 is the only way to couple v^μ to φ in a Lorentz covariant way. In order to fulfill first order gauge invariance, $d_Q T_1$ must be, according to 2.10, of divergence type i.e.

$$\begin{aligned} d_Q T_1 &= -\frac{i}{2}m_0\, a_1\partial_\mu u^\mu\, \varphi^\dagger\varphi - \frac{i}{2}m_0\,\Big(a_2 u^\mu\,\partial_\mu\,\varphi^\dagger\varphi + a_3\varphi^\dagger\partial_\mu\varphi\Big) \\ &\stackrel{!}{=} -\frac{i}{2}m_0\,\partial_\mu\Big(b_1\, u^\mu\,\varphi^\dagger\varphi\Big) \\ &= -\frac{i}{2}b_1\Big(\partial_\mu\, u^\mu\,\varphi^\dagger\varphi + u^\mu\partial_\mu\,\varphi^\dagger\varphi + u^\mu\varphi^\dagger\partial_\mu\varphi\Big) \end{aligned} \tag{3.129}$$

Then it follows that $b_1 = a_1 = a_2 = a_3$ and

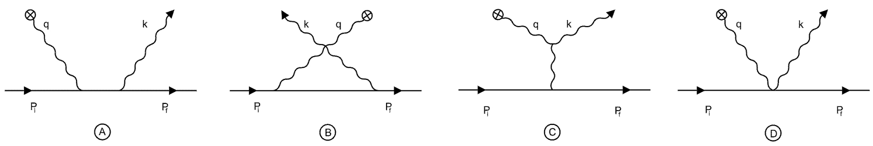
$$T_1 = a_1 \partial_\mu \left(v^\mu \varphi^\dagger \varphi \right)$$

which is of trivial divergence type and therefore physically not relevant.

To sum it up, it can be said that the different couplings relevant for the gravitational Bremsstrahl process can be used for the massless graviton as well as for the massive one without any difference when the graviton mass is assumed to be very small.

4 Quantum gravitational Bremsstrahlung

Having collected all basic objects of the theory, we are ready now to go into the calculation of the S-matrix element for gravitational Bremsstrahlung in the causal formalism. This allows us then to deduce the differential cross section and the total energy radiated. In the lowest-order non vanishing approximation, the Bremsstrahl process is described by the four second-order Feynman diagrams. They are labeled as



The asymptotic free fields, described in section 3, will now be used to construct the time ordered products T_n in the S-matrix expansion as

$$S(g) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \; T_n(x_1, \ldots, x_n) \; g(x_1) \cdots g(x_n)$$

where $g \in \mathcal{S}(\mathbb{R}^4)$ is a test function in Schwartz space. The time-ordered products T_n are operator-valued distributions and are expressed by normally ordered products of free fields. According to section 3.3, the first order coupling T_1 is given. It is obtained by the requirement of perturbative gauge invariance of the S-matrix. The second-order chronological products $T_2(x_1, x_2)$ are then inductively constructed out of T_1 according to the mechanism described in section 2.1.

In the following, we will always have an eye on the classical description of gravitational Bremsstrahlung, since the tree-level quantum calculation enables us to do so. Moreover, we are interested in comparing our results with the classical theory where the gravitational Bremsstrahl process is described by a particle of mass m, moving along an unbound asymptotic trajectory, passing under the impact parameter ρ another mass M, which is considered to induce a static gravitational potential ϕ. The particle is deflected under the influence of the potential and therefore looses energy. In the quantum calculation of gravitational Bremsstrahlung, the particle of mass m is described by a charged scalar field φ which satisfies the massive Klein-Gordon equation and is quantized with the Pauli-Jordan distribution of mass m. We will therefore call the particle, which is referred to the scalar field φ, as the φ-particle. At tree-level, the Bremsstrahl process consists basically of the fact, that the φ-particle in passing close to a central mass M, emits under the influence of its gravitational field a graviton. In this process, the central mass M takes up a part of the momentum required for the conservation of energy-momentum. We will treat the process as the gravitational scattering of the φ-particle by an external classical field. The interaction with the classical central potential

ϕ will be treated according to the external field approximation where in the Newtonian limit, ϕ becomes the Newtonian potential ϕ_N. We will therefore restrict the mass m of the φ-particle to be much smaller than the central mass M i.e. $m \ll M$. Furthermore, the φ-particle is considered to head the central mass under a large impact parameter ρ which corresponds to forward scattering under a small angle Θ'. We then are able to apply the first Born approximation method for the external field ϕ_N. By doing this, we refer to the extensive literature of Bremsstrahlung in electrodynamics where a charged particle emits electro-magnetic Bremsstrahlung under the influence of the nucleus. Worth mentioning are the pioneering works by *Bethe*, *Heitler* and collaborators [2, 4, 7, 15].

Since we want to compare our result with the classical theory, we have to apply some limiting approximation. This has the additional advantage to be prevented dealing with cumbersome expressions. Therefore, we will calculate the differential cross section in the limit of ultra relativistic particle energy ($m \ll E$) and low frequencies ($\omega \ll E$). In addition, we will impose the condition that the φ-particle be scattered by the gravitational potential through a small angle ($\Theta' \ll \frac{m}{\varepsilon}$) which in the classical picture represents scattering by large impact parameter ρ. This limit is known as the only overlapping region of the quasi-classical and Born approximation [2, 3, 4, 7, 26].

As already mentioned, the φ-particle with initial and final four-momentum p_i and p_f respectively, is represented by a charged scalar field φ with mass m obeying the massive Klein-Gordon equation

$$(\Box + m^2)\varphi(x) = 0$$

It is quantized according to

$$\left[\varphi(x_1),\, \varphi^\dagger(x_2)\right] \;=\; -\,iD_m(x_1 - x_2)$$

and Fourier-represented in Fock space as

$$\begin{aligned} \varphi(x) \;&=\; (2\pi)^{-\frac{3}{2}} \int \frac{d^3p}{\sqrt{2E_p}} \left(a_1(\vec{p})\, e^{-ipx} + a_2^\dagger(\vec{p})\, e^{ipx}\right) \\ \varphi^\dagger(x) \;&=\; (2\pi)^{-\frac{3}{2}} \int \frac{d^3p}{\sqrt{2E_p}} \left(a_2(\vec{p})\, e^{-ipx} + a_1^\dagger(\vec{p})\, e^{ipx}\right) \end{aligned} \tag{4.1}$$

where the creation and annihilation operators fulfill the commutation relations as

$$\begin{aligned} \left[a_1(\vec{p}),\, a_1^\dagger(\vec{p}\,')\right] \;&=\; \delta(\vec{p} - \vec{p}\,') \\ \left[a_2(\vec{p}),\, a_2^\dagger(\vec{p}\,')\right] \;&=\; \delta(\vec{p} - \vec{p}\,') \end{aligned}$$

4.1 Massless quantum gravitational Bremsstrahlung

4.1.1 Construction of second order $T_2(x_1,\ x_2)$

We give a step by step overview of the calculation for the S-matrix element for the process A, all other processes are treated in a similar way.

Diagram A

First, we have to construct the time-ordered distribution $T_2(x_1, x_2)$. According to the causal method, the inductive construction of T_2 is based on the causality condition 2.4. The basic objects of the theory are the time ordered products T_n of asymptotic free fields. Since the first order term T_1, the interaction, is given through perturbative quantum gauge invariance by 3.119 and 3.120, we are able to form the auxiliary distributions

$$\begin{aligned} A_2'(x_1, x_2) &= -\,T_{\text{ext}}(x_1)\,T(x_2) \\ R_2'(x_1, x_2) &= -\,T(x_2)\,T_{\text{ext}}(x_1) \end{aligned}$$

which do not have causal support, thus the ordering is relevant. Indeed, the difference $R_2' - A_2'$ has causal support, forming the distribution D_2 by

$$\begin{aligned} D_2(x_1, x_2) &= R_2'(x_1, x_2) - A_2'(x_1, x_2) \\ &= T_{\text{ext}}(x_1)\,T(x_2) - T(x_2)\,T_{\text{ext}}(x_1) \\ &= [T_{\text{ext}}(x_1),\ T(x_2)] \sim iD_m(x_1 - x_2) \end{aligned}$$

where D_m is the massive Jordan-Pauli distribution, introduced in section 2. Since A_2', R_2' and D_2 are products of normally ordered field operators we must apply Wick's theorem to get normally ordered expressions. For the Bremsstrahl process, we must pick the expressions contracting only the φ-field operators where the contractions of the $h^{\mu\nu}$-field operators are not relevant. The time ordered product T_2 is given by

$$T_2(x_1, x_2) = R_2 - R_2' \sim \left(iD_m^{\text{ret}}(x_1 - x_2) - iD_m^{(-)}(x_1 - x_2)\right) = iD_m^F(x_1 - x_2)$$

where the distribution R_2 represents the retarded part of the distribution D_2 and D_m^F is the Feynman propagator defined in [9]. At tree level, the distribution splitting of D_2 is always trivial and we have

$$D_2 \sim iD_m \overset{\text{splitting}}{=} iD_m^{\text{ret}} - iD_m^{\text{av}}$$

i.e. $R_2 \sim iD_m^{\text{ret}}$. For the explicit construction of T_2 for the process A, we use the following labeling according to 3.119 and 3.120

$$\begin{aligned} T_{\text{ext}}^m(x_1) &= i\kappa\left(h_{\text{ext}}^{\alpha\beta}(x_1) : \varphi_{,\alpha}^\dagger(x_1)\varphi_{,\beta}(x_1) : -\frac{m^2}{2} h_{\text{ext}}(x_1) : \varphi^\dagger(x_1)\varphi(x_1) :\right) \\ T^m(x_2) &= i\kappa\left(h^{\mu\nu}(x_2) : \varphi_{,\mu}^\dagger(x_2)\varphi_{,\nu}(x_2) : -\frac{m^2}{2} h(x_2) : \varphi^\dagger(x_2)\varphi(x_2) :\right) \end{aligned}$$

Then we find for the second order time-ordered distribution T_2

$$
\begin{aligned}
T_2(x_1,x_2) &= R_2 - R_2' \\
&= \kappa^2 \Bigg\{ h^{\alpha\beta}_{\text{ext}}(x_1) h^{\mu\nu}(x_2) \Big(:\varphi^\dagger_{,\alpha}(x_1)\varphi_{,\nu}(x_2): \partial^{x_1}_\beta \partial^{x_2}_\mu + :\varphi_{,\beta}(x_1)\varphi^\dagger_{,\mu}(x_2): \partial^{x_1}_\alpha \partial^{x_2}_\nu \Big) \\
&\quad - \frac{m^2}{2} h^{\alpha\beta}_{\text{ext}}(x_1) h(x_2) \Big(:\varphi^\dagger_{,\alpha}(x_1)\varphi(x_2): \partial^{x_1}_\beta + :\varphi_{,\beta}(x_1)\varphi^\dagger(x_2): \partial^{x_1}_\alpha \Big) \\
&\quad - \frac{m^2}{2} h_{\text{ext}}(x_1) h^{\mu\nu}(x_2) \big(:\varphi^\dagger(x_1)\varphi_{,\nu}(x_2): \partial^{x_2}_\mu + :\varphi(x_1)\varphi^\dagger_{,\mu}(x_2): \partial^{x_2}_\nu \big) \\
&\quad + \frac{m^4}{4} h_{\text{ext}}(x_1) h(x_2) \big(:\varphi^\dagger(x_1)\varphi(x_2): + :\varphi(x_1)\varphi^\dagger(x_2): \big) \Bigg\} iD^F_m(x_1 - x_2)
\end{aligned}
\tag{4.2}
$$

where the details are outlined in the appendix B. It is worth mentioning, that we couple the Bremsstrahl graviton only to the energy-momentum tensor $T^{\mu\nu}$ of the φ-particle according to 3.119. In a more detailed approach, one could couple the graviton to a energy-momentum tensor, including the effect of the central mass M as $\tilde{T}^{\mu\nu} = T^{\mu\nu}_m + T^{\mu\nu}_M$. But since the central mass M is considered in the limit of slow motion, we can neglect its contribution[14].

4.1.2 S-matrix element

In the following, all products of the field operators are written normally ordered. The S-matrix element is defined as

$$S_{fi}(g) = (\phi_f,\ S(g)\ \phi_i)$$

where ϕ_f and ϕ_i are the final- and initial state respectively. We write

$$(\phi_f,\ S_\bullet(g)\ \phi_i) = (\phi_f,\ [S - \delta_{fi}]\,(g)\ \phi_i)$$

where $S_\bullet$ indicates the contribution in the S-matrix expansion. In the following, we set $g(x) = 1$, since at tree level, it is always possible to perform the adiabatic limit. According to 2.1, the general second order S-matrix element takes the form

$$(\phi_f,\ S_2(1)\ \phi_i) = \frac{1}{2}\int d^4x_1 d^4x_2 (\phi_f,\ T_2(x_1,\ x_2)\ \phi_i) \tag{4.3}$$

The initial state consists of the φ-particle with mass m, represented by the charged scalar field $\varphi(x)$, according to the introductory remarks in section 4. Since in the second order causal distribution T_2, both particle species $+$ and $-$ are represented, we will for simplicity, choose the initial state to be occupied only by the $+$ particle species. Since there is no incoming graviton line, we can omit it in the initial state. The action of the external field on the φ-particle will be treated later. The initial state is then given by

$$\phi_i = a_1^\dagger(\vec{p}_i)\ \Omega$$

with $p_i^\mu = (E_i,\ \vec{p}_i)$ the initial four-momentum of the φ-particle and Ω the vacuum. The final state consists of the outgoing φ-particle (same particle species) and a free, soft

[14]The classical calculation of gravitational Bremsstrahlung by *Peters* [39] considered only the contribution from the energy-momentum tensor of the mass-current particle, too.

Bremsstrahl graviton. In the massless case, according to section 3.1.2, the gravitational scalar degree of freedom is absent. The physical, transverse polarization states, the helicity ± 2, are indicated by $\pm$. Therefore, the final state can be written for fixed three-momentum $\vec{k}_f$ as

$$\phi_f^{\pm} = \Omega\; a_1(\vec{p}_f)\; \varepsilon_{\pm}^{\rho\sigma *}(\vec{k}_f)\; A_{\rho\sigma}(\vec{k}_f)$$

where $\varepsilon_{\pm}^{\rho\sigma}$ is the polarization "tensor" in the massless case according to 3.34, $p_f^{\mu} = (E_f,\ \vec{p}_f)$ and $k^{\mu} = (E_k,\ \vec{k}_f)$ are the four-momenta of the emerging φ-particle and the Bremsstrahl graviton, respectively and we have $p_i^2 = p_f^2 = m^2$ and $k^2 = 0$. Note, that the final Bremsstrahl graviton is generated by the traceless creation operator $A_{\rho\sigma}(\vec{k}_f)$. We introduce the following abbreviations [9]

$$\begin{aligned}
\varphi_i &= a_1^{\dagger}(\vec{p}_i)\,\Omega \\
\varphi_f &= \Omega\, a_1(\vec{p}_f) \\
h_i &= \Omega \\
h_{f\rho\sigma} &= \Omega\, A_{\rho\sigma}(\vec{k}_f)
\end{aligned}$$

then

$$\begin{aligned}
\phi_i &= \varphi_i\, h_i && (4.4)\\
\phi_f^{\pm} &= \varphi_f\, h_f^{\pm} \\
&:= \varphi_f\, \varepsilon_{\pm}^{\rho\sigma *}(\vec{k}_f)\, h_{f\rho\sigma} \\
&:= \varepsilon_{\pm}^{\rho\sigma *}(\vec{k}_f)\, \phi_{f\rho\sigma} && (4.5)
\end{aligned}$$

The definition of a state by $\varphi_f,\ h_{f\rho\sigma}$ should not give rise to confusions with the definition of the field operators $\varphi(x),\ h^{\rho\sigma}(x)$.

Diagram A

For the S-matrix element, we find, according to 4.3 and 4.4, 4.5, the following form

$$\begin{aligned}
\Big(\phi_f^{\pm}, S_2^A(1)\,\phi_i\Big) &= \varepsilon_{\pm}^{\rho\sigma *}(\vec{k}_f)\Big(\phi_{f\rho\sigma}, S_2\,\phi_i\Big) \\
&= \frac{1}{2}\,\varepsilon_{\pm}^{\rho\sigma *}(\vec{k}_f)\int d^4x_1\, d^4x_2\, \Big(\varphi_f\, h_{f\rho\sigma}, T_2(x_1,x_2)\,\varphi_i\, h_i\Big) \\
&= \frac{1}{2}\,\varepsilon_{\pm}^{\rho\sigma *}(\vec{k}_f)\int d^4x_1\, d^4x_2\, \Big(\varphi_f, T_2^{\varphi}(x_1,x_2)\,\varphi_i\Big)\Big(h_{f\rho\sigma}, T_2^{h}(x_1,x_2)\, h_i\Big)
\end{aligned} \qquad (4.6)$$

where $T_2^{\varphi},\ T_2^{h}$ are the second order time-ordered distributions containing the field operators $\varphi(x)$ and $h^{\mu\nu}(x)$, respectively. Furthermore, we have set $S_2(1) \equiv S_2$. In the following, we will omit indicating the polarization "tensor" $\varepsilon_{\pm}^{\rho\sigma *}$. By inserting 4.2 for the second order T_2, we find

$$
\begin{aligned}
\Big(\phi_{f\rho\sigma}, S_2^A\,\phi_i\Big) \;=\; & \frac{i\kappa^2}{2}\int d^4x_1\, d^4x_2 \Bigg\{ \\
& \cdot \Bigg[\Big(\varphi_f, \varphi_{,\nu}(x_2)\,\partial_\beta^{x_1}\partial_\mu^{x_2} D_m^F(x_1-x_2)\,\varphi^\dagger_{,\alpha}(x_1)\,\varphi_i\Big) \\
& + \Big(\varphi_f, \varphi^\dagger_{,\mu}(x_2)\,\partial_\alpha^{x_1}\partial_\nu^{x_2} D_m^F(x_1-x_2)\,\varphi_{,\beta}(x_1)\,\varphi_i\Big)\Bigg]\Big(h_{f\rho\sigma}, h^{\mu\nu}(x_2)\,h^{\alpha\beta}_{\text{ext}}(x_1)\,h_i\Big) \\
& - \frac{m^2}{2}\Bigg[\Big(\varphi_f, \varphi(x_2)\,\partial_\beta^{x_1} D_m^F(x_1-x_2)\,\varphi^\dagger_{,\alpha}(x_1)\,\varphi_i\Big) \\
& + \Big(\varphi_f, \varphi^\dagger(x_2)\,\partial_\alpha^{x_1} D_m^F(x_1-x_2)\,\varphi_{,\beta}(x_1)\,\varphi_i\Big)\Bigg]\Big(h_{f\rho\sigma}, h(x_2)\,h^{\alpha\beta}_{\text{ext}}(x_1)\,h_i\Big) \\
& - \frac{m^2}{2}\Bigg[\Big(\varphi_f, \varphi_{,\nu}(x_2)\,\partial_\mu^{x_2} D_m^F(x_1-x_2)\,\varphi^\dagger(x_1)\,\varphi_i\Big) \\
& + \Big(\varphi_f, \varphi^\dagger_{,\mu}(x_2)\,\partial_\nu^{x_2} D_m^F(x_1-x_2)\,\varphi(x_1)\,\varphi_i\Big)\Bigg]\Big(h_{f\rho\sigma}, h^{\mu\nu}(x_2)\,h_{\text{ext}}(x_1)\,h_i\Big) \\
& + \frac{m^4}{4}\Bigg[\Big(\varphi_f, \varphi(x_2)\,D_m^F(x_1-x_2)\,\varphi^\dagger(x_1)\,\varphi_i\Big) \\
& + \Big(\varphi_f, \varphi^\dagger(x_2)\,D_m^F(x_1-x_2)\,\varphi(x_1)\,\varphi_i\Big)\Bigg]\Big(h_{f\rho\sigma}, h(x_2)\,h_{\text{ext}}(x_1)\,h_i\Big)\Bigg\}
\end{aligned}
\tag{4.7}
$$

In order to calculate the various actions of the field operators, we have to determine for example

$$
\begin{aligned}
\varphi_{,\beta}(x_1)\,\varphi_i\Big) \;&=\; \varphi^{(-)}_{,\beta}(x_1)\,\varphi_i\Big) \\
&=\; -\,i(2\pi)^{-\frac{3}{2}}\int \frac{d^3p}{\sqrt{2E_p}}\,p_\beta\, e^{-ipx_1}\,\underbrace{a_1(\vec{p})a_1^\dagger(\vec{p}_i)}_{\delta(\vec{p}-\vec{p}_i)}\,\Omega\Big) \\
&=\; -\,i\,\frac{(2\pi)^{-\frac{3}{2}}}{\sqrt{2E_i}}\,p_{i\beta}\,e^{-ip_ix_1}\Omega\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
\Big(h_{f\rho\sigma}, h^{\mu\nu}(x_2) \;&=\; \Big(h_{f\rho\sigma}, h^{\mu\nu(+)}(x_2) \\
&=\; (2\pi)^{-\frac{3}{2}}\int \frac{d^3k}{\sqrt{2E_k}}\,e^{ikx_2}\Big(\Omega, A_{\rho\sigma}(\vec{k}_f)\,\Big(\eta^{\mu\mu}\,\eta^{\nu\nu}\,A^\dagger_{\mu\nu}(\vec{k}) - \frac{1}{4}\,\eta^{\mu\nu}\,d^\dagger(\vec{k})\Big) \\
&=\; (2\pi)^{-\frac{3}{2}}\int \frac{d^3k}{\sqrt{2E_k}}\,e^{ikx_2}\Big(\Omega, \underbrace{\eta^{\mu\mu}\,\eta^{\nu\nu}\Big[A_{\rho\sigma}(\vec{k}_f), A^\dagger_{\mu\nu}(\vec{k})\Big]}_{t^{\rho\sigma\mu\nu}\,\delta(\vec{k}-\vec{k}_f)} \\
&=\; \frac{(2\pi)^{-\frac{3}{2}}}{\sqrt{2E_k}}\,e^{ik_fx_2}\,t^{\rho\sigma\mu\nu}\Big(\Omega,
\end{aligned}
$$

From now on, for simplicity, we will use the notation $\vec{k}_f = \vec{k}$. Since we have chosen only one particle species $(+)$ in the initial state, all first summands in 4.7 vanish. Moreover, since the final Bremsstrahl graviton does not carry a scalar degree of freedom, all expressions in h acting on the final state vanish, too. Therefore, we find

$$\begin{aligned}\left(\phi_f^{\rho\sigma}, S_2^A\, \phi_i\right) &= \frac{i\kappa^2 (2\pi)^{-\frac{9}{2}}}{4\sqrt{2E_i E_f E_k}} \int d^4x_1\, d^4x_2\; e^{-ip_i x_1 + i(p_f+k)x_2} \\ &\cdot \Big[p_{f\mu}\, p_{i\beta}\, \partial_\alpha^{x_1} \partial_\nu^{x_2} D_m^F(x_1 - x_2)\, t^{\rho\sigma\mu\nu}\, h_{\text{ext}}^{\alpha\beta}(x_1) \\ &\quad - i\, \frac{m^2}{2}\, p_{f\mu} \partial_\nu^{x_2} D_m^F(x_1 - x_2)\, t^{\rho\sigma\mu\nu}\, h_{\text{ext}}(x_1) \Big]\end{aligned} \tag{4.8}$$

In order to evaluate the S-matrix element in the external field approximation, we have to apply the Born approximation method for the classical external field ϕ. In our case, $h_{\text{ext}}^{\alpha\beta}(x_1)$ plays the role of the unquantized potential which we wish to relate to the Newtonian potential ϕ_N. We will restrict ourselves to the first iteration solution in powers of the coupling to the external field (first Born approximation). According to [2, 4, 7], the case of gravitational interaction is treated in close analogy to the electromagnetic case. The external field, acting at the vertex (x_1) on the φ-particle, is in momentum space described by the Fourier transform of $h_{\text{ext}}^{\alpha\beta}(x_1)$ i.e.

$$h_{\text{ext}}^{\alpha\beta}(x_1) = (2\pi)^{-\frac{3}{2}} \int d^4q\; h_{\text{ext}}^{\alpha\beta}(q)\; e^{-iqx_1}$$

where we have to determine an expression for $h_{\text{ext}}^{\alpha\beta}(q)$. In the Newtonian limit of general relativity, where the gravitational field is assumed to be weak and stationary, we may decompose the metric $g^{\alpha\beta}$ into $g^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta}$, where $\eta^{\alpha\beta}$ represents the background Minkowski metric and $h^{\alpha\beta}$ its small perturbation i.e. $\left|h^{\alpha\beta}\right| \ll 1$. This is satisfied by the assumptions of the Bremsstrahl process, namely the case of the static, central potential induced by the mass M. Furthermore, we consider the case of small Bremsstrahl energy and large impact parameter. Then, the Newtonian limit is, according to [12], given by

$$h^{00} = -2\phi_N(M)$$

We may therefore make contact to the external field approximation [4] through

$$h_{\text{ext}}^{\alpha\beta}(x_1) = (2\pi)^{-\frac{3}{2}} \int d^4q\; h_{\text{ext}}^{\alpha\beta}(\vec{q})\; e^{-iqx_1}\;\; f_\epsilon(q^0) \tag{4.9}$$

with

$$\lim_{\epsilon \to 0} f_\epsilon(q^0) = \delta(q^0)$$

and gauge $h_{\text{ext}}^{\alpha\beta}(\vec{q})$ and $h_{\text{ext}}(\vec{q})$ to

$$h_{\text{ext}}^{00}(\vec{q}) = -2\; \phi_N(\vec{q}) \tag{4.10}$$

with

$$h^{\alpha\beta}_{\rm ext}(x_1) = -2\,(2\pi)^{-\frac{3}{2}} \int d^4q\ \phi_N(\vec{q})\ e^{-iqx_1}\ \delta(q^0)\Big]_{\alpha=\beta=0}$$

and

$$h_{\rm ext}(\vec{q}) = \eta_{\alpha\beta} h^{\alpha\beta}_{\rm ext}(\vec{q}) = \eta_{00} h^{00}_{\rm ext}(\vec{q}) = -2\phi_N(\vec{q})$$

This means that the coupling to the external field is expanded in terms of $G_N Mm$. Hence, the Born-condition $G_N Mm \ll 1$ has to be satisfied if one requires that the first Born approximation is the main contribution in the expansion of the external field[15]. Then, the S-matrix element in the first Born approximation in the case of graviational interaction becomes

$$\begin{aligned}\left(\phi^{\rho\sigma}_f, S^A\,\phi_i\right) \;=\; & -\frac{i\kappa^2(2\pi)^{-6}}{2\sqrt{2E_iE_fE_k}} \int d^4q\,\phi_N(\vec{q})\,\delta(q^0) \int d^4x_1\,d^4x_2\,e^{-i(p_i+q)x_1+i(p_f+k)x_2} \\ & \cdot \Big[p_{f\mu}\,p_{i\beta=0}\,\partial^{x_1}_{\alpha=0}\partial^{x_2}_{\nu} D^F_m(x_1-x_2)\,t^{\rho\sigma\mu\nu} \\ & - i\,\frac{m^2}{2}\,p_{f\mu}\partial^{x_2}_{\nu} D^F_m(x_1-x_2)\,t^{\rho\sigma\mu\nu}\Big]\end{aligned} \tag{4.11}$$

Now, the massive Feynman propagator in the causal framework, according to [9], is

$$D^F_m(x_1-x_2) = (2\pi)^{-4}\int d^4p\ \frac{e^{-ip(x_1-x_2)}}{m^2-p^2-i\epsilon} \tag{4.12}$$

and its x-integration for $(\phi^{(\rho\sigma)}_f,\ S_{21}\ \phi_i)$ yields

$$\begin{aligned}\int d^4x_1\,d^4x_2\,\partial^{x_1}_{\alpha}\partial^{x_2}_{\nu}\,D^F_m(x_1-x_2)\,e^{iPx_1+iQx_2} &= (2\pi)^4\,\delta^4(P+Q)\,\frac{Q_\alpha Q_\nu}{m^2-Q^2-i\epsilon} \\ \int d^4x_1\,d^4x_2\,\partial^{x_2}_{\nu}\,D^F_m(x_1-x_2)\,e^{iPx_1+iQx_2} &= (2\pi)^4\,\delta^4(P+Q)\,\frac{-iQ_\nu}{m^2-Q^2-i\epsilon} \\ \int d^4x_1\,d^4x_2\,\partial^{x_1}_{\alpha}\,D^F_m(x_1-x_2)\,e^{iPx_1+iQx_2} &= (2\pi)^4\,\delta^4(P+Q)\,\frac{iQ_\alpha}{m^2-Q^2-i\epsilon}\end{aligned} \tag{4.13}$$

where in our case, $P = -(p_i+q)$ and $Q = (p_f+k_f)$. Thus, having performed the x-integration, we find for the second order S-matrix element

[15]If one wants to include nonlinear effects of the field equations of general relativity, one could have approximated the Schwarzschild geometry by a metric, which is first order in the Newtonian potential ϕ_N [39].

$$
\begin{aligned}
\left(\phi_f^{\rho\sigma}, S^A\,\phi_i\right) &= -\frac{i\kappa^2(2\pi)^{-2}}{2\sqrt{2E_iE_fE_k}}\int d^4q\,\phi_N(\vec{q})\,\delta(q^0)\,\delta^4(p_f+k-p_i-q)\\
&\quad\cdot\, t^{\rho\sigma\mu\nu}\Big(E_i(E_f+E_k)\,\frac{p_{f\mu}(p_f+k)_\nu}{m^2-(p_f+k)^2-i\epsilon}-\frac{m^2}{2}\,\frac{p_{f\mu}(p_f+k)_\nu}{m^2-(p_f+k)^2-i\epsilon}\Big)\\
&= -\frac{i\kappa^2(2\pi)^{-2}}{2\sqrt{2E_iE_fE_k}}\int d^3q\,\phi_N(\vec{q})\,\delta^3(\vec{p}_f+\vec{k}-\vec{p}_i-\vec{q})\\
&\quad\cdot\, t^{\rho\sigma\mu\nu}\,\frac{p_{f\mu}(p_f+k)_\nu}{m^2-(p_f+k)^2-i\epsilon}\Big(E_i(E_f+E_k)-\frac{m^2}{2}\Big)\\
&\quad\cdot\,\underbrace{\int dq^0\,\delta(q^0)\,\delta(p_f^0+k^0-p_i^0-q^0)}_{=\delta(E_f+E_k-E_i)}
\end{aligned}
\tag{4.14}
$$

Note, that we have found only energy conservation over the whole process. Momentum is not conserved, since the central mass M is assumed to absorb a tiny amount of momentum, but its repulsion is neglected. The Fourier transform of the Newtonian potential is given by

$$
\phi_N(\vec{q}) = -\sqrt{\frac{2}{\pi}}\frac{G_NMm}{|\vec{q}|^2}
\tag{4.15}
$$

inserting it in 4.14 leaves

$$
\begin{aligned}
\left(\phi_f^{\rho\sigma}, S^A\,\phi_i\right) &= \frac{i\kappa^2\pi^{-\frac{5}{2}}}{8\sqrt{E_iE_fE_k}}\,\frac{G_NMm}{\left|\vec{p}_f+\vec{k}-\vec{p}_i\right|^2}\,\delta(E_f+E_k-E_i)\\
&\quad\cdot\,\frac{\left(E_i(E_f+E_k)-\frac{m^2}{2}\right)}{m^2-(p_f+k)^2-i\epsilon}\,p_{f\mu}(p_f+k)_\nu\,t^{\rho\sigma\mu\nu}
\end{aligned}
$$

and with

$$
(\phi_f^{\rho\sigma},\ S^A(1)\ \phi_i) = \delta(E_f+E_k-E_i)\ \mathcal{M}^{\rho\sigma A}
$$

we find the matrix element

$$
\begin{aligned}
\mathcal{M}^{\rho\sigma A} &= \frac{i\kappa^2\pi^{-\frac{5}{2}}}{8\sqrt{E_iE_fE_k}}\,\frac{G_NMm}{\left|\vec{p}_f+\vec{k}-\vec{p}_i\right|^2}\\
&\quad\cdot\,\frac{\left(E_i(E_f+E_k)-\frac{m^2}{2}\right)}{m^2-(p_f+k)^2-i\epsilon}\,p_{f\mu}(p_f+k)_\nu\,t^{\rho\sigma\mu\nu}
\end{aligned}
\tag{4.16}
$$

By contracting the Lorentz indices μ and ν, we find

$$
p_{f\mu}Q_\nu t^{\rho\sigma\mu\nu} = \frac{1}{2}\left[p_f^\rho Q^\sigma + p_f^\sigma Q^\rho\right] - \frac{1}{4}p_{f\gamma}Q^\gamma\eta^{\rho\sigma}
$$

Then, the final matrix element for the diagram A becomes

$$\begin{aligned}\mathcal{M}^{\rho\sigma A} &= \frac{i\kappa^2\pi^{-\frac{5}{2}}}{8\sqrt{E_iE_fE_k}} \frac{G_N M m}{\left|\vec{p}_f + \vec{k} - \vec{p}_i\right|^2} \frac{E_i(E_f+E_k) - \frac{m^2}{2}}{m^2 - (p_f+k)^2 - i\epsilon} \\ &\cdot \left(\frac{1}{2}\left[p_f^\rho(p_f+k)^\sigma + p_f^\sigma(p_f+k)^\rho\right] - \frac{1}{4}p_{f\gamma}(p_f+k)^\gamma\eta^{\rho\sigma}\right)\end{aligned} \tag{4.17}$$

Diagram B

Next, we evaluate the matrix element contribution from the diagram B, which consists of only interchanging the vertex-indices $x_1 \leftrightarrow x_2$ i.e. the φ-sector remains the same whereas in the h-sector, the Lorentz indices are shifted according to $\alpha,\ \beta \leftrightarrow \mu\ \nu$. It is labeled, according to 3.119 and 3.120, by

$$\begin{aligned}T^m(x_1) &= i\kappa\left(h^{\alpha\beta}(x_1) : \varphi^\dagger_{,\alpha}(x_1)\varphi_{,\beta}(x_1) : -\frac{m^2}{2}h(x_1) : \varphi^\dagger(x_1)\varphi(x_1) :\right) \\ T^m_{\text{ext}}(x_2) &= i\kappa\left(h^{\mu\nu}_{\text{ext}}(x_2) : \varphi^\dagger_{,\mu}(x_2)\varphi_{,\nu}(x_2) : -\frac{m^2}{2}h_{\text{ext}}(x_2) : \varphi^\dagger(x_2)\varphi(x_2) :\right)\end{aligned}$$

Performing the same steps as in the case of diagram A, we find the matrix element contribution from the diagram B

$$\begin{aligned}\mathcal{M}^{\rho\sigma B} &= \frac{i\kappa^2\pi^{-\frac{5}{2}}}{8\sqrt{E_iE_fE_k}} \frac{G_N M m}{\left|\vec{p}_f + \vec{k} - \vec{p}_i\right|^2} \frac{E_f(E_i-E_k) - \frac{m^2}{2}}{m^2 - (p_i-k)^2 - i\epsilon} \\ &\cdot \left(\frac{1}{2}\left[p_i^\rho(p_i-k)^\sigma + p_i^\sigma(p_i-k)^\rho\right] - \frac{1}{4}p_{i\gamma}(p_i-k)^\gamma\eta^{\rho\sigma}\right)\end{aligned} \tag{4.18}$$

Diagram C

For the contribution from the diagram C, the labeling is according to 3.122 and 3.123 given by

$$\begin{aligned}T^h_{\text{ext}}(x_1) &= i\kappa\, h^{\alpha\beta}_{\text{ext}}(x_1)\left(-\frac{1}{4}h_{,\alpha}(x_1)h_{,\beta}(x_1) + \frac{1}{2}h_{\lambda\eta,\alpha}(x_1)h^{\lambda\eta}_{,\beta}(x_1) + h^\lambda_{\beta,\eta}(x_1)h^\eta_{\alpha,\lambda}(x_1)\right) \\ T^m(x_2) &= i\kappa\left(h^{\mu\nu}(x_2) : \varphi^\dagger_{,\mu}(x_2)\varphi_{,\nu}(x_2) : -\frac{m^2}{2}h(x_2) : \varphi^\dagger(x_2)\varphi(x_2) :\right)\end{aligned}$$

which leads to the matrix element contribution from the diagram C

$$
\begin{aligned}
\mathcal{M}^{\rho\sigma C} &= \frac{i\kappa^2\pi^{-\frac{5}{2}}}{8\sqrt{E_iE_fE_k}} \frac{G_N M m}{\left|\vec{p}_f + \vec{k} - \vec{p}_i\right|^2} \\
&\cdot \Bigg(- \frac{E_k(E_f - E_i)}{2(p_f - p_i)^2 - i\epsilon} \left[p_f^\rho p_i^\sigma + p_f^\sigma p_i^\rho - 2p_f^\nu p_{i\nu}\eta^{\rho\sigma}\right] \\
&\quad - \frac{1}{2(p_f - p_i)^2 - i\epsilon} \left[(p_i - p_f)^\sigma \eta^{0\rho} + (p_i - p_f)^\rho \eta^{0\sigma} - \frac{1}{2}(E_i - E_f)\eta^{\rho\sigma}\right] \\
&\quad \cdot \left[E_f p_i^\eta k_\eta + E_i p_f^\eta k_\eta - E_k p_f^\eta p_{i\eta} + m^2 E_k\right]\Bigg)
\end{aligned}
\tag{4.19}
$$

Diagram D

Last, we give the expression for the matrix element contribution from the diagram D. In this case of quartic coupling, T_2 is directly given according to 3.127 by

$$
\begin{aligned}
T_2^h(x_1, x_2) &= \frac{i\kappa^2 m^2}{4} \Big(h_{\text{ext}}^{\alpha\beta}(x_1) h_{\alpha\beta}(x_2) : \varphi^\dagger(x_1)\varphi(x_2) : \\
&\quad - \frac{1}{2} h_{\text{ext}}(x_1) h(x_2) : \varphi^\dagger(x_1)\varphi(x_2) : \Big) \delta(x_1 - x_2)
\end{aligned}
$$

leading to

$$
\mathcal{M}^{\rho\sigma D} = \frac{i\kappa^2 m^2 \pi^{-\frac{5}{2}}}{32\sqrt{E_f E_i E_k}} \frac{G_N M m}{\left|\vec{p}_f + \vec{k} - \vec{p}_i\right|^2} \left(\eta^{\rho 0}\eta^{\sigma 0} - \frac{1}{4}\eta^{\rho\sigma}\right) \tag{4.20}
$$

Total matrix element

The final matrix element becomes according to 4.17, 4.18, 4.19 and 4.20

$$
\begin{aligned}
\mathcal{M}^{\rho\sigma A+B+C+D} &= \frac{i\kappa^2\pi^{-\frac{5}{2}}}{8\sqrt{E_iE_fE_k}} \frac{G_N M m}{\left|\vec{p}_f+\vec{k}-\vec{p}_i\right|^2} \\
&\cdot \left\{ \frac{E_i(E_f+E_k)-\frac{m^2}{2}}{m^2-(p_f+k)^2-i\varepsilon} \left(\frac{1}{2}\left[p_f^\rho (p_f+k)^\sigma + p_f^\sigma (p_f+k)^\rho\right] - \frac{1}{4} p_{f\gamma}(p_f+k)^\gamma \eta^{\rho\sigma}\right)\right. \\
&+ \frac{E_f(E_i-E_k)-\frac{m^2}{2}}{m^2-(p_i-k)^2-i\varepsilon} \left(\frac{1}{2}\left[p_i^\rho (p_i-k)^\sigma + p_i^\sigma (p_i-k)^\rho\right] - \frac{1}{4} p_{i\gamma}(p_i-k)^\gamma \eta^{\rho\sigma}\right) \\
&- \frac{E_k(E_f-E_i)}{2(p_f-p_i)^2-i\varepsilon}\left[p_f^\rho p_i^\sigma + p_f^\sigma p_i^\rho - 2p_f^\nu p_{i\nu}\eta^{\rho\sigma}\right] \\
&- \frac{1}{2(p_f-p_i)^2-i\varepsilon}\left[(p_i-p_f)^\sigma \eta^{0\rho} + (p_i-p_f)^\rho \eta^{0\sigma} - \frac{1}{2}(E_i-E_f)\eta^{\rho\sigma}\right] \\
&\cdot \left[(E_f p_i^\eta + E_i p_f^\eta)k_\eta - E_k(p_f^\nu p_{i\nu} - m^2)\right] \\
&\left. + \frac{m^2}{4}\left(\eta^{\rho 0}\eta^{\sigma 0} - \frac{1}{4}\eta^{\rho\sigma}\right)\right\}
\end{aligned}
\tag{4.21}
$$

In order to compare it with the massive case, we outline the matrix element induced by the explicit polarizations 3.19 in momentum space in the frame $k_\mu = (\omega, 0, 0, \omega)$ parallel to the 3-axis with $k^\mu k_\mu = 0$. Moreover, only the contributions from the diagrams A and B are given since it will become clear soon, that the contributions from the diagrams C and D can be neglected in the considered limit of the Bremsstrahl process. Note that up to this point, the expressions for the matrix element are exact in the dynamical variables. As it was already mentioned in 3.19, the state is normalized to one and the diagonal operators $\tilde{A}_{\mu\nu}$ are transformed back to the $A_{\mu\nu}$ subject to 3.17. Then

$$
\vec{\mathcal{M}}^{A+B} = \frac{i\kappa^2\pi^{-\frac{5}{2}}\, G_N M m}{8\sqrt{E_iE_fE_k}\,\left|\vec{p}_f+\vec{k}-\vec{p}_i\right|^2}
\begin{pmatrix}
\begin{aligned}
\sqrt{2}&\left\{ \tfrac{E_i(E_f+E_k)-\frac{m^2}{2}}{m^2-(p_f+k)^2-i\varepsilon}\left(\tfrac{1}{2}\left[p_f^1(p_f+k)^2 + p_f^2(p_f+k)^1\right]\right)\right. \\
&\left. + \tfrac{E_f(E_i-E_k)-\frac{m^2}{2}}{m^2-(p_i-k)^2-i\varepsilon}\left(\tfrac{1}{2}\left[p_i^1(p_i-k)^2 + p_i^2(p_i-k)^1\right]\right)\right\}
\end{aligned} \\
\\
\begin{aligned}
\tfrac{1}{\sqrt{2}}&\left\{ \tfrac{E_i(E_f+E_k)-\frac{m^2}{2}}{m^2-(p_f+k)^2-i\varepsilon}\left(p_f^2(p_f+k)^2 - p_f^1(p_f+k)^1\right)\right. \\
&\left. + \tfrac{E_f(E_i-E_k)-\frac{m^2}{2}}{m^2-(p_i-k)^2-i\varepsilon}\left(p_i^2(p_i-k)^2 - p_i^1(p_i-k)^1\right)\right\}
\end{aligned}
\end{pmatrix}
\tag{4.22}
$$

4.1.3 QGB-approximation

We will now come to the more sophisticated part of the calculation. The main further steps are squaring the matrix element and integrating out the spherical angles. It turns out to be a walk on the edge between drifting into cumbersome expressions at the full tensorial level or simplifying them too much and coming up with less power of the final result. The criterion of how to impose any limiting assumptions is the requirement to find a bridge between the quantum calculation at tree level and the classical theory [2, 3, 4, 7, 26]. Since the Born approximation of quantum theory overlaps with the classical theory in the regime of high particle energy, low Bremsstrahl energy and small scattering angles, we therefore are interested in the low frequency limit and consider the case where the energy of the Bremsstrahl graviton is small compared to the energy of the φ-particle i.e. $E_k \ll E_{i,f}$. We further consider the case of the φ-particle to be of ultra relativistic energy i.e. $m \ll E_{i,f}$. In addition, we impose the condition that the particle be scattered by the gravitational potential through a small angle i.e. $\Theta' \ll \frac{m}{E_{i,f}}$ which in the classical picture corresponds to a large impact parameter ρ.

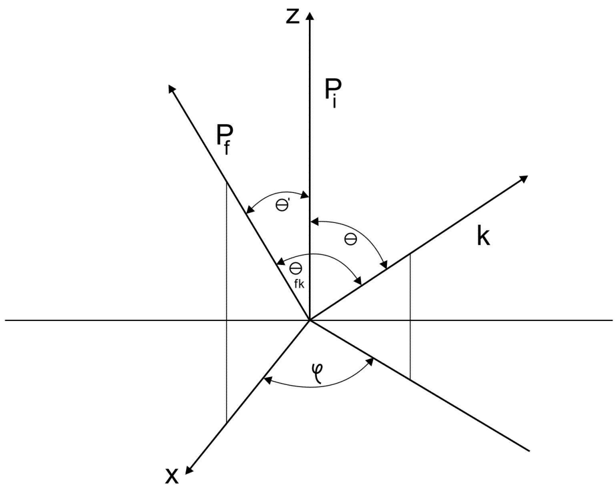
In order to perform estimates on the various expressions, we need to have an estimate on the angles involved. We make use of the analogy from electrodynamics, where in the ultra relativistic case, the Bremsstrahl graviton, seen from the lab frame, emerges mainly in the forward direction under a small angle Θ to the direction of the incident φ-particle i.e. $\Theta \sim \frac{m}{\varepsilon}$ [4, 34]. Thus, we will apply in the following small angle approximations on the various trigonometric functions i.e. $\sin\Theta \sim \Theta$ and $\cos\Theta \sim 1 - \frac{\Theta^2}{2}$.

From now on, we will use the following notation in the passage from the exact expressions to the approximated ones: $E_k = \omega \ll \varepsilon = E_i$, thus neglecting order $O(\frac{\omega}{\varepsilon})$, $E_i = \varepsilon \sim E_f$. By estimating the contributions from the different diagrams, we see when considering the diagrams A and B to be of order $O(1)$, that the contribution from the diagrams C and D are of order $O(\frac{\omega^2}{m^2})$ and $O(\frac{\omega m^2}{\varepsilon^3})$, respectively. With the assumptions mentioned above, the contribution from the process D can be neglected but in order to be able to neglect the contribution from the process C we must impose one further condition $\omega^2 \ll m^2$ which in the considered case is plausible. Under these assumptions, the matrix element becomes

$$
\begin{aligned}
\mathcal{M}^{\rho\sigma} \approx\; & \frac{i\kappa^2\pi^{-\frac{5}{2}}\, G_N M m}{8\varepsilon\sqrt{\omega}\left|\vec{p}_f + \vec{k} - \vec{p}_i\right|^2} \\
& \cdot \Bigg\{ -\frac{\varepsilon^2}{2\omega\varepsilon(1 - v\cos\Theta_{fk})}\left(\frac{1}{2}\Big[p_f^\rho(p_f+k)^\sigma + p_f^\sigma(p_f+k)^\rho\Big] - \frac{m^2}{4}\eta^{\rho\sigma}\right) \\
& + \frac{\varepsilon^2}{2\omega\varepsilon(1 - v\cos\Theta)}\left(\frac{1}{2}\Big[p_i^\rho(p_i-k)^\sigma + p_i^\sigma(p_i-k)^\rho\Big] - \frac{m^2}{4}\eta^{\rho\sigma}\right)\Bigg\}
\end{aligned}
\tag{4.23}
$$

where $\approx$ means order $O(\frac{\omega}{\varepsilon})$, $O(\frac{\omega^2}{m^2})$ and $O(\frac{m^2}{\varepsilon^2})$ are neglected and $v = \frac{|\vec{p}_i|}{E_i} \sim \frac{|\vec{p}_f|}{E_f} \sim \frac{|\vec{p}_i|}{\varepsilon}$ since $E_i \sim E_f$.

We will use a coordinate system in which the direction of the initial momentum $\vec{p}_i$ is along the z-axis and the scattering takes place in the xz-plane i.e. $\vec{p}_f$ lies in the xz-plane. The scattering angle Θ' ($\angle\vec{p}_i\vec{p}_f$) is in the considered limit of order $\Theta' \sim \frac{m^2}{\varepsilon^2} \ll \frac{m}{\varepsilon}$. The spherical angles of $\vec{k}_f$ are Θ ($\angle\vec{p}_i\vec{k}_f$), Θ_{fk} ($\angle\vec{p}_f\vec{k}_f$) and φ, the angle between the scattering plane (xz-plane) and the plane, spanned by the vectors $\vec{p}_i$ and $\vec{k}_f$.



We could have chosen the coordinate system in a different manner, but since we are interested in the explicit form of the scattering angle Θ' and the final polarizations referred to the graviton momentum $\vec{k}$ and not to $\vec{p}_i$ or $\vec{p}_f$, there is only the selection done above. The momentum transfer $\vec{q} = \vec{p}_f + \vec{k} - \vec{p}_i$ to the external potential $\phi_N(\vec{q})$ is calculated with the help of $\vec{n} = \frac{\vec{p}_i}{|\vec{p}_i|}$ and the plane cosine identity $a^2 = b^2 + c^2 - 2bc\cos\alpha$ as

$$\begin{aligned} |\vec{q}|^2 &= |\vec{n} \times \vec{q}|^2 + |\vec{n} \cdot \vec{q}|^2 \\ &= \varepsilon^2 \Theta'^2 + \omega^2 \sin^2\Theta + 2\varepsilon\omega\Theta' \sin\Theta \cos\varphi + \omega^2 \cos^2\Theta \end{aligned} \tag{4.24}$$

which under the above assumptions becomes

$$|\vec{q}|^2 \approx \varepsilon^2 \Theta'^2 + \omega^2 \cos^2\Theta \tag{4.25}$$

In order to pick the leading term out of 4.25, one has to impose $\frac{\omega^2}{\varepsilon^2 \Theta'^2} \ll 1$. Then, the expression for the momentum transfer to the external field becomes

$$|\vec{q}|^2 \approx \varepsilon^2 \Theta'^2 \tag{4.26}$$

which coincides with [26]. Since, in the course of the calculation, we will often refer to these assumptions, it is convenient to define them as the *Quantum-Gravitational-Bremsstrahl approximation,* in short QGB-approximation which are summarized as follows

$$\begin{aligned} \omega &\ll \varepsilon \\ m &\ll \varepsilon \\ \omega^2 &\ll m^2 \\ \omega^2 &\ll \varepsilon^2 \Theta'^2 \\ \Theta' &\ll m/\varepsilon \\ \Theta &\sim m/\varepsilon \\ G_N M m &\ll 1 \end{aligned} \tag{4.27}$$

The last condition is due to the Born approximation method for the classical external field. The angle Θ_{fk} is expressed through the spherical angle Θ of the graviton momentum $\vec{k}$ with the help of the spherical cosine identity as $\cos\Theta_{fk} = \cos\Theta\cos\Theta' + \sin\Theta\sin\Theta'\cos\varphi$. If we now examine the expression for the simplified matrix element 4.23, we recognize the following: if we neglect in both, the numerator and the denominator order $O(\frac{m}{\varepsilon})$, then the whole expression would vanish. We thus keep in principle in the numerator order $O(\frac{m}{\varepsilon})$ and neglect it only in the denominator as $(1-v\cos\Theta_{fk}) \to (1-v\cos\Theta - v\Theta'\sin\Theta\cos\varphi) \to (1-v\cos\Theta)$. For later convenience, we let Θ_{fk} be indicated wherever needed since it is crucial to handle Θ_{fk} properly otherwise the result diverges. The coupling constant κ is given in the Newtonian limit through, for example, the calculation of gravitational scattering of two masses [8] by $\kappa^2 = 32\pi G_N$. Then, we find for the matrix element of the two helicity states ± 2

$$\mathcal{M}_\pm \approx \frac{2iG_N^2 Mm}{\pi^{\frac{3}{2}}\omega\sqrt{\omega}\varepsilon^2\Theta'^2} \cdot \varepsilon_\pm^{\rho\sigma *}(\vec{k})\left(-\frac{p_{f\rho}(p_f+k)_\sigma}{(1-v\cos\Theta_{fk})} + \frac{p_{i\rho}(p_i-k)_\sigma}{(1-v\cos\Theta)}\right) \tag{4.28}$$

where the symmetry in the indices $\rho\sigma$ according to the discussion in section 3.1.3. is used. We now abbreviate as

$$\begin{aligned} Q^A_{\rho\sigma} &= p_{f\rho}(p_f+k)_\sigma \\ Q^B_{\rho\sigma} &= p_{i\rho}(p_i-k)_\sigma \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} A &= \frac{2iG_N^2 Mm}{\pi^{\frac{3}{2}}\omega\sqrt{\omega}\varepsilon^2\Theta'^2} \\ a_1 &= \frac{1}{(1-v\cos\Theta_{fk})} \\ a_2 &= \frac{1}{(1-v\cos\Theta)} \end{aligned} \tag{4.30}$$

Then, in the limit of the QGB-approximation, the matrix element for massless gravitational Bremsstrahlung becomes

$$\mathcal{M}_\pm \approx A\ \varepsilon_\pm^{\rho\sigma *}(\vec{k})\left\{-a_1 Q^A_{\rho\sigma} + a_2 Q^B_{\rho\sigma}\right\} \tag{4.31}$$

4.1.4 Differential cross section and radiated energy

The scattering cross section in the laboratory frame is given, according to [1], by

$$d\sigma(k) = (2\pi)^2\frac{E_i}{|\vec{p}_i|}\ d^3p_f\ d^3k\ \delta(E_f + E_k - E_i)\,|\mathcal{M}(p_i,\ p_f,\ k)|^2 \tag{4.32}$$

The differentials are written as

$$\begin{aligned} d^3p_f &= \vec{p}_f^{\,2}\, d|\vec{p}_f|\, d\Omega_f \;=\; E_f\,|\vec{p}_f|\, dE_f\, d\Omega_f \\ d^3k &= E_k^2\, dE_k\, d\Omega_k \end{aligned}$$

with the spherical angles

$$d\Omega_k = \sin\Theta \; d\Theta \; d\varphi$$

and

$$d\Omega_f = 2\pi \sin\Theta' \; d\Theta'$$

since $d\varphi_f$ can be directly integrated out. Inserting this into the scattering cross section 4.32, we can perform the dE_{p_f} integration

$$\begin{aligned} d\sigma\left(E_k\right) &= (2\pi)^2 \frac{E_i E_k^2 \left|\vec{p}_f\right|}{\left|\vec{p}_i\right|} dE_k \int dE_f \; E_f \; \delta\left(E_f + E_k - E_i\right) \left|\mathcal{M}\left(p_i, \; p_f, \; k\right)\right|^2 \; d\Omega_f \, d\Omega_k \\ &= (2\pi)^2 \frac{E_i\left(E_i - E_k\right) E_k^2 \left|\vec{p}_f\right|}{\left|\vec{p}_i\right|} dE_k \; \left|\mathcal{M}\left(\left(E_i - E_k\right), \; E_k, \; E_i\right)\right|^2 \; d\Omega_f \, d\Omega_k \end{aligned} \tag{4.33}$$

In the case of real Bremsstrahlung, we have to sum over the two polarization degrees of freedom when evaluating the cross section. In the limit of the QGB-approximation subject to 3.33, the scattering cross section becomes

$$d\sigma(\omega) \approx (2\pi)^2 \varepsilon^2 \omega^2 \; d\omega \; d\Omega_f \; d\Omega_k \sum_{\pm} \left|\mathcal{M}_{\pm}\right|^2$$

Integrating over the spherical angles leaves

$$\begin{aligned} \frac{d\sigma\left(\omega\right)}{d\omega} &\approx (2\pi)^2 \varepsilon^2 \omega^2 \, d\Omega_f \int \sum_{\pm} \left|\mathcal{M}_{\pm}\right|^2 \; d\Omega_k \\ &\approx (2\pi)^2 \varepsilon^2 \omega^2 \, 2\pi \, \Theta' \, d\Theta' \, \frac{8 v^2 G_N^4 M^2 m^2}{3\pi^2 \omega^3 \Theta'^2} \\ &\approx \frac{64\pi G_N^4 M^2 m^2 \varepsilon^2}{3\omega\Theta'} \, d\Theta' \end{aligned} \tag{4.34}$$

where $\int \sum_{\pm} \left|\mathcal{M}_{\pm}\right|^2 \; d\Omega_k$ is the squared matrix element according to B.15, integrated over the spherical angles in the limit $v^2 \to 1$. See appendix C, for details of the calculation.

In order to make contact to the classical theory, we impose the relation between the scattering angle Θ' and the impact parameter ρ in the presence of a Newtonian field. In the relativistic regime, the relation is given by [6]

$$\Theta' = \frac{2 G_N M}{\rho \beta^2} (1 + \beta^2)$$

which reduces in the ultra relativistic case to

$$\Theta' \approx \frac{4 G_N M}{\rho}$$

Inserting into 4.34 gives

$$\frac{d\sigma(\omega)}{d\omega} \approx \frac{64\pi G_N^4 M^2 m^2 \varepsilon^2}{3\omega\rho}\ d\rho$$

The energy cross section, defined as

$$\frac{d\mathcal{E}}{d\omega} = \frac{\omega\ d\sigma}{2\pi\rho\ d\rho\ d\omega}$$

becomes

$$\frac{d\mathcal{E}}{d\omega} \approx \frac{32 G_N^4 M^2 m^2 \varepsilon^2}{3\rho^2}$$

and with $\gamma = \frac{\varepsilon}{m}$, we arrive at the final result for the spectral distribution in the massless case

$$\boxed{\frac{d\mathcal{E}}{d\omega} \approx \frac{32 G_N^4 M^2 m^4 \gamma^2}{3\rho^2}} \tag{4.35}$$

which does not depend on the frequency ω. In order to integrate 4.35 to the total energy radiated

$$\mathcal{E}^{\text{tot}} = \int_0^{\omega_{\text{th}}} \frac{d\mathcal{E}}{d\omega} d\omega \tag{4.36}$$

we must find a value for the threshold frequency ω_{th} which, from the point of view of a forward observer, determines the range between the low- and high-frequency regime. It is clear that for a general discussion, we cannot deduce ω_{th} from kinematical arguments. In order to compare with classical results, we insert the classical value ([34]) $\omega_{\text{th}} \approx \gamma/\rho$ in 4.36, and find the total energy radiated by a single encounter

$$\boxed{\mathcal{E}^{\text{tot}} \approx \frac{32 G_N^4 M^2 m^4 \gamma^3}{3\rho^3}} \tag{4.37}$$

Our result 4.35 for the spectral distribution can be compared with the work by *Galtsov* et al. [26]. In order to avoid cumbersome tensorial expressions, they treated a toy-model of scalar quantum gravitational Bremsstrahlung. Their result for the spectral distribution

$$\frac{d\mathcal{E}}{d\omega} \approx \frac{16(f G_N M m)^2 \gamma^2}{3\pi\rho^2}$$

where f is the coupling constant for the Bremsstrahl graviton, agrees with 4.35 up to a factor of two and the coupling constant. The factor of two originates from the degrees of freedom of the Bremsstrahl graviton, a single one in the scalar treatment by *Galtsov* et al. and two in the tensorial calculation. Since *Galtsov* et al. treated a toy model of scalar Bremsstrahlung, they were not allowed to compare their result with a realistic classical calculation. They alternatively performed an independent classical calculation and recovered their result of the spectral distribution in the Born approximation of quantum theory.

So, pushing the arguments to the extreme, already on the level of comparing our result with the work by *Galtsov* et al, we may say that our result agrees with the classical theory up to a numerical factor of order $O(1)$. We will discuss in the next section various classical approaches and the correspondence to our result.

4.1.5 Comparison with the classical theory

According to the discussion in section 4.1.3, the results in the QGB-approximation allow for the comparison with the classical theory. The key point of this comparison lies in the expression for the threshold frequency $\omega_{\rm th}$ which bounds the low frequency regime. In other words, $\omega_{\rm th}$ acts as a cut-off parameter for the low frequency approach. There have been investigations on classical gravitational Bremsstrahlung by many authors in the sixties and seventies. The classical picture mainly consists of two massive bodies (stars, planets, black holes), flying past each other in an unbound orbit, deflecting their trajectories and radiating gravitational waves. In the following, we will discuss only the most important works.

Peters [39] derived the total energy radiated by a single encounter in the limit of a small mass m, moving through the gravitational field induced by a heavy mass M i.e. $m \ll M$ and of small deflection angle i.e. large impact parameter. He studied the perturbation in the Schwarzschild metric of the body with mass M by the small mass m which was allowed to move with any velocity less than c. *Kovacs/Thorne* [34] studied the problem in a more general way, in different velocity regimes and seen from different observer points without restrictions on the masses. They used the "postlinear wave-generation formalism", a weak field formalism, approximating the stress-energy tensor and the gravitational field. Their results can only be compared on the level of the total energy radiated $\mathcal{E}^{\rm tot}$. Both treatments came up with $\mathcal{E}^{\rm tot} \sim \gamma^3/\rho^3$, i.e. the functional dependence of third order in the gamma factor and third order in the inverse impact parameter. This agrees with 4.37.

There have been other approaches[16] to gravitational Bremsstrahlung, namely the "Zero-Frequency-Limit" (ZFL) by *Smarr* [41], "The Method of Virtual Quanta" (MVQ) by *Matzner/Nutku* [37] and the "Colliding Plane Waves" (CPW) approach by *D'Eath* [22]. They all derived the total energy radiated by a single encounter and received the $\mathcal{E}^{\rm tot} \sim \gamma^3/\rho^3$ dependence. Moreover, in the various approaches, the threshold frequency $\omega_{\rm th}$ turned out to be $\omega_{\rm th} \sim \frac{\gamma}{\rho}$ for a forward region observer.

One important point to mention is the range of validity of the results obtained in the Born approximation of quantum theory. Since the external field is expanded in orders of $G_N Mm$, the condition $G_N Mm \ll 1$ must hold in order the first Born approximation to be the main contribution in the expansion of the external field. According to the discussion in [34], there are no astrophysical configurations satisfying the condition $G_N Mm \ll 1$ since reasonable encounter velocities and values of stellar masses (stars, planets, black holes) lie in different regimes. Thus, the experimental data of astrophysical sources of gravitational waves do not provide a range overlapping with the result obtained from the Born approximation. One finds, that the validity of the results in the Born approximation lie in the range of atomic masses.

Nevertheless, we may conclude, that massless gravity as a quantum gauge theory in the sense of causal perturbation theory allows us to recover the theoretical results of the classical theory independently.

[16] See for example the work by *Kovacs/Thorne* [34] for an illustrative overview.

4.2 Massive quantum gravitational Bremsstrahlung

The derivation of the S-matrix element for massive gravitational Bremsstrahlung is done in complete analogy to the massless case. According to section 3.3, the massive, asymptotic spin-2 field, representing the massive graviton, couples to matter in the same way as the massless field does. The same is true for the trilinear coupling and the quartic coupling in the limit of small graviton mass[17]. Therefore, the massive causal distribution $T_2(x_1 - x_2)$ is identical to the one in the massless case and we can directly go over to the derivation of the S-matrix element. The main difference in its calculation to the massless case lies in the definition of the final state ϕ_f. According to section 3.2.2, the massive graviton as the asymptotic spin-2 particle carries six physical degrees of freedom. They consist of admixtures of traceless-, trace- and bosonic vector ghost degrees of freedom. Since, subject to the discussion in section 3.2.3, the polarization sum over this six physical degrees of freedom cannot be written in terms of k_μ, $\tilde{k}_\mu$ and $\eta_{\mu\nu}$ with the right transformation properties as it was the case in the massless framework, we have to derive for each entire massive physical degree of freedom its matrix element. In doing this, we will be able to perform the limit of zero graviton mass and to compare the result with the massless matrix element according to 4.22 when choosing an explicit coordinate system. This is, in analogy to the massless case, $k_\mu = (\omega, 0, 0, k_3)$ parallel to the 3-axis with $k^\mu k_\mu = m_0^2$.

The final state for a fixed, normalized, massive degree of freedom $\hat{a} = 1, \ldots, 6$ can be written for fixed three-momentum $\vec{k}_f$ as

$$h_{f\hat{a}} \; = \; \Omega \, G_{\hat{a}} \left(A_{\rho\sigma}, \, d, \, b_\delta \right) (\vec{k}_f) \; := \; \Omega \, G_{\hat{a}(\rho\sigma\delta)} \, (\vec{k}_f) \qquad (4.38)$$

where $A_{\rho\sigma}$, d, b_ρ are the annihilation operators for the traceless- (spin-2), the trace- (spin-0) and the bosonic vector ghost degree of freedom, respectively. $G_{\hat{a}(\rho\sigma\delta)}$ is a linear function with subscript $(\rho\sigma\delta)$ indicating the indices of the corresponding annihilation operators. According to 3.88, the final state may be written as a six tuple

$$\vec{h}_f \; = \; \Omega \begin{pmatrix} c_1 \, A_{12}^\dagger \\ c_2 \left(\frac{im_1}{k_3} \left(A_{33} - A_{11} \right) + b_3 \right) \\ c_3 \left(\frac{2im_1}{k_3} \, A_{23} + b_2 \right) \\ c_4 \left(\frac{2im_1}{k_3} \, A_{13} + b_1 \right) \\ c_5 \left(\frac{im_1}{2(m_1^2 + k_3^2)} \left(2k_3 \, A_{03} + \sqrt{2m_1^2 + k_3^2} \; d \right) + b_0 \right) \\ c_6 \left(A_{22} - A_{11} \right) \end{pmatrix} (\vec{k}_f) \; := \; \Omega \, \vec{G}_{(\rho\sigma\delta)} \, (\vec{k}_f) \qquad (4.39)$$

where the diagonal annihilation operators $\tilde{A}_{\mu\nu}$ are transformed back to the regular

[17] In the context of gravitational Bremsstrahlung, the mass m_0 of the graviton is small compared to the mass of the scalar field $\varphi(x)$.

$A_{\mu\nu}$ ones. The constants $c_1 \dots c_6$ are derived by normalizing the physical states $\psi_{\hat{1}} \dots \psi_{\hat{6}}$ and are given by 3.82 - 3.87.

Using the analogy from 4.4 and 4.5, the initial and final state in the massive case take the form

$$
\begin{aligned}
\phi_i &= a_1^\dagger(\vec{p}_i)\Omega \\
\vec{\phi}_f &= \Omega\, a_1(\vec{p}_f)\, \vec{G}_{(\rho\sigma\delta)}\,(\vec{k}_f)
\end{aligned} \tag{4.40}
$$

where $p_i^\mu = (E_i,\ \vec{p}_i)$, $p_f^\mu = (E_f,\ \vec{p}_f)$ and $k^\mu = (E_k,\ \vec{k}_f)$ are again the four-momenta of the scalar field $\varphi(x)$ and the Bremsstrahl graviton, respectively. If we now treat again the φ-sector and the h-sector separately, i.e. $\varphi_i = a_1^\dagger(\vec{p}_i)\Omega$, $\varphi_f = \Omega a_1(\vec{p}_f)$, $h_i = \Omega$ and $\vec{h}_f = \Omega\vec{G}_{(\rho\sigma\delta)}(\vec{k}_f)$, we have

$$
\begin{aligned}
\phi_i &= \varphi_i\, h_i \\
\vec{\phi}_f &= \varphi_f\, \vec{h}_f
\end{aligned}
$$

In consequence of 4.39, the S-matrix element becomes a six tuple, too. Again we start with the calculation for the diagram A. According to 4.6, the massive S-matrix element is

$$
\begin{aligned}
\left(\vec{\phi}_f, S_2^A \phi_i\right) &= \frac{1}{2}\int d^4x_1\, d^4x_2\, \left(\varphi_f \vec{h}_f, T_2(x_1,x_2)\,\varphi_i h_i\right) \\
&= \frac{1}{2}\int d^4x_1\, d^4x_2\, \left(\varphi_f, T_2^\varphi(x_1,x_2)\,\varphi_i\right)\left(\vec{h}_f, T_2^h(x_1,x_2)\,h_i\right)
\end{aligned}
$$

where T_2^φ, T_2^h are again the second order causal distributions containing the field operators $\varphi(x)$ and $h^{\mu\nu}(x)$, respectively. The φ-sector takes the same form as in the massless case whereas the h-sector becomes a six tuple, representing the six massive degrees of freedom. We will pursue by representing the field operator $h^{\mu\nu}(x)$ according to 3.55, to be able to compare the resulting expressions with the corresponding massless ones. As already mentioned, the action on the field operators in the φ-sector is identical to the one in the massless case and the action in the h-sector takes the form

$$
\begin{aligned}
\left(\vec{h}_f, h^{\mu\nu}(x_2) + v^\mu(x_2)\right) &= \left(\vec{h}_f, H^{\mu\nu}(x_2) + \frac{1}{4}\eta^{\mu\nu}h(x_2) + v^\mu(x_2)\right) \\
&= (2\pi)^{-\frac{3}{2}}\int \frac{d^3k}{\sqrt{2E_k}}\, e^{ikx_2}\left(\Omega, \vec{G}_{(\rho\sigma\delta)}(\vec{k}_f)\left(\eta^{\mu\mu}\,\eta^{\nu\nu}\,A_{\mu\nu}^\dagger(\vec{k}) - \frac{1}{4}\,\eta^{\mu\nu}\,d^\dagger(\vec{k})\right)\right. \\
&= (2\pi)^{-\frac{3}{2}}\int \frac{d^3k}{\sqrt{2E_k}}\, e^{ikx_2}\Big(\Omega, \underbrace{\eta^{\mu\mu}\,\eta^{\nu\nu}\left[\vec{G}_{(\rho\sigma\delta)}(\vec{k}_f), A_{\mu\nu}^\dagger(\vec{k})\right]}_{:=\vec{F}_1(t^{(\rho\sigma)\mu\nu})\,\delta(\vec{k}-\vec{k}_f)} \\
&\qquad - \underbrace{\frac{1}{4}\,\eta^{\mu\nu}\left[\vec{G}_{(\rho\sigma\delta)}(\vec{k}_f), d^\dagger(\vec{k})\right]}_{:=\vec{F}_2(\eta^{\mu\nu})\,4\delta(\vec{k}-\vec{k}_f)}
\end{aligned} \tag{4.41}
$$

where the contribution from the bosonic vector ghost $v^\mu(x_2)$ vanishes because it does not couple to the scalar field in consequence of 3.129 . The linear "commutation function" $\vec{F}_1(t^{(\rho\sigma)\mu\nu})$ is given by

$$\vec{F}_1(t^{(\rho\sigma)\mu\nu}) \;=\; \begin{pmatrix} c_1\, t^{12\mu\nu} \\ c_2\, \frac{im_1}{k_3}\left(t^{33\mu\nu} - t^{11\mu\nu}\right) \\ c_3\, \frac{2im_1}{k_3}\, t^{23\mu\nu} \\ c_4\, \frac{2im_1}{k_3}\, t^{13\mu\nu} \\ c_5\, \frac{im_1 k_3}{(m_1^2+k_3^2)}\, t^{03\mu\nu} \\ c_6\left(t^{22\mu\nu} - t^{11\mu\nu}\right) \end{pmatrix} \tag{4.42}$$

and $\vec{F}_2(\eta^{\mu\nu})$ by

$$\vec{F}_2(\eta^{\mu\nu}) \;=\; \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ c_5\, \frac{im_1\sqrt{2m_1^2+k_3^2}}{2(m_1^2+k_3^2)}\, \eta^{\mu\nu} \\ 0 \end{pmatrix} \tag{4.43}$$

One recognizes that the contributions in $\vec{F}_1(t^{(\rho\sigma)\mu\nu})$ have already been calculated in the massless case, substituting the specific indices for $(\rho\sigma)$. Therefore we have to calculate additionally the contribution in $\vec{F}_2(\eta^{\mu\nu})$ from the trace (spin-0) annihilation operator d in order to determine the entire matrix element for each massive degrees of freedom. Following the same procedure as in the massless case, we find the massive S-matrix element for the diagram A

$$\begin{aligned}\left(\vec{\phi}_f, S_2^A\phi_i\right) \;=\; & \delta(E_f+E_k-E_i)\,\frac{i\kappa^2\pi^{-\frac{5}{2}}\,G_N M m}{8\sqrt{E_iE_fE_k}\,\left|\vec{p}_f+\vec{k}-\vec{p}_i\right|^2}\cdot\frac{E_i(E_f+E_k)-\frac{m^2}{2}}{m^2-(p_f+k)^2-i\epsilon} \\ & \cdot\left(\vec{F}_1(t^{(\rho\sigma)\mu\nu})\,p_{f\mu}(p_f+k)_\nu - \vec{F}_2(\eta^{\mu\nu})\left[p_{f\mu}(p_f+k)_\nu - \frac{m^2}{2}\,\eta^{\mu\nu}\right]\right)\end{aligned} \tag{4.44}$$

where in the following, we will abbreviate

$$B := \frac{i\kappa^2\pi^{-\frac{5}{2}}\, G_N M m}{8\sqrt{E_i E_f E_k}\left|\vec{p}_f + \vec{k} - \vec{p}_i\right|^2} \tag{4.45}$$

In consequence of 4.44, we may write the massive matrix element as a six-tuple $\vec{\mathcal{M}}$, too. In the considered limit of the QGB-approximation where the diagrams C and D do not contribute to the total matrix element, the individual entries of $\vec{\mathcal{M}}^{A+B}$ from the diagrams A and B, still exact in the dynamical variables, are

$$\begin{aligned}\mathcal{M}_1^{A+B} &= c_1\, B \left\{ \frac{E_i(E_f + E_k) - \frac{m^2}{2}}{m^2 - (p_f + k)^2 - i\varepsilon} \left(\frac{1}{2}\left[p_f^1 (p_f + k)^2 + p_f^2 (p_f + k)^1 \right] \right) \right. \\ &\quad \left. + \frac{E_f(E_i - E_k) - \frac{m^2}{2}}{m^2 - (p_i - k)^2 - i\varepsilon} \left(\frac{1}{2}\left[p_i^1 (p_i - k)^2 + p_i^2 (p_i - k)^1 \right] \right) \right\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_2^{A+B} &= c_2\, B\, \frac{im_1}{k_3} \left\{ \frac{E_i(E_f + E_k) - \frac{m^2}{2}}{m^2 - (p_f + k)^2 - i\varepsilon} \left(p_f^3 (p_f + k)^3 - p_f^1 (p_f + k)^1 \right) \right. \\ &\quad \left. + \frac{E_f(E_i - E_k) - \frac{m^2}{2}}{m^2 - (p_i - k)^2 - i\varepsilon} \left(p_i^3 (p_i - k)^3 - p_i^1 (p_i - k)^1 \right) \right\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_3^{A+B} &= c_3\, B\, \frac{2im_1}{k_3} \left\{ \frac{E_i(E_f + E_k) - \frac{m^2}{2}}{m^2 - (p_f + k)^2 - i\varepsilon} \left(\frac{1}{2}\left[p_f^2 (p_f + k)^3 + p_f^3 (p_f + k)^2 \right] \right) \right. \\ &\quad \left. + \frac{E_f(E_i - E_k) - \frac{m^2}{2}}{m^2 - (p_i - k)^2 - i\varepsilon} \left(\frac{1}{2}\left[p_i^2 (p_i - k)^3 + p_i^3 (p_i - k)^2 \right] \right) \right\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_4^{A+B} &= c_4\, B\, \frac{2im_1}{k_3} \left\{ \frac{E_i(E_f + E_k) - \frac{m^2}{2}}{m^2 - (p_f + k)^2 - i\varepsilon} \left(\frac{1}{2}\left[p_f^1 (p_f + k)^3 + p_f^3 (p_f + k)^1 \right] \right) \right. \\ &\quad \left. + \frac{E_f(E_i - E_k) - \frac{m^2}{2}}{m^2 - (p_i - k)^2 - i\varepsilon} \left(\frac{1}{2}\left[p_i^1 (p_i - k)^3 + p_i^3 (p_i - k)^1 \right] \right) \right\}\end{aligned}$$

and

$$
\begin{aligned}
\mathcal{M}_5^{A+B} &= c_5\, B \left\{ \frac{E_i(E_f+E_k) - \frac{m^2}{2}}{m^2 - (p_f+k)^2 - i\varepsilon} \left(\frac{i m_1 k_3}{m_1^2 + k_3^2} \Big[E_f (p_f+k)^3 + p_f^3 (E_f + E_k) \Big] \right.\right. \\
&\qquad \left. - \frac{i m_1 \sqrt{2 m_1^2 + k_3^2}}{2(m_1^2 + k_3^2)} \Big[p_{f\gamma} (p_f+k)^\gamma - 2m^2 \Big] \right) \\
&+ \frac{E_f(E_i-E_k) - \frac{m^2}{2}}{m^2 - (p_i-k)^2 - i\varepsilon} \left(\frac{i m_1 k_3}{m_1^2 + k_3^2} \Big[E_i (p_i-k)^3 + p_i^3 (E_i - E_k) \Big] \right. \\
&\qquad \left.\left. - \frac{i m_1 \sqrt{2 m_1^2 + k_3^2}}{2(m_1^2 + k_3^2)} \Big[p_{f\gamma} (p_f+k)^\gamma - 2m^2 \Big] \right) \right\}
\end{aligned}
$$

The last entry of the matrix element is

$$
\begin{aligned}
\mathcal{M}_6^{A+B} &= c_6\, B \left\{ \frac{E_i(E_f+E_k) - \frac{m^2}{2}}{m^2 - (p_f+k)^2 - i\varepsilon} \Big(p_f^2 (p_f+k)^2 - p_f^1 (p_f+k)^1 \Big) \right. \\
&\qquad \left. + \frac{E_f(E_i-E_k) - \frac{m^2}{2}}{m^2 - (p_i-k)^2 - i\varepsilon} \Big(p_i^2 (p_i-k)^2 - p_i^1 (p_i-k)^1 \Big) \right\}
\end{aligned}
$$

If we now perform the limit of zero graviton mass in the massive matrix element $\vec{\mathcal{M}}^{A+B}$, one recovers the contribution from the massless case 4.22 and four vanishing entries. This is due to the factor m_0 in front of $\mathcal{M}_2$, $\mathcal{M}_3$, $\mathcal{M}_4$ and $\mathcal{M}_5$ whereas the normalization constants $c_1 \ldots c_6$ go to unity in the limit of zero graviton mass. It is remarkable, that the smooth limit in the physical degrees of freedom according to 3.89 is not spoiled by the derivation of the matrix element for each massive degree of freedom. Therefore, we find the massive matrix element in the limit of zero graviton mass as

$$\lim_{m_1 \searrow 0} \vec{\mathcal{M}}^{A+B} = B \begin{pmatrix} \sqrt{2}\Bigg\{ \frac{E_i(E_f+E_k)-\frac{m^2}{2}}{m^2-(p_f+k)^2-i\varepsilon} \left(\frac{1}{2}\left[p_f^1(p_f+k)^2 + p_f^2(p_f+k)^1 \right] \right) \\ + \frac{E_f(E_i-E_k)-\frac{m^2}{2}}{m^2-(p_i-k)^2-i\varepsilon} \left(\frac{1}{2}\left[p_i^1(p_i-k)^2 + p_i^2(p_i-k)^1 \right] \right) \Bigg\} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}}\Bigg\{ \frac{E_i(E_f+E_k)-\frac{m^2}{2}}{m^2-(p_f+k)^2-i\varepsilon} \left(p_f^2(p_f+k)^2 - p_f^1(p_f+k)^1 \right) \\ + \frac{E_f(E_i-E_k)-\frac{m^2}{2}}{m^2-(p_i-k)^2-i\varepsilon} \left(p_i^2(p_i-k)^2 - p_i^1(p_i-k)^1 \right) \Bigg\} \end{pmatrix} \tag{4.46}$$

Note, that we only compare the massive and the massless matrix element contributions from the diagrams A and B. The contributions from the diagrams C and D would give similar expressions but are not relevant in the considered limit of the Bremsstrahl process.

5 Discussion, conclusion & outlook

We have analyzed the nature of quantum gravitational Bremsstrahlung in the causal formalism of perturbation theory. The underlying concept of gravity as the spin-2 quantum gauge theory, elaborated for the massless and massive case, provided us with the basic principles to fully develop the theory without giving up fundamental principles. We have shown that the six physical degrees of freedom of the massive graviton[18] go over smoothly into the two of the massless graviton whereas four massive bosonic vector ghost degrees of freedom decouple in the limit of zero graviton mass. In calculating the scattering cross section and the total energy radiated, there have arisen no surprises when applying the Born approximation method for the classical external field. The matrix element for gravitational Bremsstrahlung was calculated for each physical degree of freedom in both, the massless and the massive framework. It has been confirmed that they go over smoothly into each other in the limit of zero graviton mass.

As it was mentioned at the end of section 4.1, the results obtained in consequence of the Born condition, lie in the range of atomic masses. On the other hand, it may, in principle, be possible that astrophysical configurations exist which amplify the very tiny effect of gravitational Bremsstrahlung to a macroscopic, observable effect. Possible candidates are double star systems or galactic fluids, plasma- or gas clouds. It must be stressed that due to the smooth limit between the massive and the massless Bremsstrahl

[18] A massive physical state, as an element in $Ker\{Q, Q^\dagger\}$, is an admixture of traceless- (spin-2), trace- (spin-0) and bosonic vector ghost degrees of freedom.

matrix element, there is no discontinuity between the two approaches and therefore it will be extremely hard to rule out the massive theory. This means that massive gravity as a quantum gauge theory is a legitimate candidate to be tested by experimental observations.

By discussing the different bounds on the graviton mass, one distinguishes between estimates from the radiative and non-radiative regime [36, 43, 45, 51]. Bounds from the radiative regime are based on double star systems and inspiralling compact binaries, where frequency-shifts of gravitational waves due to a non-zero graviton mass are expected to be measured. Bounds from the non-radiative regime originate from the modification of the gravitational potential to a Yukawa type or through Kepler's third law. The experimental graviton mass upper bound ranges between $m_g \sim 10^{-19}eV - 10^{-32}eV$. These bounds show that up to now, massive gravity passes all experimental tests of classical gravity.

In recent years [20, 29, 38], the relation between the cosmological constant and the mass of the graviton has been lifted to a new discussion on the subject [50]. Massive gravity as the quantum gauge theory will provide a consistent framework to further elaborate the problem of the cosmological constant where the discrepancies in the classical approach, mentioned at the beginning, may be sailed round. The solution of Einstein's equation including a cosmological term is the de Sitter or anti de Sitter metric. In contrast, we have constructed massive gravity as the quantum gauge theory on Minkowski background. Since, locally, all this metrics behave Minkowskian, the massive spin-2 quantum gauge theory describes the phenomenon of gravity on this small scale where the very concept of graviton is restricted to. In summarizing the features mentioned above, one may guess that the physical consequences originating out of massive gravity are spread over different regimes of nowadays physics.

It is worth mentioning a few more conceptual features. In comparison to massive electrodynamics one may ask what theory, the massless or the massive, would be the fundamental one. From the point of view of the Hilbert space structure induced by the gauge charge Q and the emerging numbers of degrees of freedom, the situation is similar to the electromagnetic case. But when taking the relation between the mass of the graviton and the cosmological constant into account, one may argue that massive gravity as a quantum gauge theory is the fundamental one and that massless gravity with cosmological constant Λ is its classical limit. This opinion is supported by the fact that classical massive gravity cannot be formulated consistently without giving up basic principles.

As a closing statement, we may stress that on the level of quantum gravitational Bremsstrahlung, massless and massive gravity as quantum gauge theories are two nearby frameworks with equally legitimation which goes over smoothly into each other in the limit of zero graviton mass.

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Appendices

A Construction of $T_2(x_1, x_2)$

The construction of the causal distribution T_2 is the same in the massless as in the massive case. We give an example of its construction for the process A. The second order T_2 for the other processes is calculated completely analogous. Recall from section 4.1.1, the causal distribution D_2, given by

$$\begin{aligned} D_2(x_1,x_2) &= R'_2(x_1,x_2) - A'_2(x_1,x_2) \\ &= T_{\text{ext}}(x_1)\,T(x_2) - T(x_2)\,T_{\text{ext}}(x_1) \\ &= [T_{\text{ext}}(x_1), T(x_2)] \sim D_m(x_1 - x_2) \end{aligned} \tag{A.1}$$

where we will use the following labeling, according to 3.119 and 3.120,

$$\begin{aligned} T^m_{\text{ext}}(x_1) &= i\kappa\left(h^{\alpha\beta}_{\text{ext}}(x_1) : \varphi^\dagger_{,\alpha}(x_1)\varphi_{,\beta}(x_1) : -\frac{m^2}{2}h_{\text{ext}}(x_1) : \varphi^\dagger(x_1)\varphi(x_1) :\right) \\ T^m(x_2) &= i\kappa\left(h^{\mu\nu}(x_2) : \varphi^\dagger_{,\mu}(x_2)\varphi_{,\nu}(x_2) : -\frac{m^2}{2}h(x_2) : \varphi^\dagger(x_2)\varphi(x_2) :\right) \end{aligned}$$

then, D_2 is given according to A.1 by

$$\begin{aligned} D_2(x_1,x_2) &= \Big[T_{\text{ext}}(x_1), T(x_2)\Big] \\ &= -\kappa^2\Big\{\Big[h^{\alpha\beta}_{\text{ext}}(x_1) : \varphi^\dagger_{,\alpha}(x_1)\varphi_{,\beta}(x_1) :, h^{\mu\nu}(x_2) : \varphi^\dagger_{,\mu}(x_2)\varphi_{,\nu}(x_2) :\Big] \\ &\quad -\frac{m^2}{2}\Big[h^{\alpha\beta}_{\text{ext}}(x_1) : \varphi^\dagger_{,\alpha}(x_1)\varphi_{,\beta}(x_1) :, h(x_2) : \varphi^\dagger(x_2)\varphi(x_2) :\Big] \\ &\quad -\frac{m^2}{2}\Big[h_{\text{ext}}(x_1) : \varphi^\dagger(x_1)\varphi(x_1) :, h^{\mu\nu}(x_2) : \varphi^\dagger_{,\mu}(x_2)\varphi_{,\nu}(x_2) :\Big] \\ &\quad +\frac{m^4}{4}\Big[h_{\text{ext}}(x_1) : \varphi^\dagger(x_1)\varphi(x_1) :, h(x_2) : \varphi^\dagger(x_2)\varphi(x_2) :\Big]\Big\} \end{aligned} \tag{A.2}$$

Now we have to choose the relevant contractions between the expressions of normally ordered field operators. Since in the Bremsstrahl process, there is only the massive propagator from the φ-particle, we can leave the contractions of the h-fields aside. In general, we have

$$\overline{\chi\ \chi}^\dagger := \Big[\chi^{(-)}, \chi^{(+)}\Big]_\pm$$

where the $+$ *sign* stands for fermions and the $-$ *sign* for bosons. The contraction over a commutator is given by

$$\Big[\overline{\varphi(x_1), \varphi}^\dagger(x_2)\Big] = \Big[\overline{\varphi^\dagger(x_1), \varphi}(x_2)\Big] = -iD_m(x_1 - x_2)$$

where D_m is the Jordan-Pauli distribution for mass m, introduced in section 2. Thus we find, contracting the expressions containing the scalar field,

$$\begin{aligned} D_2(x_1,x_2) \;=\; -\kappa^2 \Big\{ & \Big[h^{\alpha\beta}_{\rm ext}(x_1) : \varphi^{\dagger}_{,\alpha}(x_1)\varphi_{,\beta}(x_1) :, h^{\mu\nu}(x_2) : \varphi^{\dagger}_{,\mu}(x_2)\varphi_{,\nu}(x_2) : \Big] \\ & - \frac{m^2}{2}\Big[h^{\alpha\beta}_{\rm ext}(x_1) : \varphi^{\dagger}_{,\alpha}(x_1)\varphi_{,\beta}(x_1) :, h(x_2) : \varphi^{\dagger}(x_2)\varphi(x_2) : \Big] \\ & - \frac{m^2}{2}\Big[h_{\rm ext}(x_1) : \varphi^{\dagger}(x_1)\varphi(x_1) :, h^{\mu\nu}(x_2) : \varphi^{\dagger}_{,\mu}(x_2)\varphi_{,\nu}(x_2) : \Big] \\ & + \frac{m^4}{4}\Big[h_{\rm ext}(x_1) : \varphi^{\dagger}(x_1)\varphi(x_1) :, h(x_2) : \varphi^{\dagger}(x_2)\varphi(x_2) : \Big] \Big\} \end{aligned} \tag{A.3}$$

Since we deal with bosons, there is no minus sign appearing by permutating the fields. The field operators $h^{\alpha\beta}(x_i)$ become normally ordered, since its contraction vanishes for the process in question i.e.

$$h^{\alpha\beta}(x_1)h^{\mu\nu}(x_2) \;=\; : h^{\alpha\beta}(x_1)h^{\mu\nu}(x_2) : + \underbrace{h^{\alpha\beta}(x_1)h^{\mu\nu}(x_2)}_{\text{not considered}}$$

Thus we find

$$\begin{aligned} D_2(x_1,x_2) \;=\; -\kappa^2 \Big\{ & - h^{\alpha\beta}_{\rm ext}(x_1)h^{\mu\nu}(x_2)\Big(: \varphi^{\dagger}_{,\alpha}(x_1)\varphi_{,\nu}(x_2) : \partial^{x_1}_{\beta}\partial^{x_2}_{\mu} + : \varphi_{,\beta}(x_1)\varphi^{\dagger}_{,\mu}(x_2) : \partial^{x_1}_{\alpha}\partial^{x_2}_{\nu}\Big) \\ & + \frac{m^2}{2} h^{\alpha\beta}_{\rm ext}(x_1)h(x_2)\Big(: \varphi^{\dagger}_{,\alpha}(x_1)\varphi(x_2) : \partial^{x_1}_{\beta} + : \varphi_{,\beta}(x_1)\varphi^{\dagger}(x_2) : \partial^{x_1}_{\alpha}\Big) \\ & + \frac{m^2}{2} h_{\rm ext}(x_1)h^{\mu\nu}(x_2)\Big(: \varphi^{\dagger}(x_1)\varphi_{,\nu}(x_2) : \partial^{x_2}_{\mu} + : \varphi(x_1)\varphi^{\dagger}_{,\mu}(x_2) : \partial^{x_2}_{\nu}\Big) \\ & - \frac{m^4}{4} h_{\rm ext}(x_1)h(x_2)\Big(: \varphi^{\dagger}(x_1)\varphi(x_2) : + : \varphi(x_1)\varphi^{\dagger}(x_2) : \Big) \Big\}\, iD_m(x_1-x_2) \end{aligned} \tag{A.4}$$

At tree-level, the splitting of the causal distribution $D_m(x_1,x_2)$ is always trivial, i.e.

$$iD_m(x_1-x_2) = iD^{\rm ret}_m(x_1-x_2) - iD^{\rm av}_m(x_1-x_2)$$

Thus we find

$$R_2(x_1,x_2) \sim iD^{\rm ret}_m(x_1-x_2)$$

Next, we have to determine R'_2

$$
\begin{aligned}
R_2'(x_1,x_2) &= -T(x_2)T_{\text{ext}}(x_1) \\
&= \kappa^2 \Bigg\{ h^{\mu\nu}(x_2)h^{\alpha\beta}_{\text{ext}}(x_1)\Big(:\varphi^\dagger_{,\mu}(x_2)\varphi_{,\beta}(x_1): \partial^{x_2}_\nu \partial^{x_1}_\alpha + :\varphi_{,\nu}(x_2)\varphi^\dagger_{,\alpha}(x_1): \partial^{x_2}_\mu \partial^{x_1}_\beta \Big) \\
&- \frac{m^2}{2} h^{\mu\nu}(x_2)h_{\text{ext}}(x_1)\Big(:\varphi^\dagger_{,\mu}(x_2)\varphi(x_1): \partial^{x_2}_\nu + :\varphi_{,\nu}(x_2)\varphi^\dagger(x_1): \partial^{x_2}_\mu \Big) \\
&- \frac{m^2}{2} h(x_2)h^{\alpha\beta}_{\text{ext}}(x_1)\Big(:\varphi^\dagger(x_2)\varphi_{,\beta}(x_1): \partial^{x_1}_\alpha + :\varphi(x_2)\varphi^\dagger_{,\alpha}(x_1): \partial^{x_1}_\beta \Big) \\
&+ \frac{m^4}{4} h(x_2)h_{\text{ext}}(x_1)\Big(:\varphi^\dagger(x_2)\varphi(x_1): + :\varphi(x_2)\varphi^\dagger(x_1): \Big) \Bigg\} iD^{(-)}_m(x_1-x_2)
\end{aligned}
\tag{A.5}
$$

Then, the second order distribution is given by

$$
T_2(x_1,x_2) = R_2 - R_2' \sim \Big(iD^{\text{ret}}_m(x_1-x_2) - iD^{(-)}_m(x_1-x_2) \Big) = iD^F_m(x_1-x_2)
$$

where the massive Feynman propagator is given according to 4.12. Then we find

$$
\begin{aligned}
T_2(x_1,x_2) &= R_2 - R_2' \\
&= \kappa^2 \Bigg\{ h^{\alpha\beta}_{\text{ext}}(x_1)h^{\mu\nu}(x_2)\Big(:\varphi^\dagger_{,\alpha}(x_1)\varphi_{,\nu}(x_2): \partial^{x_1}_\beta \partial^{x_2}_\mu + :\varphi_{,\beta}(x_1)\varphi^\dagger_{,\mu}(x_2): \partial^{x_1}_\alpha \partial^{x_2}_\nu \Big) \\
&- \frac{m^2}{2} h^{\alpha\beta}_{\text{ext}}(x_1)h(x_2)\Big(:\varphi^\dagger_{,\alpha}(x_1)\varphi(x_2): \partial^{x_1}_\beta + :\varphi_{,\beta}(x_1)\varphi^\dagger(x_2): \partial^{x_1}_\alpha \Big) \\
&- \frac{m^2}{2} h_{\text{ext}}(x_1)h^{\mu\nu}(x_2)\Big(:\varphi^\dagger(x_1)\varphi_{,\nu}(x_2): \partial^{x_2}_\mu + :\varphi(x_1)\varphi^\dagger_{,\mu}(x_2): \partial^{x_2}_\nu \Big) \\
&+ \frac{m^4}{4} h_{\text{ext}}(x_1)h(x_2)\Big(:\varphi^\dagger(x_1)\varphi(x_2): + :\varphi(x_1)\varphi^\dagger(x_2): \Big) \Bigg\} iD^F_m(x_1-x_2)
\end{aligned}
\tag{A.6}
$$

Note, that the dynamics do not mix particle species.

B Squaring & spherical integral

In this part of the appendix, the steps of squaring the matrix element and integrating over the spherical angles $d\Omega_k$ are performed. For the squaring procedure $\sum_\pm |\mathcal{M}_\pm|^2$, we refer to the discussion in section 3.1.3. In the limit of the QGB-approximation, the matrix element is given, according to 4.31, by

$$
\mathcal{M}_\pm \approx A\; \varepsilon^{\rho\sigma *}_\pm(\vec{k}) \left\{ -a_1 Q^A_{\rho\sigma} + a_2 Q^B_{\rho\sigma} \right\}
$$

where the $Q's$ are given by 4.29

$$
\begin{aligned}
Q^A_{\rho\sigma} &= p_{f\rho}(p_f + k)_\sigma \\
Q^B_{\rho\sigma} &= p_{i\rho}(p_i - k)_\sigma
\end{aligned}
$$

and the A and $a's$ by 4.30

$$
\begin{aligned}
A &= \frac{2iG_N^2 Mm}{\pi^{\frac{3}{2}}\omega\sqrt{\omega}\varepsilon^2\Theta'^2} \\
a_1 &= \frac{1}{(1 - v\cos\Theta_{fk})} \\
a_2 &= \frac{1}{(1 - v\cos\Theta)}
\end{aligned}
$$

Recall from section 3.1.3, 3.33, squaring the matrix element consists of acting with the polarization sum function $P_{(0)}^{\mu\nu\mu'\nu'}$ for the two physical polarization states $\pm$ on the square of the matrix element as

$$
\begin{aligned}
\sum_{\pm} \left|\mathcal{M}_{\pm}\right|^2 &= \sum_{\pm} \left|\varepsilon_{\pm}^{\rho\sigma *}(\vec{k})\,\mathcal{M}_{\rho\sigma}\right|^2 \\
&= \sum_{\pm} \left|\varepsilon_{\pm}^{\rho\sigma *}(\vec{k})\right|^2 \left|\mathcal{M}_{\rho\sigma}\right|^2 \\
&= \sum_{\pm} \varepsilon_{\pm}^{\rho\sigma}(\vec{k})\,\varepsilon_{\pm}^{\rho'\sigma' *}(\vec{k})\,\mathcal{M}_{\rho\sigma}\,\mathcal{M}_{\rho'\sigma'}^{*} \\
&= P_{(0)}^{\rho\sigma\rho'\sigma'}\,\mathcal{M}_{\rho\sigma}\,\mathcal{M}_{\rho'\sigma'}^{*}
\end{aligned}
$$

thus

$$
\begin{aligned}
\sum_{\pm} |\mathcal{M}_{\pm}|^2 &= P_{(0)}^{\rho\sigma\rho'\sigma'}\mathcal{M}_{\rho\sigma}\mathcal{M}_{\rho'\sigma'}^{*} \\
&\approx AA^*\,P_{(0)}^{\rho\sigma\rho'\sigma'}\Big\{-a_1 Q_{\rho\sigma}^{A} + a_2 Q_{\rho\sigma}^{B}\Big\}\left\{-a_1 Q_{\rho'\sigma'}^{A} + a_2 Q_{\rho'\sigma'}^{B}\right\} \\
&\approx AA^*\left\{a_1^2\,\underbrace{P_{(0)}^{\rho\sigma\rho'\sigma'}\,Q_{\rho\sigma}^{A}Q_{\rho'\sigma'}^{A}}_{A_1} - a_1 a_2\,\underbrace{P_{(0)}^{\rho\sigma\rho'\sigma'}\,Q_{\rho\sigma}^{A}Q_{\rho'\sigma'}^{B}}_{A_{23}}\right. \\
&\qquad \left. - a_1 a_2\,\underbrace{P_{(0)}^{\rho\sigma\rho'\sigma'}\,Q_{\rho'\sigma'}^{A}Q_{\rho\sigma}^{B}}_{A_{32}} + a_2^2\,\underbrace{P_{(0)}^{\rho\sigma\rho'\sigma'}\,Q_{\rho'\sigma'}^{B}Q_{\rho\sigma}^{B}}_{A_4}\right\} \\
&\approx AA^*\left\{a_1^2 A_1 - 2a_1 a_2 A_{2,3} + a_2^2 A_4\right\}
\end{aligned}
\tag{B.1}
$$

where we label the contributions from the terms A_{23} and A_{32} by $A_{23} = A_{32} := A_{2,3}$ since they are identical. The massless polarization sum function $P_{(0)}^{\mu\nu\mu'\nu'}$ is given by 3.43

$$
P_{(0)}^{\rho\sigma\rho'\sigma'} = \frac{1}{2}\Big\{\Pi^{\rho\rho'}\Pi^{\sigma\sigma'} + \Pi^{\rho\sigma'}\Pi^{\sigma\rho'} - \Pi^{\rho\sigma}\Pi^{\rho'\sigma'}\Big\}
$$

where the "Π-tensor" is given according to 3.46 by

$$\Pi^{\rho\sigma} = \frac{\tilde{k}^\rho k^\sigma + k^\sigma \tilde{k}^\rho}{k\tilde{k}} - \eta^{\rho\sigma}$$

In order to simplify the expressions, we make use of the following notation

$$\begin{aligned} \varepsilon_i^+ &= (1 + v\cos\Theta) \\ \varepsilon_i^- &= (1 - v\cos\Theta) \\ \varepsilon_f^+ &= (1 + v\cos\Theta_{fk}) \\ \varepsilon_f^- &= (1 - v\cos\Theta_{fk}) \end{aligned} \tag{B.2}$$

and

$$\begin{aligned} k\tilde{k} &= 2\omega^2 \\ p_{i,f}\tilde{k} &= \varepsilon\omega\varepsilon_{i,f}^+ \\ p_{i,f}k &= \varepsilon\omega\varepsilon_{i,f}^- \end{aligned} \tag{B.3}$$

Contribution A_1

The contribution in B.1 from the term A_1 is

$$\begin{aligned} a_1^2\, P_{(0)}^{\rho\sigma\rho'\sigma'} Q_{\rho\sigma}^A Q_{\rho'\sigma'}^A &= a_1^2\, P_{(0)}^{\rho\sigma\rho'\sigma'} \Big(p_{f\rho}(p_f+k)_\sigma\Big)\Big(p_{f\rho'}(p_f+k)_{\sigma'}\Big) \\ &\approx \frac{a_1^2}{2}\Big[\varepsilon_f^4\varepsilon_f^{+2}\varepsilon_f^{-2} - 2m^2\varepsilon_f^2\varepsilon_f^+\varepsilon_f^- + m^4\Big] \end{aligned} \tag{B.4}$$

where we have neglected only order $O\left(\frac{\omega}{\varepsilon}\right)$, all orders of $\frac{m}{\varepsilon}$ were kept. In the present context, it is clear that ε^{-2} means $(\varepsilon^-)^2$.

Contribution $A_{2,3}$

The contribution in B.1 from the term A_{23} and A_{32} is

$$\begin{aligned} -a_1a_2\, P_{(0)}^{\rho\sigma\rho'\sigma'} Q_{\rho\sigma}^A Q_{\rho'\sigma'}^B &= -\,a_1a_2\, P_{(0)}^{\rho\sigma\rho'\sigma'} \Big(p_{f\rho}(p_f+k)_\sigma\Big)\Big(p_{i\rho'}(p_i-k)_{\sigma'}\Big) \\ &\approx -\,\frac{a_1a_2}{2}\Bigg[\frac{\varepsilon_i^2\varepsilon_f^2}{2}\Big[\varepsilon_i^{-2}\varepsilon_f^{+2} + \varepsilon_i^{+2}\varepsilon_f^{-2}\Big] - 2m^2\varepsilon_f\varepsilon_i\Big(\varepsilon_i^-\varepsilon_f^+ + \varepsilon_i^+\varepsilon_f^-\Big) \\ &\qquad + m^2\varepsilon_f\varepsilon_i\Big(\varepsilon_f^-\varepsilon_f^+ + \varepsilon_i^+\varepsilon_i^-\Big) + m^4\Bigg] \end{aligned} \tag{B.5}$$

where again order $O\left(\frac{\omega}{\varepsilon}\right)$ is neglected while $O\left(\frac{m^2}{\varepsilon^2}\right) < 1$ is taken to be small.

Contribution A_4

The contribution in B.1 from the term A_4 is

$$\begin{aligned} a_2^2\, P_{(0)}^{\rho\sigma\rho'\sigma'} Q^B_{\rho'\sigma'} Q^B_{\rho\sigma} &= a_2^2\, P_{(0)}^{\rho\sigma\rho'\sigma'} \Big(p_{i\rho'}(p_i-k)_{\sigma'}\Big)\Big(p_{i\rho}(p_i-k)_{\sigma}\Big) \\ &\approx \frac{a_2^2}{2}\Big[\varepsilon_i^4\varepsilon_i^{-2}\varepsilon_i^{+2} - 2m^2\varepsilon_i^2\varepsilon_i^{+}\varepsilon_i^{-} + m^4\Big] \end{aligned} \tag{B.6}$$

where $\approx$ means order $O\left(\frac{\omega}{\varepsilon}\right)$ is neglected. In the following, we set $\varepsilon_i = \varepsilon_f = \varepsilon$, apply the QGB-approximation and make use of $(1-v\cos\Theta_{fk}) \to (1-v\cos\Theta - v\Theta'\sin\Theta\cos\varphi)$ i.e. keeping order $O(\frac{m}{\varepsilon})$. Then the squared matrix element according to B.1 is

$$\begin{aligned} \sum_{\pm}|\mathcal{M}_\pm|^2 &= P_{(0)}^{\rho\sigma\rho'\sigma'}\mathcal{M}_{\rho\sigma}\mathcal{M}^*_{\rho'\sigma'} \\ &\approx \frac{4G_N^4M^2m^2}{\pi^3\omega^3\varepsilon^4\Theta'^4}\Bigg\{a_1^2A_1 - 2a_1a_2A_{2,3} + a_2^2A_4\Bigg\} \\ &\approx \frac{4G_N^4M^2m^2}{\pi^3\omega^3\varepsilon^4\Theta'^4}\Bigg\{\frac{A_1}{(1-v\cos\Theta - v\,\Theta'\sin\Theta\cos\varphi)^2} \\ &\quad - \frac{2A_{2,3}}{(1-v\cos\Theta)(1-v\cos\Theta - v\,\Theta'\sin\Theta\cos\varphi)} + \frac{A_4}{(1-v\cos\Theta)^2}\Bigg\} \\ &\approx \frac{4G_N^4M^2m^2}{\pi^3\omega^3\varepsilon^4\Theta'^4}\Bigg\{\underbrace{\frac{(1-v\cos\Theta)^2A_1}{(1-v\cos\Theta)^2(1-v\cos\Theta - v\,\Theta'\sin\Theta\cos\varphi)^2}}_{I_{A_1}} \\ &\quad - 2\underbrace{\frac{(1-v\cos\Theta)(1-v\cos\Theta - v\,\Theta'\sin\Theta\cos\varphi)A_{2,3}}{(1-v\cos\Theta - v\,\Theta'\sin\Theta\cos\varphi)^2}}_{I_{A_{2,3}}} \\ &\quad + \underbrace{\frac{(1-v\cos\Theta - v\,\Theta'\sin\Theta\cos\varphi)^2A_4}{(1-v\cos\Theta - v\,\Theta'\sin\Theta\cos\varphi)^2}}_{I_{A_4}}\Bigg\} \\ &\approx \frac{4G_N^4M^2m^2}{\pi^3\omega^3\varepsilon^4\Theta'^4}\Bigg\{I_{A_1} - 2I_{A_{2,3}} + I_{A_4}\Bigg\} \end{aligned} \tag{B.7}$$

The terms containing trigonometric functions are treated as follows: Since the expression for the squared matrix element B.7 sum up to zero, if we neglect order $O(\frac{m}{\varepsilon})$ i.e. by assigning $\cos\Theta_{fk} \sim \cos\Theta$, we basically keep order $O(\frac{m}{\varepsilon})$ in the numerator and neglect it in the denominator. But in order to have control over the various expressions, we will use the following notation

$$
\begin{aligned}
t &:= 1 + v\cos\Theta \\
s &:= 1 - v\cos\Theta \\
c &:= v\,\Theta'\sin\Theta\cos\varphi
\end{aligned}
\tag{B.8}
$$

i.e. with the spherical cosine, we have $(1-v\cos\Theta_{fk}) = (1-v\cos\Theta - v\Theta'\sin\Theta\cos\varphi) = (s-c)$, then we find for the squared matrix element according to B.7

$$
\sum_{\pm} |\mathcal{M}_{\pm}|^2 \;\approx\; \frac{4G_N^4 M^2 m^2}{\pi^3\omega^3\varepsilon^4\Theta'^4}\left\{\frac{s^2\,A_1 - 2s(s-c)\,A_{2,3} + (s-c)^2\,A_4}{s^2(s-c)^2}\right\} \tag{B.9}
$$

Since we are facing the integration over the spherical angles $d\Omega_k = \sin\Theta d\Theta d\varphi$, we will indicate it from now on.

Contribution I_{A_1}

According to B.4 and with the help of B.2, the contribution in B.7 from the term A_1, using B.8 is

$$
\begin{aligned}
I_{A_1}\,d\Omega_k &= \frac{s^2\,A_1}{s^2(s-c)^2}\,d\Omega_k \\
&= \frac{A_1}{(s-c)^2}\,d\Omega_k \\
&= \frac{\frac{1}{2}\left[\varepsilon_f^4\varepsilon_f^{+2}\varepsilon_f^{-2} - 2m^2\varepsilon_f^2\varepsilon_f^{+}\varepsilon_f^{-} + m^4\right]}{(s-c)^2}\,d\Omega_k \\
&= \frac{\frac{1}{2}\left[\varepsilon^4(1-v^2\cos^2\Theta_{fk})^2 - 2m^2\varepsilon^2(1-v^2\cos^2\Theta_{fk}) + m^4\right]}{(s-c)^2}\,d\Omega_k \\
&= \frac{1}{2}\left\{\varepsilon^4(t+c)^2 - \frac{2m^2\varepsilon^2(t+c)}{(s-c)} + \frac{m^4}{(s-c)^2}\right\}d\Omega_k
\end{aligned}
\tag{B.10}
$$

Contribution $I_{A_{2,3}}$

According to B.5 and with the help of B.2, the contribution in B.7 from the term $A_{2,3}$, using B.8 is

$$
\begin{aligned}
I_{A_{2,3}}\, d\Omega_k &= \frac{s(s-c)\, A_{2,3}}{s^2(s-c)^2}\, d\Omega_k \\
&= \frac{A_{2,3}}{s(s-c)}\, d\Omega_k \\
&= \frac{1}{2}\left\{ \frac{\frac{\varepsilon_i^2\varepsilon_f^2}{2}\left[\varepsilon_i^{-2}\varepsilon_f^{+2} + \varepsilon_i^{+2}\varepsilon_f^{-2}\right] - 2m^2\varepsilon_f\varepsilon_i\left(\varepsilon_i^-\varepsilon_f^+ + \varepsilon_i^+\varepsilon_f^-\right)}{s(s-c)} \right. \\
&\quad \left. + \frac{m^2\varepsilon_f\varepsilon_i\left(\varepsilon_f^-\varepsilon_f^+ + \varepsilon_i^+\varepsilon_i^-\right) + m^4}{s(s-c)} \right\} d\Omega_k \\
&= \frac{1}{2}\left\{ \frac{\frac{\varepsilon^4}{2}(1 - v\cos\Theta)^2(1 + v\cos\Theta_{fk})^2}{s(s-c)} \right. \\
&\qquad + \frac{\frac{\varepsilon^4}{2}(1 + v\cos\Theta)^2(1 - v\cos\Theta_{fk})^2}{s(s-c)} \\
&\qquad - \frac{2m^2\varepsilon^2\left[(1 - v\cos\Theta)(1 + v\cos\Theta_{fk}) + (1 + v\cos\Theta)(1 - v\cos\Theta_{fk})\right]}{s(s-c)} \\
&\qquad + \frac{m^2\varepsilon^2\left[(1 - v\cos\Theta_{fk})(1 + v\cos\Theta_{fk}) + (1 + v\cos\Theta)(1 - v\cos\Theta)\right]}{s(s-c)} \\
&\qquad \left. + \frac{m^4}{s(s-c)} \right\} d\Omega_k \\
&= \frac{1}{2}\left\{ \frac{\varepsilon^4}{2}\left[\frac{s(t+c)^2}{(s-c)} + \frac{t^2(s-c)}{s}\right] - 2m^2\varepsilon^2\left[\frac{(t+c)}{(s-c)} + \frac{t}{s}\right] \right. \\
&\qquad \left. + m^2\varepsilon^2\left[\frac{(t+c)}{s} + \frac{t}{(s-c)}\right] + \frac{m^4}{s(s-c)} \right\} d\Omega_k
\end{aligned}
\tag{B.11}
$$

Contribution I_{A_4}

According to B.6 and with the help of B.2, the contribution in B.7 from the term A_4, using B.8 is

$$
\begin{aligned}
I_{A_4}\, d\Omega_k &= \frac{(s-c)^2\, A_4}{s^2(s-c)^2}\, d\Omega_k \\
&= \frac{A_4}{s^2}\, d\Omega_k \\
&= \frac{\frac{1}{2}\Big[\varepsilon_i^4 \varepsilon_i^{+2} \varepsilon_i^{-2} - 2m^2 \varepsilon_i^2 \varepsilon_i^{+} \varepsilon_i^{-} + m^4\Big]}{s^2}\, d\Omega_k \\
&= \frac{\frac{1}{2}\Big[\varepsilon^4(1-v^2\cos^2\Theta)^2 - 2m^2\varepsilon^2(1-v^2\cos^2\Theta) + m^4\Big]}{s^2}\, d\Omega_k \\
&= \frac{1}{2}\left\{\varepsilon^4 t^2 - \frac{2m^2\varepsilon^2 t}{s} + \frac{m^4}{s^2}\right\} d\Omega_k
\end{aligned}
\tag{B.12}
$$

Now, all contributions according to B.10, B.11 and B.12 together are

$$
\begin{aligned}
\left\{I_{A_1} - 2I_{A_{2,3}} + I_{A_4}\right\} d\Omega_k &= \left\{\frac{s^2\, A_1 - 2s(s-c)\, A_{2,3} + (s-c)^2\, A_4}{s^2(s-c)^2}\right\} d\Omega_k \\
&= \frac{1}{2}\Bigg\{\varepsilon^4(t+c)^2 - 2m^2\varepsilon^2\frac{(t+c)}{(s-c)} + \frac{m^4}{(s-c)^2} \\
&\quad - 2\Big[\frac{\varepsilon^4}{2}\frac{s(t+c)^2}{(s-c)} + \frac{\varepsilon^4}{2}\frac{t^2(s-c)}{s} - 2m^2\varepsilon^2\frac{(t+c)}{(s-c)} \\
&\quad - 2m^2\varepsilon^2\frac{t}{s} + m^2\varepsilon^2\frac{(t+c)}{s} + m^2\varepsilon^2\frac{t}{(s-c)} + \frac{m^4}{s(s-c)}\Big] \\
&\quad + \varepsilon^4 t^2 - 2m^2\varepsilon^2\frac{t}{s} + \frac{m^4}{s^2}\Bigg\} d\Omega_k
\end{aligned}
\tag{B.13}
$$

As a check, if we neglect $c = v\Theta' \sin\Theta\cos\varphi$ of order $O(\frac{m}{\varepsilon})$ in B.13, the entire expression vanishes as it should. This implies, that the surviving expression is of order Θ'^2, because the terms, linear in $\cos\varphi$ would vanish by the integration $[0, 2\pi]$. Thus, the leading term in B.13, integrated over the spherical angles, is

$$
\begin{aligned}
\int \left\{I_{A_1} - 2I_{A_{2,3}} + I_{A_4}\right\} d\Omega_k &\approx \frac{\varepsilon^4 v^2 \Theta'^2}{2}\int_0^{\pi}\int_0^{2\pi} \sin^2\Theta\cos^2\varphi\,\sin\Theta\, d\Theta\, d\varphi \\
&\approx \frac{2\pi\varepsilon^4 v^2\Theta'^2}{3}
\end{aligned}
\tag{B.14}
$$

For the squared and integrated matrix element $\mathcal{M}_{\pm}$, in the limit of the *QGB*-approximation, according to B.7 and B.14 we find

$$
\begin{aligned}
\int \sum_{\pm} |\mathcal{M}_{\pm}|^2\, d\Omega_k &\approx \frac{4G_N^4 M^2 m^2}{\pi^3\omega^3\varepsilon^4\Theta'^4}\int \left\{I_{A_1} - 2I_{A_{2,3}} + I_{A_4}\right\} d\Omega_k \\
&\approx \frac{4G_N^4 M^2 m^2}{\pi^3\omega^3\varepsilon^4\Theta'^4}\left\{\frac{2\pi\varepsilon^4 v^2\Theta'^2}{3}\right\} \\
&\approx \frac{8v^2 G_N^4 M^2 m^2}{3\pi^2\omega^3\Theta'^2}
\end{aligned}
\tag{B.15}
$$

This expression will be inserted into the differential cross section 4.34.

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