Qualitative Features of Solutions of KdV and mKdV on the Circle

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Zusammenfassung

In dieser Arbeit beweisen wir neue qualitative Eigenschaften von Lösungen der Korteweg-de Vries Gleichung (KdV) auf dem Kreis. Das Hauptresultat besagt, dass die Fourier Koeffizienten einer Lösung im Sobolev Raum H^N, $N \geq 0$, einer WKB-Entwicklung erster Ordung mit stark oszillierenden Phasenfaktoren in Abhänigkeit der KdV Frequenzen genügen.

Abstract

In this thesis we prove new qualitative features of solutions of KdV on the circle. The first result says that the Fourier coefficients of a solution of KdV in Sobolev space H^N, $N \geq 0$, admit a WKB type expansion up to first order with strongly oscillating phase factors defined in terms of the KdV frequencies. The second result provides estimates for the approximation of such a solution by trigonometric polynomials of sufficiently large degree. With the help of the Miura map we prove analogous results for the mKdV equation.

To prove these results we need asymptotic expansions of spectral data of Schrödinger operators. Mostly importantly we need new asymptotic estimates of the Floquet exponents of Schrödinger operators on the circle, which are discussed in this thesis. By the same techniques, known asymptotic estimates of various other spectral quantities are improved.

Contents

Introduction

Consider the Korteweg-de Vries equation (KdV)

$$\partial_t u = -\partial_x^3 u + 6u\partial_x u \tag{1}$$

on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. It is globally in time well-posed on the Sobolev spaces $H^N \equiv H^N(\mathbb{T}, \mathbb{R})$ with $N \geq -1$. The aim of this thesis is to describe new qualitative features of periodic solutions of KdV. First note that in contrast to solutions on the real line, periodic solutions do not have a special profile decomposition as $t \to \pm\infty$. Our main point of interest, related to the numerical experiments of Fermi, Pasta, and Ulam of particle chains, is to know how the distribution of energy among the Fourier modes evolves. A partial result in this direction says that due to the integrals provided by the KdV hierarchy, the Sobolev norms of smooth solutions stay bounded uniformly in time. In this thesis we make further contributions to the study of how the Fourier coefficients $\hat{u}_n(t) = \int_0^1 u(t,x)e^{-2\pi inx}dx$ of a solution $u(t,x)$ of (1) evolve in time. Our first result aims at describing dispersion phenomena for solutions of KdV by studying how $\hat{u}_n(t)$ evolve for $|n|$ large. More precisely, we want to investigate if $\hat{u}_n(t)$ admits a WKB type expansion of the form

$$\hat{u}_n(t) = e^{iw_n t}\left(a_n(t) + \frac{b_n(t)}{n} + \dots\right), \tag{2}$$

where $e^{iw_n t}$ is a strongly oscillating phase factor with frequency w_n and the coefficients $a_n(t), b_n(t), \dots$ vary more slowly and satisfy the estimates

$$\sum n^{2N}|a_n(t)|^2 < \infty \quad \text{and} \quad \sum n^{2N}|b_n(t)|^2 < \infty.$$

To state our result more precisely, denote by ω_n, $n \geq 1$, the KdV frequencies of $u(t)$. Let us recall how they are defined. The KdV equation can be written as a Hamiltonian PDE with phase space L^2 and Poisson bracket

$$\{F, G\}(q) := \int_0^1 \partial F \, \partial_x \, \partial G dx \tag{3}$$

where F, G are C^1-functionals on L^2 and ∂F denotes the L^2-gradient of F. Then KdV takes the form $\partial_t u = \partial_x \partial_u \mathcal{H}$ where $\mathcal{H}$ is the KdV Hamiltonian

$$\mathcal{H}(q) := \int_0^1 \left(\frac{1}{2}(\partial_x q)^2 + q^3\right) dx.$$

In terms of this set-up, the ω_n's are given by

$$\omega_n = \partial_{I_n} \mathcal{H}.$$

Here we use that $\mathcal{H}$ can be expressed as a real analytic function of the action variables I_n, $n \geq 1$, so that the partial derivatives $\partial_{I_n}\mathcal{H}$ are well defined – see below for more details. Alternatively, ω_n can be viewed as a function of q, which by a slight abuse of terminology, we also denote by ω_n. Clearly, for any $n \geq 1$, $\omega_n(u(t))$ is independent of t and depends in a nonlinear fashion on $u(0)$. It is convenient to introduce

$$\omega_{-n} := -\omega_n \quad \forall n \in \mathbb{Z}_{\geq 1} \quad \text{and} \quad \omega_0 := 0$$

and to denote the KdV flow by S^t, i.e., $S^t(u(0)) = u(t)$. In addition, let

$$R^t(u(0)) := S^t(u(0)) - \sum_{n\in\mathbb{Z}} e^{i\omega_n t}\hat{u}_n(0)e^{2\pi i n x}$$

where for any $n \in \mathbb{Z}$, $\omega_n = \omega_n(u(0))$.

Theorem 0.1. *For $q = u(0) \in H^N$, $N \in \mathbb{Z}_{\geq 0}$, the error $R^t(q)$ of the approximation $\sum_{n\in\mathbb{Z}} e^{i\omega_n t}\hat{q}_n e^{2\pi i n x}$ of the flow $S^t(q)$ has the following properties:*

(i) $R^t : H^N \to H^{N+1}$ is continuous;

(ii) for any $q \in H^N$, the orbit $\{R^t(q) |\, t \in \mathbb{R}\}$ is relatively compact in H^{N+1};

(iii) for any $M > 0$, the set of orbits $\{R^t(q) |\, t \in \mathbb{R},\, q \in H^N, \|q\|_{H^N} \leq M\}$ is bounded in H^{N+1};

(iv) if in addition $N \in \mathbb{Z}_{\geq 1}$, then $\partial_t R^t : H^N \to H^{N-1}$ is continuous and for any $q \in H^N$, the orbit $\{\partial_t R^t(q) |\, t \in \mathbb{R}\}$ is relatively compact in H^{N-1}. Moreover for any $M > 0$ the set of orbits $\{\partial_t R^t(q) |\, t \in \mathbb{R},\, q \in H^N, \|q\|_{H^N} \leq M\}$ is bounded in H^{N-1}.

Remark 0.1. *Actually, one can prove that for any $c \in \mathbb{R}$, the restrictions of R^t and $\partial_t R^t$ to the affine subspace $H^N_c = \{q \in H^N |\, \int_0^1 q(x)dx = c\}$ are real analytic. See Remark 4.1 for a precise statement.*

Remark 0.2. *In case $u(0)$ is a finite gap potential, there are formulas, due to Its-Matveev [7], for the frequencies ω_n in terms of periods of an Abelian differential, defined on the spectral curve associated to $u(0)$. These formulas can be extended to potentials in H^N, $N \geq -1$, – cf [17]. Alternative formulas can be found in [12], Appendix F.*

Remark 0.3. *Note that the frequencies ω_n depend on the initial conditions in a nonlinear way. The statement of Theorem 0.1 no longer holds if the KdV frequencies ω_n are replaced by their linearization at 0, i.e., by $(2\pi n)^3$, confirming the belief of experts in the field that solutions of KdV on the circle are not approximated by linear evolution over a time interval of infinite length – see [2] for results on linear approximations of solutions over* finite *time intervals.*

In terms of the above WKB ansatz (2), Theorem 0.1 says that with $w_n := \omega_n$ and $a_n(t) := \hat{u}_n(0)$, the remainder term

$$\rho_n(t) := b_n(t) + \cdots := n \cdot \left(e^{-i\omega_n t}\hat{u}_n(t) - \hat{u}_n(0)\right) = n\widehat{R}^t_n(u(0))e^{-i\omega_n t}$$

satisfies $\sum n^{2N}|\rho_n(t)|^2 < \infty$ and in case $N \in \mathbb{Z}_{\geq 1}$,

$$\sum n^{2(N-2)}|\partial_t \rho_n(t)|^2 < \infty. \tag{4}$$

As the asymptotics of the KdV frequencies are given by $\omega_n = 8\pi^3 n^3 + O(n)$ (cf formula (4.28) in Section 4.1) estimate (4) quantifies the assertion that $(\rho_n(t))_{n\in\mathbb{Z}}$ varies more slowly than $(\hat{u}_n(t))_{n\in\mathbb{Z}}$.
A second result of this thesis concerns the approximation of KdV solutions by trigonometric polynomials. For any $L \in \mathbb{Z}_{\geq 1}$, denote by $P_L : L^2 \to L^2$ the L^2-orthogonal projection of $L^2 = H^0(\mathbb{T}, \mathbb{R})$ onto the $2L+1$ dimensional $\mathbb{R}$-vector space generated by $e^{2\pi i n x}$, $|n| \leq L$.

Theorem 0.2. *Let $N \in \mathbb{Z}_{\geq 0}$ be arbitrary. Then for any $M > 0$ and $\epsilon > 0$ there exists $L_{\epsilon,M} \geq 1$ such that for any $u(0) \in H^N$, with $\|u(0)\|_{H^N} \leq M$, $L \geq L_{\epsilon,M}$ and any $t \in \mathbb{R}$*

$$\|(Id - P_L)u(0)\|_{H^N} - \epsilon \leq \|(Id - P_L)u(t)\|_{H^N} \leq \|(Id - P_L)u(0)\|_{H^N} + \epsilon.$$

In particular, if $u(0)$ with $\|u(0)\|_{H^N} \leq M$ is a trigonometric polynomial of order $L_\star$, then for any $L \geq \max(L_\star, L_{\epsilon,M})$, $P_L u(t)$ approximates $u(t)$ uniformly in $t \in \mathbb{R}$ up an error of size ϵ.

Remark 0.4. *The proof of Theorem 0.2 shows that for any $|n| > L_{\epsilon,M}$ and $\|u(0)\|_{H^N} \leq M$,*

$$|\hat{u}_n(0)| - \epsilon \leq |\hat{u}_n(t)| \leq |\hat{u}_n(0)| + \epsilon \quad \forall t \in \mathbb{R}.$$

It means that for $|n|$ sufficiently large, the amplitude of the n'th Fourier mode is approximately constant, uniformly on bounded sets of H^N.

Remark 0.5. *It follows from the proof of Theorem 0.1 that corresponding results hold for the flow of any Hamiltonian in the Poisson algebra of KdV. In particular, this is true for the flows of Hamiltonians in the KdV hierarchy.*

The main ingredient of the proofs of Theorem 0.1 and Theorem 0.2 are refined asymptotics of the Birkhoff map of KdV. This map provides normal coordinates, allowing to solve KdV by quadrature. Let us recall its set-up. First note that the average of any solution $u(t) \equiv u(t,x)$ of KdV in H^N is a conserved quantity. In particular, for any $c \in \mathbb{R}$, KdV leaves the subspace $H^N_c \equiv H^N_c(\mathbb{T}, \mathbb{R})$ of H^N invariant where

$$H^N_c = \left\{p(x) = \sum \hat{p}_n e^{2\pi i n x} \mid \hat{p}_0 = c;\ \|p\|_N < \infty;\ \hat{p}_{-n} = \overline{\hat{p}_n}\ \forall n \in \mathbb{Z}\right\}$$

with

$$\|p\|_N := \Big(\sum |n|^{2N} |\hat{p}_n|^2\Big)^{\frac{1}{2}}.$$

In the case $N = 0$, we often write L^2_c for H^0_c and $\|p\|$ instead of $\|p\|_0$. To describe the normal coordinates of KdV, let us introduce for any $\alpha \in \mathbb{R}$ the $\mathbb{R}$-subspace $\mathfrak{h}^\alpha$ of $\ell^{2,\alpha}$, given by

$$\mathfrak{h}^\alpha := \big\{ z = (z_n)_{n \neq 0} \in \ell^{2,\alpha} |\, z_{-n} = \overline{z}_n \forall n \geq 1 \big\}$$

where

$$\ell^{2,\alpha} \equiv \ell^{2,\alpha}(\mathbb{Z}_0, \mathbb{C}) := \{ z = (z_n)_{n \neq 0} |\ \|z\|_\alpha < \infty \},$$

$\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$, and

$$\|z\|_\alpha := \Big(\sum_{n \neq 0} |n|^{2\alpha} |z_n|^2 \Big)^{\frac{1}{2}}.$$

The space $\mathfrak{h}^\alpha$ is endowed with the standard Poisson bracket for which $\{z_n, z_{-n}\} = -\{z_{-n}, z_n\} = 2i$ for any $n \geq 1$ whereas all other brackets between coordinate functions vanish. Furthermore we denote by $H^N_{0,\mathbb{C}} \equiv H^N_0(\mathbb{T}, \mathbb{C})$, $L^2_{0,\mathbb{C}} \equiv L^2_0(\mathbb{T}, \mathbb{C})$ and $\mathfrak{h}^\alpha_\mathbb{C}$ the complexification of the spaces H^N_0, L^2_0, and $\mathfrak{h}^\alpha$. Note that $\mathfrak{h}^\alpha_\mathbb{C} = \ell^{2,\alpha}(\mathbb{Z}_0, \mathbb{C})$. A detailed proof of the following result can be found in [12] – cf also [10].

Theorem 0.3. *There exist an open neighbourhood W of L^2_0 in $L^2_{0,\mathbb{C}}$ and a real analytic map $\Phi : W \to \mathfrak{h}^{1/2}_\mathbb{C}$ with the following properties:*

(BC1) For any $N \in \mathbb{Z}_{\geq 0}$, the restriction of Φ to H^N_0 is a canonical, bianalytic diffeomorphism onto $\mathfrak{h}^{N+1/2}$.

(BC2) When expressed in the new coordinates, the KdV-Hamiltonian $\mathcal{H} \circ \Phi^{-1}$, defined on $\mathfrak{h}^{3/2}$, is a real analytic function of the action variables $I_n = (z_n z_{-n})/2$, $n \geq 1$ alone.

(BC3) The differential $\Phi_0 \equiv d_0\Phi$ of Φ at 0 is the weighted Fourier transform,

$$\Phi_0(h) = \left(\frac{1}{\sqrt{|n|\pi}} \hat{h}_n \right)_{n \neq 0} \tag{5}$$

The coordinates z_n, $n \neq 0$, are referred to as (complex) Birkhoff coordinates whereas Φ is called Birkhoff map. Note that in [12] the Birkhoff map is defined slightly differently by setting $\Phi(q)$ to be $(x_n, y_n)_{n \geq 1}$ where $x_n = (z_n + z_{-n})/2$ and $y_n = i(z_n - z_{-n})/2$. The fact that KdV admits globally defined Birkhoff coordinates is a very special feature of KdV. In more physical terms it says that KdV, when considered with periodic boundary conditions, is a system of infinitely many coupled oscillators. The key ingredient of the proofs of Theorem 0.1 and Theorem 0.2 is the following result on the asymptotics of the Birkhoff map, which has an interest in its own.

Theorem 0.4. *For $N \in \mathbb{Z}_{\geq 0}$, there exists an open neighbourhood W_N of H_0^N in $H_{0,\mathbb{C}}^N \cap W$ so that $\Phi - \Phi_0$ maps W_N into $\mathfrak{h}_{\mathbb{C}}^{N+3/2}$ and, as a map from W_N to $\mathfrak{h}_{\mathbb{C}}^{N+3/2}$, is analytic. Here W is the neighbourhood of L_0^2 in $L_{0,\mathbb{C}}^2$ of Theorem 0.3. Furthermore, the restriction $A := (\Phi - \Phi_0)_{|H_0^N} : H_0^N \to \mathfrak{h}^{N+3/2}$ is a bounded map, i.e. it is bounded on bounded subsets of H_0^N.*

The asymptotic estimates obtained to prove Theorem 0.4 can be used to get a short proof of the Fredholm property of the differential of Φ at any $q \in H_0^N$ (Corollary 2.5).

In Section 3.1 a result similar to the one stated in Theorem 0.4 is proved for the Birkhoff map of KdV constructed in [9], where the phase space is endowed with the Poisson bracket introduced by Magri. As an application, a corresponding result is then derived for the modified Korteweg-de Vries equation (mKdV) on H^N with $N \geq 1$ – see Section 3.2. To this end we use the result in [14] that the Miura map $f \mapsto \partial_x f + f^2$ canonically embeds the symplectic leaves of the phase space of mKdV, endowed with the Poisson bracket (3), into the phase space of KdV, endowed with the Magri bracket. (For a detailed study of the Miura map see [16].) Finally, in Section 4.2 we prove results similar to the ones of Theorem 0.1 and Theorem 0.2 for mKdV .

Related results Recently, Kuksin and Piatnitski initiated a study of random perturbations with damping of the KdV equation [21], [19]. More precisely, they seek to determine how the KdV-action variables evolve under certain perturbed equations. For this purpose they express the perturbed KdV equation in normal coordinates. Up to highest order, it is a linear differential equation if the nonlinear part $\Phi - \Phi_0$ of the Birkhoff map is 1-smoothing, i.e. if it maps H_0^N to $\mathfrak{h}^{N+3/2}$ for any $N \geq 0$. In their recent paper, Kuksin and Perelman [20] succeeded in showing that on a neighbourhood U of the equilibrium point $q = 0$, there exists a canonical, real analytic diffeomorphism $\Psi : U \to V$ with $V \subseteq \mathfrak{h}^{1/2}$ a neighbourhood of 0 in $\mathfrak{h}^{1/2}$ providing Birkhoff coordinates for KdV so that $\Psi - \Psi_0$ is 1-smoothing where Ψ_0 denotes the linearisation of Ψ at $q = 0$ and coincides with Φ_0. They obtain the map Ψ by generalizing Eliasson's construction of a Birkhoff map near an equilibrium point of a finite dimensional integrable system to a class of integrable PDEs including the KdV equation. In order to apply Eliasson's construction, Kuksin and Perelman need coordinates for the KdV equation, provided in [8], as a starting point. Eliasson's construction is based on Moser's path-method and, in general, cannot be extended to get global coordinates. However, for the study of random perturbations of KdV in [19], global Birkhoff coordinates for KdV are needed. In [20], it was conjectured that there exists a globally defined Birkhoff map Ψ so that $\Psi - \Psi_0$ is 1-smoothing. Note that Birkhoff maps are not uniquely determined. Theorem 0.4 confirms that this conjecture holds true and that Ψ can be chosen to be the Birkhoff map of Theorem 0.3.

To prove Theorem 0.4 we need asymptotics of various spectral quantities of Schrödinger operators. For q in $L_{0,\mathbb{C}}^2$ the Schrödinger operator is given by $L_q := -d_x^2 + q$, where q can be extended periodically to $\mathbb{R}$. For $\lambda \in \mathbb{C}$ the equation $L_q(y) = \lambda y$ allows two linearly independent C^1-solutions $y_1(x, \lambda), y_2(x, \lambda)$ which are uniquely determined by the initial conditions $y_1(0, \lambda) = y_2'(0, \lambda) := \partial_x y_2'(0, \lambda) = 1$ and $y_1'(0, \lambda) = y_2(0, \lambda) = 0$. Consider now the Dirichlet spectrum on $[0, 1]$ of L_q. It consists of a sequence of complex numbers bounded from below. We list them lexicographically and with algebraic multiplicities,

$$\mu_0 \preceq \mu_1 \preceq \mu_2 \preceq \mu_3 \preceq \dots$$

where two complex numbers a, b, are ordered lexicographically, $a \preceq b$, if $[\operatorname{Re} a < \operatorname{Re} b]$ or $[\operatorname{Re} a = \operatorname{Re} b$ and $\operatorname{Im} a \leq \operatorname{Im} b]$. The first result concerns the asymptotic estimates for the Floquet exponent

$$\kappa_n = \log\left((-1)^n y_2'(1, \mu_n)\right) \tag{6}$$

when $n \to \infty$. Here log denotes the principal branch of the logarithm. Note that $y_2'(1, \mu_n) \neq 0$. By deforming the potential q to the zero potential one sees that $(-1)^n y_2'(1, \mu_n) > 0$ for any $q \in L_0^2$. Hence $\log\left((-1)^n y_2'(1, \mu_n)\right)$ is well-defined. In fact, there exists a neighbourhood W of L_0^2 in $L_{0,\mathbb{C}}^2$ so that for $q \in W$ and any $n \geq 1$, $\kappa_n(q)$ is well-defined by (6). Actually for any bounded subset B in $H_{0,\mathbb{C}}^N$ there exists n_0 so that κ_n is well-defined on B for any $n \geq n_0$. The κ_n's have been introduced in [4] and studied for square integrable potentials in [26]. Note that for $\lambda = \mu_n$ the Floquet matrix

$$M(\lambda) := \begin{pmatrix} y_1(1, \lambda) & y_2(1, \lambda) \\ y_1'(1, \lambda) & y_2'(1, \lambda) \end{pmatrix}$$

is lower triangular. Hence $y_2'(1, \mu_n)$ is one of the two eigenvalues or so called Floquet multipliers of $M(\mu_n)$, the other one being $y_1(1, \mu_n)$ which by the Wronskian identity equals $1/y_2'(1, \mu_n)$. In particular it follows that

$$\kappa_n = -\log\left((-1)^n y_1(1, \mu_n)\right). \tag{7}$$

We want to figure out an asymptotic expansion of κ_n in case the potential q is of higher regularity.

Theorem 0.5. *Let $N \geq 0$. Then for any q in $W \cap H_{0,\mathbb{C}}^N$,*

$$\kappa_n = \frac{1}{2\pi n}\left(\langle q, \sin 2\pi n x\rangle + \frac{1}{n^{N+1}} \ell_n^2\right)$$

uniformly on bounded subsets of $W \cap H_{0,\mathbb{C}}^N$.

The next results concern the asymptotics of the Dirichlet eigenvalues $(\mu_n)_{n\geq 1}$, the Neumann eigenvalues $(\eta_n)_{n\geq 0}$, and the periodic eigenvalues $(\lambda_n)_{n\geq 0}$ of the Schrödinger operator $-d_x^2 + q$ for a potential q in $H_{0,\mathbb{C}}^N$. (Note that the periodic and Neumann eigenvalues can be lexicographically ordered as well.) Recall that for q in $L_{0,\mathbb{C}}^2$,

$$\mu_n = n^2\pi^2 + \ell_n^2, \quad \eta_n = n^2\pi^2 + \ell_n^2 \quad \text{and} \quad \lambda_{2n}, \lambda_{2n-1} = n^2\pi^2 + \ell_n^2.$$

For f, g in $L^2_{\mathbb{C}}$, let

$$\langle f, g\rangle = \int_0^1 f(x)\bar{g}(x)dx.$$

In particular, $\langle q, e^{2\pi ikx}\rangle = \int_0^1 q(x)e^{-2\pi ikx}dx$ denotes the k'th Fourier coefficient of q.

Theorem 0.6. *Let q be in $H^N_{0,\mathbb{C}}$ with $N \in \mathbb{Z}_{\geq 0}$. Then*

$$\mu_n = m_n - \langle q, \cos 2\pi nx\rangle + \frac{1}{n^{N+1}}\ell^2_n \tag{8}$$

$$\eta_n = m_n + \langle q, \cos 2\pi nx\rangle + \frac{1}{n^{N+1}}\ell^2_n \tag{9}$$

uniformly on bounded subsets of potentials in $H^N_{0,\mathbb{C}}$. The quantity m_n is of the form

$$m_n = n^2\pi^2 + \sum_{2\leq 2j\leq N+1} c_{2j}\frac{1}{n^{2j}} \tag{10}$$

with coefficients c_{2j} which are independent of n and N and given by integrals of polynomials in q and its derivatives up to order $2j-2$.

Remark 0.6. *The asymptotics* (8) *are proved in [23] and the uniform boundedness of the error in* (8) *is shown in [27]. We present a simple and self-contained proof of Theorem 0.6 in Section 4.*

Note that in contrast to the asymptotics of the Dirichlet eigenvalues or the Neumann eigenvalues, the size of κ_n for any $q \in H^N_0$ is of the order of $\frac{1}{n^{N+1}}\ell^2_n$. As we will see in the course of the proof of Theorem 0.5, this results from somewhat surprising cancellations in the case $N \geq 1$. The case $N = 0$ has been treated in [26], p 60, and is special as such cancellations do not yet appear.
Using similar arguments as in the proof of Theorem 0.6 one obtains corresponding estimates for the periodic eigenvalues.

Theorem 0.7. *Let q be in $H^N_{0,\mathbb{C}}$ with $N \in \mathbb{Z}_{\geq 0}$. Then*

$$\{\lambda_{2n}, \lambda_{2n-1}\} = \{m_n \pm \sqrt{\langle q, e^{2\pi inx}\rangle\langle q, e^{-2\pi inx}\rangle + \frac{1}{n^{2N+1}}\ell^2_n} + \frac{1}{n^{N+1}}\ell^2_n\} \tag{11}$$

uniformly on bounded subsets of potentials in $H^N_{0,\mathbb{C}}$. Again, m_n is the expression defined in (10)*.*

A proof of the asymptotic estimate (11), but not of the uniform boundedness of the error terms stated above, can be found in [23].
Unfortunately, the asymptotics of Theorem 0.7 do not lead to satisfactory asymptotic estimates of the sequence $(\tau_n)_{n\geq 1}$ where $\tau_n = (\lambda_{2n} + \lambda_{2n-1})/2$. In Theorem 1.2 of Section 1.8, the asymptotic estimates of Theorem 0.7 are improved on H^N_0 and lead to the desired estimates for τ_n. However in the complex case different arguments have to be used to obtain stronger asymptotics of τ_n.

Theorem 0.8. *(i) For any* $q \in H_0^N$, $N \in \mathbb{Z}_{\geq 0}$,

$$\tau_n(q) = m_n + \frac{1}{n^{N+1}} \ell_n^2 \tag{12}$$

where m_n *is given by* (10) *and the error term is uniformly bounded on bounded sets of potentials in* H_0^N*.*

(ii) For any $N \in \mathbb{Z}_{\geq 0}$*, there exists an open neighbourhood* $W_N \subseteq H_{0,\mathbb{C}}^N$ *of* H_0^N *so that* (12) *holds on* W_N *with a locally uniformly bounded error term.*

The thesis is organised as follows. In Chapter 1 we prove Theorem 0.5, Theorem 0.6, Theorem 0.7, and Theorem 0.8. In Chapter 2, these estimates are used to improve on asymptotic estimates of actions, angles, and Birkhoff coordinates for KdV. In Section 2.3 we prove Theorem 0.4 and in Section 4.1, Theorem 0.1 and Theorem 0.2 are proved. In Section 3.2 and Section 4.2 corresponding results for the modified Korteweg-de Vries Gleichung (mKdV) are proved.

Chapter 1

Asymptotics of spectral quantities

In this chapter we discuss the various asymptotics of spectral quantities of Schrödinger operators stated in the Introduction.

1.1 Notions and notations

If not stated otherwise we will use the notions and notations introduced in [12]. For the convenience of the reader we now recall the ones most frequently used in this thesis. For q in $L^2_{0,\mathbb{C}}$, the Schrödinger operator $L_q := -d_x^2 + q$, considered on the interval $[0,2]$ with periodic boundary conditions, has a discrete spectrum, consisting of a sequence of complex numbers bounded from below. We list them lexicographically and with algebraic multiplicities,

$$\lambda_0 \preceq \lambda_1 \preceq \lambda_2 \preceq \lambda_3 \preceq \dots$$

where two complex numbers a, b, are ordered lexicographically, $a \preceq b$, if $[\operatorname{Re} a < \operatorname{Re} b]$ or $[\operatorname{Re} a = \operatorname{Re} b$ and $\operatorname{Im} a \leq \operatorname{Im} b]$. These eigenvalues satisfy the asymptotics

$$(\lambda_{2n} - n^2\pi^2)_{n\geq 1}, (\lambda_{2n-1} - n^2\pi^2)_{n\geq 1} \in \ell^2$$

or, expressed in a more convenient form,

$$\lambda_{2n-1}, \lambda_{2n} = n^2\pi^2 + \ell^2_n,$$

valid uniformly on bounded subsets of $L^2_{0,\mathbb{C}}$. In particular, this means that for any $R > 0$ there exists $r > 0$ so that for any $q \in L^2_{0,\mathbb{C}}$ with $\|q\| \leq R$,

$$|\lambda_k - n^2\pi^2| \leq r\pi^2 \quad \forall k \in \{2n, 2n-1\}, \forall n \geq 1. \tag{1.1}$$

For real q, the periodic eigenvalues are real and satisfy

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots .$$

Restricting ourselves to a sufficiently small neigbourhood W of L^2_0 in $L^2_{0,\mathbb{C}}$, we can always ensure that the closed intervals

$$G_n = \{(1-t)\lambda_{2n-1} + t\lambda_{2n} | \ 0 \le t \le 1\}, \quad n \ge 1,$$

as well as

$$G_0 = \{t + \lambda_0 | \ -\infty < t \le 0\}$$

are disjoint from each other. By a slight abuse of terminology, for any $n \ge 1$, we refer to the closed interval G_n as the n'th gap and to $\gamma_n := \lambda_{2n} - \lambda_{2n-1}$, as the n'th gap length. We denote by τ_n the middle point of G_n, $\tau_n = (\lambda_{2n} + \lambda_{2n-1})/2$. Due to the asymptotic behaviour of the periodic eigenvalues, the G_n's admit mutually disjoint neighbourhoods $U_n \subseteq \mathbb{C}$ with $G_n \subseteq U_n$ called *isolating neighbourhoods*. Moreover, inside each U_n, we choose a circuit Γ_n around G_n with counterclockwise orientation. Both U_n and Γ_n can be chosen to be locally independent of q. For q in a sufficiently small neighbourhood W of L^2_0 in $L^2_{0,\mathbb{C}}$, the U_n's with $n \ge n_0$, $n_0 = n_0(q)$ sufficiently large, can be chosen to be discs, $U_n = \{\lambda \in \mathbb{C} | \ |\lambda - n^2\pi^2| < r\pi^2\}$, where $0 < r \le n_0$ and n_0, r are chosen so large that (1.1) holds. Such neighbourhoods will be called isolating neighbourhoods with parameters $n_0 \ge 1$ and $r > 0$. In the course of this thesis, W will be shrunk several times, but we continue to denote it by W.
By $\Delta(\lambda) \equiv \Delta(\lambda, q)$ we denote the discriminant of $-d_x^2 + q$,

$$\Delta(\lambda) = \operatorname{tr} M(1, \lambda)$$

where $M(x, \lambda)$ is the 2×2 matrix whose columns $(y_i(x, \lambda), y_i'(x, \lambda))^T$, $i = 1, 2$, are solutions of $-y'' + qy = \lambda y$ with $M(0, \lambda) = Id_{2\times 2}$. The function $\Delta(\lambda)$ is entire and $\Delta^2(\lambda) - 4$ has a product representation (see [12], Proposition B.10)

$$\Delta^2(\lambda) - 4 = 4(\lambda_0 - \lambda) \prod_{k \ge 1} \frac{(\lambda_{2k} - \lambda)(\lambda_{2k-1} - \lambda)}{\pi_k^4} \tag{1.2}$$

where $\pi_k = k\pi$ for any $k \ge 1$. For real q, the Dirichlet and the Neumann eigenvalues are real and satisfy

$$\lambda_1 \le \mu_1 \le \lambda_2 < \lambda_3 \le \mu_2 \le \lambda_4 < \dots \quad \text{and} \quad \eta_0 \le \lambda_0 < \lambda_1 \le \eta_1 \le \lambda_2 < \dots .$$

Restricting ourselves to a sufficiently small neighbourhood W of L^2_0 in $L^2_{0,\mathbb{C}}$, we can assure that for any $q \in W$ there exist isolating neighbourhoods $U_n \subset \mathbb{C}$ so that $\mu_n, \eta_n \in U_n$, $n \ge 1$, whereas λ_0 and η_0 are not contained in any of the U_n's. Isolating neighbourhoods with this additional property can be chosen to be locally independent of q.
Finally let us recall the notion of the s-root, introduced in [12]. For $a, b \in \mathbb{C}$, we define on $\mathbb{C} \setminus \{(1-t)a + tb \,|\, 0 \le t \le 1\}$ the s-root of $(b - \lambda)(\lambda - a)$, determined by setting for $\lambda \in \mathbb{C}$ with $|\lambda - \tau| > |b - a|$

$$\sqrt[s]{(b-\lambda)(\lambda-a)} = i(\lambda - \tau) \sqrt[+]{1 - w^2}$$

where $\tau = (b+a)/2$ and $w = (b-a)/2(\lambda - \tau)$ – see figure 2, p 62 in [12], showing a sign table. Here $\sqrt[+]{\lambda}$ denotes the principal branch of the square root on $\mathbb{C} \setminus (-\infty, 0]$ characterised by,

$$\sqrt[+]{\lambda} > 0 \quad \text{for } \lambda > 0.$$

Throughout the thesis, $\log \lambda$ denotes the principal branch of the logarithm, defined on $\mathbb{C} \setminus (-\infty, 0]$. In particular, $\log 1 = 0$.

1.2 Infinite products

In this section we provide asymptotic estimates for infinite products of complex numbers needed to prove the asymptotic estimates of spectral quantities claimed in the Introduction. For a given sequence $(a_m)_{m\geq 1}$ in $\mathbb{C}$, the infinite product $\prod_{m\geq 1}(1+a_m)$ is said to converge if the sequence $\left(\prod_{1\leq m\leq M}(1+a_m)\right)_{M\geq 1}$ is convergent. In such a case we set

$$\prod_{m\geq 1}(1+a_m) = \lim_{M\to\infty} \prod_{1\leq m\leq M}(1+a_m).$$

It is said to be absolutely convergent if $\prod_{m\geq 1}(1+|a_m|) < \infty$. Note that an absolutely convergent infinite product is convergent. A sufficient condition for $\prod_{m\geq 1}(1+a_m)$ being absolutely convergent is that $\|a\|_{\ell^1} := \sum_{m\geq 1}|a_m| < \infty$. Indeed, as $0 \leq \log(1+x) \leq x$ for any $x \geq 0$ one has

$$\prod_{m\geq 1}(1+|a_m|) = \exp\left(\sum_{m\geq 1}\log(1+|a_m|)\right) \leq \exp\left(\sum_{m\geq 1}|a_m|\right). \tag{1.3}$$

We will improve on the estimates of infinite products of Appendix L in [12] by using (a version of) the discrete Hilbert transform, defined for an arbitrary sequence $\alpha = (\alpha_m)_{m\geq 1} \in \ell^2_{\mathbb{C}}$ by

$$H\alpha := \left(\sum_{m\neq n}\alpha_m\left(\frac{1}{n-m}+\frac{1}{n+m}\right)\right)_{n\geq 1}.$$

The following result is due to Hilbert – see e.g. [6], p 213 for a proof.

Lemma 1.1. *H defines a bounded linear operator on $\ell^2_{\mathbb{C}}$, with $\|H\| \leq 2\pi$.*

Later we will need the following auxiliary result.

Lemma 1.2. *For any ℓ^1-sequence $(a_m)_{m\geq 1} \subseteq \mathbb{C}$ with $|a_m| \leq \frac{1}{2}$ for any $m \geq 1$, one has*

$$\left|\prod_{m\geq 1}(1+a_m) - 1\right| \leq |A|e^S + |B|e^{S+S^2}$$

where $A = \sum_{m\geq 1} a_m$, $B = \sum_{m\geq 1}|a_m|^2$, and $S = \sum_{m\geq 1}|a_m|$.

Proof. As $|a_m| \le \frac{1}{2}$, the logarithm $\log(1 + a_m)$ is well defined and one has

$$\prod_{m\ge1}(1+a_m) = \exp\left(\sum_{m\ge1}\log(1+a_m)\right) = e^A \exp\left(\sum_{m\ge1}\log(1+a_m) - a_m\right).$$

The estimate $|\log(1+z) - z| \le |z|^2$, $z \in \mathbb{C}$ with $|z| \le \frac{1}{2}$, then leads to the following bound for $R := \sum_{m\ge1}(\log(1+a_m) - a_m)$,

$$|R| \le \sum_{m\ge1}|a_m|^2 = B \le S^2.$$

With $|A| \le S$ it then follows that

$$\left|\prod_{m\ge1}(1+a_m) - 1\right| = \left|(e^A - 1) + e^A(e^R - 1)\right| \le \left|(e^A - 1)\right| + e^S\left|(e^R - 1)\right|.$$

As $|e^z - 1| \le |z|e^{|z|}$, $z \in \mathbb{C}$, this leads to the claimed statement,

$$\left|\prod_{m\ge1}(1+a_m) - 1\right| \le |A|e^S + |B|e^{S+S^2}.$$

□

To state the results on infinite products coming up in our study we need to introduce some more notation. First let us introduce another notion of isolating neighbourhoods. We say that the discs $U_n \subseteq \mathbb{C}$, $n \ge 1$, are a family of *isolating neighbourhoods with parameters* $n_0 \ge 1$, $r > 0$, $\rho > 0$ if the U_n's are mutually disjoint open discs with centers $z_n \in \mathbb{R}$ satisfying $z_1 < z_2 < \dots$, so that

$$U_n \subseteq D_r^n := \{\lambda \in \mathbb{C} |\ |\lambda - n^2\pi^2| < r\pi^2\} \quad \forall n \ge 1 \tag{1.4}$$

and

$$U_n = D_r^n \quad \forall n \ge n_0 + 1$$

so that for any $n, m \ge 1$

$$|\lambda - \mu| \ge \frac{1}{\rho}|n^2 - m^2| \quad \forall \lambda \in U_n, \forall \mu \in U_m. \tag{1.5}$$

We remark that the results stated below continue to hold for a weaker notion of isolating neighbourhoods, but they suffice for our purposes. Let $a^0 := (n^2\pi^2)_{n\ge1}$.

Proposition 1.1. *Assume that $(U_n)_{n\geq 1}$ is a sequence of isolating neighbourhoods with parameters n_0, r, ρ. Then for arbitrary sequences $(a_m)_{m\geq 1}, (b_m)_{m\geq 1} \subseteq \mathbb{C}$ with $\alpha := a - a^0, \beta := b - a^0$ in $\ell^2_{\mathbb{C}}$, $b_m \in U_m$ for any $m \geq 1$, and any sequence of complex numbers $\Lambda := (\lambda_n)_{n\geq 1}$ with $\lambda_n \in U_n$ for any $n \geq 1$*

$$f_n(\lambda_n) = \prod_{m\neq n} \frac{a_m - \lambda_n}{b_m - \lambda_n} = 1 + \frac{1}{n}\ell^2_n$$

uniformly on bounded subsets of $\alpha, \beta \in \ell^2_{\mathbb{C}}$ with $b_m \in U_m$ for any $m \geq 1$ and uniformly in Λ with $\lambda_n \in U_n$ for any $n \geq 1$. More precisely,

$$\sum_{n\geq 1} n^2 |f_n(\lambda_n) - 1|^2 \leq K^2_{\alpha,\beta,\Lambda}$$

with a constant $K_{\alpha,\beta,\Lambda} > 0$ which can be chosen uniformly for bounded subsets of $\alpha, \beta \in \ell^2_{\mathbb{C}}$ with $b_m \in U_m$ for any $m \geq 1$ and uniformly for $\Lambda = (\lambda_n)_{n\geq 1}$ with $\lambda_n \in U_n$ for any $n \geq 1$.

Proof. For any $n, m \geq 1$ with $n \neq m$,

$$\frac{a_m - \lambda_n}{b_m - \lambda_n} = 1 + a_{nm} \quad \text{and} \quad a_{nm} = \frac{a_m - b_m}{b_m - \lambda_n}.$$

As by assumption

$$|b_m - \lambda_n| \geq \frac{1}{\rho}|m^2 - n^2| \quad \forall m \neq n$$

and $|m^2 - n^2| \geq n|m - n|$, one has by the Cauchy-Schwarz inequality

$$\sum_{m\neq n} |a_{nm}| \leq \rho \sum_{m\neq n} \frac{|a_m - b_m|}{|m^2 - n^2|} \leq \frac{\pi\rho}{n}\|a - b\| \tag{1.6}$$

where we used that

$$\sum_{m\neq n} \frac{1}{|m - n|^2} \leq 2\sum_{k\geq 1} \frac{1}{k^2} = \frac{\pi^2}{3}.$$

By (1.3) this leads to the following estimate

$$\left| \prod_{m\neq n} \frac{a_m - \lambda_n}{b_m - \lambda_n} \right| \leq \prod_{m\neq n} (1 + |a_{nm}|) \leq \exp\left(\frac{\pi\rho}{n}\|a - b\|\right).$$

Now let us consider the asymptotics of $\prod_{m\neq n} \frac{a_m - \lambda_n}{b_m - \lambda_n}$ as $n \to \infty$. Note that for any $m \neq n$,

$$\left| \frac{a_m - b_m}{b_m - \lambda_n} \right| \leq \rho \frac{\|a - b\|}{|m^2 - n^2|} \leq \frac{\rho}{n}\|a - b\|.$$

Choose $n_1 \geq n_0$ so that $\frac{\rho}{n_1}\|a - b\| \leq \frac{1}{2}$. Hence for $n \geq n_1$, Lemma 1.2 can be applied to $(a_{nm})_{m\neq n}$. To obtain the claimed estimates we need to show that $A_n := \sum_{m\neq n} a_{nm} = \frac{1}{n}\ell_n^2$ and $B_n := \sum_{m\neq n} |a_{nm}|^2 = \frac{1}{n}\ell_n^2$. We begin by estimating A_n. Note that for $n \geq 1$ and $m \neq n$,

$$a_{nm} = \frac{b_m - a_m}{n^2\pi^2 - m^2\pi^2} + \frac{b_m - a_m}{n^2\pi^2 - m^2\pi^2}\left(\frac{n^2\pi^2 - m^2\pi^2}{\lambda_n - b_m} - 1\right). \tag{1.7}$$

The two terms on the right hand side of the latter identity are treated separately. Use that $(n^2 - m^2)^{-1} = \frac{1}{2n}\left(\frac{1}{n-m} + \frac{1}{n+m}\right)$ to conclude from Lemma 1.1 that

$$\left(\sum_{n\geq 1} n^2 \left|\sum_{m\neq n} \frac{b_m - a_m}{n^2\pi^2 - m^2\pi^2}\right|^2\right)^{\frac{1}{2}} \leq \|b - a\|. \tag{1.8}$$

To estimate the second term on the right hand side of (1.7), use that $b_m, \lambda_m \in U_m$ so that in view of (1.4)

$$\begin{aligned}\left|\frac{n^2\pi^2 - m^2\pi^2}{\lambda_n - b_m} - 1\right| &\leq \frac{|b_m - m^2\pi^2| + |\lambda_n - n^2\pi^2|}{|\lambda_n - b_m|} \\ &\leq \rho\frac{2r\pi^2}{|n^2 - m^2|},\end{aligned}$$

to obtain the following estimate for the weighted ℓ^1-norm – and hence the weighted ℓ^2-norm –

$$\begin{aligned}&\sum_{n\geq 1} n \sum_{m\neq n} \left|\frac{b_m - a_m}{n^2\pi^2 - m^2\pi^2}\right| \left|\frac{n^2\pi^2 - m^2\pi^2}{\lambda_n - b_m} - 1\right| \\ \leq& 2r\rho \sum_{n\geq 1}\sum_{m\neq n} \frac{n}{n^2}\frac{|b_m - a_m|}{(n-m)^2} \\ \leq& 2r\rho\left(\sum_{n\geq 1}\sum_{m\neq n}\frac{1}{n^2}\frac{1}{|n-m|^2}\right)^{\frac{1}{2}}\left(\sum_{n\geq 1}\sum_{m\neq n}\frac{|b_m - a_m|^2}{|n-m|^2}\right)^{\frac{1}{2}} \\ \leq& 4\pi r\rho\|b - a\|.\end{aligned} \tag{1.9}$$

The estimates for the B_n's are simpler as we do not need to split a_{nm}. Indeed, as

$$B_n \leq \sum_{m\neq n} \rho^2 \frac{|a_m - b_m|^2}{|n^2 - m^2|^2}$$

we get the following estimate for its weighted ℓ^1-norm – and hence for its weighted ℓ^2-norm –

$$\begin{aligned}\sum_{n\geq 1} n^2 B_n &\leq \rho^2 \sum_{n\geq 1}\frac{n^2}{n^2}\sum_{m\neq n}\frac{|a_m - b_m|^2}{(n-m)^2} \\ &\leq \rho^2\|b - a\|^2 2\sum_{k\geq 1}\frac{1}{k^2} \\ &\leq 4\rho^2\|b - a\|^2.\end{aligned} \tag{1.10}$$

The claimed estimates then follow from Lemma 1.2, (1.6), and the statement on the uniformity of the estimates follow from the explicit bounds (1.8)–(1.10). □

Corollary 1.1. *Assume that $(U_n)_{n\geq 1}$ is a sequence of isolating neighbourhoods with parameters n_0, r, ρ. Then for any sequence $a = (a_m)_{m\geq 1}$ with $a - a^0 \in \ell^2_{\mathbb{C}}$ and any sequence $\Lambda = (\lambda_n)_{n\geq 1}$ with $\lambda_n \in U_n$ for any $n \geq 1$, the infinite product $\prod_{m\neq n} \frac{a_m-\lambda_n}{m^2\pi^2}$ is absolutely convergent for any $n \geq 1$ and*

$$\prod_{m\neq n} \frac{a_m - \lambda_n}{m^2\pi^2} = \frac{(-1)^{n+1}}{2} + \frac{1}{n}\ell^2_n$$

uniformly on bounded subsets of $a - a^0 \in \ell^2_{\mathbb{C}}$ and uniformly with respect to Λ with $\lambda_n \in U_n$ for any $n \geq 1$.

Proof. For any $m \geq 1, n \geq 1$ write

$$\frac{a_m - \lambda_n}{m^2\pi^2} = 1 + \frac{a_m - m^2\pi^2 - \lambda_n}{m^2\pi^2}$$

and

$$\sum_{m\geq 1} \left| \frac{a_m - m^2\pi^2 - \lambda_n}{m^2\pi^2} \right| \leq \frac{\|\alpha\| + |\lambda_n|}{\pi^2} \sum_{m\geq 1} \frac{1}{m^2} = \frac{\|\alpha\| + |\lambda_n|}{6},$$

where $\alpha := a - a^0$. Hence according to (1.3), $\prod_{m\neq n} \frac{a_m-\lambda_n}{m^2\pi^2}$ is absolutely convergent and bounded in terms of $\|\alpha\|$ and $|\lambda_n|$. It remains to estimate the product for $n \geq n_0 + 1$. Recall that $\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}$ has the product expansion,

$$\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} = \prod_{m\geq 1} \frac{m^2\pi^2 - \lambda}{m^2\pi^2}.$$

Hence for any $\lambda_n \in U_n$ with $n \geq n_0 + 1$ one has

$$\prod_{m\neq n} \frac{a_m - \lambda_n}{m^2\pi^2} = \frac{\sin\sqrt{\lambda_n}}{n^2\pi^2 - \lambda_n} \frac{n^2\pi^2}{\sqrt{\lambda_n}} \prod_{m\neq n} \frac{a_m - \lambda_n}{m^2\pi^2 - \lambda_n}. \tag{1.11}$$

In order to apply Proposition 1.1 to the product $\prod_{m\neq n} \frac{a_m-\lambda_n}{m^2\pi^2-\lambda_n}$ we replace for $1 \leq m \leq n_0$ the disc U_m by the disc with center $m^2\pi^2$ and radius 1. Then the parameters n_0 and r can be left as is whereas ρ is replaced by $\rho_1 \geq \rho$ so that

$$|m^2\pi^2 - \lambda_n| \geq \frac{1}{\rho_1}|m^2 - n^2| \quad \forall\lambda_n \in U_n, \forall m \neq n, \forall n \geq n_0 + 1.$$

By Proposition 1.1

$$\prod_{m\neq n} \frac{a_m - \lambda_n}{m^2\pi^2 - \lambda_n} = 1 + \frac{1}{n}\ell^2_n \tag{1.12}$$

where the asymptotics are uniform in the sense stated there. Now let us estimate the remaining terms of the right hand side of (1.11). For this purpose write

$$\frac{\sin\sqrt{\lambda_n}}{n^2\pi^2-\lambda_n} = (-1)^{n+1}\frac{\sin(\sqrt{\lambda_n}-n\pi)}{\sqrt{\lambda_n}-n\pi}\frac{1}{\sqrt{\lambda_n}+n\pi}.$$

Note that $\sqrt{\lambda_n} = n\pi + O(\frac{1}{n})$ and hence by Taylor expansion,

$$\begin{aligned}
\sin(\sqrt{\lambda_n}-n\pi) &= (\sqrt{\lambda_n}-n\pi)\left(1+O\left(\frac{1}{n^2}\right)\right)\\
\frac{1}{\sqrt{\lambda_n}+n\pi} &= \frac{1}{2n\pi}\left(1+O\left(\frac{1}{n^2}\right)\right)\\
\frac{1}{\sqrt{\lambda_n}} &= \frac{1}{n\pi}\left(1+O\left(\frac{1}{n^2}\right)\right).
\end{aligned}$$

Altogether one obtains in this way

$$\frac{\sin\sqrt{\lambda_n}}{n^2\pi^2-\lambda_n}\frac{n^2\pi^2}{\sqrt{\lambda_n}} = \frac{(-1)^{n+1}}{2} + O\left(\frac{1}{n^2}\right). \tag{1.13}$$

One easily sees that the error term $O(\frac{1}{n^2})$ in (1.13) can be bounded by $C\frac{1}{n^2}$ where $C > 0$ is a constant, only depending on r and n_0. Combining estimates (1.12) and (1.13) then leads to the claimed statement.

□

1.3 Special solutions

In this section we prove asymptotics of special solutions of $-y'' + qy = \lambda y$ as $|\lambda| \to \infty$ for potentials q in $H^N_{0,\mathbb{C}}$ with $N \geq 0$ which are needed to derive the asymptotics of various spectral quantities stated in the Introduction. These solutions are obtained with a WKB ansatz and are a version, suited for our purposes, of solutions introduced and studied by Marchenko [23], p 50 ff.

As $L(q) = -\partial_x^2 + q$ is a differential operator of second order it is convenient to introduce ν as a new spectral parameter with ν^2 playing the role of λ. The special solutions considered are denoted by $z_N(x,\nu)$ and defined for $\nu \neq 0$ by

$$z_N(x,\nu) = y_1(x,\nu^2) + \alpha_N(0,\nu)y_2(x,\nu^2) \tag{1.14}$$

where $y_1(x,\lambda)$, $y_2(x,\lambda)$ denote the standard fundamental solutions of $-y''+qy = \lambda y$ and q is assumed to be in $H^N_{0,\mathbb{C}} \equiv H^N_0(\mathbb{T},\mathbb{C})$. The function $\alpha_N(x,\nu)$ is given by

$$\alpha_N(x,\nu) = i\nu + \sum_{k=1}^{N}\frac{s_k(x)}{(2i\nu)^k} \tag{1.15}$$

where

$$s_1(x) = q(x), \quad s_2(x) = -\partial_x q(x) \tag{1.16}$$

and, for $2 \le k \le N$, s_{k+1} is determined by the recursion relation

$$s_{k+1}(x) = -\partial_x s_k(x) - \sum_{j=1}^{k-1} s_{k-j}(x) s_j(x). \tag{1.17}$$

(Note that (1.17) remains true for $N = 1$: In this case, the sum in (1.17) is not present.) By an induction argument one sees that for any $q \in H^N_{0,\mathbb{C}}$, and any $0 \le k \le N$, s_{k+1} is a universal isobaric polynomial homogeneous of degree $1 + \frac{k}{2}$. Here the adjective 'isobaric' signifies that q is considered to be of degree 1 and differentiation ∂_x of degree 1/2. Furthermore, for any $1 \le k \le N$,

$$s_{k+1}(x) = (-1)^k \partial_x^k q(x) + S_{k-2}(x) \tag{1.18}$$

where $S_{k-2}(x)$ is a polynomial in $q(x), \partial_x q(x), \ldots, \partial_x^{k-2} q(x)$ with constant coefficients and where $S_{-1} \equiv 0$. As a consequence, $\alpha_N(\cdot, \nu) \in H^1_{\mathbb{C}}$ and by the Sobolev embedding theorem, $\alpha_N(x, \nu)$ is continuous in x and hence (1.14) well-defined. Moreover, as $s_j \in H^{N+1-j}_{\mathbb{C}}$ for any $1 \le j \le N+1$, $\sum_{j=1}^N s_{N+1-j}(x) s_j(x)$ is in $H^1_{\mathbb{C}}$ and one may define s_{N+2} by formula (1.17) as an element in $H^{-1}_{\mathbb{C}}$

$$s_{N+2}(x) = -\partial_x s_{N+1} - \sum_{j=1}^{N} s_{N+1-j} s_j = (-1)^{N+1} \partial_x^{N+1} q(x) + H^1_{\mathbb{C}}. \tag{1.19}$$

Clearly, for any $\nu \in \mathbb{C} \setminus \{0\}$, $z_N(x, \nu)$ and $z_N(x, -\nu)$ are solutions of $-y'' + qy = \lambda y$ with $\lambda = \nu^2$ which both are 1 at $x = 0$ and are linearly dependent solutions iff

$$\alpha_N(0, \nu) - \alpha_N(0, -\nu) = 0,$$

i.e. ν is a zero of the following polynomial of degree $N + 1$

$$\begin{aligned} p_N(\nu) &= (2i\nu)^N \left(\alpha_N(0, \nu) - \alpha_N(0, -\nu) \right) \\ &= (2i\nu)^{N+1} + \sum_{k=1}^{N} s_k(0) \left(1 - (-1)^k\right) (2i\nu)^{N-k}. \end{aligned}$$

It then follows that there exists $\nu_0 > 0$ so that

$$|p_N(\nu)| \ge 1 \quad \forall\, |\nu| \ge \nu_0.$$

By the Sobolev embedding theorem, the number ν_0 can be chosen uniformly on bounded sets of potentials in $H^N_{0,\mathbb{C}}$. In particular, for $|\nu| \ge \nu_0$ one has

$$y_1(x, \nu^2) = \frac{1}{\alpha_N(0, \nu) - \alpha_N(0, -\nu)} \Big(\alpha_N(0, \nu) z_N(x, -\nu) - \alpha_N(0, -\nu) z_N(x, \nu) \Big)$$

and

$$y_2(x,\nu^2) = \frac{1}{\alpha_N(0,\nu)-\alpha_N(0,-\nu)}\left(z_N(x,\nu)-z_N(x,-\nu)\right).$$

Furthermore note that for any $x \in \mathbb{R}$, $z_N(x,\nu)$ is analytic on $\mathbb{C}^* \times H^N_{0,\mathbb{C}}$. As mentioned above, the solutions $z_N(x,\nu)$ are determined by a WKB ansatz,

$$z_N(x,\nu) = w_N(x,\nu) + \frac{r_N(x,\nu)}{(2i\nu)^{N+1}}$$

with

$$w_N(x,\nu) = \exp\left(\int_0^x \alpha_N(t,\nu)dt\right). \tag{1.20}$$

By the considerations above it follows that $w_N(\cdot,\nu)$ is in $H^2_{\mathbb{C}}[0,1]$. As $y_1(\cdot,\nu^2)$ and $y_2(\cdot,\nu^2)$, and hence $z_N(\cdot,\nu)$, are in $H^{N+2}_{\mathbb{C}}[0,1]$ one then concludes that $r_N(\cdot,\nu)$ is in $H^2_{\mathbb{C}}[0,1]$ as well. To study the asymptotics of $r_N(x,\nu)$ as $|\nu| \to \infty$, first note that

$$w_N' = \alpha_N w_N \quad \text{and} \quad w_N'' = (\alpha_N' + \alpha_N^2)w_N.$$

When substituting the latter expression into $-y'' + qy = \lambda y$ one obtains

$$\left(-\alpha_N' - \alpha_N^2 + q - \nu^2\right)w_N + \frac{1}{(2i\nu)^{N+1}}\left(-r_N'' + qr_N - \nu^2 r_N\right) = 0. \tag{1.21}$$

Expanding α_N^2 in powers of ν^{-1} leads to

$$\alpha_N^2 = -\nu^2 + \sum_{k=0}^{N-1}\frac{s_{k+1}}{(2i\nu)^k} + \sum_{k=2}^{2N}\frac{1}{(2i\nu)^k}\sum_{1\le l\le k-1,\, 1\le k-l,\, l\le N} s_{k-l}s_l.$$

Taking into account the identity $s_1 = q$ and the relations (1.17) one then gets

$$\begin{aligned}
-\alpha_N' - \alpha_N^2 + q - \nu^2 = &-\frac{s_1' + s_2}{2i\nu} - \sum_{k=2}^{N-1}\left(s_k' + s_{k+1} + \sum_{1\le l\le k-1} s_{k-l}s_l\right)\frac{1}{(2i\nu)^k}\\
&-\left(s_N' + \sum_{1\le l\le N-1} s_{N-l}s_l\right)\frac{1}{(2i\nu)^N}\\
&-\sum_{k=1}^{N}\left(\sum_{k\le l\le N} s_{N+k-l}s_l\right)\frac{1}{(2i\nu)^{N+k}}\\
=&\frac{s_{N+1}}{(2i\nu)^N} - \sum_{k=1}^{N}\left(\sum_{k\le l\le N} s_{N+k-l}s_l\right)\frac{1}{(2i\nu)^{N+k}}.
\end{aligned}$$

Substituting this expression into (1.21) yields

$$-r_N'' + qr_N - \nu^2 r_N = -2i\nu f_N(x,\nu) \tag{1.22}$$

where

$$f_N(x,\nu) := s_{N+1}w_N - \sum_{k=1}^{N}\left(\sum_{k\le l\le N} s_{N+k-l}s_l\right)\frac{w_N}{(2i\nu)^k}. \tag{1.23}$$

Recall that

$$s_{N+1}(x) - (-1)^N\partial_x^N q(x) \in H^2_{\mathbb{C}}, \tag{1.24}$$

$s_{N+k-l}s_l \in H^1_{\mathbb{C}}$ for any $k \le l \le N$ and $w_N(\cdot,\nu) \in H^2_{\mathbb{C}}[0,1]$. Hence $f_N(\cdot,\nu) \in L^2_{\mathbb{C}}[0,1]$. More precisely,

$$f_N(\cdot,\nu) = (-1)^N\partial_x^N q \cdot w_N(\cdot,\nu) + H^1_{\mathbb{C}}[0,1]$$

uniformly on bounded subsets of $H^N_{0,\mathbb{C}}$ and uniformly for $\nu \in \mathbb{C}$ with $|\nu| \ge 1$ and $|\operatorname{Im}\nu| \le C$. Further note that

$$r_N(0,\nu) = (2i\nu)^{N+1}\Big(z_N(0,\nu) - w_N(0,\nu)\Big) = 0 \tag{1.25}$$

and

$$r_N'(0,\nu) = (2i\nu)^{N+1}\Big(z_N'(0,\nu) - \alpha(0,\nu)w_N(0,\nu)\Big) = 0. \tag{1.26}$$

The estimates for r_N are obtained by using that it satisfies the inhomogeneous Schrödinger equation (1.22). Given $q \in L^2_{0,\mathbb{C}}$ and $\nu \in \mathbb{C}$ denote by $r(x,\nu)$ the unique solution of the initial value problem

$$-r'' + qr - \nu^2 r = -2i\nu f(x,\nu) \tag{1.27}$$

$$r(0,\nu) = 0 \quad \text{and} \quad r'(0,\nu) = 0 \tag{1.28}$$

where the inhomogeneous term on the right hand side of (1.27) is assumed to be in $L^2_{\mathbb{C}}([0,1])$ for any value of ν. Note that by assumption

$$Q(1) = 0 \quad \text{where} \quad Q(x) = \int_0^x q(t)dt \quad (0 \le x \le 1). \tag{1.29}$$

The solution $r(x,\nu)$ of (1.27)-(1.28) satisfies the following standard estimates.

Lemma 1.3. *Let $q \in L^2_{0,\mathbb{C}}$ and $f(x,\nu) = h(x)e^{ix\nu}$ with $h \in L^2_{\mathbb{C}}[0,1]$. Then for any $\nu \in \mathbb{C}\setminus\{0\}$ and $0 \le x \le 1$, the solution $r(x,\nu)$ of (1.27)-(1.28) satisfies the estimates*

$$|r(x,\nu)| \le \left(\frac{R^2}{|\nu|} + \frac{4R^3}{|\nu|^2}\left(1 + \frac{1}{|\nu|}\right)\right)\|h\|$$

$$|r'(x,\nu)| \le \left(R^2 + \frac{2\,(1+\|q\|)\,R^3}{|\nu|}\left(1 + \frac{1}{|\nu|}\right)\right)\|h\|$$

where

$$R \equiv R(\nu,q) := \exp\left(|\operatorname{Im}\nu| + \|q\|\right).$$

Proof. By the method of the variation of constants, the solution $r(x,\nu)$ has the following integral representation – see e.g. [26], Theorem 2, p 12

$$r(x,\nu) = \int_0^x \left(y_1(t,\nu^2)y_2(x,\nu^2) - y_1(x,\nu^2)y_2(t,\nu^2)\right) f(t,\nu)dt \tag{1.30}$$

where $y_i = y_i(x,\nu^2,q)$, $i = 1,2$, denote the fundamental solutions of $-y'' + qy = \nu^2 y$. They satisfy the following estimates on $[0,1] \times \mathbb{C} \times L^2_{0,\mathbb{C}}$ – see e.g. [26], Theorem 3, p 13

$$\left|y_1(x,\nu^2,q) - \cos\nu x\right| \le \frac{R}{|\nu|}; \qquad \left|y_2(x,\nu^2,q) - \frac{\sin\nu x}{\nu}\right| \le \frac{R}{|\nu|^2}$$

$$\left|y_1'(x,\nu^2,q) + \nu\sin\nu x\right| \le \|q\|R; \qquad \left|y_2'(x,\nu^2,q) - \cos\nu x\right| \le \frac{\|q\|R}{|\nu|}.$$

Formula (1.30) then leads to the following estimates for $0 \le x \le 1$, $\nu \in \mathbb{C}\setminus\{0\}$, $q \in L^2_{0,\mathbb{C}}$

$$\left|r(x,\nu) - \int_0^x \frac{\sin\nu(x-t)}{\nu} f(t,\nu)dt\right| \le \frac{4R^3}{|\nu|^2}\left(1 + \frac{1}{|\nu|}\right)\|h\|$$

$$\left|r'(x,\nu) - \int_0^x \cos\nu(x-t) f(t,\nu)dt\right| \le \frac{2\left(1+\|q\|\right)R^3}{|\nu|}\left(1 + \frac{1}{|\nu|}\right)\|h\|.$$

As

$$\left|\int_0^x \frac{\sin\nu(x-t)}{\nu} f(t,\nu)dt\right| \le \frac{1}{|\nu|}\int_0^1 e^{2|\operatorname{Im}\nu|}|h(t)|dt \le \frac{1}{|\nu|}R^2\|h\|$$

and

$$\left|\int_0^x \cos\nu(x-t) f(t,\nu)dt\right| \le \int_0^1 e^{2|\operatorname{Im}\nu|}|h(t)|dt \le R^2\|h\|$$

it then follows that

$$|r(x,\nu)| \le \left(\frac{R^2}{|\nu|} + \frac{4R^3}{|\nu|^2}\left(1 + \frac{1}{|\nu|}\right)\right)\|h\|$$

$$|r'(x,\nu)| \le \left(R^2 + \frac{2\left(1+\|q\|\right)R^3}{|\nu|}\left(1 + \frac{1}{|\nu|}\right)\right)\|h\|.$$

□

To obtain the claimed asymptotics of the periodic and Dirichlet eigenvalues, the estimates of Lemma 1.3 have to be refined.

Lemma 1.4. *Assume that* $q \in L^2_{0,\mathbb{C}}$. *Then for any sequence* $\nu_n = n\pi + \frac{1}{n}\ell^2_n$, $n \ge 1$, *the solution* $r(x,\nu)$ *of* (1.27)-(1.28) *satisfies the following estimates:*

(i) If $f(x,\nu) = h(x)e^{ix\nu}$ *with* $h \in L^2_{\mathbb{C}}[0,1]$,

$$r(1,\pm\nu_n) = (-1)^n \int_0^1 h(x)dx + (-1)^{n+1}\int_0^1 h(x)e^{\pm 2in\pi x}dx$$
$$\pm \frac{(-1)^{n+1}}{2in\pi}\int_0^1 Q(x)h(x)dx + \frac{1}{n}\ell^2_n$$

and

$$r'(1,\pm\nu_n) = \pm\, in\pi(-1)^n \int_0^1 h(x)dx \pm in\pi(-1)^n \int_0^1 h(x)e^{\pm 2in\pi x}dx + \frac{(-1)^{n+1}}{2}\int_0^1 Q(x)h(x)dx + \ell^2_n.$$

(ii) If $f(x,\nu) = \frac{h(x,\nu)}{\nu^2}e^{i\nu x}$ *and the family* $h(\cdot,\nu)$ *is bounded in* $L^2_{\mathbb{C}}[0,1]$*, then*

$$r(1,\pm\nu_n) = O(\frac{1}{n^2}) \quad \text{and} \quad r'(1,\pm\nu_n) = O\left(\frac{1}{n}\right).$$

The estimates in (i) and (ii) are uniform on bounded sets of $q's$*,* $h's$ *and* $\ell^2_{\mathbb{C}}$*-sequences* $\left(\frac{\nu_n - n\pi}{n}\right)_{n\geq 1}$.

Proof. As in the proof of Lemma 1.3, the solution r of (1.27)-(1.28) is written in the following integral form

$$r(x,\nu) = 2i\nu\int_0^x G(x,t;\nu)f(t,\nu)dt \tag{1.31}$$

where, with $\lambda = \nu^2$,

$$G(x,t;\nu) = y_1(t,\lambda)y_2(x,\lambda) - y_1(x,\lambda)y_2(t,\lambda).$$

As a consequence

$$r'(x,\nu) = 2i\nu\int_0^x \partial_x G(x,t;\nu)f(t,\nu)dt.$$

According to [26] p 14, the solutions $y_i(x,\lambda)$, $i = 1,2$, and their derivatives $y'_i(x,\lambda)$ admit the following expansion

$$y_1(x,\lambda) = \cos\nu x + \frac{1}{\nu}\int_0^x \sin\nu(x-t)\cdot\cos\nu t\cdot q(t)dt + O(\nu^{-2})$$
$$y_2(x,\lambda) = \frac{\sin\nu x}{\nu} + \frac{1}{\nu^2}\int_0^x \sin\nu(x-t)\cdot\sin\nu t\cdot q(t)dt + O(\nu^{-3})$$

and

$$y'_1(x,\lambda) = -\nu\sin\nu x + \int_0^x \cos\nu(x-t)\cdot\cos\nu t\cdot q(t)dt + O(\nu^{-1})$$
$$y'_2(x,\lambda) = \cos\nu x + \frac{1}{\nu}\int_0^x \cos\nu(x-t)\cdot\sin\nu t\cdot q(t)dt + O(\nu^{-2}).$$

These estimates are uniform on the strip $|\operatorname{Im}\nu| \leq C$, with $C > 0$ arbitrary. Hence

$$G(x,t;\nu) = \frac{\sin\nu(x-t)}{\nu} + \frac{1}{\nu^2}\int_0^x \sin\nu(x-s)\cdot\sin\nu(s-t)\cdot q(s)ds - \frac{1}{\nu^2}\int_0^t \sin\nu(s-t)\cdot\sin\nu(x-s)\cdot q(s)ds + O(\nu^{-3})$$

and

$$\partial_x G(x,t;\nu) = \cos\nu(x-t) + \frac{1}{\nu}\int_0^x \cos\nu(x-s)\cdot\sin\nu(s-t)\cdot q(s)ds$$
$$-\frac{1}{\nu}\int_0^t \sin\nu(s-t)\cdot\cos\nu(x-s)\cdot q(s)ds + O(\nu^{-2}).$$

When substituted into (1.31) one gets, up to an error term which is uniform on bounded sets of $q's$ and $f's$,

$$r(x,\nu) = I + II + III + O(\nu^{-2})$$

where

$$I \equiv I(x,\nu) = 2i\int_0^x \sin\nu(x-t)\cdot f(t,\nu)dt$$
$$II \equiv II(x,\nu) = \frac{2i}{\nu}\int_0^x dt f(t,\nu)\cdot\int_0^x ds\sin\nu(x-s)\cdot\sin\nu(s-t)\cdot q(s)$$
$$III \equiv III(x,\nu) = -\frac{2i}{\nu}\int_0^x dt f(t,\nu)\cdot\int_0^t ds\sin\nu(x-s)\cdot\sin\nu(s-t)\cdot q(s)$$

After regrouping the terms $\partial_x II$ and $\partial_x III$, the derivative $r'(x,\nu)$ can be written in the form

$$r'(x,\nu) = I_1 + II_1 + III_1 + O(\nu^{-1})$$

where

$$I_1 \equiv I_1(x,\nu) = \partial_x I(x,\nu) = 2i\nu\int_0^x \cos\nu(x-t)\cdot f(t,\nu)dt$$
$$II_1 \equiv II_1(x,\nu) = 2i\int_0^x dt f(t,\nu)\cdot\int_0^x ds\cos\nu(x-s)\cdot\sin\nu(s-t)\cdot q(s)$$
$$III_1 \equiv III_1(x,\nu) = -2i\int_0^x dt f(t,\nu)\cdot\int_0^t ds\cos\nu(x-s)\cdot\sin\nu(s-t)\cdot q(s).$$

To prove item (i), each of the three terms is treated separately. Recall that in (i), $f(x,\nu)$ is of the form $f(t,\nu) = h(t)e^{i\nu t}$. Using that

$$2i\sin\nu(x-t) = e^{i\nu x}e^{-i\nu t} - e^{-i\nu x}e^{i\nu t}$$

term I can be computed to be

$$I = e^{i\nu x}\int_0^x h(t)dt - e^{-i\nu x}\int_0^x h(t)e^{2i\nu t}dt.$$

As by assumption, $\nu_n = n\pi + \frac{1}{n}\ell_n^2$, one has

$$e^{\pm i\nu_n t} = e^{\pm in\pi t}\left(1 + \frac{1}{n}\ell_n^2\right).$$

Thus

$$I(1,\pm\nu_n) = (-1)^n \int_0^1 h(t)dt - (-1)^n \int_0^1 h(t)e^{\pm 2in\pi t}dt + \frac{1}{n}\ell_n^2.$$

Similarly one shows that

$$I_1(x,\nu) = i\nu e^{i\nu x} \int_0^x h(t)dt + i\nu e^{-i\nu x} \int_0^x h(t)e^{2i\nu t}dt$$

and therefore

$$I_1(1,\pm\nu_n) = \pm in\pi(-1)^n \int_0^1 h(t)dt \pm in\pi(-1)^n \int_0^1 h(t)e^{\pm 2in\pi t}dt + \ell_n^2.$$

To treat the term II, write

$$\begin{aligned} 2i\sin\nu(x-s)\cdot 2i\sin\nu(s-t) =& e^{i\nu x}e^{-i\nu t} + e^{-i\nu x}e^{i\nu t} \\ & - e^{-i\nu x}e^{-i\nu t}e^{2i\nu s} - e^{i\nu x}e^{i\nu t}e^{-2i\nu s} \end{aligned}$$

to get, with the notation of (1.29),

$$\begin{aligned} II(x,\nu) =& \frac{e^{i\nu x}}{2i\nu}\int_0^x h(t)dt\cdot Q(x) + \frac{e^{-i\nu x}}{2i\nu}\int_0^x h(t)e^{2i\nu t}dt\cdot Q(x) \\ & - \frac{e^{-i\nu x}}{2i\nu}\int_0^x h(t)dt\int_0^x q(s)e^{2i\nu s}ds \\ & - \frac{e^{i\nu x}}{2i\nu}\int_0^x h(t)e^{2i\nu t}dt\int_0^x q(s)e^{-2i\nu s}ds. \end{aligned}$$

As by assumption (1.29), $Q(1) = 0$ one gets, arguing as above

$$II(1,\pm\nu_n) = \frac{1}{n}\ell_n^2.$$

Similarly,

$$\begin{aligned} II_1(x,\nu) =& \frac{e^{i\nu x}}{2}\int_0^x h(t)dt\cdot Q(x) - \frac{e^{-i\nu x}}{2}\int_0^x h(t)e^{2i\nu t}dt\cdot Q(x) \\ & + \frac{e^{-i\nu x}}{2}\int_0^x h(t)dt\int_0^x q(s)e^{2i\nu s}ds \\ & - \frac{e^{i\nu x}}{2}\int_0^x h(t)e^{2i\nu t}dt\int_0^x q(s)e^{-2i\nu s}ds \end{aligned}$$

leading to

$$\begin{aligned} II_1(1,\pm\nu_n) =& \frac{(-1)^n}{2}\int_0^1 h(t)dt\cdot\int_0^1 q(s)e^{\pm 2in\pi s}ds \\ & + \frac{(-1)^{n+1}}{2}\int_0^1 h(t)e^{2in\pi t}dt\cdot\int_0^1 q(s)e^{\mp 2in\pi s}ds + \frac{1}{n}\ell_n^2. \end{aligned}$$

The term III is treated similarly, to get

$$
\begin{aligned}
III(1,\nu) = &-\frac{e^{i\nu}}{2i\nu}\int_0^1 Q(t)h(t)dt - \frac{e^{-i\nu}}{2i\nu}\int_0^1 Q(t)h(t)e^{2i\nu t}dt \\
&+ \frac{e^{i\nu}}{2i\nu}\int_0^1\int_0^1 1_{[0,t]}(s)q(s)h(t)e^{2i(\nu t-\nu s)}dtds \\
&+ \frac{e^{-i\nu}}{2i\nu}\int_0^1 q(s)\left(\int_s^1 h(t)dt\right)e^{2i\nu s}ds.
\end{aligned}
$$

As $Q(t)h(t)$ and $q(s)\cdot\int_s^1 h(t)dt$ are in $L^2_{\mathbb{C}}[0,1]$ and $1_{[0,t]}(s)q(s)h(t)$ is in $L^2_{\mathbb{C}}[0,1]^2$ it then follows that

$$
III(1,\pm\nu_n) = \pm\frac{(-1)^{n+1}}{2in\pi}\int_0^1 Q(t)h(t)dt + \frac{1}{n}\ell^2_n.
$$

Finally, in the same way, one obtains

$$
\begin{aligned}
&III_1(1,\pm\nu_n) \\
=&\frac{(-1)^{n+1}}{2}\int_0^1 h(t)Q(t)dt + \frac{(-1)^{n+1}}{2}\int_0^1 h(t)\left(\int_0^t q(s)e^{\pm 2i\pi ns}ds\right)dt \\
&+ \frac{(-1)^n}{2}\int_0^1 h(t)e^{\pm 2in\pi t}\left(\int_0^t q(s)e^{\mp 2in\pi s}ds\right)dt \\
&+ \frac{(-1)^n}{2}\int_0^1 h(t)e^{\pm 2in\pi t}Q(t)dt + \frac{1}{n}\ell^2_n.
\end{aligned}
$$

Towards item (ii) recall that in this case $f(x,\nu) = \frac{h(x,\nu)}{\nu^2}e^{i\nu x}$ and in a straightforward way each of the terms I,II, and III can be bounded pointwise by $O(\nu^{-2})$ whereas each of the terms I_1, II_1, III_1 can be bounded pointwise by $O(\nu^{-1})$, leading to the claimed estimates. Going through the various steps of the proof one verifies in a straightforward way that the claimed uniformity of the estimates (i) and (ii) hold. □

Lemma 1.4 will now be applied to get the desired estimates for $r_N(1,\pm\sqrt{\mu_n})$ as $n\to\infty$. Here $r_N(1,\pm\sqrt{\mu_n})$ is given by (1.22)-(1.26). Actually we formulate our results in a slightly more general form. For q in $H^N_{0,\mathbb{C}}$ and $1\le k\le N+2$, let us introduce $a_k = \int_0^1 s_k(x)dx$. Recall that $s_{N+2} = (-1)^{N+1}\partial_x^{N+1}q + L^2_{\mathbb{C}}$. Hence s_{N+2} is in $H^{-1}_{\mathbb{C}}$ and the integral $\int_0^1 s_{N+2}(x)dx$ is well-defined. By (1.19)

$$
a_{N+2} = -\sum_{j=1}^{N}\int_0^1 s_{N+1-j}(x)s_j(x)dx. \tag{1.32}
$$

Proposition 1.2. *Let q be in $H^N_{0,\mathbb{C}}$ with $N \geq 0$. Then for any sequence $\nu_n = n\pi + \frac{1}{n}\ell^2_n$ one has*

$$\begin{aligned}
(i) \quad r_N(1, \pm\nu_n) \quad =&(-1)^n a_{N+1} + (-1)^{n+1}(\pm 2in\pi)^N \int_0^1 q(x)e^{\pm 2in\pi x}dx \\
&\pm \frac{(-1)^n}{2in\pi} a_{N+2} + \frac{1}{n}\ell^2_n \\
(ii) \quad r'_N(1, \pm\nu_n) \quad =& \pm in\pi(-1)^n a_{N+1} \pm in\pi(-1)^n(\pm 2in\pi)^N \int_0^1 q(x)e^{\pm 2in\pi x}dx \\
&+ \frac{(-1)^n}{2} a_{N+2} + \ell^2_n,
\end{aligned}$$

uniformly for q's in bounded subsets of $H^N_{0,\mathbb{C}}$ and $(\nu_n)_{n\geq 1}$ in sets of sequences such that $(n(\nu_n - n\pi))_{n\geq 1}$ is uniformly bounded in $\ell^2_{\mathbb{C}}$.

Proof. (i) Let us first treat the case $N = 0$. Then the right hand side of (1.22) is given by $f_0(t, \nu_n) = q(x)e^{i\nu_n x}$. Hence one has by Lemma 1.4 (i) with $h = q$,

$$r_0(1, \pm\nu_n) = (-1)^{n+1} \int_0^1 q(x)e^{\pm 2in\pi x}dx + \frac{1}{n}\ell^2_n$$

where we used that, by assumption, $Q(1) = 0$ and that

$$\int_0^1 Q(x)q(x)dx = \int_0^1 \frac{1}{2}\partial_x Q(x)^2 dx = 0.$$

To apply Lemma 1.4 for $N \geq 1$ we write f_N on the right hand side of (1.22) in the form

$$f_N(x, \nu) = h_1(x)e^{i\nu x} + \frac{h_2(x)}{2i\nu}e^{i\nu x} + \frac{h_3(x, \nu)}{\nu^2}e^{i\nu x}.$$

To determine h_1, h_2, and h_3 we need to analyse the expression (1.23) defining $f_N(x, \nu)$ in more detail. Recall that by (1.20), $w_N(x, \nu) = \exp\left(\int_0^x \alpha_N(t, \nu)dt\right)$ and

$$\alpha_N(t, \nu) = i\nu + \frac{q(t)}{2i\nu} + O(\nu^{-2}).$$

Hence, with $Q(x) = \int_0^x q(t)dt$,

$$w_N(x, \nu) = e^{i\nu x}\left(1 + \frac{Q(x)}{2i\nu} + O(\nu^{-2})\right)$$

and

$$\begin{aligned}
f_N(x, \nu) =& s_{N+1}(x)e^{i\nu x} + \left(Q(x)s_{N+1}(x) - \sum_{l=1}^N s_{N+1-l}(x)s_l(x)\right)\frac{e^{i\nu x}}{2i\nu} \\
&+ \frac{h_3(x, \nu)}{\nu^2}e^{i\nu x}
\end{aligned} \tag{1.33}$$

where $h_3(x,\nu)$ can be explicitly computed from (1.23) and $h_3(\cdot,\nu_n)$ is bounded in $L^2_{\mathbb{C}}[0,1]$ uniformly. Thus by Lemma 1.4 one gets

$$\begin{aligned} r_N(1,\pm\nu_n) =&(-1)^n \int_0^1 s_{N+1} dx + (-1)^{n+1} \int_0^1 s_{N+1} e^{\pm 2in\pi x} dx \\ &\pm \frac{(-1)^{n+1}}{2in\pi} \cdot \int_0^1 Q \cdot s_{N+1} dx + \frac{1}{n}\ell^2_n \\ &\pm \frac{(-1)^n}{2in\pi} \left(\int_0^1 Q \cdot s_{N+1} dx - \int_0^1 \sum_{l=1}^N s_{N+1-l} \cdot s_l dx \right) \\ &\pm \frac{(-1)^{n+1}}{2in\pi} \int_0^1 Q \cdot s_{N+1} e^{\pm 2in\pi x} dx \\ &\pm \frac{(-1)^n}{2in\pi} \int_0^1 \left(\sum_{l=1}^N s_{N+1-l} \cdot s_l \right) e^{\pm 2in\pi x} dx + \frac{1}{n}\ell^2_n. \end{aligned}$$

To continue, note that by (1.18)

$$s_{N+1}(x) = (-1)^N \partial_x^N q(x) + H^2_{\mathbb{C}}.$$

As q is periodic, integrating by parts yields

$$\begin{aligned} \int_0^1 s_{N+1}(x) e^{\pm 2in\pi x} dx =&(-1)^N \int_0^1 \partial_x^N q(x) e^{\pm 2in\pi x} dx + O(\frac{1}{n^2}) \\ =&(\pm 2in\pi)^N \int_0^1 q(x) e^{\pm 2in\pi x} dx + O(\frac{1}{n^2}). \end{aligned}$$

Moreover, as $Q(x)s_{N+1}(x)$ and $\sum_{l=1}^N s_{N+1-l}(x)s_l(x)$ are in $L^2_{\mathbb{C}}$, their sequences of Fourier coefficients are in $\ell^2_{\mathbb{C}}$. Taking into account identity (1.32) and that the terms containing $\int_0^1 Q(x)s_{N+1}(x)dx$ cancel each other we then get

$$\begin{aligned} r_N(1,\pm\nu_n) =&(-1)^n \int_0^1 s_{N+1} dx + (-1)^{n+1} (\pm 2in\pi)^N \int_0^1 q(x) e^{\pm 2in\pi x} dx \\ &\pm \frac{(-1)^n}{2in\pi} a_{N+2} + \frac{1}{n}\ell^2_n. \end{aligned}$$

(ii) Again we treat the case $N = 0$ first. Then $f_0(t,\nu_n) = q(x)e^{i\nu_n x}$ and hence by Lemma 1.4 (i),

$$r_0'(1,\pm\nu_n) = \pm in\pi(-1)^n \int_0^1 q(x) e^{\pm 2in\pi x} dx + \ell^2_n$$

where we again used that $\int_0^1 Q(x)q(x)dx = 0$. As $a_1 = 0$ and $a_2 = 0$, the obtained asymptotics coincide with the claimed ones. If $N \geq 1$, then we again use the representation (1.33) of $f_N(x,\nu)$ to conclude from Lemma 1.4 that

$$r'(1,\pm\nu_n) = \pm\, in\pi(-1)^n a_{N+1} \pm in\pi(-1)^n \int_0^1 (-1)^N \partial_x^N q(x) e^{\pm 2in\pi x} dx + \frac{(-1)^n}{2} a_{N+2} + \ell_n^2$$

which leads to the claimed asymptotic estimate. Going through the various steps of the proof one verifies in a straightforward way that the claimed uniformity holds. □

At various occasions we will need the following property of the coefficients s_k for k even.

Lemma 1.5. *For q in $H^N_{0,\mathbb{C}}$ with $N \geq 0$, and $1 \leq k \leq N+2$, $a_k = \int_0^1 s_k(x)dx$ is equal to an integral of a polynomial of q and its derivatives up to order $k-3$. Moreover*

$$a_{2k} = \int_0^1 s_{2k}(x)dx = 0 \quad \forall\, 2 \leq 2k \leq N+2. \tag{1.34}$$

Proof. The first statement follows from the definition (1.16)-(1.17) of s_k. Indeed by (1.18), for $2 \leq k \leq N+1$, $a_k = \int_0^1 s_{k-2}(x)dx$. For the case $k = N+2$ see (1.32). The identities (1.34) hold in the case $N = 0$ or $N = 1$ as $a_2 = \int_0^1 -\partial_x q dx = 0$ by the definition (1.17) of s_2. It therefore suffices to consider the case where $N \geq 2$. By approximating $q \in H^N_{0,\mathbb{C}}$ with a sequence in $H^{4N}_{0,\mathbb{C}}$, it suffices to proof (1.34) for q in $H^{4N}_{0,\mathbb{C}}$. For $\nu \neq 0$, denote by $Y_{4N}(x,\nu)$ the solution matrix

$$Y_{4N}(x,\nu) = \begin{pmatrix} z_{4N}(x,-\nu) & z_{4N}(x,\nu) \\ z'_{4N}(x,-\nu) & z'_{4N}(x,\nu) \end{pmatrix}.$$

As $z_{4N}(0,\nu) = 1$ and $z'_{4N}(0,\nu) = \alpha_{4N}(0,\nu)$ one has

$$\det Y_{4N}(0,\nu) = \alpha_{4N}(0,\nu) - \alpha_{4N}(0,-\nu) = 2i\nu + \sum_{1\leq 2l+1\leq 4N} \frac{2s_{2l+1}(0)}{(2i\nu)^{2l+1}}.$$

Therefore, $\det Y_{4N}(0,\nu) \neq 0$ for $|\nu|$ sufficiently large. Furthermore, by the Wronskian identity, $\det Y_{4N}(1,\nu) = \det Y_{4N}(0,\nu)$ and hence, for $|\nu|$ sufficiently large,

$$\frac{z_{4N}(1,-\nu)z'_{4N}(1,\nu) - z_{4N}(1,\nu)z'_{4N}(1,-\nu)}{\alpha_{4N}(0,\nu) - \alpha_{4N}(0,-\nu)} = 1 \tag{1.35}$$

Write $z_{4N}(1,\pm\nu)$ as a product with an error term as follows. Introduce

$$\begin{aligned}A :=& \exp\left(i\nu + \sum_{1\le 2l+1\le 4N} \frac{a_{2l+1}}{(2i\nu)^{2l+1}}\right)\\ B :=& \exp\left(\sum_{2\le 2l\le 4N} \frac{a_{2l}}{(2i\nu)^{2l}}\right)\\ R_\pm :=& \frac{r_{4N}(1,\pm\nu)}{(\pm 2i\nu)^{4N+1}} \quad \text{and} \quad R'_\pm := \frac{r'_{4N}(1,\pm\nu)}{(\pm 2i\nu)^{4N+1}}\end{aligned}$$

As $\alpha_{4N}(x,\nu)$ is periodic in x, $\alpha_{4N}(1,\nu) = \alpha_{4N}(0,\nu)$, one concludes

$$\begin{aligned}z_{4N}(1,\nu) =& A\cdot B + R_+, & z_{4N}(1,-\nu) =& A^{-1}B + R_-\\ z'_{4N}(1,\nu) =& \alpha_{4N}(0,\nu)A\cdot B + R'_+, & z'_{4N}(1,-\nu) =& \alpha_{4N}(0,-\nu)A^{-1}B + R'_-.\end{aligned}$$

Substituting these expressions into (1.35) yields

$$B^2 = 1 - \frac{R}{\alpha_{4N}(0,\nu) - \alpha_{4N}(0,-\nu)} \tag{1.36}$$

where

$$\begin{aligned}R =& \alpha_{4N}(0,\nu)R_-AB + R'_+A^{-1}B + R_-R'_+\\ &- \alpha_{4N}(0,-\nu)R_+A^{-1}B - R'_-AB - R_+R'_-.\end{aligned}$$

By Lemma 1.3, applied to $f(x,\nu) = -2i\nu f_{4N}(x,\nu)$ with $f_{4N}(x,\nu)$ defined as in (1.23), one concludes that for ν real,

$$R_\pm = O\left(\frac{1}{\nu^{4N}}\right) \quad \text{and} \quad R'_\pm = O\left(\frac{1}{\nu^{4N-1}}\right).$$

Furthermore, $\alpha_{4N}(0,\nu) - \alpha_{4N}(0,-\nu) = 2i\nu + O\left(\frac{1}{\nu}\right)$. Taking the logarithm of both sides of (1.36) then yields, for $\nu \longrightarrow \infty$,

$$\begin{aligned}2\sum_{2\le 2l\le 4N} \frac{a_{2l}}{(2i\nu)^{2l}} =& \log\left(1 - \frac{R}{\alpha_{2N}(0,\nu) - \alpha_{2N}(0,-\nu)}\right)\\ =& O\left(\frac{1}{\nu^{4N}}\right)\end{aligned}$$

which implies that

$$a_{2l} = 0 \quad \forall 2 \le 2l \le 4N-2.$$

As $N+2 \le 4N-2$ for $N \ge 2$, the claimed result follows.

□

1.4 Asymptotics of μ_n and η_n

In this section we prove the asymptotic estimates for the Dirichlet and Neumann eigenvalues as stated in Theorem 0.6 in the Introduction.

Proof of Theorem 0.6. Let us first prove the asymptotics of the Dirichlet eigenvalues. The main ingredient of the proof are the special solutions $z_N(x, \pm\nu)$ of $-y'' + qy = \nu^2 y$ for q in $H^N_{0,\mathbb{C}}$, constructed in Section 1.3,

$$z_N(x,\nu) = \exp\left(\int_0^x \alpha_N(t,\nu)dt\right) + \frac{r_N(x,\nu)}{(2i\nu)^{N+1}} \tag{1.37}$$

where $\alpha_N(t,\nu) = i\nu + \sum_{k=1}^N \frac{s_k(t)}{(2i\nu)^k}$ and the functions $s_k(t)$ are given by (1.16)-(1.17). Note that $z_N(0, \pm\nu) = 1$ and recall that for $|\nu|$ sufficiently large, $z_N(x,\nu)$ and $z_N(x,-\nu)$ are linearly independent. Hence $z_N(x,\nu) - z_N(x,-\nu)$ is a scalar multiple of $y_2(x,\nu^2)$. The n'th Dirichlet eigenvalue μ_n therefore satisfies

$$z_N(1,\nu_n) - z_N(1,-\nu_n) = 0 \tag{1.38}$$

where $\nu_n = \sqrt[+]{\mu_n}$. To analyse (1.38) note that by Lemma 1.5, $\exp\left(\int_0^1 \alpha_N(t,\nu)dt\right)$ equals

$$\begin{aligned} A &= \exp\left(i\nu_n + \sum_{1\le 2l+1\le N} \frac{a_{2l+1}}{(2i\nu_n)^{2l+1}}\right) \\ &= (-1)^n \exp\left(i(\nu_n - n\pi) + \sum_{1\le 2l+1\le N} \frac{a_{2l+1}}{(2i\nu_n)^{2l+1}}\right) \end{aligned}$$

where $a_k = \int_0^1 s_k(t)dt$. Combining (1.37) and (1.38) we therefore get the equation

$$A - A^{-1} = R \quad \text{where} \quad R = \frac{r_N(1,-\nu_n)}{(-2i\nu_n)^{N+1}} - \frac{r_N(1,\nu_n)}{(2i\nu_n)^{N+1}}$$

i.e., A satisfies the quadratic equation

$$A^2 - RA - 1 = 0. \tag{1.39}$$

As $\mu_n = n^2\pi^2 + \ell^2_n$ one has $\nu_n = n\pi + \frac{1}{n}\ell^2_n$ and hence A is the solution of (1.39) given by

$$(-1)^n A = (-1)^n R/2 + \sqrt{1 + R^2/4} = 1 + R^2/8 + O(R^4) + (-1)^n R/2. \tag{1.40}$$

According to Proposition 1.2 and in view of the asymptotics $\nu_n = n\pi + \frac{1}{n}\ell^2_n$ one has

$$\begin{aligned} -(2i\nu_n)^{N+1}R &= r_N(1,\nu_n) + (-1)^N r_N(1,-\nu_n) \\ &= (-1)^n(1 + (-1)^N)a_{N+1} + 2(-1)^{n+1}(2in\pi)^N \langle q, \cos 2n\pi x\rangle \\ &\quad + \frac{(-1)^n}{2in\pi}(1 + (-1)^{N+1})a_{N+2} + \frac{1}{n}\ell^2_n. \end{aligned}$$

As by Lemma 1.5, $(1+(-1)^N)a_{N+1} = 2a_{N+1}$ as well as $(1+(-1)^{N+1})a_{N+2} = 2a_{N+2}$, and

$$(2i\nu_n)^{-(N+1)} = (2in\pi)^{-(N+1)}(1 + \frac{1}{n^2}\ell_n^2) \tag{1.41}$$

one gets

$$\frac{(-1)^n}{2}R = -\frac{1}{(2in\pi)^{N+1}}\Big(a_{N+1} - (2in\pi)^N\langle q, \cos 2n\pi x\rangle + \frac{1}{2in\pi}a_{N+2} + \frac{1}{n}\ell_n^2\Big).$$

As $a_1 = \int_0^1 q(x)dx = 0$ we then conclude that in the case $N = 0$,

$$(-1)^n R = \frac{1}{in\pi}\left(\langle q, \cos 2n\pi x\rangle + \frac{1}{n}\ell_n^2\right)$$

and $R^2 = \frac{1}{n^2}\ell_n^2$ whereas for $N \geq 1$, using that $(2in\pi)^N\langle q, \cos 2n\pi x\rangle = \ell_n^2$

$$R^2 = \frac{1}{(2in\pi)^{N+1}}O\left(\frac{1}{n^2}\right).$$

Substituting these estimates into (1.40) therefore yields in both cases, $N = 0$ and $N \geq 1$,

$$\exp\left(i(\nu_n - n\pi) + \sum_{1\leq 2l+1\leq N}\frac{a_{2l+1}}{(2i\nu_n)^{2l+1}}\right) = 1 + \frac{(-1)^n}{2}R + O(R^2)$$
$$= 1 - \frac{1}{(2in\pi)^{N+1}}\Big(a_{N+1} - (2in\pi)^N\langle q, \cos 2n\pi x\rangle + \frac{1}{2in\pi}a_{N+2} + \frac{1}{n}\ell_n^2\Big).$$

Taking the principal branch of the logarithm of both sides of the latter identity and multiplying by $-i$ we thus obtain in view of (1.41), that $\rho_n := \nu_n - n\pi$ equals

$$\rho_n = \sum_{1\leq 2l+1\leq N+2}(-1)^l\frac{a_{2l+1}}{(2\nu_n)^{2l+1}} - \frac{1}{2n\pi}\langle q, \cos 2n\pi x\rangle + \frac{1}{n^{N+2}}\ell_n^2. \tag{1.42}$$

Unfortunately, ν_n appears also on the right hand side of (1.42). To address this issue, we follow an approach found by Marchenko [23]. Let

$$F(z) = \sum_{1\leq 2l+1\leq N+2}\frac{(-1)^l}{2^{2l+1}}a_{2l+1}z^{2l+1} \tag{1.43}$$

and write

$$\frac{1}{\nu_n} = \frac{1}{n\pi + \rho_n} = \frac{1/n}{\pi + \rho_n/n}.$$

We approximate $F(\frac{1}{\nu_n})$ by approximating ρ_n by $\rho(1/n)$ in the above expression where ρ is an analytic function so that near $z = 0$,

$$\rho(z) - F\left(\frac{z}{\pi + z\rho(z)}\right) = 0.$$

To find ρ introduce

$$G(z,w) := w - F\left(\frac{z}{\pi + zw}\right).$$

Note that $G(0,0) = 0$ and $\partial_w G(0,0) = 1$. Hence by the implicit function theorem there exists near $z = 0$ a unique analytic function $\rho = \rho(z)$ so that $\rho(0) = 0$ and $G(z, \rho(z)) = 0$ for z near 0. Note that F is an odd function; hence

$$G(-z,-w) = -w + F\left(\frac{z}{\pi + zw}\right) = -G(z,w)$$

and as a consequence, $G(-z, -\rho(z)) = 0$ near $z = 0$. On the other hand, $G(-z, \rho(-z)) = 0$ and therefore, by the uniqueness of $\rho(z)$, one has $\rho(-z) = -\rho(z)$. It follows that ρ has an expansion of the form

$$\rho(z) = \sum_{k=0}^{\infty} b_{2k+1} z^{2k+1}. \tag{1.44}$$

The coefficients b_{2k+1} can be computed recursively from the identity

$$\rho(z) = F\left(\frac{z}{\pi + z\rho(z)}\right).$$

In this way one sees that for any $k \geq 0$, b_{2k+1} is a polynomial in the coefficients a_{2l+1} of F with $0 \leq l \leq k$. The Taylor expansion of $F(z)$ at $z_n = \frac{1}{n\pi + \rho(\frac{1}{n})}$ with Lagrange's remainder term reads

$$F\left(\frac{1}{\nu_n}\right) = \rho\left(\frac{1}{n}\right) + F_n \cdot \left(\rho\left(\frac{1}{n}\right) - \rho_n\right) \tag{1.45}$$

where

$$F_n = \int_0^1 F'\left(z_n + t\left(\frac{1}{\nu_n} - z_n\right)\right) dt \cdot \frac{1}{\nu_n \cdot (n\pi + \rho(1/n))} = O(\frac{1}{n^2}) \tag{1.46}$$

as $\rho(0) = 0$ and $\rho_n = \frac{1}{n}\ell_n^2 \to 0$ as $n \to \infty$. Subtracting $\rho(1/n)$ on both sides of the identity (1.42) then yields

$$(1 + F_n) \cdot (\rho_n - \rho(1/n)) = -\frac{1}{2n\pi}\langle q, \cos 2n\pi x\rangle + \frac{1}{n^{N+2}}\ell_n^2$$

or

$$\rho_n - \rho(1/n) = -\frac{1}{2n\pi}\langle q, \cos 2n\pi x\rangle + \frac{1}{n^{N+2}}\ell_n^2.$$

Thus we have shown that

$$\nu_n = n\pi + \sum_{1 \leq 2k+1 \leq N+2} b_{2k+1}\frac{1}{n^{2k+1}} - \frac{1}{2n\pi}\langle q, \cos 2n\pi x\rangle + \frac{1}{n^{N+2}}\ell_n^2.$$

By taking squares on both sides of the latter identity we obtain the asymptotics (8) with the claimed properties of the expression m_n of (10). Going through

the arguments of the proof one verifies the stated uniformity property of the error term in (8).
The asymptotic estimates for the Neumann eigenvalues $(\eta_n)_{n\geq 0}$ are derived in a similar way as the ones for the Dirichlet eigenvalues. Note that the special solutions $z_N(x,\pm\nu)$ satisfy $z_N(0,\pm\nu)=1$ and $z_N'(0,\pm\nu)=\alpha_N(0,\pm\nu)$. Hence $y(x,\nu):=\alpha_N(0,-\nu)z_N(x,\nu)-\alpha_N(0,\nu)z_N(x,-\nu)$ satisfies $y'(0,\nu)=0$. As $\alpha_N(0,\pm\nu)=\pm i\nu+\sum_{1\leq k\leq N}\frac{s_k(0)}{(\pm 2i\nu)^k}$, for $|\nu|$ sufficiently large,

$$y(0,\nu)=\alpha_N(0,-\nu)-\alpha_N(0,\nu)=-2i\nu+O\left(\frac{1}{\nu}\right)\neq 0.$$

Therefore $y(x,\nu)$ is parallel to $y_1(x,\nu^2)$. Again it is convenient to introduce $\nu_n=\sqrt{\eta_n}=n\pi+\frac{1}{n}\ell_n^2$. The n'th Neumann eigenvalue η_n is then characterised by

$$\alpha_N(0,-\nu_n)z_N'(1,\nu_n)-\alpha_N(0,\nu_n)z_N'(1,-\nu_n)=0. \tag{1.47}$$

To analyze (1.47) note that

$$z_N'(1,\pm\nu_n)=\alpha_N(1,\pm\nu_n)A^{\pm 1}+\frac{r_N'(1,\nu_n)}{(\pm 2i\nu_n)^{N+1}},$$

where $A^{\pm 1}=\exp\left(\pm i\nu_n+\sum_{1\leq 2l+1\leq N}\frac{a_{2l+1}}{(\pm 2i\nu_n)^{2l+1}}\right)$. As $\alpha_N(x,\pm\nu_n)$ is 1-periodic in x, (1.47) reads

$$\begin{aligned}&\alpha_N(0,-\nu_n)\alpha_N(0,\nu_n)A-\alpha_N(0,-\nu_n)\alpha_N(0,\nu_n)A^{-1}\\ =&\alpha_N(0,\nu_n)\frac{r_N'(1,-\nu_n)}{(-2i\nu_n)^{N+1}}-\alpha_N(0,-\nu_n)\frac{r_N'(1,\nu_n)}{(2i\nu_n)^{N+1}}.\end{aligned}$$

Thus A satisfies the quadratic equation

$$A^2-2RA-1=0$$

where

$$R=\frac{1}{2\alpha_N(0,-\nu_n)}\frac{r_N'(1,-\nu_n)}{(-2i\nu_n)^{N+1}}-\frac{1}{2\alpha_N(0,\nu_n)}\frac{r_N'(1,\nu_n)}{(2i\nu_n)^{N+1}}.$$

As $\nu_n=n\pi+\frac{1}{n}\ell_n^2$, we write

$$(-1)^nA=\exp\left(i(\nu_n-n\pi)+\sum_{1\leq 2l+1\leq N}\frac{a_{2l+1}}{(2i\nu_n)^{2l+1}}\right)$$

leading to

$$(-1)^nA=(-1)^nR+\sqrt[+]{1+R^2}=1+(-1)^nR+O(R^2). \tag{1.48}$$

By Proposition 1.2, $(-1)^n r'_N(1, \pm\nu_n)$ equals

$$\pm in\pi a_{N+1} \pm in\pi(\pm 2in\pi)^N \int_0^1 q(x) e^{\pm 2\pi inx} dx + \frac{a_{N+2}}{2} + \ell_n^2.$$

Further

$$\alpha_N(0, \pm\nu_n) = \pm i\nu_n + \sum_{1\le k\le N} \frac{s_k(0)}{(\pm 2i\nu_n)^k} = \pm in\pi \left(1 + O\left(\frac{1}{n^2}\right)\right),$$

and thus

$$\frac{(-1)^n r'_N(1, \pm\nu_n)}{\alpha_N(0, \pm\nu_n)} = a_{N+1} + (\pm 2in\pi)^N \int_0^1 q(x) e^{\pm 2\pi inx} dx + \frac{a_{N+2}}{\pm 2in\pi} + \frac{1}{n}\ell_n^2$$

yielding the asymptotic estimate

$$\begin{aligned}(-1)^n R = &\frac{1}{2}\frac{(-1)^{N+1}}{(2in\pi)^{N+1}} a_{N+1} - \frac{1}{2}\frac{1}{2in\pi}\int_0^1 q(x) e^{-2\pi inx} dx + \frac{a_{N+2}}{2}\frac{(-1)^{N+2}}{(2in\pi)^{N+2}} \\ -&\frac{1}{2}\frac{1}{(2in\pi)^{N+1}} a_{N+1} - \frac{1}{2}\frac{1}{2in\pi}\int_0^1 q(x) e^{2\pi inx} dx - \frac{a_{N+2}}{2}\frac{1}{(2in\pi)^{N+2}} + \frac{1}{n^{N+2}}\ell_n^2.\end{aligned}$$

Using that by Lemma 1.5,

$$a_{N+1}(1 + (-1)^N) = 2a_{N+1} \quad \text{and} \quad a_{N+2}(1 + (-1)^{N+1}) = 2a_{N+2},$$

one then gets

$$(-1)^n R = -a_{N+1}\frac{1}{(2in\pi)^{N+1}} - \frac{1}{2in\pi}\langle q, \cos 2\pi nx\rangle - a_{N+2}\frac{1}{(2in\pi)^{N+2}} + \frac{1}{n^{N+2}}\ell_n^2.$$

In view of $a_1 = a_2 = 0$ one then gets for any $N \ge 0$

$$R^2 = \frac{1}{n^{N+2}}\ell_n^2.$$

Substituting the latter two estimates into (1.48) one concludes

$$\begin{aligned}&\exp\left(i(\nu_n - n\pi) + \sum_{1\le 2l+1\le N} \frac{a_{2l+1}}{(2i\nu_n)^{2l+1}}\right) \\ =&1 - a_{N+1}\frac{1}{(2in\pi)^{N+1}} - \frac{1}{2in\pi}\langle q, \cos 2\pi nx\rangle - a_{N+2}\frac{1}{(2in\pi)^{N+2}} + \frac{1}{n^{N+2}}\ell_n^2\end{aligned}$$

and taking the principal branch of the logarithm on both sides one gets after multiplying by $-i$ for any $N \ge 0$,

$$\rho_n := \nu_n - n\pi = \sum_{1\le 2l+1\le N+2} \frac{(-1)^l a_{2l+1}}{(2\nu_n)^{2l+1}} + \frac{1}{2n\pi}\langle q, \cos 2\pi nx\rangle + \frac{1}{n^{N+2}}\ell_n^2.$$

Arguing as in the case of the Dirichlet eigenvalues one then concludes that

$$\nu_n = n\pi + \sum_{1\le 2l+1\le N+2} \frac{b_{2l+1}}{n^{2l+1}} + \frac{1}{2n\pi}\langle q, \cos 2\pi n x\rangle + \frac{1}{n^{N+2}}\ell^2_n.$$

Squaring both sides of the latter identity yields the claimed asymptotics for the Neumann eigenvalues. Going through the arguments of the proof one verifies the stated uniformity property of the error term. □

1.5 Asymptotics of λ_n

The main purpose of this section is to prove Theorem 0.7 and Theorem 1.1.

Proof of Theorem 0.7. As for the proof of Theorem 0.6, the main ingredient are the special solutions $z_N(x, \pm\nu_n)$ of $-y'' + qy = \nu^2 y$ for potentials $q \in H^N_{0,\mathbb{C}}$, constructed in Section 1.3,

$$z_N(x,\nu) = \exp\left(\int_0^x \alpha_N(t,\nu)dt\right) + \frac{r_N(x,\nu)}{(2i\nu)^{N+1}}$$

where $\alpha_N(t,\nu) = i\nu + \sum_{1\le k\le N} \frac{s_k(t)}{(2i\nu)^k}$. (Without further reference, we use the notation introduced in Section 1.3.) Recall from Section 1.3 that for $|\nu|$ sufficiently large, $z_N(x,\nu)$ and $z_N(x,-\nu)$ are linearly independent. Denote by $Y_N(x,\nu)$ the solution matrix

$$Y_N(x,\nu) = \begin{pmatrix} z_N(x,-\nu) & z_N(x,\nu) \\ z'_N(x,-\nu) & z'_N(x,\nu) \end{pmatrix}$$

and recall that $r_N(0,\nu) = 0$ and $r'_N(0,\nu) = 0$ so that

$$Y_N(0,\nu) = \begin{pmatrix} 1 & 1 \\ \alpha_N(0,-\nu) & \alpha_N(0,\nu). \end{pmatrix}$$

The large periodic eigenvalues of $-d_x^2 + q$ on the interval $[0,1]$ are thus given by the zeros of the characteristic function

$$\chi_p(\nu) := \det\left(Y_N(1,\nu) - Y_N(0,\nu)\right)$$

whereas the large antiperiodic eigenvalues of $-d_x^2 + q$ on the interval $[0,1]$ are given by the zeros of

$$\chi_{ap}(\nu) := \det\left(Y_N(1,\nu) + Y_N(0,\nu)\right).$$

The two cases are treated in a similar fashion and hence we concentrate on the periodic case only. Recall that for $q = 0$, they are given by $\lambda_0 = 0$, $\lambda_{4n} = \lambda_{4n-1} = (2n\pi)^2$. For arbitrary q one then knows that the large periodic eigenvalues are λ_{2n}, λ_{2n-1} with n large and *even*. It is convenient to introduce the notation

$$\nu_n^+ = \sqrt[+]{\lambda_{2n}}, \quad \nu_n^- = \sqrt[+]{\lambda_{2n-1}}$$

and to write ν_n if we do not need to specify our choice among ν_n^+ and ν_n^-. Let us now compute the asymptotics of $\chi_p(\nu)$. First note that $\alpha_N(1,\nu) = \alpha_N(0,\nu)$ as q is 1-periodic, leading to the formula

$$z'_N(1,\nu) = \alpha_N(0,\nu)\exp\left(\int_0^1 \alpha_N(t,\nu)dt\right) + \frac{r'_N(1,\nu)}{(2i\nu)^{N+1}}.$$

Then we have

$$z_N(1,\pm\nu_n) = A^{\pm 1} + R_\pm$$

and

$$z'_N(1,\pm\nu_n) = \alpha^\pm A^{\pm 1} + R'_\pm$$

where

$$A^{\pm 1} := \exp\left(\pm\int_0^1 \alpha_N(t,\nu_n)dt\right) = \exp\left(\int_0^1 \alpha_N(t,\pm\nu_n)dt\right)$$

(the latter identity follows from Lemma 1.5)

$$R_\pm := \frac{r_N(1,\pm\nu_n)}{(\pm 2i\nu_n)^{N+1}} \quad R'_\pm := \frac{r'_N(1,\pm\nu_n)}{(\pm 2i\nu_n)^{N+1}}$$

and

$$\alpha^\pm := \alpha_N(0,\pm\nu_n) = \pm i\nu_n + \sum_{k=1}^N \frac{s_k(0)}{(\pm 2i\nu_n)^k}.$$

Hence $\chi_p(\nu_n)$ equals

$$\left(A^{-1} + R_- - 1\right)\left(\alpha^+ A + R'_+ - \alpha^+\right) - \left(\alpha^- A^{-1} + R'_- - \alpha^-\right)\left(A + R_+ - 1\right)$$

or

$$\chi_p(\nu_n) = \xi A + \zeta + \eta A^{-1}$$

where

$$\begin{aligned}
\xi =& \alpha^+ R_- - R'_- - \alpha^+ + \alpha^- \\
\zeta =& 2\alpha^+ - 2\alpha^- + R'_- - \alpha^+ R_- - R'_+ + \alpha^- R_+ + R_- R'_+ - R'_- R_+ \\
\eta =& -\alpha^- R_+ + R'_+ - \alpha^+ + \alpha^-.
\end{aligned}$$

Note that

$$\zeta = -(\xi + \eta) + R \quad \text{with} \quad R = R_- R'_+ - R'_- R_+.$$

As $\chi_p(\nu_n) = 0$, one has $\xi A + \zeta + \eta A^{-1} = 0$ or

$$A = \frac{\xi + \eta - R + \epsilon\sqrt{(\xi + \eta - R)^2 - 4\xi\eta}}{2\xi} \tag{1.49}$$

with an appropriate choice of the sign $\epsilon = \epsilon_n^{\pm} \in \{\pm 1\}$. We want to estimate the terms on the right hand side of (1.49). By Proposition 1.2 and the assumption that n is *even*, $(-1)^n = 1$ and hence

$$r_N(1, \pm\nu_n) = a_{N+1} - e_n^{\pm} \pm \frac{1}{2in\pi} a_{N+2} + \frac{1}{n}\ell_n^2$$

where

$$e_n^{\pm} = (\pm 2in\pi)^N \int_0^1 q(x) e^{\pm 2in\pi x} dx.$$

Similarly, the asymptotics of $r_N'(1, \pm\nu_n)$ are given by

$$r_N'(1, \pm\nu_n) = \pm in\pi a_{N+1} \pm in\pi e_n^{\pm} + \frac{1}{2} a_{N+2} + \ell_n^2.$$

Furthermore, as $a_1 = \int_0^1 q dx = 0$ by assumption, and $\nu_n = n\pi + \frac{1}{n}\ell_n^2$

$$\alpha^{\pm} = \pm in\pi + \frac{1}{n}\ell_n^2.$$

These asymptotics yield the following estimates

$$\xi = \alpha^- - \alpha^+ + \frac{-a_{N+1} + a_{N+2}/(2in\pi) + \ell_n^2/n}{(-2in\pi)^N} \tag{1.50}$$

$$\eta = \alpha^- - \alpha^+ + \frac{a_{N+1} + a_{N+2}/(2in\pi) + \ell_n^2/n}{(2in\pi)^N}. \tag{1.51}$$

Furthermore, $R = R_- R_+' - R_-' R_+$ has an expansion of the form

$$R = \frac{i}{(2n\pi)^{2N+1}} \left(a_{N+1}^2 - e_n^+ e_n^- + \frac{1}{n}\ell_n^2 \right). \tag{1.52}$$

Now we are ready to estimate the terms on the right hand side of (1.49). By Lemma 1.5, $a_{N+1} = 0$ $[a_{N+2} = 0]$ for N odd [even]. Hence for any N, $(1 + (-1)^{N+1})a_{N+1} = 0$ and $(1 + (-1)^N)a_{N+2} = 0$. As a consequence, (1.50)-(1.51) yield

$$\xi + \eta = 2(\alpha^- - \alpha^+) + \frac{1}{n^{N+1}}\ell_n^2$$

and

$$\eta - \xi = \frac{1 + (-1)^N}{(2in\pi)^N} a_{N+1} + \frac{1 + (-1)^{N+1}}{(2in\pi)^{N+1}} a_{N+2} + \frac{1}{n^{N+1}}\ell_n^2.$$

Furthermore, if $N = 0$, then use $a_1 = \int_0^1 q(x)dx = 0$ to conclude that by (1.50), $R = \frac{1}{n}\ell_n^2$, whereas for $N = 1$, the fact that $a_2 = 0$ leads to the estimate $R = \frac{1}{n^2}\ell_n^2$. For $N \geq 2$, one gets from (1.50) that $R = \frac{1}{n^{N+1}}\ell_n^2$. Altogether we have established that for any $N \geq 0$

$$R = \frac{1}{n^{N+1}}\ell_n^2$$

and thus

$$\xi + \eta - R = 2(\alpha^- - \alpha^+) + \frac{1}{n^{N+1}} \ell_n^2.$$

To estimate the square root in (1.49), the term R will play a role. First note that

$$(\xi + \eta - R)^2 - 4\xi\eta = (\eta - \xi)^2 - 2(\xi + \eta)R + R^2.$$

As $a_{N+1} \cdot a_{N+2} = 0$ for any $N \geq 0$ by Lemma 1.5, we get

$$\begin{aligned}(\eta - \xi)^2 =& \frac{(1 + (-1)^N)^2}{(2in\pi)^{2N}} a_{N+1}^2 + \frac{1}{n^{2N+1}} \ell_n^2 \\ -2(\xi + \eta)R =& \left(4(\alpha^+ - \alpha^-) + \frac{1}{n^{N+1}} \ell_n^2\right) \frac{(-1)^{N+1}}{(2in\pi)^{2N+1}} \left(a_{N+1}^2 - e_n^+ e_n^- + \frac{1}{n} \ell_n^2\right) \\ =& \frac{(-1)^{N+1} 4}{(2in\pi)^{2N}} \left(a_{N+1}^2 - e_n^+ e_n^-\right) + \frac{1}{n^{2N+1}} \ell_n^2.\end{aligned}$$

Clearly $R^2 = \frac{1}{n^{2N+1}} \ell_n^2$ and thus

$$\begin{aligned}(\xi + \eta - R)^2 - 4\xi\eta =& \frac{(1 + (-1)^N)^2 - (-1)^N 4}{(2in\pi)^{2N}} a_{N+1}^2 + \frac{(-1)^N 4}{(2in\pi)^{2N}} e_n^+ e_n^- \\ &+ \frac{1}{n^{2N+1}} \ell_n^2.\end{aligned}$$

Using once more that $a_{N+1} = 0$ for N odd it follows that for any $N \geq 0$,

$$\left((1 + (-1)^N)^2 - (-1)^N 4\right) a_{N+1}^2 = 0$$

leading to

$$\sqrt{(\xi + \eta - R)^2 - 4\xi\eta} = \frac{2}{(2n\pi)^N} \sqrt{e_n^+ e_n^- + \frac{1}{n} \ell_n^2}.$$

Finally we need to estimate $1/2\xi$. As $(\alpha^- - \alpha^+) = -2in\pi + O(\frac{1}{n})$ it follows from (1.50) that

$$2\xi = 2(\alpha^- - \alpha^+) \left(1 + \frac{(-1)^N a_{N+1}}{(2in\pi)^{N+1}} - \frac{(-1)^N a_{N+2}}{(2in\pi)^{N+2}} + \frac{1}{n^{N+2}} \ell_n^2\right).$$

As $a_{N+1} = 0$ for $N = 0$, we get from the Taylor expansion $(1 + x)^{-1} = 1 - x + O(x^2)$, for any $N \geq 0$,

$$\frac{1}{2\xi} = \frac{1}{2(\alpha^- - \alpha^+)} \left(1 + \frac{(-1)^{N+1} a_{N+1}}{(2in\pi)^{N+1}} + \frac{(-1)^N a_{N+2}}{(2in\pi)^{N+2}} + \frac{1}{n^{N+2}} \ell_n^2\right).$$

Combining the estimates obtained so far and substituting them into the identity (1.49) we get for any $N \geq 0$, after dividing nominator and denominator of the right hand side by $2(\alpha^- - \alpha^+)$,

$$\begin{aligned}
A =& \left(1 \pm \frac{i\epsilon_n}{(2n\pi)^{N+1}}\sqrt{e_n^+ e_n^- + \frac{1}{n}\ell_n^2} + \frac{1}{n^{N+2}}\ell_n^2\right) \\
&\cdot \left(1 - \frac{(-1)^N a_{N+1}}{(2in\pi)^{N+1}} + \frac{(-1)^N a_{N+2}}{(2in\pi)^{N+2}} + \frac{1}{n^{N+2}}\ell_n^2\right) \\
=& 1 + \frac{(-1)^{N+1} a_{N+1}}{(2in\pi)^{N+1}} + \frac{(-1)^{N+2} a_{N+2}}{(2in\pi)^{N+2}} \\
&\pm \frac{i\epsilon_n}{(2n\pi)^{N+1}}\sqrt{e_n^+ e_n^- + \frac{1}{n}\ell_n^2} + \frac{1}{n^{N+2}}\ell_n^2.
\end{aligned}$$

Here we used that $\left(\sqrt{e_n^+ e_n^- + \frac{1}{n}\ell_n^2}\right)_{n\geq 1}$ is in ℓ^2. Taking the principal branch of the logarithm of both sides and taking into account that $\log(1+x) = x + O(x^2)$ as well as $\nu_n - n\pi = \frac{1}{n}\ell_n^2$ and $e^{in\pi} = 1$ as n is *even* one gets

$$\begin{aligned}
i(\nu_n - n\pi) + \sum_{1\leq 2l+1\leq N} \frac{a_{2l+1}}{(2i\nu_n)^{2l+1}} =& \frac{(-1)^{N+1} a_{N+1}}{(2in\pi)^{N+1}} + \frac{(-1)^{N+2} a_{N+2}}{(2in\pi)^{N+2}} \\
&\pm \frac{i\epsilon_n}{(2n\pi)^{N+1}}\sqrt{e_n^+ e_n^- + \frac{1}{n}\ell_n^2} + \frac{1}{n^{N+2}}\ell_n^2.
\end{aligned}$$

Using once more that by Lemma 1.5, $a_k = 0$ for k even we get, after multiplying both sides by $-i$, the following estimate for $\rho_n := \nu_n - n\pi$

$$\rho_n = \sum_{1\leq 2l+1\leq N+2} (-1)^l \frac{a_{2l+1}}{(2\nu_n)^{2l+1}} + \frac{\epsilon_n}{(2n\pi)^{N+1}}\sqrt{e_n^+ e_n^- + \frac{1}{n}\ell_n^2} + \frac{1}{n^{N+2}}\ell_n^2.$$

Arguing as in the proof of Theorem 0.6 (Section 1.4), one has by (1.43)-(1.46)

$$\sum_{1\leq 2l+1\leq N+2} (-1)^l \frac{a_{2l+1}}{(2\nu_n)^{2l+1}} = \rho\left(\frac{1}{n}\right) + F_n \cdot \left(\rho\left(\frac{1}{n}\right) - \rho_n\right)$$

with $\rho(z)$ given by (1.44) and F_n by (1.46). Therefore

$$(1 + F_n)\cdot\left(\rho_n - \rho\left(\frac{1}{n}\right)\right) = \frac{\epsilon_n}{(2n\pi)^{N+1}}\sqrt{e_n^+ e_n^- + \frac{1}{n}\ell_n^2} + \frac{1}{n^{N+2}}\ell_n^2.$$

By (1.46), $F_n = O(1/n^2)$ and thus

$$\nu_n = n\pi + \sum_{1\leq 2l+1\leq N+2} b_{2l+1}\frac{1}{n^{2l+1}} + \frac{\epsilon_n}{(2n\pi)^{N+1}}\sqrt{e_n^+ e_n^- + \frac{1}{n}\ell_n^2} + \frac{1}{n^{N+2}}\ell_n^2.$$

Squaring the latter expression yields

$$\nu_n^2 = m_n + \frac{\epsilon_n}{(2n\pi)^N}\sqrt{e_n^+ e_n^- + \frac{1}{n}\ell_n^2} + \frac{1}{n^{N+1}}\ell_n^2$$

with m_n given by (10). Going through the arguments of the proof one verifies that the error terms are uniformly bounded on bounded subsets of potentials in $H^N_{0,\mathbb{C}}$. □

We have also the following result on the coefficients c_{2j} in (10).

Corollary 1.2. *Let $N \in \mathbb{Z}_{\geq 0}$. Then for any $2 \leq 2j \leq N$, c_{2j} is a spectral invariant on $H^N_{0,\mathbb{C}}$, i.e., for any two potentials p, q in $H^N_{0,\mathbb{C}}$ so that $-d_x^2 + p$ and $-d_x^2 + q$ have the same periodic spectrum, one has $c_{2j}(p) = c_{2j}(q)$. In addition, if $N + 1$ is even (otherwise c_{N+1} vanishes on $H^N_{0,\mathbb{C}}$) there exists an open neighbourhood $\tilde{W}_N \subseteq H^N_{0,\mathbb{C}}$ of H^N_0 so that on $\tilde{W}_N$, c_{N+1} is a spectral invariant as well.*

Proof of Corollary 1.2. Let p, q in $H^N_{0,\mathbb{C}}$ be isospectral, i.e., $-d_x^2 + p$ and $-d_x^2 + q$ have the same periodic spectrum. By the asymptotic estimates (11) of Theorem 0.7 we have $c_{2j}(p) = c_{2j}(q)$ for any $2 \leq 2j \leq N$. The case of the coefficient c_{N+1}, $N + 1$ even, is more subtle as the factor in the asymptotic estimate (11) containing Fourier coefficients of q, is of comparable size.

By [12], Theorem 11.10 and Theorem 11.11, there exists an open neighbourhood $\tilde{W}_N \subseteq H^N_{0,\mathbb{C}}$ of H^N_0 so that any two isospectral potentials p, q in in $\tilde{W}_N$ can be approximated by isospectral finite gap potentials. In particular it follows that there exist sequences $(p_l)_{l\geq 1}, (q_l)_{l\geq 1}$ in $\tilde{W}_N$ with the following properties

(i) $(p_l)_{l\geq 1}, (q_l)_{l\geq 1} \subset H^k_{0,\mathbb{C}}$ for any $k \in \mathbb{Z}_{\geq 0}$;

(ii) $\lim_{l\to\infty} p_l = p$, $\lim_{l\to\infty} q_l = q$ in $H^N_{0,\mathbb{C}}$;

(iii) the periodic spectra of $-d_x^2 + p_l$ and $-d_x^2 + q_l$ coincide for any l in $\mathbb{Z}_{\geq 1}$.

By (i) and the first part of this proof it then follows that

$$c_{N+1}(p_l) = c_{N+1}(q_l) \quad \forall l \geq 1.$$

As c_{N+1} is the integral of a polynomial in q and its derivatives up to order $N-1$, it follows that, $c_{N+1}(p) = \lim_{l\to\infty} c_{N+1}(p_l)$ and $c_{N+1}(q) = \lim_{l\to\infty} c_{N+1}(q_l)$. Hence $c_{N+1}(p) = c_{N+1}(q)$ as claimed. □

Remark 1.1. *Denote by $\Delta(\lambda) \equiv \Delta(\lambda, q)$ the discriminant of $-d_x^2 + q$, i.e., $\Delta(\lambda) = y_1(1, \lambda) + y_2'(1, \lambda)$. Using Corollary 1.2 one can prove that the Poisson bracket $\{c_{N+1}, \Delta(\lambda)\} = \int_0^1 \partial_q c_{N+1} \partial_x \partial_q \Delta(\lambda)$ vanishes on H^N_0 for any $\lambda \in \mathbb{C}$. As c_{N+1} and $\Delta(\lambda)$ are defined on all of $H^N_{0,\mathbb{C}}$ and are analytic there $\{c_{N+1}, \Delta(\lambda)\} = 0$ on $H^N_{0,\mathbb{C}}$ for any $\lambda \in \mathbb{C}$.*

Unfortunately, the asymptotics of Theorem 0.7 do not suffice for our purposes. Actually we need estimates of $\gamma_n^2 = (\lambda_{2n} - \lambda_{2n-1})^2$ and $\tau_n = (\lambda_{2n} + \lambda_{2n-1})/2$, $n \geq 1$ which are better than the ones obtained from Theorem 0.7.

Theorem 1.1. *Let $N \in \mathbb{Z}_{\geq 0}$. Then for any q in $H^N_{0,\mathbb{C}}$,*

$$\gamma_n^2 = 4\hat{q}_n\hat{q}_{-n} + \frac{1}{n^{2N+1}}\ell_n^1 \tag{1.53}$$

uniformly on bounded subsets in $H^N_{0,\mathbb{C}}$.

Proof. The claimed estimate follows from Theorem 1.2 in [11]. In the case at hand it says that

$$\min_{\pm}\left|\gamma_n \pm 2\sqrt{\rho(n)\rho(-n)}\right| = \frac{1}{n^{N+1}}\ell_n^2 \tag{1.54}$$

uniformly on bounded sets of $H^N_{0,\mathbb{C}}$. Here the sequence $(\rho(n))_{n\in\mathbb{Z}}$ is given by

$$\rho(n) := \langle q, e^{2\pi i n x}\rangle + \beta_1(n)$$

with

$$\beta_1(n) := \frac{1}{\pi^2}\sum_{k\neq \pm n}\frac{\langle q, e^{2\pi i(n-k)x}\rangle}{n-k}\frac{\langle q, e^{2\pi i(n+k)x}\rangle}{n+k}.$$

Then

$$\begin{aligned}&\left|\gamma_n^2 - 4\langle q, e^{2\pi i n x}\rangle\langle q, e^{-2\pi i n x}\rangle\right| \\ \leq\ &\left|\gamma_n^2 - 4\rho(n)\rho(-n)\right| + 4\left|\rho(n)\rho(-n) - \langle q, e^{2\pi i n x}\rangle\langle q, e^{-2\pi i n x}\rangle\right|.\end{aligned} \tag{1.55}$$

Note that

$$\gamma_n^2 - 4\rho(n)\rho(-n) = \left(\gamma_n + \epsilon_n 2\sqrt{\rho(n)\rho(-n)}\right)\left(\gamma_n - \epsilon_n 2\sqrt{\rho(n)\rho(-n)}\right),$$

where $\epsilon_n \in \{+,-\}$ is chosen such that

$$\left|\gamma_n + \epsilon_n 2\sqrt{\rho(n)\rho(-n)}\right| = \min_{\pm}\left|\gamma_n \pm 2\sqrt{\rho(n)\rho(-n)}\right|.$$

By Cauchy's inequality

$$\begin{aligned}&\sum_{n\geq 1} n^{2N+1}\left|\gamma_n^2 - 4\rho(n)\rho(-n)\right| \\ \leq\ &\left(\sum_{n\geq 1} n^{2N+2}\left|\gamma_n + \epsilon_n 2\sqrt{\rho(n)\rho(-n)}\right|^2\right)^{1/2}\left(\sum_{n\geq 1} n^{2N}\left|\gamma_n - \epsilon_n 2\sqrt{\rho(n)\rho(-n)}\right|^2\right)^{1/2}.\end{aligned}$$

By (1.54), the first factor of the latter product is uniformly bounded on bounded sets of q's in $W \cap H^N_{0,\mathbb{C}}$, whereas the second factor can be estimated by

$$\left(\sum_{n\geq 1} n^{2N}|\gamma_n|^2\right)^{1/2} + 2\left(\sum_{n\geq 1} n^{2N}|\rho(n)|^2\right)^{1/2}\left(\sum_{n\geq 1} n^{2N}|\rho(-n)|^2\right)^{1/2}.$$

By [11], Theorem 1.1, $\left(\sum_{n\geq 1} n^{2N}|\gamma_n|^2\right)^{1/2}$ is uniformly bounded on bounded sets of q's in W whereas by the definition of $\rho(\pm n)$

$$\left(\sum_{n\geq 1} n^{2N}|\rho(\pm n)|^2\right)^{1/2} \leq \|q\|_N + \|\beta_1\|_N \leq \|q\|_N + \|q\|_N^2.$$

For the latter inequality we used that by [11], Lemma 2.10, $\|\beta_1\|_{N+1} \leq \|q\|_N^2$. It remains to estimate the second summand on the right hand side of (1.55). By the definition of $\rho(n)$

$$\begin{aligned}&\sum_{n\geq 1} n^{2N+1}\left|\rho(n)\rho(-n) - \langle q, e^{2\pi inx}\rangle\langle q, e^{-2\pi inx}\rangle\right|\\ \leq& 2\|\beta_1\|_{N+1}\|q\|_N + \|\beta_1\|_{N+1}^2\\ \leq& 2\|q\|_N^3(1+\|q\|_N),\end{aligned}$$

where we again used [11], Lemma 2.10. □

1.6 Asymptotics of the κ_n

In this section we prove Theorem 0.5.

Proof of Theorem 0.5. In Section 1.3, for $q \in H^N_{0,\mathbb{C}}$, we consider solutions of $-y'' + qy = \nu^2 y$ of the form

$$z_N(x,\pm\nu) = y_1(x,\nu^2) + \alpha_N(0,\pm\nu)y_2(x,\nu^2)$$

where

$$\alpha_N(x,\pm\nu) = \pm i\nu + \sum_{k=1}^{N} \frac{s_k(x)}{(\pm 2i\nu)^k}.$$

(Note that for $N = 0$, the latter sum is zero.) Hence for $\nu_n = \sqrt[+]{\mu_n}$ one gets

$$z_N(1,\nu_n) = y_1(1,\mu_n) = z_N(1,-\nu_n).$$

In view of (7) it then follows that

$$\kappa_n(q) = -\frac{1}{2}\log\left(z_N(1,\nu_n)z_N(1,-\nu_n)\right). \tag{1.56}$$

In Section 1.3 we show that

$$z_N(x,\pm\nu) = w_N(x,\pm\nu) + \frac{r_N(x,\pm\nu)}{(\pm 2i\nu)^{N+1}}$$

where

$$w_N(x,\pm\nu) = \exp\left(\int_0^x \alpha_N(t,\pm\nu)dt\right).$$

Hence $z_N(1,\nu_n)z_N(1,-\nu_n) = I + II + III$ where

$$\begin{aligned} I =& \exp\left(\sum_{k=1}^{N}(1+(-1)^k)\int_0^1 s_k(x)dx\cdot(2i\nu_n)^{-k}\right) \\ II =& w_N(1,-\nu_n)\frac{r_N(1,\nu_n)}{(2i\nu_n)^{N+1}} + w_N(1,\nu_n)\frac{r_N(1,-\nu_n)}{(-2i\nu_n)^{N+1}} \\ III =& r_N(1,\nu_n)r_N(1,-\nu_n)(-1)^{N+1}(2i\nu_n)^{-2N-2}. \end{aligned}$$

The three terms are analyzed separately. Let us begin with I. By Lemma 1.5

$$(1+(-1)^k)\int_0^1 s_k(x)dx = 0 \quad \forall 1 \le k \le N+2. \tag{1.57}$$

(These are the cancellations alluded to above.) Therefore

$$I = 1. \tag{1.58}$$

Towards II, note that in view of the assumption $\int_0^1 q(x)dx = 0$ one has $\mu_n = n^2\pi^2 + \ell_n^2$ and thus

$$\nu_n = n\pi + \frac{1}{n}\ell_n^2 = n\pi\left(1 + \frac{1}{n^2}\ell_n^2\right).$$

It implies that

$$(2i\nu_n)^{-N-1} = (2in\pi)^{-N-1} + n^{-N-3}\ell_n^2$$

and

$$w_N(1,\pm\nu_n) = (-1)^n\left(1 + \frac{1}{n}\ell_n^2\right).$$

Furthermore, by Proposition 1.2, $r_N(1,\pm\nu_n)$ is given by

$$(-1)^n\left(a_{N+1} - (\pm 2in\pi)^N\int_0^1 q(x)e^{\pm 2in\pi x}dx \pm \frac{1}{2in\pi}a_{N+2} + \frac{1}{n}\ell_n^2\right).$$

Hence

$$\begin{aligned} II =&(-1)^n(2in\pi)^{-N-1}\left(r_N(1,\nu_n) + (-1)^{N+1}r_N(1,-\nu_n)\right) + \frac{1}{n^{N+2}}\ell_n^2 \\ =&(2in\pi)^{-N-1}\Bigg((1+(-1)^{N+1})a_{N+1} - 2i\langle q, \sin 2\pi nx\rangle \\ &+ \frac{1+(-1)^{N+2}}{2in\pi}a_{N+2}\Bigg) + \frac{1}{n^{N+2}}\ell_n^2. \end{aligned}$$

Hence again by (1.57) (Lemma 1.5)

$$(1+(-1)^{N+2})a_{N+2} = 0 \quad \text{and} \quad (1+(-1)^{N+1})a_{N+1} = 0.$$

Therefore

$$II = -\frac{1}{n\pi}\langle q, \sin 2\pi nx\rangle + \frac{1}{n^{N+2}}\ell_n^2. \tag{1.59}$$

Finally,

$$III = \frac{1}{n^{N+2}}\ell_n^2. \tag{1.60}$$

(If $N \geq 1$ one has the stronger estimate $III = O(n^{-N-3})$.) Combining (1.58)-(1.60) then yields

$$z_N(1, \nu_n)z_N(1, -\nu_n) = 1 - \frac{1}{n\pi}\langle q, \sin 2\pi nx\rangle + \frac{1}{n^{N+2}}\ell_n^2$$

which, in view of (1.56), leads to

$$\begin{aligned}\kappa_n =& -\frac{1}{2}\log\left(z_N(1, \nu_n)z_N(1, -\nu_n)\right)\\ =& \frac{1}{2n\pi}\langle q, \sin 2\pi nx\rangle + \frac{1}{n^{N+2}}\ell_n^2.\end{aligned}$$

Going through the various steps of the proof one verifies in a straightforward way that the claimed uniformity holds. □

1.7 Asymptotics of τ_n, $\dot{\lambda}_n$, and σ_m^n

First we prove the following asymptotics for $(\tau_n)_{n\geq 1}$ stated in the Introduction.

Theorem 0.8. *(i) For any $q \in H_0^N$, $N \in \mathbb{Z}_{\geq 0}$,*

$$\tau_n(q) = m_n + \frac{1}{n^{N+1}}\ell_n^2 \tag{1.61}$$

where m_n is given by (10) *and the error term is uniformly bounded on bounded sets of potentials in H_0^N.*

(ii) For any $N \in \mathbb{Z}_{\geq 0}$, there exists an open neighbourhood $W_N \subseteq H_{0,\mathbb{C}}^N$ of H_0^N so that (12) *holds on W_N with a locally uniformly bounded error term.*

Remark 1.2. *We expect that the asymptotics* (12) *hold on all of $H_{0,\mathbb{C}}^N$, and that the error term in* (12) *is bounded on bounded sets of W_N. However, for our purposes, the result as stated suffices.*

Proof of Theorem 0.8 (i). Theorem 0.8 (i) is a direct consequence of the asymptotics of the periodic eigenvalues stated in Section 1.8 Theorem 1.2. □

In the case of complex valued potentials the arguments are more involved. The strategy to prove Theorem 0.8 (ii) is to try to find for any given element q in $H_{0,\mathbb{C}}^N$ an isospectral potential p with the property that the periodic spectrum consists of the disjoint union of the Dirichlet and Neumann spectrum. Such a

p is found with the help of the so called Birkhoff map (cf [12] for a detailed construction) for any q in $H^N_{0,\mathbb{C}}$ sufficiently close to the real subspace H^N_0. The asymptotics of τ_n are then obtained by applying Theorem 0.6 and Corollary 1.2. First we need to establish some auxiliary results. Introduce the following subset of $L^2_{0,\mathbb{C}}$

$$E := \{q \in L^2_{0,\mathbb{C}} |\ spec_D(-d_x^2 + q) \subseteq spec_p(-d_x^2 + q)\}.$$

We begin by examining some properties of $E \cap L^2_0$. It turns out that for q in $E \cap L^2_0$, each Neumann eigenvalue is a periodic one as well. To prove this fact we first need to establish the following result for even potentials. We say that $q \in L^2_{0,\mathbb{C}}$ is even if $q(x) = q(1-x)$ for a.e. $0 < x < 1$.

Lemma 1.6. *Assume that $q \in L^2_{0,\mathbb{C}}$ is even. Then each Dirichlet and each Neumann eigenvalue of $-d_x^2 + q$ is also a periodic eigenvalue.*

Proof. Consider first the Dirichlet eigenvalues $(\mu_n)_{n\geq 1}$. For any $n \geq 1$, the function $g(x) := y_2(1-x, \mu_n)$ satisfies the equation $(0 < x < 1)$

$$\begin{aligned} -g''(x) + q(x)g(x) =& - y_2''(1-x,\mu_n) + q(1-x)y_2(1-x,\mu_n) \\ =& \mu_n y_2(1-x,\mu_n) \end{aligned}$$

where we used that by assumption $q(x) = q(1-x)$ for a.e. $0 < x < 1$. As $g(0) = y_2(1,\mu_n)$ it then follows that $g(x) = g'(0)y_2(x,\mu_n)$ for any $0 \leq x \leq 1$. But $g'(0) = -y_2'(1,\mu_n)$ and therefore

$$-1 = g'(1) = g'(0)y_2'(x,\mu_n)|_{x=1} = -y_2'(1,\mu_n)^2.$$

As a consequence $y_2'(1,\mu_n) = \pm 1$, implying that μ_n is a periodic eigenvalue of $-d_x^2 + q$ (when considered of [0,2]). For the Neumann eigenvalues $\eta_n,\ n \geq 0$, one argues similarly. Recall that $y_1'(1,\eta_n) = 0$. Consider $h(x) := y_1(1-x,\eta_n)$. Then $h'(0) = 0$ and a.e. $0 < x < 1$,

$$-h''(x) + q(x)h(x) = \eta_n h(x).$$

Hence $h(x) = h(0)y_1(x,\eta_n)$, or, when evaluated at $x = 1$, $1 = y_1(1,\eta_n)^2$, again implying that η_n is a periodic eigenvalue. □

Lemma 1.6 allows us to prove the following result for elements in $E \cap L^2_0$, mentioned above.

Lemma 1.7. *For any $q \in E \cap L^2_0$, the Neumann spectrum $(\eta_n)_{n\geq 0}$ of $-d_x^2 + q$ is contained in the periodic spectrum as well and one has*

$$\eta_0 = \lambda_0, \quad \{\eta_n, \mu_n\} = \{\lambda_{2n-1}, \lambda_{2n}\} \quad \forall n \geq 1.$$

Proof. As $\mu_n(q)$ is assumed to be a periodic eigenvalue, one has $y_2'(1,\mu_n) = (-1)^n$ and hence $\kappa_n = \log(-1)^n y_2'(1,\mu_n) = 0$ for any $n \geq 1$. By [26], Lemma 3.4, it then follows that q is even and thus by Lemma 1.6 the Neumann eigenvalues $(\eta_n)_{n\geq 0}$ are also periodic ones. Furthermore, as q is assumed to be real, one has $\eta_0 \leq \lambda_0$ and, for any $n \geq 1$, $\lambda_{2n-1} \leq \mu_n, \eta_n \leq \lambda_{2n}$. Hence $\eta_0 = \lambda_0$ and $\{\eta_n, \mu_n\} = \{\lambda_{2n-1}, \lambda_{2n}\}$ for any $n \geq 1$ as claimed. □

Recall that the Birkhoff map is a map from an open neighbourhood W of L_0^2 to $\mathfrak{h}_{\mathbb{C}}^{1/2}$. In the following argument we use the coordinates $x_n = (z_n + z_{-n})/2$, $y_n = i(z_n - z_{-n})/2, n \geq 1$ – cf Introduction. Denote by $(x_n(q), y_n(q))_{n\geq 1}$ the image $\Phi(q)$. First we characterise the image of $E \cap W$ by Φ. For this purpose introduce the closed subspace Z of $\mathfrak{h}_{\mathbb{C}}^{1/2}$,

$$Z := \{(x_k, y_k)_{k\geq 1} \in \mathfrak{h}_{\mathbb{C}}^{1/2} |\; y_k = 0 \;\forall k \geq 1\}.$$

Lemma 1.8. *Elements of $E \cap W$ are mapped by Φ to $Z \cap \Phi(W)$ and*

$$\Phi|_{E\cap L_0^2} : E \cap L_0^2 \to Z \cap \mathfrak{h}^{1/2}$$

is bijective.

Proof. By a straightforward computation it follows from the definition of the Birkhoff map, analyzed in [12], that $\Phi(E \cap L_0^2) \subset Z \cap \mathfrak{h}^{1/2}$. Moreover, as $\Phi|_{L_0^2} : L_0^2 \to \mathfrak{h}^{1/2}$ is a diffeomorphism the restriction of Φ to $L_0^2 \cap E$ is $1 - 1$. To show that $\Phi(L_0^2 \cap E) = \mathfrak{h}^{1/2} \cap Z$, let $(x_k, 0)_{k\geq 1}$ be an arbitrary element in $\mathfrak{h}^{1/2} \cap Z$, and define $p = \Phi^{-1}((x_k, 0)_{k\geq 1})$. Denote by q the potential in $L_0^2 \cap E$ with the same periodic spectrum as p and determined uniquely by the conditions

$$\mu_k(q) = \begin{cases} \lambda_{2k} & \text{if } x_k \leq 0 \\ \lambda_{2k-1} & \text{if } x_k > 0. \end{cases}$$

By a straightforward calculation one then concludes that $\Phi(q) = \Phi(p)$ and hence $p = q$.

□

According to [12], for any $N \in \mathbb{Z}_{\geq 0}$, there exists an open neighbourhood $\tilde{W}_N$ of H_0^N in $W \cap H_{0,\mathbb{C}}^N$ such that $\Phi|_{\tilde{W}_N} : \tilde{W}_N \to \Phi(\tilde{W}_N) \subseteq \mathfrak{h}_{\mathbb{C}}^{N+1/2}$ is a diffeomorphism. We denote $\Phi(\tilde{W}_N)$ by $\tilde{V}_N$.

Proposition 1.3. *For any $N \in \mathbb{Z}_{\geq 0}$ and any $q \in \Phi^{-1}(\tilde{V}_N \cap Z)$.*

$$\eta_0(q) = \lambda_0(q) \quad \{\mu_n(q), \eta_n(q)\} = \{\lambda_{2n}(q), \lambda_{2n-1}(q)\} \;\forall n \geq 1.$$

Proof. As $\Phi|_{\tilde{W}_N}$ is a real analytic diffeomorphism onto its image $\tilde{V}_N$ and $Z \cap \tilde{V}_N$ is a real analytic submanifold of $\tilde{V}_N$ it follows that $\Phi^{-1}(\tilde{V}_N \cap Z)$ is a real analytic submanifold of $\tilde{W}_N$. Both, Dirichlet and Neumann eigenvalues are all simple for $q \in W$. It follows that they are real analytic functions on W and so are their restrictions to $\tilde{W}_N$. Using that the discriminant $\Delta(\lambda, q)$, given by $\Delta(\lambda, q) = y_1(1, \lambda) + y_2'(1, \lambda)$, is analytic on $\mathbb{C} \times L_{0,\mathbb{C}}^2$ it then follows that the compositions

$$F_n : \tilde{W}_N \to \mathbb{C},\; q \mapsto \Delta(\mu_n(q), q)^2 - 4 \quad (n \geq 1)$$

and

$$G_n : \tilde{W}_N \to \mathbb{C},\; q \mapsto \Delta(\eta_n(q), q)^2 - 4 \quad (n \geq 0)$$

are analytic. One verifies easily that they are real on $\tilde{W}_N \cap H_0^N$. By the definition of E, F_n $(n \geq 1)$ vanishes on $E \cap H_0^N$, and by Lemma 1.7, so does G_n $(n \geq 0)$. Furthermore, by Lemma 1.8, one concludes that $E \cap H_0^N = \Phi^{-1}(Z \cap \mathfrak{h}^{N+1/2})$. Hence, being analytic, the functions F_n $(n \geq 1)$ and G_n $(n \geq 0)$ vanish on $\Phi^{-1}(\tilde{V}_N \cap Z)$. By the choice of W (see end of the Introduction), for any $q \in W$ and $n \geq 1$, the eigenvalues $\mu_n(q), \eta_n(q), \lambda_{2n-1}(q), \lambda_{2n}(q)$ are contained in an isolating neighbourhood U_n. The U_n's are pairwise disjoint and none of them contains λ_0 or η_0. This implies the claimed statement. □

Using Proposition 1.3 we want to find a neighbourhood $W_N \subseteq \tilde{W}_N$ of H_0^N so that for any $q \in W_N$ there exists an isospectral potential $p \in W_N \cap E$. First we establish the following elementary result. For $\beta \in \mathbb{R}_{\geq 0}$ define

$$\ell^{1,\beta}_{\mathbb{C}} = \left\{ v = (v_k)_{k\geq 1} \subseteq \mathbb{C} | \; \|v\|_{\ell^{1,\beta}} = \sum_{k=1}^{\infty} k^{\beta} |v_k| < \infty \right\}.$$

Introduce the map

$$Q : \ell^{2,\alpha}_{\mathbb{C}} \to \ell^{1,2\alpha}_{\mathbb{C}}, \quad (u_k)_{k\geq 1} \mapsto Q((u_k)_{k\geq 1}) = (u_k^2)_{k\geq 1}.$$

Obviously one has $\|Q(u)\|_{\ell^{1,2\alpha}} = \|u\|^2_{\ell^{2,\alpha}}$. Denote by $B_{\ell^{1,\beta}}(v;\epsilon)$ $[B_{\ell^{2,\alpha}}(u;\epsilon)]$ the open ball of radius ϵ in $\ell^{1,\beta}_{\mathbb{C}}$ $[\ell^{2,\alpha}_{\mathbb{C}}]$, centered at $v \in \ell^{1,\beta}_{\mathbb{C}}$ $[u \in \ell^{2,\alpha}_{\mathbb{C}}]$.

Lemma 1.9. *For any $\alpha \geq 0$, $\epsilon > 0$, and $u \in \ell^2_{\alpha,\mathbb{C}}$*

$$Q(B_{\ell^{2,\alpha}}(u;\sqrt{\epsilon})) \supseteq B_{\ell^{1,2\alpha}}(Q(u);\epsilon).$$

Proof. Let $u \in \ell^{2,\alpha}_{\mathbb{C}}$ be arbitrary and consider $v = (v_k)_{k\geq 1} \in B_{\ell^{1,2\alpha}}(Q(u);\epsilon)$. We would like to find $(h_k)_{k\geq 1} \in \ell^{2,\alpha}_{\mathbb{C}}$ so that for any $k \geq 1$,

$$(u_k + h_k)^2 = v_k \text{ implying that } \|(u_k + h_k)^2 - u_k^2\|_{\ell^{1,2\alpha}} < \epsilon.$$

For any $k \geq 1$ we obtain the following quadratic equation for h_k

$$h_k^2 + 2u_k h_k - b_k \quad \text{where} \quad b_k = v_k - u_k^2.$$

Choose the sign $\sigma_k \in \{\pm 1\}$ of the root in $h_k = -u_k + \sigma_k \sqrt{v_k^2 + b_k}$ in such a way that

$$|h_k| = \min_{\pm} |-u_k \pm \sqrt{u_k^2 + b_k}|.$$

Then

$$h_k^2 \leq |(-u_k + \sigma_k \sqrt{u_k^2 + b_k})(-u_k - \sigma_k \sqrt{u_k^2 + b_k})| = |b_k|$$

and therefore

$$\|(h_k)_{k\geq 1}\|_{\ell^{2,\alpha}} \leq \|v - Q(u)\|_{\ell^{1,2\alpha}} < \epsilon.$$

□

The following lemma will allow us to deal with complex potentials.

Lemma 1.10. *For any $z^0 \in \mathfrak{h}^{N+1/2}$ there exists a neighbourhood $V(z^0)$ of z^0 in $\tilde V_N$ so that for any $z = (x_k, y_k)_{k\ge 1} \in V(z_0)$ there exists an element $(u_k, 0)_{k\ge1}$ in $\tilde V_N$ with $u_k^2 = x_k^2 + y_k^2$ for any $k \ge 1$.*

Proof. Let $z^0 = (x_k^0, y_k^0)_{k\ge1}$ be an arbitrary real sequence in $\mathfrak{h}^{N+1/2}$. Define $u^0 = (u_k^0)_{k\ge1}$ in $\ell^{2,N+1/2}$ by setting $u_k^0 = \sqrt[+]{v_k^0}$ for any $k \ge 1$ where $v_k^0 = (x_k^0)^2 + (y_k^0)^2$. As $(u_k^0, 0)_{k\ge1} \in \mathfrak{h}^{N+1/2}$ and $\tilde V_N (= \Phi(\tilde W_N))$ is open in $\mathfrak{h}_{\mathbb{C}}^{N+1/2}$, we can choose $\epsilon > 0$ so that

$$\{(u_k, 0)_{k\ge1} | \ (u_k)_{k\ge1} \in B_{\ell^{2,N+1/2}}(u^0; \sqrt{\epsilon})\} \subseteq \tilde V_N.$$

Now consider the map

$$P : \tilde V_N \to \ell_{\mathbb{C}}^{1,2N+1}, \quad (x_k, y_k)_{k\ge1} \mapsto (x_k^2 + y_k^2)_{k\ge1}.$$

Clearly, P is continuous and therefore there exists a neighbourhood $V(z^0)$ of z^0 in $\tilde V_N$ so that

$$P(V(z^0)) \subseteq B_{\ell^{1,2N+1}}(v^0; \epsilon).$$

By Lemma 1.9, $B_{\ell^{1,2\alpha}}(Q(v^0); \epsilon) \subseteq Q(B_{\ell^{2,\alpha}}(v^0; \sqrt{\epsilon}))$ and hence the neighbourhood $V(z^0)$ has the claimed property. □

Having made these preparations we are ready to prove Theorem 0.8.

Proof of Theorem 0.8 (ii). Let $W_N \subseteq \tilde W_N$ be the open neighbourhood of H_0^N in $W \cap H_{0,\mathbb{C}}^N$ given by

$$W_N = \bigcup_{q \in H_0^N} \Phi^{-1}(V(\Phi(q)))$$

where $V(\Phi(q)) \subseteq \tilde V_N$ is the neighbourhood of Lemma 1.10. Then for any $q \in W_N$ there exists an element $p \in \tilde W_N$ so that $\Phi(p) = (u_k, 0)_{k\ge1} \in \tilde V_N$ satisfies $u_k^2 = x_k^2 + y_k^2$ where $\Phi(q)) = (x_k, y_k)_{k\ge1}$. Hence p and q are isospectral – see [12], Theorem 11.10. In particular $\tau_n(q) = \tau_n(p)$ for any $n \ge 1$. By Proposition 1.3 $\{\mu_n(p), \eta_n(p)\} = \{\mu_n(q), \eta_n(q)\}$ for any $n \ge 1$ and hence by Theorem 0.6

$$\tau_n(p) = m_n(p) + \frac{1}{n^{N+1}} \ell_n^2.$$

As by Corollary 1.2, $m_n(p) = m_n(q)$, one then gets

$$\tau_n(q) = m_n(q) + \frac{1}{n^{N+1}} \ell_n^2. \tag{1.62}$$

Going through the arguments of the proof one verifies the error term in (1.62) is locally uniformly bounded. □

Combining Theorem 0.6 and Theorem 0.8 one obtains

Corollary 1.3. *(i) For any $q \in H_0^N$, $N \in \mathbb{Z}_{\geq 0}$,*

$$\tau_n - \mu_n = \langle q, \cos 2\pi n x\rangle + \frac{1}{n^{N+1}}\ell_n^2 \tag{1.63}$$

where the error term is uniformly bounded on bounded sets of potentials in H_0^N.

(ii) For any $N \in \mathbb{Z}_{\geq 0}$, there exists an open neighbourhood $W_N \subseteq H_{0,\mathbb{C}}^N$ of H_0^N so that (1.63) *holds on W_N with a locally uniformly bounded error term.*

In the remainder of this section we study some applications of the asymptotics of the periodic eigenvalues. For your purposes it suffices to consider potentials q in a sufficiently small neighbourhood W of L_0^2 in $L_{0,\mathbb{C}}^2$. Recall that we denote by $\Delta(\lambda)$ the discriminant of $-d_x^2 + q$ and that $\Delta^2(\lambda) - 4$ has the product representation

$$\Delta^2(\lambda) - 4 = 4(\lambda_0 - \lambda)\prod_{n\geq 1}\frac{(\lambda_{2n}-\lambda)(\lambda_{2n-1}-\lambda)}{\pi_n^4}$$

where $\pi_n = n\pi$ for any $n \geq 1$. Let $\dot{\Delta}(\lambda) = \partial_\lambda \Delta(\lambda)$. According to Proposition B.13 of [12] it also admits a product representation,

$$\dot{\Delta}(\lambda) = -\prod_{n\geq 1}\frac{\dot{\lambda}_n - \lambda}{\pi_n^2},$$

and the zeros $\dot{\lambda}_n$ satisfy

$$\dot{\lambda}_n - \tau_n = O(\gamma_n^2) \tag{1.64}$$

locally uniformly for q in W. Shrinking W, if necessary, we can assume without loss of generality that for any $n \geq 1$, $\dot{\lambda}_n \in U_n$ locally uniformly in W. The following estimate improves on the one of Proposition B.13 in [12].

Proposition 1.4. *For q in W,*

$$\dot{\lambda}_n - \tau_n = \frac{\gamma_n^2}{n}\ell_n^2 \tag{1.65}$$

locally uniformly on W. On L_0^2, (1.65) *is uniformly bounded on bounded subsets of L_0^2.*

Proof. For any given $n \geq 1$, write

$$\Delta^2(\lambda) - 4 = \frac{(\lambda_{2n}-\lambda)(\lambda - \lambda_{2n-1})}{\pi_n^2}\Delta_n(\lambda) \tag{1.66}$$

where

$$\Delta_n(\lambda) = 4\,\frac{\lambda - \lambda_0}{\pi_n^2}\left(\prod_{m\neq n}\frac{\lambda_{2m}-\lambda}{\pi_m^2}\right)\left(\prod_{m\neq n}\frac{\lambda_{2m-1}-\lambda}{\pi_m^2}\right). \tag{1.67}$$

Uniformly for $\lambda \in U_n$,

$$\frac{\lambda - \lambda_0}{\pi_n^2} = 1 + O\left(\frac{1}{n^2}\right) = 1 + \frac{1}{n}\ell_n^2.$$

By Corollary 1.1, uniformly for $\lambda \in U_n$,

$$\begin{aligned}\left(\prod_{m\neq n}\frac{\lambda_{2m}-\lambda}{\pi_m^2}\right)\left(\prod_{m\neq n}\frac{\lambda_{2m-1}-\lambda}{\pi_m^2}\right) &= \left(\frac{(-1)^{n+1}}{2} + \frac{1}{n}\ell_n^2\right)^2 \\ &= \frac{1}{4} + \frac{1}{n}\ell_n^2,\end{aligned}$$

hence

$$\Delta_n(\lambda) = 1 + \frac{1}{n}\ell_n^2 \tag{1.68}$$

and by shrinking the size of the isolating neighbourhoods one obtains from Cauchy's estimate that

$$\dot\Delta_n(\lambda) = \frac{1}{n}\ell_n^2$$

uniformly in $n \geq 1$, $\lambda \in U_n$. By (1.66)

$$\begin{aligned} 0 &= \frac{d}{d\lambda}|_{\lambda=\dot\lambda_n}(\Delta^2(\lambda) - 4) \\ &= \frac{\lambda_{2n-1} - \dot\lambda_n + \lambda_{2n} - \dot\lambda_n}{\pi_n^2}\Delta_n(\dot\lambda_n) \\ &\quad + \frac{(\lambda_{2n} - \dot\lambda_n)(\dot\lambda_n - \lambda_{2n-1})}{\pi_n^2}\dot\Delta_n(\dot\lambda_n)\end{aligned}$$

or

$$0 = 2(\tau_n - \dot\lambda_n)(1 + \frac{1}{n}\ell_n^2) + \left(\frac{\gamma_n^2}{4} - (\tau_n - \dot\lambda_n)^2\right)\frac{1}{n}\ell_n^2.$$

Hence

$$(\tau_n - \dot\lambda_n)\left(1 + \frac{1}{n}\ell_n^2 + (\tau_n - \dot\lambda_n)\frac{1}{n}\ell_n^2\right) = \frac{\gamma_n^2}{n}\ell_n^2.$$

As by (1.64), $\tau_n - \dot\lambda_n = O(1)$ it then follows that

$$\tau_n - \dot\lambda_n = \frac{\gamma_n^2}{n}\ell_n^2$$

as claimed. Going through the arguments of the proof one sees that (1.65) holds locally uniformly on W and uniformly on bounded subsets of L_0^2. □

By Theorem D.1 in [12] there exists a sequence $(\psi_n)_{n\geq 1}$ of entire functions,

$$\psi_n(\lambda) = \frac{2}{\pi_n}\prod_{m\neq n}\frac{\sigma_m^n - \lambda}{\pi_m^2} \tag{1.69}$$

so that

$$\frac{1}{2\pi}\int_{\Gamma_m}\frac{\psi_n(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}d\lambda=\delta_{mn}\quad\forall m\geq 1,\tag{1.70}$$

where $\sqrt[c]{\Delta^2(\lambda)-4}$ denotes the canonical root introduced in [12], Section 6. Recall that Γ_m denotes a counterclockwise oriented circuit in U_m around the interval G_m and $\tau_m=(\lambda_{2m}+\lambda_{2m-1})/2$. By Theorem D.1 in [12], for any $n\geq 1$, the zeros σ_m^n, $m\neq n$, are real analytic functions of $q\in W$ so that $\sigma_m^n-\tau_m=O\left(\gamma_m^2/m\right)$ uniformly for $n\geq 1$, and locally uniformly in $q\in W$. Moreover one can choose W so that $\sigma_m^n\in U_m$ for any $n\geq 1$, $m\neq n$, and locally uniformly in W.

Proposition 1.5. *For q in W,*

$$\sigma_m^n-\tau_m=\frac{\gamma_m^2}{m}\ell_m^2\tag{1.71}$$

locally uniformly on W and uniformly in n. On L_0^2, (1.71) *is uniformly bounded on bounded subsets of L_0^2.*

Proof. We drop the superscript in $\sigma^n=(\sigma_m^n)_{m\neq n}$ for the course of this proof. For $m\neq n$ one has

$$0=\int_{\Gamma_m}\frac{\sigma_m-\lambda}{\sqrt[s]{(\lambda_{2m}-\lambda)(\lambda-\lambda_{2m-1})}}\chi_m^n(\sigma,\lambda)d\lambda\tag{1.72}$$

with

$$\chi_m^n(\sigma,\lambda)=(-1)^{m-1}\frac{\sigma_n-m^2\pi^2}{\sigma_n-\lambda}\frac{m\pi}{\sqrt[+]{\lambda-\lambda_0}}\prod_{j\neq m}\frac{\sigma_j-\lambda}{\sqrt[+]{(\lambda_{2j}-\lambda)(\lambda_{2j-1}-\lambda)}}$$

and $\sigma_n=\tau_n$. For $\lambda=m^2\pi^2+\ell_m^2$, one has uniformly in $n\geq 1$, $m\neq n$,

$$\frac{\sigma_n-m^2\pi^2}{\sigma_n-\lambda}=1+\frac{\lambda-m^2\pi^2}{\sigma_n-\lambda}=1+\frac{1}{m}\ell_m^2$$

and $\sqrt[+]{\lambda-\lambda_0}=m\pi\sqrt[+]{1+\frac{1}{m}\ell_m^2}=m\pi\left(1+\frac{1}{m}\ell_m^2\right)$. Furthermore, by Proposition 1.1,

$$\left(\prod_{j\neq m}\frac{\sigma_j-\lambda}{\sqrt[+]{(\lambda_{2j}-\lambda)(\lambda_{2j-1}-\lambda)}}\right)^2=1+\frac{1}{m}\ell_m^2.\tag{1.73}$$

Taking square roots on both sides of (1.73), one has

$$(-1)^{m-1}\prod_{j\neq m}\frac{\sigma_j-\lambda}{\sqrt[+]{(\lambda_{2j}-\lambda)(\lambda_{2j-1}-\lambda)}}=1+\frac{1}{m}\ell_m^2.$$

Altogether we have for $\lambda=m^2\pi^2+\ell_m^2$, $\lambda\in U_m$,

$$\chi_m^n(\lambda)=1+\frac{1}{m}\ell_m^2\tag{1.74}$$

uniformly in n. The integral

$$\int_{\Gamma_m} \frac{\sigma_m - \lambda}{\sqrt[s]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} d\lambda \tag{1.75}$$

can be explicitly computed. In the case where $\lambda_{2m} = \lambda_{2m-1}$, one gets from the definition of the s-root and Cauchy's formula that the integral equals $2\pi(\sigma_m - \tau_m)$. In the case $\lambda_{2m} \neq \lambda_{2m-1}$,

$$\int_{\Gamma_m} \frac{\sigma_m - \lambda}{\sqrt[s]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} d\lambda = 2 \int_{\lambda_{2m-1}}^{\lambda_{2m}} \frac{\sigma_m - \lambda}{\sqrt[+]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} d\lambda.$$

By the change of coordinate $\lambda = \tau_m + t\frac{\gamma_m}{2}$ we get

$$\begin{aligned} \int_{\Gamma_m} \frac{\sigma_m - \lambda}{\sqrt[s]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} d\lambda &= 2 \int_{-1}^{1} \frac{\sigma_m - \tau_m}{\sqrt[+]{1 - t^2}} dt - \gamma_m \int_{-1}^{1} \frac{t}{\sqrt[+]{1 - t^2}} dt \\ &= 2(\sigma_m - \tau_m) \int_{-1}^{1} \frac{1}{\sqrt[+]{1 - t^2}} dt. \end{aligned}$$

Therefore in both cases

$$\frac{1}{2\pi} \int_{\Gamma_m} \frac{\sigma_m - \lambda}{\sqrt[s]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} d\lambda = \sigma_m - \tau_m.$$

Hence, by (1.72)

$$0 = (\sigma_m - \tau_m)\chi_m^n(\tau_m) + \frac{1}{2\pi} \int_{\Gamma_m} \frac{(\sigma_m - \lambda)(\chi_m^n(\lambda) - \chi_m^n(\tau_m))}{\sqrt[s]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} d\lambda. \tag{1.76}$$

As $\tau_m = m^2\pi^2 + \ell_m^2$ it follows from (1.74) that

$$\chi_m^n(\tau_m) = 1 + \frac{1}{m}\ell_m^2 \tag{1.77}$$

and, by the Taylor expansion of χ_m^n at τ_m and Cauchy's estimate, for $\lambda = m^2\pi^2 + \ell_m^2$, $\lambda \in U_m$

$$\frac{\chi_m^n(\lambda) - \chi_m^n(\tau_m)}{\lambda - \tau_m} = \frac{1}{m}\ell_m^2.$$

Finally as $\sigma_m - \lambda = O(\gamma_m), \lambda \in G_m$, one has $(\sigma_m - \lambda)(\lambda - \tau_m) = O(\gamma_m^2)$ for $\lambda \in G_m$. Hence by Lemma M.1 in [12]

$$\begin{aligned} &\int_{\Gamma_m} \frac{(\sigma_m - \lambda)(\chi_m^n(\lambda) - \chi_m^n(\tau_m))}{\sqrt[s]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} d\lambda \\ =&2 \int_{G_m} \frac{(\sigma_m - \lambda)(\lambda - \tau_m)\frac{\chi_m^n(\lambda) - \chi_m^n(\tau_m)}{\lambda - \tau_m}}{\sqrt[+]{(\lambda_{2m} - \lambda)(\lambda - \lambda_{2m-1})}} d\lambda \\ =&\frac{\gamma_m^2}{m}\ell_m^2 \end{aligned}$$

This together with (1.76) gives

$$\sigma_m^n - \tau_m = \frac{\gamma_m^2}{m}\ell_m^2$$

uniformly in n. Going through the arguments of the proof one sees that (1.71) holds locally uniformly on W and uniformly on bounded subsets of L_0^2. □

1.8 Asymptotics of λ_n for real potentials

The aim of this section is to prove the following asymptotics of the periodic eigenvalues of $-d_x^2 + q$ for q *real valued.*

Theorem 1.2. *For any $N \in \mathbb{Z}_{\geq 0}$ and $q \in H_0^N$,*

$$\lambda_{2n} = m_n + |\langle q, e^{2\pi i n x}\rangle| + \frac{1}{n^{N+1}}\ell_n^2 \tag{1.78}$$

and

$$\lambda_{2n-1} = m_n - |\langle q, e^{2\pi i n x}\rangle| + \frac{1}{n^{N+1}}\ell_n^2 \tag{1.79}$$

uniformly on bounded subsets of H_0^N. Here m_n is the expression given by (10).

Remark 1.3. *For $N \geq 1$, the asymptotics* (1.78)-(1.79) *are due to Marchenko [23], whereas for the uniformity statement we could not find any reference.*

Proof. We again use the special solutions $z_N(x,\nu)$ introduced in Section 1.3. Following Marchenko we represent these special solutions for $|\nu|$ sufficiently large by an exponential function , $z_N(x,\nu) = \exp\left(i\nu x + \int_0^x \sigma(t,\nu)dt\right)$, where

$$\sigma(t,\nu) = \sum_1^N \frac{s_k(t)}{(2i\nu)^k} + \frac{\sigma_N(t,\nu)}{(2i\nu)^N} \tag{1.80}$$

and $s_k(t)$ are given as in Section 1.3. Recall that

$$z_N(x,\nu) = \exp\left(i\nu x + \sum_{k=1}^N \int_0^x \frac{s_k(t)}{(2i\nu)^k}dt\right) + \frac{r_N(x,\nu)}{(2\pi i\nu)^{N+1}}.$$

Hence for $|\nu|$ large,

$$\int_0^x \frac{\sigma_N(t,\nu)}{(2i\nu)^N}dt = \log\left(1 + \exp\left(-i\nu x - \sum_{k=1}^N \int_0^x \frac{s_k(t)}{(2i\nu)^k}dt\right) \cdot \frac{r_N(x,\nu)}{(2\pi i\nu)^{N+1}}\right).$$

Furthermore, as $r_N(0,\nu) = 0$ and $r'(0,\nu) = 0$ it follows that

$$\sigma_N(0,\nu) = 0.$$

As $z_N'(x,\nu) = z_N(x,\nu)(i\nu + \sigma(x,\nu))$, the determinant of the solution matrix $Y_N(x,\nu)$, given by

$$Y_N(x,\nu) = \begin{pmatrix} z_N(x,-\nu) & z_N(x,\nu) \\ z_N'(x,-\nu) & z_N'(x,\nu) \end{pmatrix}$$

can be computed to be

$$\det Y_N(x,\nu) = z_N(x,-\nu) z_N(x,\nu) \cdot w(x,\nu)$$

where

$$w(x,\nu) = 2i\nu + \sigma(x,\nu) - \sigma(x,-\nu). \tag{1.81}$$

Note that $\det Y_N(0,\nu) = w(0,\nu)$. As the Wronskian is x- independent we get

$$z_N(1,-\nu) z_N(1,\nu) \cdot w(1,\nu) = w(0,\nu)$$

or

$$z_N(1,-\nu) z_N(1,\nu) = \frac{w(0,\nu)}{w(1,\nu)}. \tag{1.82}$$

This identity allows to express $z_N(1,-\nu)$ in terms of $z_N(1,\nu)$,

$$z_N(1,-\nu) = \frac{w(0,\nu)}{w(1,\nu)} \frac{1}{z_N(1,\nu)}. \tag{1.83}$$

Further note that the fundamental matrix is given by $Y_N(x,\nu)Y_N(0,\nu)^{-1}$. The condition that ν^2 be a periodic eigenvalue can be expressed by

$$\chi_p(\nu) = \det\left(Y_N(1,\nu)Y_N(0,\nu)^{-1} - Id_{2\times 2}\right) = 0$$

whereas anti-periodic eigenvalues are characterised by

$$\chi_{ap}(\nu) = \det\left(Y_N(1,\nu)Y_N(0,\nu)^{-1} + Id_{2\times 2}\right) = 0.$$

The two cases are treated similary and so we concentrate on the first one. Clearly, $\det\left(Y_N(1,\nu)Y_N(0,\nu)^{-1} - Id_{2\times 2}\right) = \frac{\det(Y_N(1,0) - Y_N(0,\nu))}{\det(Y_N(0,\nu))}$ and

$$\begin{aligned}
&\det(Y_N(1,0) - Y_N(0,\nu)) \\
=&\det\begin{pmatrix} z_N(1,-\nu) - 1 & z_N(1,\nu) - 1 \\ z_N'(1,-\nu) - z_N'(0,-\nu) & z_N'(1,\nu) - z_N'(0,\nu) \end{pmatrix} \\
=&\Big(z_N(1,-\nu) - 1\Big)\Big(z_N(1,\nu)\Big(i\nu + \sigma(1,\nu)\Big) - (i\nu + \sigma(0,\nu))\Big) \\
&- \Big(z_N(1,\nu) - 1\Big)\Big(z_N(1,-\nu)(-i\nu + \sigma(1,-\nu)) - (-i\nu + \sigma(0,-\nu))\Big) \\
=&- \Big(2i\nu + \sigma(1,\nu) - \sigma(0,-\nu)\Big) z_N(1,\nu) \\
&+ \Big(-2i\nu + \sigma(1,-\nu) - \sigma(0,\nu)\Big) z_N(1,-\nu) \\
&+ z_N(1,-\nu) z_N(1,\nu)\Big(2i\nu + \sigma(1,\nu) - \sigma(1,-\nu)\Big) + 2i\nu + \sigma(0,\nu) - \sigma(0,-\nu).
\end{aligned}$$

Introduce $G(\nu) := 2i\nu + \sigma(1,\nu) - \sigma(0,-\nu)$. With $z_N(1,-\nu)z_N(1,\nu) = \frac{w(0,\nu)}{w(1,\nu)}$ one then gets

$$\begin{aligned}\det(Y_N(1,0) - Y_N(0,\nu)) = & -G(\nu)z_N(1,\nu) + G(-\nu)z_N(1,-\nu) \\ & + \frac{w(0,\nu)}{w(1,\nu)} \cdot w(1,\nu) + w(0,\nu).\end{aligned}$$

The equation $\chi_p(\nu) = 0$ is thus equivalent to

$$G(\nu)z_N(1,\nu) - 2w(0,\nu) - G(-\nu)z_N(1,-\nu) = 0.$$

Dividing this equation by $w(1,\nu)z_N(1,-\nu) = \frac{w(0,\nu)}{z_N(1,\nu)}$ leads to the following quadratic equation

$$\frac{G(\nu)}{w(0,\nu)} z_N(1,\nu)^2 - 2z_N(1,\nu) - \frac{G(-\nu)}{w(1,\nu)} = 0$$

or

$$z_N(1,\nu) = \frac{w(0,\nu)}{G(\nu)} \pm \frac{w(0,\nu)}{G(\nu)} \sqrt{1 + \frac{G(\nu)G(-\nu)}{w(0,\nu)w(1,\nu)}}. \tag{1.84}$$

We now bring the right hand side of this identity into a more convenient form. For this purpose introduce

$$D(\nu) = \frac{\sigma(1,\nu) - \sigma(0,\nu)}{w(0,\nu)}. \tag{1.85}$$

Using that $w(0,\nu) = 2i\nu + \sigma(0,\nu) - \sigma(0,-\nu)$ we get

$$G(\nu) = 2i\nu + \sigma(1,\nu) - \sigma(0,-\nu) = w(0,\nu) + \sigma(1,\nu) - \sigma(0,\nu)$$

or

$$G(\nu) = w(0,\nu)(1 + D(\nu)). \tag{1.86}$$

As $w(0,-\nu) = -w(0,\nu)$ by the definition of $w(0,\nu)$ one gets

$$\begin{aligned}\frac{G(\nu)G(-\nu)}{w(0,\nu)w(1,\nu)} = & \frac{w(0,\nu)(1+D(\nu))w(0,-\nu)(1+D(-\nu))}{w(0,\nu)w(1,\nu)} \\ = & -\frac{1 + D(\nu) + D(-\nu) + D(\nu)D(-\nu)}{1 + \frac{w(1,\nu)-w(0,\nu)}{w(0,\nu)}}.\end{aligned}$$

As

$$\begin{aligned}w(1,\nu) - w(0,\nu) = & 2i\nu + \sigma(1,\nu) - \sigma(1,-\nu) - 2i\nu + \sigma(0,\nu) + \sigma(0,-\nu) \\ = & w(0,\nu)(D(\nu) + D(-\nu))\end{aligned}$$

we then get

$$\frac{G(\nu)G(-\nu)}{w(0,\nu)w(1,\nu)} = -\frac{1+D(\nu)+D(-\nu)+D(\nu)D(-\nu)}{1+D(\nu)+D(-\nu)}$$
$$= -1 - \frac{D(\nu)D(-\nu)}{1+D(\nu)+D(-\nu)}$$

Substituting this identity as well as (1.86) into (1.84) yields

$$z_N(1,\nu) = \frac{1}{1+D(\nu)}\left(1 \pm i\sqrt{\frac{D(\nu)D(-\nu)}{1+D(\nu)+D(-\nu)}}\right). \tag{1.87}$$

We have to take a closer look at $D(\nu)$. Recall that $D(\nu) = \frac{\sigma(1,\nu)-\sigma(0,\nu)}{w(0,\nu)}$ where

$$\sigma(x,\nu) = i\nu + \sum_1^N \frac{s_k(x)}{(2i\nu)^k} + \frac{\sigma_N(x,\nu)}{(2i\nu)^N}.$$

As the s_k's are 1-periodic and $\sigma_N(0,\nu) = 0$ it follows that

$$\sigma(1,\nu) - \sigma(0,\nu) = \frac{\sigma_N(1,\nu)}{(2i\nu)^N}.$$

Writing

$$w(0,\nu) = 2i\nu + \sigma(0,\nu) - \sigma(0,-\nu) = 2i\nu\left(1 + \frac{\sigma(0,\nu)-\sigma(0,-\nu)}{2i\nu}\right)$$

then leads to

$$D(\nu) = \frac{\sigma_N(1,\nu)}{(2i\nu)^{N+1}}\left(1 + \frac{\sigma(0,\nu)-\sigma(0,-\nu)}{2i\nu}\right)^{-1}$$
$$= \frac{\sigma_N(1,\nu)}{(2i\nu)^{N+1}}\left(1 + O\left(\frac{\sigma(0,\nu)-\sigma(0,-\nu)}{2i\nu}\right)\right).$$

As $\sigma(0,\nu) = \sum_1^N \frac{s_k(0)}{(2i\nu)^k}$ one has

$$\sigma(0,-\nu) - \sigma(0,\nu) = \sum_{k=1}^N \frac{s_k(0)}{(2i\nu)^k}((-1)^k - 1) = O\left(\frac{1}{\nu}\right)$$

and hence

$$D(\nu) = \frac{\sigma_N(1,\nu)}{(2i\nu)^{N+1}}\left(1 + O\left(\frac{1}{\nu^2}\right)\right). \tag{1.88}$$

In view of this estimate for $D(\nu)$ we can take the logarithm of (1.87) for $\nu = \nu_n$ where $\nu_n = \sqrt[+]{\lambda_{2n}}$ or $\nu_n = \sqrt[+]{\lambda_{2n-1}}$ with n *even.* (Recall that the periodic eigenvalues of $-d_x^2 + q$ when considered on $[0,1]$ are given by $\lambda_0 < \lambda_3 \leq \lambda_4 <$

$\lambda_7 \leq \lambda_8 < \ldots$.) Due to the asymptotics $\lambda_{2n}, \lambda_{2n-1} = n^2\pi^2 + \ell_n^2$ it follows that $\nu_n = n\pi + \frac{1}{n}\ell_n^2$. Taking the logarithm of

$$z_N(1, \nu_n) = \exp\left(i(\nu_n - n\pi) + \sum_1^N \int_0^1 \frac{s_k(t)}{(2i\nu_n)^k} dt + \int_0^1 \frac{\sigma_N(t, \nu_n)}{(2i\nu_n)^N} dt \right).$$

As n is even, the identity (1.87) together with Lemma 1.5 leads to

$$\begin{aligned}
&i(\nu_n - n\pi) - i \sum_{1 \leq 2l+1 \leq N} \frac{(-1)^l a_{2l+1}}{(2\nu_n)^{2l+1}} + \int_0^1 \frac{\sigma_N(t, \nu_n)}{(2i\nu_n)^N} dt \\
&= -\log\left(1 + D(\nu_n)\right) + \log\left(1 \pm i \frac{\sqrt{D(\nu_n)D(-\nu_n)}}{\sqrt{1 + D(\nu_n) + D(-\nu_n)}}\right) \\
&= -\frac{\sigma_N(1, \nu_n)}{(2i\nu_n)^{N+1}} + \frac{1}{2} \frac{\sigma(1, \nu_n)^2}{(2i\nu_n)^{2N+2}} + O\left(\left(\frac{1}{n^{N+3}}\right)\right) \pm i \frac{\sqrt{D(\nu_n)D(-\nu_n)}}{\sqrt{1 + D(\nu_n) + D(-\nu_n)}} \\
&\quad + \frac{1}{2} \frac{D(\nu_n)D(-\nu_n)}{1 + D(\nu_n) + D(-\nu_n)} + O\left(\left(\frac{D(\nu_n)D(-\nu_n)}{1 + D(\nu_n) + D(-\nu_n)}\right)^{3/2}\right).
\end{aligned}$$

By the estimate (1.88) and the expansion $(1+x)^{-1/2} = 1 - x/2 + 3x^2/8 + O(x^3)$ one gets

$$\begin{aligned}
(1 + D(\nu_n) + D(-\nu_n))^{-1/2} =& 1 - \frac{1}{2} \frac{1}{(2i\nu_n)^{N+1}} \Big(\sigma_N(1, \nu_n) \\
&+ (-1)^{N+1} \sigma_N(1, -\nu_n) \Big) + O\left(\frac{1}{n^{2N+2}}\right)
\end{aligned}$$

As $\sqrt{D(\nu_n)D(-\nu_n)} = O\left(\frac{1}{n^{N+1}}\right)$ it then follows that

$$\begin{aligned}
\frac{\sqrt{D(\nu_n)D(-\nu_n)}}{\sqrt{1 + D(\nu_n) + D(-\nu_n)}} &= \sqrt{D(\nu_n)D(-\nu_n)} - \frac{1}{2} \frac{\sqrt{D(\nu_n)D(-\nu_n)}}{(2i\nu_n)^{N+1}} \\
&\quad \cdot \Big(\sigma_N(1, \nu_n) + (-1)^{N+1} \sigma_N(1, -\nu_n) \Big) + O\left(\frac{1}{n^{N+3}}\right).
\end{aligned}$$

Combining the estimates above leads to

$$\begin{aligned}
&\nu_n - n\pi - \sum_{1 \leq 2l+1 \leq N} \frac{(-1)^l a_{2l+1}}{(2\nu_n)^{2l+1}} - i \int_0^1 \frac{\sigma_N(t, \nu_n)}{(2i\nu_n)^N} dt \\
&= i \frac{\sigma_N(1, \nu_n)}{(2i\nu_n)^{N+1}} - \frac{i}{2} \frac{\sigma_N(1, \nu_n)^2}{(2i\nu_n)^{2N+2}} \pm \Big(\sqrt{D(\nu_n)D(-\nu_n)} \\
&\quad - \frac{1}{2} \frac{\sqrt{D(\nu_n)D(-\nu_n)}}{(2i\nu_n)^{N+1}} \left(\sigma_N(1, \nu_n) + (-1)^{N+1} \sigma_N(1, -\nu_n) \right) \Big) \\
&\quad - \frac{i}{2} D(\nu_n)D(-\nu_n) + O\left(\frac{1}{n^{N+3}}\right). \qquad (1.89)
\end{aligned}$$

We need now to consider $\sigma_N(1,\nu_n)$ and $\int_0^1 \sigma_N(t,\nu_n)dt$. For this purpose introduce the following notation

$$A := \exp\left(i\nu - i\sum_{1\le 2l+1\le N}\frac{(-1)^l a_{2l+1}}{(2\nu)^{2l+1}}\right)$$
$$B := \exp\left(\int_0^1 \frac{\sigma_N(t,\nu)}{(2i\nu)^N}dt\right).$$

By the definition of z_N and σ_N,

$$z_N(1,\nu) = A\cdot B = A + \frac{r_N(1,\nu)}{(2i\nu)^{N+1}}.$$

Hence

$$B = 1 + A^{-1}\frac{r_N(1,\nu)}{(2i\nu)^{N+1}}$$

and, by taking logarithm,

$$\int_0^1 \frac{\sigma_N(t,\nu)}{(2i\nu)^N}dt = A^{-1}\frac{r_N(1,\nu)}{(2i\nu)^{N+1}} - \frac{1}{2}A^{-2}\frac{r_N(1,\nu)^2}{(2i\nu)^{2N+2}} + O\left(\frac{1}{\nu^{3N+3}}\right). \tag{1.90}$$

Furthermore, with $z_N(1,\nu) = A\cdot B$,

$$\begin{aligned} z_N'(1,\nu) =& z_N(1,\nu)\left(i\nu + \sum_1^N \frac{s_k(0)}{(2i\nu)^k} + \frac{\sigma_N(1,\nu)}{(2i\nu)^N}\right)\\ =& A\left(i\nu + \sum_1^N \frac{s_k(0)}{(2i\nu)^k}\right) + \frac{r_N'(1,\nu)}{(2i\nu)^{N+1}} \end{aligned}$$

or, as $z_N(1,\nu) - A = r_N(1,\nu)/(2i\nu)^{N+1}$,

$$z_N(1,\nu)\frac{\sigma_N(1,\nu)}{(2i\nu)^N} = -\frac{r_N(1,\nu)}{(2i\nu)^{N+1}}\left(i\nu + \sum_1^N \frac{s_k(0)}{(2i\nu)^k}\right) + \frac{r_N'(1,\nu)}{(2i\nu)^{N+1}}.$$

It leads to the formula

$$z_N(1,\nu)\sigma_N(1,\nu) = -\frac{r_N(1,\nu)}{2} - \frac{r_N(1,\nu)}{2i\nu}\sum_1^N \frac{s_k(0)}{(2i\nu)^k} + \frac{r_N'(1,\nu)}{2i\nu}. \tag{1.91}$$

Recall that by Proposition 1.2 (keep in mind that n is *even*)

$$r_N(1,\pm\nu_n) = a_{N+1} - (\pm 2in\pi)^N\langle q, e^{\mp 2in\pi x}\rangle \pm \frac{1}{2in\pi}a_{N+2} + \frac{1}{n}\ell_n^2$$

and

$$r_N'(1,\pm\nu_n) = \pm in\pi a_{N+1} \pm in\pi(\pm 2in\pi)^N\langle q, e^{\mp 2in\pi x}\rangle + \frac{a_{N+2}}{2} + \frac{1}{n}\ell_n^2.$$

Using that $a_1 = 0$ and $\nu_n - n\pi = \frac{1}{n}\ell_n^2$ it follows that

$$z_N(1, \nu_n) = \exp\left(i(\nu_n - n\pi) + \sum_{1 \le 2l+1 \le N} \frac{a_{2l+1}}{(2i\nu_n)^{2l+1}} \right) = 1 + \frac{1}{n}\ell_n^2.$$

Hence

$$\begin{aligned} z_N(1, \pm\nu_n)\sigma_N(1, \pm\nu_n) = &- \frac{a_{N+1}}{2} + \frac{(\pm 2in\pi)^N}{2} \langle q, e^{\mp 2in\pi x} \rangle \mp \frac{1}{2in\pi} \frac{a_{N+2}}{2} + \frac{1}{n}\ell_n^2 \\ &+ \frac{a_{N+1}}{2} + \frac{(\pm 2in\pi)^N}{2} \langle q, e^{\mp 2in\pi x} \rangle \pm \frac{1}{2in\pi} \frac{a_{N+2}}{2} + \frac{1}{n}\ell_n^2 \end{aligned}$$

and thus

$$z_N(1, \pm\nu_n)\sigma_N(1, \pm\nu_n) = (\pm 2in\pi)^N \langle q, e^{\mp 2in\pi x} \rangle + \frac{1}{n}\ell_n^2. \tag{1.92}$$

As $z_N(1, \pm\nu_n) = 1 + O\left(\frac{1}{n}\right)$

$$\frac{\sigma_N(1, \pm\nu_n)}{(\pm 2i\nu_n)^{N+1}} = \frac{1}{\pm 2in\pi} \langle q, e^{\mp 2\pi inx} \rangle + \frac{1}{n^{N+2}}\ell_n^2. \tag{1.93}$$

Hence also

$$\frac{\sigma_N(1, \pm\nu_n)^2}{(\pm 2i\nu_n)^{2N+2}} = \frac{1}{n^{2N+2}}\ell_n^2$$

and

$$\frac{\sqrt{D(\nu_n)D(-\nu_n)}}{(2i\nu_n)^{N+1}} \left(\sigma_N(1, \nu_n) + (-1)^{N+1}\sigma_N(1, -\nu_n) \right) = \frac{1}{n^{2N+2}}\ell_n^2$$

as well as

$$D(\nu_n)D(-\nu_n) = \frac{\sigma_N(1, \nu_n)\sigma_N(1, -\nu_n)}{(2\nu_n)^{2N+2}} \left(1 + O\left(\frac{1}{n^2}\right) \right) = \frac{1}{n^{2N+2}}\ell_n^2.$$

In view of all this and (1.90), (1.89) reads

$$\begin{aligned} \nu_n = &n\pi + \sum_{1 \le 2l+1 \le N} \frac{(-1)^l a_{2l+1}}{(2\nu_n)^{2l+1}} + iA^{-1} \frac{r_N(1, \nu_n)}{(2i\nu_n)^{N+1}} \\ &- \frac{i}{2}A^{-2} \frac{r_N(1, \nu_n)^2}{(2i\nu_n)^{2N+2}} + \frac{1}{2n\pi} \int_0^1 q(x) e^{2\pi inx} dx \\ &+ \sqrt{D(\nu_n)D(-\nu_n))} + \frac{1}{n^{N+2}}\ell_n^2. \end{aligned}$$

But

$$A^{-1} = 1 + O\left(\frac{1}{n^2}\right)$$

hence

$$iA^{-1}\frac{r_N(1,\nu_n)}{(2i\nu_n)^{N+1}} = i\frac{a_{N+1}}{(2i\nu_n)^{N+1}} - \int_0^1 \frac{q(x)e^{2\pi inx}}{2n\pi}dx + i\frac{a_{N+2}}{(2i\nu_n)^{N+2}}$$

and using that $a_{N+1} = 0$ in the case $N = 0$ we get for any $N \geq 0$ that

$$A^{-2}\frac{r_N(1,\nu_n)}{(2i\nu_n)^{2N+2}} = O\left(\frac{1}{n^{N+3}}\right).$$

We then get in view of Lemma 1.5

$$\nu_n = n\pi + \sum_{1\leq 2l+1\leq N+2} \frac{(-1)^l a_{2l+1}}{(2\nu_n)^{2l+1}} \pm \sqrt{D(\nu_n)D(-\nu_n))} + \frac{1}{n^{N+2}}\ell_n^2.$$

Note that the periodic spectrum of $-d_x^2+q$ is real and hence $\sqrt[+]{\lambda_{2n}}$, $\sqrt[+]{\lambda_{2n-1}}$ are real for n sufficiently large. It then follows from Lemma 1.11 below that

$$\sqrt[+]{D(\nu_n)D(-\nu_n)} = |D(\nu_n)|$$

But by (1.88), $D(\nu_n) = \frac{\sigma_N(1,\nu_n)}{(2in\pi)^{N+1}} + O\left(\frac{1}{n^{N+3}}\right)$ and by (1.93)

$$\frac{\sigma_N(1,\nu_n)}{(2in\pi)^{N+1}} = \frac{1}{2in\pi}\int_0^1 q(x)e^{2in\pi x}dx + \frac{1}{n^{N+2}}\ell_n^2.$$

Hence, for q real we have by Cauchy's inequality

$$\begin{aligned}&\left| |\frac{\sigma_N(1,\nu_n)}{(2in\pi)^{N+1}}| - \frac{1}{2n\pi}|\int_0^1 q(x)e^{2\pi inx}dx|\right| \\ \leq &\left|\frac{\sigma_N(1,\nu_n)}{(2in\pi)^{N+1}} - \frac{1}{2in\pi}\int_0^1 q(x)e^{2\pi inx}dx\right| = \frac{1}{n^{N+2}}\ell_n^2.\end{aligned}$$

We therefore have established that

$$\nu_n = n\pi + \sum_{1\leq 2l+1\leq N+2} \frac{(-1)^l a_{2l+1}}{(2\nu_n)^{2l+1}} \pm \left|\frac{1}{2n\pi}\int_0^1 q(x)e^{2\pi inx}dx\right| + \frac{1}{n^{N+2}}\ell_n^2.$$

Arguing as for the asymptotics of the Dirichlet eigenvalues one gets

$$\nu_n = n\pi + \sum_{1\leq 2l+1\leq N+2} \frac{(-1)^l b_{2l+1}}{(2n\pi)^{2l+1}} \pm \left|\frac{1}{2n\pi}\int_0^1 q(x)e^{2\pi inx}dx\right| + \frac{1}{n^{N+2}}\ell_n^2.$$

Then for $q \in H_0^N$

$$\nu_n^2 = m_n \pm \left|\int_0^1 q(x)e^{2\pi inx}dx\right| + \frac{1}{n^{N+1}}\ell_n^2$$

where m_n is given by (10). Going through the arguments of the proof one concludes that the error term has the claimed uniformity property. Finally we need to determine the signs $\pm$ in the asymptotics of λ_{2n} and λ_{2n-1}. It is convenient to introduce $\lambda_n^+ = \lambda_{2n}$, $\lambda_n^- = \lambda_{2n-1}$. Choose $\epsilon_n^\pm \in \{-1, 1\}$ so that

$$\lambda_n^\pm = m_n + \epsilon_n^\pm \left| \int_0^1 q(x) e^{2\pi i n x} dx \right| + \frac{1}{n^{N+1}} \ell_n^2.$$

We note that $\epsilon_n^\pm$ are not uniquely determined and that as $\lambda_n^- \leq \lambda_n^+$, we may choose $\epsilon_n^\pm$ so that $\epsilon_n^- \leq \epsilon_n^+$. We claim that $\epsilon_n^\pm$ can be chosen in such a way that $\epsilon_n^+ = 1$ and $\epsilon_n^- = -1$ for any $n \geq 1$. Indeed, let $J = \{n \geq 1 | \ \epsilon_n^+ = \epsilon_n^-\}$. If J is finite we can change $\epsilon_n^+, \epsilon_n^-$ as claimed without changing the asymptotics. If J is infinite, then the n's in J form a subsequence $(n_k)_{k \geq 1}$ in $\mathbb{N}$ such that $\epsilon_{n_k}^+ = \epsilon_{n_k}^-$ for any $k \geq 1$. As

$$\tau_{n_k} = \frac{\lambda_{n_k}^+ + \lambda_{n_k}^-}{2} = m_{n_k} + \epsilon_{n_k}^+ \left| \int_0^1 q(x) e^{2\pi i n_k x} dx \right| + \frac{1}{n_k^{N+1}} \ell_k^2$$

it then follows from the asymptotics of the τ_n's that

$$\left| \int_0^1 q(x) e^{2\pi i n_k x} dx \right|_{k \geq 1} = \frac{1}{n_k^{N+1}} \ell_k^2$$

and hence we can again change the $\epsilon_{n_k}^\pm$ so that now $\epsilon_{n_k}^\pm = \pm 1$. This proves the claimed asymptotics. □

Lemma 1.11. *For any $\nu \in \mathbb{R}$ and any $q \in L_0^2$ (in particular q real valued)*

(i) $z_N(x, -\nu) = \overline{z_N(x, \nu)} \quad \forall x \in \mathbb{R}$.

(ii) $\sigma(x, -\nu) = \overline{\sigma(x, \nu)} \quad \forall x \in \mathbb{R}$

(iii) $D(-\nu) = \overline{D(\nu)}$.

Proof. (i) It is easy to check that $z(x, -\nu)$ and $\overline{z(x, \nu)}$ both satisfy the equation

$$-y'' + qy = \nu^2 y. \tag{1.94}$$

Further recall that

$$z(0, -\nu) = 1 = \overline{z(0, \nu)}$$

and

$$z'(0, -\nu) = -i\nu + \sum_1^N \frac{s_k(0)}{(-2i\nu)^k} = \overline{i\nu + \sum_1^N \frac{s_k(0)}{(2i\nu)^k}} = \overline{z'(0, \nu)}$$

By the uniqueness of solutions of (1.94) with given initial values it then follows that

$$z(x, -\nu) = \overline{z(x, \nu)} \quad \forall x \in \mathbb{R}, \forall \nu \in \mathbb{R}.$$

(ii) By (1.16)-(1.18) one sees that $s_k(x)$ is real for any $x \in \mathbb{R}$ and any $1 \leq k \leq N$. Hence by (i) and the definition (1.80) of $\sigma(t, \nu)$ it follows that

$$\sigma(t, -\nu) = \overline{\sigma(t, \nu)}.$$

(iii) In view of the definition of $D(\nu)$ ((1.85), (1.81)), the claimed identity follows from (ii). □

Chapter 2

Asymptotics of the Birkhoff map I

In this chapter we proof Theorem 0.4. First recall the construction of the Birkhoff coordinates, presented in detail in [12]. It is based on action and angle variables defined as follows. On a neighbourhood W of L_0^2 in $L_{0,\mathbb{C}}^2$ action variables are defined by the formula

$$I_n = \frac{1}{\pi}\int_{\Gamma_n} \lambda \frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}} d\lambda \qquad (n \geq 1), \tag{2.1}$$

where $\Delta(\lambda)$ is the discriminant, $\dot{\Delta}(\lambda) = \partial_\lambda \Delta(\lambda)$ and Γ_n is a contour around the interval $G_n := \{t\lambda_{2n-1} + (1-t)\lambda_{2n} |\ 0 \leq t \leq 1\}$ in the isolating neighbourhood U_n of G_n. Angles conjugate to these actions are defined in terms of the entire functions $(\psi_n)_{n\geq 1}$, introduced in Section 1.7 – see (1.69)-(1.70). For $q \in W \setminus Z_n$ with $Z_n = \{q \in W |\ \gamma_n(q) = 0\}$ the n'th angle variable is given by

$$\theta_n(q) = \beta_{n,n}(q) + \beta_n(q) \quad \text{and} \quad \beta_n(q) = \sum_{k \neq n} \beta_{n,k}(q)$$

where

$$\beta_{n,n}(q) = \int_{\lambda_{2n-1}}^{\mu_n^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} d\lambda \quad \mathrm{mod} 2\pi,$$

and, for $k \neq n$,

$$\beta_{n,k}(q) = \int_{\lambda_{2k-1}}^{\mu_k^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} d\lambda.$$

Here μ_n^* is the point $(\mu_n, \sqrt[*]{\Delta^2(\mu_n) - 4})$ on the affine curve Σ_q,

$$\Sigma_q = \left\{(\lambda, z) \in \mathbb{C}^2 | z^2 = \Delta^2(\lambda) - 4\right\},$$

with $\mu_n = \mu_n(q)$ denoting the n'th Dirichlet eigenvalue of $-d_x^2 + q$ and $\sqrt[*]{\Delta^2(\mu_n) - 4}$ denoting the square root of $\Delta^2(\mu_n) - 4$ given by

$$\sqrt[*]{\Delta^2(\mu_n) - 4} = y_1(1, \mu_n) - y_2'(1, \mu_n).$$

The integral in the formula for $\beta_{n,k}$ is a straight line integral from $(\lambda_{2n-1}, 0)$ to μ_n^* in Σ_q. In Lemma 8.2 and Lemma 8.3 of [12], it is shown that $\beta_{n,k}$ $(k \neq n)$ is a well defined analytic function on W. In Theorem 8.5 of [12], it is shown that $\beta_n = \sum_{k \neq 0} \beta_{n,k}$ is absolutely summable on W. The $\pm n$'th complex Birkhoff coordinate is then given on $W \setminus Z_n$ by

$$z_{\pm n} = \sqrt{2I_n} e^{\mp i\theta_n} = \sqrt{2I_n} e^{\mp i(\beta_{n,n} + \beta_n)}, \quad n \in \mathbb{Z}_{\geq 1}.$$

In [12] it is shown that after shrinking W if necessary, the complex coordinates $z_{\pm n}$ can be analytically extended for any $n \in \mathbb{Z}_{\geq 1}$ to W.
In Section 2.1 we derive asymptotics of the actions $I_n, n \in \mathbb{Z}_{\geq 1}$ and $e^{i\beta_n}, n \in \mathbb{Z}_{\geq 1}$ as n tends to infinity. In Section 2.2 we derive asymptotics for $z_n^{\pm} := \gamma_n e^{\pm i\beta_{n,n}}$. Finally in Section 2.3 we prove Theorem 0.4.

2.1 Asymptotics of actions and angles

In this section we improve on estimates of the actions and angles obtained in [12], Section 7 respectively Section 8. Let us begin with asymptotics of the actions (2.1). First we need to derive improved estimates of the quotient I_n/γ_n^2, given in [12], Theorem 7.3.

Proposition 2.1. *Locally uniformly on W, the quotient I_n/γ_n^2 satisfies*

$$8\pi n \frac{I_n}{\gamma_n^2} = 1 + \frac{1}{n}\ell_n^2. \tag{2.2}$$

Moreover

$$\xi_n = \sqrt[+]{8I_n/\gamma_n^2} = \frac{1}{\sqrt{\pi n}}(1 + \frac{1}{n}\ell_n^2) \tag{2.3}$$

is well-defined as a real analytic, non-vanishing function on W. In particular, at $q = 0$, we have $\xi_n = \frac{1}{\sqrt{\pi n}}$ for all $n \geq 1$. On L_0^2, (2.3) holds uniformly on bounded subsets of L_0^2.

Proof. We refer to [12], Section 7, for all notions, notations (and results) not explained here. In view of Theorem 7.3 in [12] it only remains to be shown the improved asymptotics (2.2). Recall the product expansions

$$\Delta^2(\lambda) - 4 \quad = \quad 4(\lambda_0 - \lambda) \prod_{n \geq 1} \frac{(\lambda_{2n} - \lambda)(\lambda_{2n-1} - \lambda)}{\pi_n^4}$$

and

$$\dot{\Delta}(\lambda) \quad = \quad -\prod_{n\geq 1}\frac{\dot{\lambda}_n-\lambda}{\pi_n^2}.$$

For λ on Γ_n write

$$\frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}=\frac{1}{2\pi n}\frac{\lambda-\dot{\lambda}_n}{\sqrt[s]{(\lambda_{2n}-\lambda)(\lambda-\lambda_{2n-1})}}\chi_n(\lambda) \tag{2.4}$$

where for $\lambda \in U_n$,

$$\chi_n(\lambda)=(-1)^{n-1}\frac{n\pi}{\sqrt[+]{\lambda-\lambda_0}}\prod_{m\neq n}\frac{\dot{\lambda}_m-\lambda}{\sqrt[+]{(\lambda_{2m}-\lambda)(\lambda_{2m-1}-\lambda)}}, \tag{2.5}$$

and where the canonical root $\sqrt[c]{\Delta^2(\lambda)-4}$ has been introduced earlier. (For questions of signs, see Section 6 in [12].) Note that for $\lambda \in U_n$,

$$\frac{n\pi}{\sqrt[+]{\lambda-\lambda_0}}=1+\frac{1}{n}\ell_n^2. \tag{2.6}$$

Furthermore, by Proposition 1.4, the roots $\dot{\lambda}_n$ of $\dot{\Delta}$ satisfy $\dot{\lambda}_n=\tau_n+\frac{\gamma_n^2}{n}\ell_n^2$ and hence, by Proposition 1.1, one has for $\lambda\in U_n$,

$$\begin{aligned}\left(\prod_{m\neq n}\frac{\dot{\lambda}_m-\lambda}{\sqrt[+]{(\lambda_{2m}-\lambda)(\lambda_{2m-1}-\lambda)}}\right)^2 &= \prod_{m\neq n}\frac{\dot{\lambda}_m-\lambda}{\lambda_{2m}-\lambda}\prod_{m\neq n}\frac{\dot{\lambda}_m-\lambda}{\lambda_{2m-1}-\lambda}\\ &=1+\frac{1}{n}\ell_n^2\end{aligned}$$

or

$$(-1)^{n-1}\prod_{m\neq n}\frac{\dot{\lambda}_m-\lambda}{\sqrt[+]{(\lambda_{2m}-\lambda)(\lambda_{2m-1}-\lambda)}}=1+\frac{1}{n}\ell_n^2. \tag{2.7}$$

Combining (2.6) and (2.7) leads to the estimate

$$\chi_n(\lambda)=1+\frac{1}{n}\ell_n^2 \tag{2.8}$$

uniformly for $\lambda\in U_n$. Introduce

$$Z_n=\{q\in W \mid \lambda_{2n}(q)=\lambda_{2n-1}(q)\}.$$

Arguing as in Section 7 of [12] one gets for q in $W\setminus Z_n$

$$\begin{aligned}I_n &= \frac{1}{\pi}\int_{\Gamma_n}(\lambda-\dot{\lambda}_n)\frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta^2(\lambda)-4}}d\lambda\\ &=\frac{1}{2\pi^2 n}\int_{\Gamma_n}\frac{(\lambda-\dot{\lambda}_n)^2}{\sqrt[s]{(\lambda_{2n}-\lambda)(\lambda-\lambda_{2n-1})}}\chi_n(\lambda)d\lambda.\end{aligned}$$

Substituting $\lambda = \tau_n + \zeta\gamma_n/2$ and setting $\delta_n = 2(\dot{\lambda}_n - \tau_n)/\gamma_n$ yields

$$I_n = \frac{\gamma_n^2}{8\pi^2 n} \int_{\Gamma_n'} (\zeta - \delta_n)^2 \chi_n(\tau_n + \zeta\frac{\gamma_n}{2}) \frac{d\zeta}{\sqrt[s]{1-\zeta^2}}$$

where Γ_n' is some circuit in $\mathbb{C}$ around $[-1, 1]$. Thus on $W \setminus Z_n$,

$$\begin{aligned} 8\pi n \frac{I_n}{\gamma_n^2} &= \frac{1}{\pi} \int_{\Gamma_n'} (\zeta - \delta_n)^2 \chi_n\left(\tau_n + \zeta\frac{\gamma_n}{2}\right) \frac{d\zeta}{\sqrt[s]{1-\zeta^2}} \\ &= \frac{1}{\pi} \int_{\Gamma_n'} (\zeta - \delta_n)^2 \chi_n(\tau_n) \frac{d\zeta}{\sqrt[s]{1-\zeta^2}} \\ &+ \frac{1}{\pi} \int_{\Gamma_n'} (\zeta - \delta_n)^2 \left(\chi_n\left(\tau_n + \zeta\frac{\gamma_n}{2}\right) - \chi_n(\tau_n)\right) \frac{d\zeta}{\sqrt[s]{1-\zeta^2}}. \end{aligned}$$

By Proposition 1.4, δ_n is well defined on all of W, hence so is the r.h.s. of the latter identity. As

$$(\zeta - \delta_n)^2 = \zeta^2 + \frac{\gamma_n}{n}\ell_n^2$$

Lemma M.1 in [12] and (2.8) then imply that

$$\begin{aligned} \frac{1}{\pi} \int_{\Gamma_n'} (\zeta - \delta_n)^2 \chi_n(\tau_n) \frac{d\zeta}{\sqrt[s]{1-\zeta^2}} &= \frac{1}{\pi} \int_{\Gamma_n'} \frac{\zeta^2 d\zeta}{\sqrt[s]{1-\zeta^2}} + \frac{1}{n}\ell_n^2 \\ &= 1 + \frac{1}{n}\ell_n^2. \end{aligned}$$

By the Taylor expansion of order 0 of χ_n at $\lambda = \tau_n$, Cauchy's estimate, and (2.8) to bound $\dot{\chi}_n(\lambda)$, one gets

$$\frac{1}{\pi} \int_{\Gamma_n'} (\zeta - \delta_n)^2 \left(\chi_n(\tau_n + \zeta\frac{\gamma_n}{2}) - \chi_n(\tau_n)\right) \frac{d\zeta}{\sqrt[s]{1-\zeta^2}} = \frac{1}{n}\ell_n^2.$$

Altogether,

$$8\pi n I_n/\gamma_n^2 = 1 + \frac{1}{n}\ell_n^2$$

and thus

$$\xi_n = \frac{1}{\sqrt{\pi n}}(1 + \frac{1}{n}\ell_n^2).$$

Going through the arguments of the proof one sees that the estimate (2.3) holds locally uniformly on W and uniformly on bounded subsets of L_0^2. □

Proposition 2.1 leads to the following asymptotics of the action variables.

Proposition 2.2. *Locally uniformly on* $W \cap H_{0,\mathbb{C}}^N$

$$2I_n = \frac{1}{\pi n}\left(\hat{q}_n \hat{q}_{-n} + \frac{1}{n^{2N+1}}\ell_n^1\right). \tag{2.9}$$

On H_0^N*, the error in* (2.9) *is uniformly bounded on bounded subsets of* H_0^N*.*

Proof. By Theorem 1.1 one has

$$\gamma_n^2 = 4\hat{q}_n\hat{q}_{-n} + \frac{1}{n^{2N+1}}\ell_n^1.$$

Combined with the asymptotics (2.2) we then get,

$$2I_n = \frac{1}{4n\pi}\gamma_n^2\left(1 + \frac{1}{n}\ell_n^2\right) = \frac{1}{\pi n}(1 + \frac{1}{n}\ell_n^2)\left(\hat{q}_n\hat{q}_{-n} + \frac{1}{n^{2N+1}}\ell_n^1\right)$$

or

$$2I_n = \frac{1}{\pi n}\left(\hat{q}_n\hat{q}_{-n} + \frac{1}{n^{2N+1}}\ell_n^1\right).$$

Going through the arguments of the proof one sees that (2.9) holds locally uniformly on $W \cap H_{0,\mathbb{C}}^N$ and uniformly on bounded subsets of H_0^N. □

Next we improve on the estimates of the angle variables obtained in [12], Section 8. To this end we use the improved estimate of the zeros $(\sigma_m^n)_{m\neq n}$ of the entire function ψ_n,

$$\sigma_m^n = \tau_m + \frac{\gamma_m^2}{m}\ell_m^2$$

of Proposition 1.5. Recall that $Z_n = \{q \in W|\ \gamma_n(q) = 0\}$ where W is the neighbourhood of L_0^2 in $L_{0,\mathbb{C}}^2$ of Theorem 0.3. For $q \in W \setminus Z_n$ denote, as in [12], by $\theta_n(q)$ the n'th angle variable

$$\theta_n(q) = \beta_{n,n}(q) + \beta_n(q) \quad \text{and} \quad \beta_n(q) = \sum_{k\neq n}\beta_{n,k}(q)$$

where

$$\beta_{n,n}(q) = \int_{\lambda_{2n-1}}^{\mu_n^*}\frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2-4}}d\lambda \quad \text{mod} 2\pi, \tag{2.10}$$

and, for $k \neq n$,

$$\beta_{n,k}(q) = \int_{\lambda_{2k-1}}^{\mu_k^*}\frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2-4}}d\lambda. \tag{2.11}$$

Here μ_n^* is the point $(\mu_n, \sqrt[*]{\Delta^2(\mu_n)-4})$ on the affine curve Σ_q,

$$\Sigma_q = \left\{(\lambda, z) \in \mathbb{C}^2 | z^2 = \Delta^2(\lambda) - 4\right\}, \tag{2.12}$$

with $\mu_n = \mu_n(q)$ denoting the n'th Dirichlet eigenvalue of $-d_x^2 + q$ and $\sqrt[*]{\Delta^2(\mu_n)-4}$ denoting the square root of $\Delta^2(\mu_n) - 4$ given by

$$\sqrt[*]{\Delta^2(\mu_n)-4} = y_1(1,\mu_n) - y_2'(1,\mu_n). \tag{2.13}$$

The integral in (2.10) is a straight line integral from $(\lambda_{2n-1}, 0)$ to μ_n^* in Σ_q and the one in (2.11) is defined similarly. See Section 6 in [12] for more explanations concerning the notation. Let us begin by analyzing $\beta_{n,k}$ in more detail. In Lemma 8.2 and Lemma 8.3 of [12], it is shown that $\beta_{n,k}$ $(k \neq n)$ is a well defined analytic function on W. In Theorem 8.5 of [12], it is shown that $\beta_n = \sum_{k\neq 0} \beta_{n,k}$ is absolutely summable on W and with the help of Lemma 8.4 in [12] one proves that the following estimate holds.

Lemma 2.1. *Locally uniformly on W,*

$$\beta_n = O\left(\frac{1}{n}\right). \tag{2.14}$$

On L_0^2, (2.14) holds uniformly on bounded subsets of L_0^2.

Next let us consider $\beta_{n,n}$ in more detail. Recall that $\beta_{n,n}$ is defined on $W \setminus Z_n$ mod 2π and is analytic on $W \setminus Z_n$ mod π. Similarly as in (2.4), we write for λ near G_n

$$\frac{\psi_n(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} = \frac{\zeta_n(\lambda)}{\sqrt[s]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} \tag{2.15}$$

where

$$\zeta_n(\lambda) = (-1)^{n-1} \frac{\pi n}{\sqrt[+]{\lambda - \lambda_0}} \prod_{m\neq n} \frac{\sigma_m^n - \lambda}{\sqrt[+]{(\lambda_{2m} - \lambda)\,(\lambda_{2m-1} - \lambda)}}. \tag{2.16}$$

The following results improve on the asymptotic estimates of Lemma 9.2 in [12].

Lemma 2.2. *(i) Uniformly for $\lambda \in U_n$*

$$\zeta_n(\lambda) = 1 + \frac{1}{n}\ell_n^2.$$

(ii) Uniformly for $\mu \in G_n$

$$\zeta_n(\mu) = 1 + \frac{\gamma_n}{n}\ell_n^2.$$

(iii) The error estimates in (i) and (ii) hold locally uniformly on W and are uniformly bounded on bounded subsets of L_0^2.

Proof. (i) Note that by Proposition 1.1

$$(-1)^{n-1} \prod_{m\neq n} \frac{\sigma_m^n - \lambda}{\sqrt[+]{(\lambda_{2m} - \lambda)\,(\lambda_{2m-1} - \lambda)}} = 1 + \frac{1}{n}\ell_n^2 \tag{2.17}$$

uniformly for $\lambda \in U_n$. With

$$\lambda - \lambda_0 = n^2\pi^2 \left(1 + \frac{\lambda - n^2\pi^2 - \lambda_0}{n^2\pi^2}\right)$$

one gets

$$\frac{\sqrt[+]{\lambda - \lambda_0}}{n\pi} = \sqrt[+]{1 + \frac{\lambda - n^2\pi^2 - \lambda_0}{n^2\pi^2}} = 1 + O\left(\frac{1}{n^2}\right). \tag{2.18}$$

Together, (2.17) and (2.18) yield

$$\zeta_n(\lambda) = 1 + \frac{1}{n}\ell_n^2 \tag{2.19}$$

uniformly for $\lambda \in U_n$.

(ii) Assume that $\gamma_n \neq 0$. By the normalisation of the ψ_n-function,

$$\int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{\psi_n(\lambda)}{\sqrt[c]{\Delta(\lambda)^2 - 4}} d\lambda = \pi$$

for the line integral from λ_{2n-1} to λ_{2n} obtained by deforming Γ_n to the interval $[\lambda_{2n-1}, \lambda_{2n}]$. By (2.15) one then gets for any $\mu \in U_n$

$$\begin{aligned}
\pi &= \int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{\zeta_n(\lambda)}{\sqrt[s]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda \\
&= \int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{\zeta_n(\mu)}{\sqrt[s]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda + \int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{\zeta_n(\lambda) - \zeta_n(\mu)}{\sqrt[s]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda.
\end{aligned}$$

By a straightforward computation,

$$\int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{d\lambda}{\sqrt[s]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} = \pi. \tag{2.20}$$

Combined with Lemma M.1 in [12], we then obtain

$$\begin{aligned}
|\zeta_n(\mu) - 1| &= \frac{1}{\pi}\left|\int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{\zeta_n(\lambda) - \zeta_n(\mu)}{\sqrt[s]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda\right| \\
&\leq \max_{\lambda \in G_n} |\zeta_n(\lambda) - \zeta_n(\mu)|. \qquad (2.21)
\end{aligned}$$

This bound holds no matter if $\gamma_n \neq 0$ or not. Recall that $\zeta_n(\lambda)$ is analytic for λ in U_n. Hence one concludes from (2.19) and Cauchy's estimate that

$$\dot{\zeta}_n(\mu) = \frac{1}{n}\ell_n^2$$

uniformly for $\mu \in U_n$ and thus by the mean value theorem, for $q \in W$ and $\mu \in G_n$,

$$\max_{\lambda \in G_n} |\zeta_n(\lambda) - \zeta_n(\mu)| \leq \frac{|\gamma_n|}{n}\ell_n^2$$

uniformly for $\mu \in G_n$ as claimed. This combined with (2.21) shows (ii).

(iii) Going through the arguments of the proofs of (i) and (ii) one sees that the estimates hold locally uniformly on W and uniformly on bounded subsets of L_0^2.

□

Instead of describing the improved asymptotics of $\beta_{n,n}$, we directly study the asymptotics of $z_n^{\pm}$, defined on $W \setminus Z_n$ by

$$z_n^{\pm} = \gamma_n e^{\pm i\beta_{n,n}}. \tag{2.22}$$

This is the topic of the following section.

2.2 Asymptotics of $z_n^{\pm}$

The purpose of this section is to prove sharp asymptotic estimates of $z_n^{\pm}$. First note that it was shown in [12] that $z_n^{\pm}$, given by (2.22), analytically extend to all of W – see formula (9.4) in [12].

Proposition 2.3. *For $N \in \mathbb{Z}_{\geq 0}$, let $W_N \subseteq W \cap H^N_{0,\mathbb{C}}$ be the neighbourhood of H_0^N given in Theorem 0.8. For any $q \in W_N$,*

$$z_n^{\pm} = 2\hat{q}_{\mp n} + \frac{1}{n^{N+1}}\ell_n^2$$

locally uniformly on $W \cap H^N_{0,\mathbb{C}}$. On H_0^N, the error is uniformly bounded on bounded subsets of H_0^N.

First we need to make some preparations. Assume that $q \in W \setminus Z_n$. Then

$$z_n^{\pm} = \gamma_n \exp\left(\pm i \int_{\lambda_{2n-1}}^{\mu_n^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2-4}} d\lambda\right). \tag{2.23}$$

If $\mu_n = \lambda_{2n-1}$, then $z_n^{\pm} = \gamma_n$ whereas if $\mu_n = \lambda_{2n}$, then $z_n^{\pm} = -\gamma_n$ by the normalization of ψ_n. By Lemma 9.1 in [12], $z_n^{\pm}$ are analytic functions on $W \setminus Z_n$. In the case where $\mu_n \notin \{\lambda_{2n}, \lambda_{2n-1}\}$, we choose as path of integration the interval $[\lambda_{2n-1}, \mu_n]$ in $\mathbb{C}$ and obtain the following formula

$$z_n^{\pm} = \gamma_n \exp\left(\pm i \int_{\lambda_{2n-1}}^{\mu_n} \frac{\psi_n(\lambda)}{\sqrt[*]{\Delta(\lambda)^2-4}} d\lambda\right) \tag{2.24}$$

where for λ in $[\lambda_{2n-1}, \mu_n]$, $\sqrt[*]{\Delta(\lambda)^2-4}$ is defined to be the continuous function on $[\lambda_{2n-1}, \lambda_{2n}]$ with sign determined by

$$\sqrt[*]{\Delta(\lambda)^2-4}|_{\lambda=\mu_n} = \sqrt[*]{\Delta(\mu_n)^2-4}.$$

As, by assumption, $\mu_n \notin \{\lambda_{2n}, \lambda_{2n-1}\}$, the root $\sqrt[*]{\Delta(\lambda)^2-4}$ is well-defined. In analogy with formula (2.15) define

$$\sqrt[*]{(\lambda_{2n}-\lambda)(\lambda-\lambda_{2n-1})} := \sqrt[*]{\Delta(\lambda)^2-4} \cdot \frac{\zeta_n(\lambda)}{\psi_n(\lambda)}, \qquad \lambda \in [\lambda_{2n-1}, \mu_n]. \tag{2.25}$$

As $\sqrt[*]{\Delta(\mu_n)^2-4} = y_1(1,\mu_n) - y_2'(1,\mu_n)$ is defined on all of W and analytic there, $\sqrt[*]{(\lambda_{2n}-\mu_n)(\mu_n-\lambda_{2n-1})}$ analytically extends to W as well and

$$z_n^\pm = \gamma_n \exp\left(\pm i \int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda)}{\sqrt[*]{(\lambda_{2n}-\lambda)(\lambda-\lambda_{2n-1})}} d\lambda\right). \tag{2.26}$$

To obtain the claimed estimates for $z_n^\pm$, we write $z_n^\pm$ as a product

$$z_n^+ = u_n^+ v_n^+ \quad \text{and} \quad z_n^- = u_n^- v_n^- \tag{2.27}$$

where

$$u_n^\pm = \gamma_n \exp\left(\pm i \int_{\lambda_{2n-1}}^{\mu_n} \frac{1}{\sqrt[*]{(\lambda_{2n}-\lambda)(\lambda-\lambda_{2n-1})}} d\lambda\right) \tag{2.28}$$

and

$$v_n^\pm = \exp\left(\pm i \int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda)-1}{\sqrt[*]{(\lambda_{2n}-\lambda)(\lambda-\lambda_{2n-1})}} d\lambda\right) \tag{2.29}$$

Arguing as in the proof of Lemma 9.1 in [12] one concludes that $u_n^\pm$ are analytic on $W \setminus Z_n$. Together with the analyticity of $z_n^\pm$ on $W \setminus Z_n$ it then follows that $v_n^\pm$ are analytic on $W \setminus Z_n$ as well. For $q \in W \setminus Z_n$ with $\mu_n = \lambda_{2n-1}$, one easily sees that

$$u_n^\pm = \gamma_n \quad \text{and} \quad v_n^\pm = 1. \tag{2.30}$$

A straightforward computation shows that for $q \in W \setminus Z_n$ with $\mu_n = \lambda_{2n}$

$$u_n^\pm = -\gamma_n \quad v_n^\pm = 1. \tag{2.31}$$

We will see that both, $u_n^\pm$ and $v_n^\pm$, continuously extend to all of W and that they admit asymptotics for $n \to \infty$ which allow to prove the asymptotics of $z_n^\pm$, claimed in Proposition 2.3. The quantities $u_n^\pm$ and $v_n^\pm$ will be studied separately. Let us begin with the $u_n^\pm$'s. To this end introduce for any q in W,

$$\Delta_n(\lambda) = 4\,\frac{\lambda-\lambda_0}{\pi_n^2}\left(\prod_{m\neq n} \frac{\lambda_{2m}-\lambda}{\pi_m^2}\right)\left(\prod_{m\neq n} \frac{\lambda_{2m-1}-\lambda}{\pi_m^2}\right). \tag{2.32}$$

Hence for λ in U_n, the principal branch of the square root $\sqrt[+]{\Delta_n(\lambda)}$ is well-defined. Furthermore recall that in Section 4, we have introduced for any $n \geq 1$ and $q \in W$

$$\kappa_n(q) = \log\,(-1)^n y_2'(1,\mu_n) \tag{2.33}$$

where $\mu_n = \mu_n(q)$ is the n'th Dirichlet eigenvalue. Here log denotes the principal branch of the logarithm.

Proposition 2.4. *For any $n \geq 1$ and $q \in W \setminus Z_n$*

$$u_n^{\pm} = 2(\tau_n - \mu_n) \pm i \frac{2\pi n}{\sqrt[+]{\Delta_n(\mu_n)}} 2 \sinh \kappa_n. \tag{2.34}$$

In particular, for any $n \geq 1$, $u_n^{\pm}$ extend analytically to all of W. For $q \in Z_n$,

$$u_n^{\pm} = 2(\tau_n - \mu_n) \pm 2i \sqrt[*]{-(\tau_n - \mu_n)^2}. \tag{2.35}$$

Remark 2.1. *In Appendix A, Proposition 2.4 is used to derive the formula for the differential of the Birkhoff map at $q = 0$ by a short calculation.*

Proof of Proposition 2.4. By the definition (2.33) of κ_n,

$$2 \sinh \kappa_n = (-1)^n y_2'(1, \mu_n) - ((-1)^n y_2'(1, \mu_n))^{-1} = (-1)^{n-1} \sqrt[*]{\Delta(\mu_n)^2 - 4}.$$

Recall that by (1.66),

$$\Delta^2(\lambda) - 4 = \frac{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}{\pi_n^2} \Delta_n(\lambda). \tag{2.36}$$

By the definition (2.16) of $\zeta_n(\lambda)$ and the definition (2.10) of ψ_n one sees that for q real , $\zeta_n(\mu_n) > 0$ and $(-1)^{n-1}\psi_n(\mu_n) > 0$. Hence by the definition (2.25) of the root $\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}$ it follows that for q real

$$(-1)^{n-1} \sqrt[*]{\Delta(\mu_n)^2 - 4} = \sqrt[+]{\Delta_n(\mu_n)} \frac{\sqrt[*]{(\lambda_{2n} - \mu_n)(\mu_n - \lambda_{2n-1})}}{\pi_n}.$$

As both sides of the last identity are analytic on W it holds for any $q \in W$. Hence it is to prove that

$$u_n^{\pm} = 2(\tau_n - \mu_n) \pm 2i \sqrt[*]{(\lambda_{2n} - \mu_n)(\mu_n - \lambda_{2n-1})}. \tag{2.37}$$

In the case where $\mu_n = \lambda_{2n-1}$, one obtains from (2.30) that the left and right hand side of (2.37) are equal to γ_n. If $\mu_n = \lambda_{2n}$, by (2.31), both sides of (2.37) equal $-\gamma_n$. It remains to verify (2.37) for $q \in W \setminus Z_n$ with $\mu_n \notin \{\lambda_{2n}, \lambda_{2n-1}\}$. Without loss of generality we may assume that $|\mu_n - \lambda_{2n-1}| \leq |\mu_n - \lambda_{2n}|$. Otherwise, we exchange the role of λ_{2n-1} and λ_{2n} and note that this will not change the value of $u_n^{\pm}$. Denote by $V_n \subseteq U_n$ a (small) open neighbourhood of $(\lambda_{2n-1}, \mu_n]$ which does not contain λ_{2n-1} nor λ_{2n}. There, the root $\sqrt[*]{(\lambda_{2n} - \mu)(\mu - \lambda_{2n-1})}$, defined on $[\lambda_{2n-1}, \mu_n]$, continuously extends to $V_n \cup \{\lambda_{2n-1}\}$. We again denote it by $\sqrt[*]{(\lambda_{2n} - \mu)(\mu - \lambda_{2n-1})}$. Furthermore, for $\mu \in V_n$, introduce

$$f_n^{\pm}(\mu) = \gamma_n \exp\left(\pm i \int_{\lambda_{2n-1}}^{\mu} \frac{d\lambda}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} \right) \tag{2.38}$$

and

$$g_n^{\pm}(\mu) = 2(\tau_n - \mu) \pm 2i \sqrt[*]{(\lambda_{2n} - \mu)(\mu - \lambda_{2n-1})}. \tag{2.39}$$

Then

$$\lim_{\mu\to\lambda_{2n-1}} f_n^{\pm}(\mu) = \gamma_n = \lim_{\mu\to\lambda_{2n-1}} g_n^{\pm}(\mu),$$

$$\partial_\mu f_n^{\pm}(\mu) = \pm f_n^{\pm}(\mu) \frac{i}{\sqrt[*]{(\lambda_{2n}-\mu)(\mu-\lambda_{2n-1})}}$$

and

$$\partial_\mu g_n^{\pm}(\mu) = \pm g_n^{\pm}(\mu) \frac{i}{\sqrt[*]{(\lambda_{2n}-\mu)(\mu-\lambda_{2n-1})}}.$$

Hence f_n^+ and g_n^+ satisfy the same 1st order differential equation and have the same value at $\mu = \lambda_{2n-1}$. Hence

$$f_n^+(\mu) = g_n^+(\mu) \quad \forall \mu \in V_n. \tag{2.40}$$

In particular $f_n^+(\mu_n) = g_n^+(\mu_n)$. Similarly one has $f_n^-(\mu_n) = g_n^-(\mu_n)$ and the identity (2.37) is proved.

Note that the right hand side of (2.34) is defined on all of W. By the analyticity of $y_j(1,\lambda,q)$ $(j = 1,2)$ on $\mathbb{C}\times W$, the analyticity of τ_n, μ_n, κ_n on W, and the analyticity of $\Delta_n(\lambda, q)$ on $U_n \times W$, it then follows that the right hand side of (2.34) is analytic on W. Formula (2.35) is obtained from (2.37). □

As an application of Proposition 2.4 and the asymptotics of Section 3 and Section 4 we obtain

Corollary 2.1. *For q in W_N with $W_N \subseteq W \cap H^N_{0,\mathbb{C}}$ given as in Corollary 1.3,*

$$u_n^{\pm} = 2\langle q, \cos 2\pi n x\rangle \pm 2i\langle q, \sin 2\pi n x\rangle + \frac{1}{n^{N+1}}\ell_n^2$$

locally uniformly on W_N. On H_0^N, the error is uniformly bounded on bounded subsets of H_0^N.

Proof of Corollary 2.1. By Proposition 2.4, for $q \in W$,

$$u_n^{\pm} = 2(\tau_n - \mu_n) \pm i \frac{2\pi n}{\sqrt[+]{\Delta_n(\mu_n)}} 2\sinh\kappa_n.$$

By Theorem 0.5, for q in $W \cap H^N_{0,\mathbb{C}}$,

$$\kappa_n = \frac{1}{2\pi n}\left(\langle q, \sin 2\pi n x\rangle + \frac{1}{n^{N+1}}\ell_n^2\right)$$

and the Taylor expansion of $\sinh z$ at $z = 0$ then yields

$$\sinh\kappa_n = \frac{1}{2\pi n}\left(\langle q, \sin 2\pi n x\rangle + \frac{1}{n^{N+1}}\ell_n^2\right).$$

According to Corollary 1.1, $\Delta_n(\lambda) = 1 + \frac{1}{n}\ell_n^2$ and hence

$$\left(\sqrt[+]{\Delta_n(\lambda)} \right)^{-1} = 1 + \frac{1}{n}\ell_n^2$$

uniformly for $\lambda \in U_n$. Furthermore, by Corollary 1.3, for $q \in W_N$,

$$2(\tau_n - \mu_n) = 2\langle q, \cos 2\pi nx\rangle + \frac{1}{n^{N+1}}\ell_n^2.$$

Altogether we get for q in W_N

$$\begin{aligned} u_n^{\pm} =& 2\langle q, \cos 2\pi nx\rangle + \frac{1}{n^{N+1}}\ell_n^2 \pm 2i \left(1 + \frac{1}{n}\ell_n^2\right)\left(\langle q, \sin 2\pi nx\rangle + \frac{1}{n^{N+1}}\ell_n^2\right) \\ =& 2\langle q, \cos 2\pi nx\rangle \pm 2i\langle q, \sin 2\pi nx\rangle + \frac{1}{n^{N+1}}\ell_n^2 \end{aligned}$$

as claimed. Going through the arguments of the proof one sees that the estimate holds locally uniformly on W_N and uniformly on bounded subsets of H_0^N.

□

It remains to analyze the asymptotics for $v_n^{\pm}$. We need the following auxiliary result.

Lemma 2.3. *For q in $W \setminus Z_n$ with $|\mu_n - \lambda_{2n-1}| \le |\mu_n - \lambda_{2n}|$*

$$\left| \int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\lambda_{2n-1})}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda \right| \le (|\mu_n - \tau_n| + |\gamma_n|) \sup_{\lambda \in [\lambda_{2n-1}, \mu_n]} |\dot{\zeta}_n(\lambda)| \tag{2.41}$$

and

$$\left| \int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda_{2n-1}) - 1}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda \right| \le 3\left(|\mu_n - \tau_n| + |\gamma_n|\right) \sup_{\lambda \in G_n} \frac{|\zeta_n(\lambda) - 1|}{|\gamma_n|} \tag{2.42}$$

If $|\mu_n - \lambda_{2n-1}| \ge |\mu_n - \lambda_{2n}|$, (2.41) *and* (2.42) *hold if the roles of λ_{2n-1} and λ_{2n} are interchanged.*

Proof of Lemma 2.3. First, assume that $q \in W \setminus Z_n$ and

$$|\mu_n - \lambda_{2n-1}| \le |\mu_n - \lambda_{2n}|. \tag{2.43}$$

This implies that $\mu_n \neq \lambda_{2n}$. If $\mu_n = \lambda_{2n-1}$ then (2.41) and (2.42) hold as their left-hand sides vanish. Now assume that $\mu_n \neq \lambda_{2n-1}$. By the mean value theorem

$$\zeta_n(\lambda) = f_n(\lambda)(\lambda - \lambda_{2n-1}) + \zeta_n(\lambda_{2n-1}) \tag{2.44}$$

where for λ in U_n

$$f_n(\lambda) = \int_0^1 \dot{\zeta}_n \left(\lambda_{2n-1} + s(\lambda - \lambda_{2n-1})\right) ds.$$

Hence

$$\sup_{\lambda \in [\lambda_{2n-1}, \mu_n]} |f_n(\lambda)| \leq \sup_{\lambda \in [\lambda_{2n-1}, \mu_n]} \left| \dot{\zeta}_n(\lambda) \right|. \tag{2.45}$$

(Here we assume (without loss of generality) that U_n is convex.) It follows from (2.44) that, with $\lambda(t) = \lambda_{2n-1} + t(\mu_n - \lambda_{2n-1})$,

$$\left| \int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\lambda_{2n-1})}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda \right| = \left| \int_0^1 f_n(\lambda) \frac{\sqrt{\lambda - \lambda_{2n-1}}}{\sqrt{\lambda_{2n} - \lambda}} (\mu_n - \lambda_{2n-1}) dt \right|.$$

In view of (2.43), $|\lambda(t) - \lambda_{2n-1}| \leq |\lambda(t) - \lambda_{2n}|$ for any $0 \leq t \leq 1$ and thus

$$\begin{aligned} &\left| \int_0^1 \frac{f_n(\lambda)\sqrt{\lambda - \lambda_{2n-1}}}{\sqrt{\lambda_{2n} - \lambda}} (\mu_n - \lambda_{2n-1}) dt \right| \\ \leq &|\mu_n - \lambda_{2n-1}| \sup_{\lambda \in [\lambda_{2n-1}, \mu_n]} |f_n(\lambda)|. \end{aligned}$$

As $|\mu_n - \lambda_{2n-1}| \leq |\mu_n - \tau_n| + |\gamma_n|/2$, the claimed estimate (2.41) then follows from (2.45). Next consider the term

$$(\zeta_n(\lambda_{2n-1}) - 1) \int_{\lambda_{2n-1}}^{\mu_n} \frac{d\lambda}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}}.$$

To estimate the integral, let $\lambda(t) = \lambda_{2n-1} + t(\mu_n - \lambda_{2n-1})$ to get

$$\begin{aligned} \left| \int_{\lambda_{2n-1}}^{\mu_n} \frac{d\lambda}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} \right| &= \left| \int_0^1 \frac{\sqrt{\mu_n - \lambda_{2n-1}}}{\sqrt{\lambda_{2n} - \lambda}} \frac{dt}{\sqrt{t}} \right| \\ &\leq \sqrt{2} \frac{2|\mu_n - \lambda_{2n-1}|^{\frac{1}{2}}}{|\gamma_n|^{\frac{1}{2}}} \end{aligned} \tag{2.46}$$

where we used again that by (2.43), $|\lambda_{2n} - \lambda(t)| \geq |\gamma_n/2|$ for $0 \leq t \leq 1$. In view of Lemma 2.2 (ii) and as

$$2|\mu_n - \lambda_{2n-1}|^{\frac{1}{2}} |\gamma_n|^{\frac{1}{2}} \leq |\mu_n - \lambda_{2n-1}| + |\gamma_n| \leq |\mu_n - \tau_n| + \frac{3}{2}|\gamma_n|$$

the claimed estimate (2.42) follows. The case when $|\mu_n - \lambda_{2n-1}| \geq |\mu_n - \lambda_{2n}|$ is treated in a similar way. □

Corollary 2.2. *For any $n \geq 1$, v_n^+ and v_n^- continuously extend to all of W. These extensions are again denoted by $v_n^{\pm}$. For $q \in Z_n$ with $\mu_n \neq \tau_n$*

$$v_n^{\pm} = \exp\left(\pm i \int_{\tau_n}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\tau_n)}{\sqrt[*]{-(\tau_n - \lambda)^2}} d\lambda \right) \tag{2.47}$$

whereas for $q \in Z_n$ with $\mu_n = \tau_n$, $v_n^{\pm} = 1$.

Proof of Corollary 2.2. Let $q \in Z_n$ with $\mu_n \neq \tau_n$. It follows from (1.70), (2.15), and (2.20) that for $q \in W \setminus Z_n$,

$$\int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{\zeta_n(\lambda) - 1}{\sqrt{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda = 0 \tag{2.48}$$

for any choice of the sign of the root. In particular, (2.48) holds for the $*$-root and thus

$$\int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda) - 1}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda = \int_{\lambda_{2n}}^{\mu_n} \frac{\zeta_n(\lambda) - 1}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda.$$

Hence without loss of generality we may assume that $|\mu_n - \lambda_{2n-1}| \leq |\mu_n - \lambda_{2n}|$ for q, as otherwise we simply interchange the role of λ_{2n} and λ_{2n-1} in (2.29). Moreover, we may assume that the isolating neighbourhood U_n of q is also an isolating neighbourhood of p for any p in some neighbourhood W_q of q. To compute the limit $\lim_{p \to q,\, p \in W_q \setminus Z_n} v_n^{\pm}$ split

$$\int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda) - 1}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda$$

into two parts by writing

$$\zeta_n(\lambda) - 1 = (\zeta_n(\lambda) - \zeta_n(\lambda_{2n-1})) + (\zeta_n(\lambda_{2n-1}) - 1).$$

Then

$$\lim_{p \to q,\, p \in W_q \setminus Z_n} \int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\lambda_{2n-1})}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda = \int_{\tau_n}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\tau_n)}{\sqrt[*]{-(\tau_n - \lambda)^2}} d\lambda \Big|_{p=q}$$

and in view of Lemma 2.2 and (2.46),

$$\lim_{p \to q,\, p \in W_q \setminus Z_n \,|\mu_n - \lambda_{2n-1}| \leq |\lambda_{2n} - \mu_n|} \int_{\lambda_{2n-1}}^{\mu_n} \frac{\zeta_n(\lambda_{2n-1}) - 1}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda = 0.$$

By the same reason the integral

$$\int_{\lambda_{2n}}^{\mu_n} \frac{\zeta_n(\lambda_{2n}) - 1}{\sqrt[*]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda$$

converges to zero as $p \to q$ for $p \in W_q$ with $|\mu_n - \lambda_{2n-1}| \geq |\lambda_{2n} - \mu_n|$. Altogether we thus have shown that

$$\lim_{p \to q,\, p \in W_q \setminus Z_n} v_n^{\pm}(p) = \exp\left(\pm i \int_{\tau_n}^{\mu_n} \frac{\zeta_n(\lambda) - \zeta_n(\tau_n)}{\sqrt[*]{-(\tau_n - \lambda)^2}} d\lambda \right)$$

as claimed. Now let us consider the case where $q \in Z_n$ with $\mu_n = \tau_n$. From (2.47) one concludes that

$$\lim_{p \to q,\, p \in W_q \cap Z_n,\, \mu_n \neq \tau_n} v_n^{\pm}(p) = 1$$

and from Lemma 2.3 and Lemma 2.2 one sees that

$$\lim_{p \to q,\, p \in W_q \setminus Z_n,\, \mu_n \neq \tau_n} v_n^{\pm}(p) = 1.$$

□

It remains to study the asymptotics of $v_n^{\pm}$, now defined on all of W.

Corollary 2.3. *For q in W*

$$v_n^{\pm} = 1 + \frac{1}{n}\ell_n^2 \tag{2.49}$$

locally uniformly on W. On L_0^2, (2.49) *holds uniformly on bounded subsets of L_0^2.*

Proof of Corollary 2.3. We want to apply Lemma 2.3. First note that in view of Corollary 2.2, the estimates of Lemma 2.3 hold on all of W, not only on $W \setminus Z_n$. Furthermore, by shrinking the isolating neighbourhoods we get from Lemma 2.2 and Cauchy's estimate

$$\sup_{\lambda \in U_n} |\dot{\zeta}_n(\lambda)| = \frac{1}{n}\ell_n^2.$$

By Lemma 2.2,

$$\sup_{\lambda \in G_n} \frac{|\zeta_n(\lambda) - 1|}{|\gamma_n|} = \frac{1}{n}\ell_n^2.$$

Using that $|e^x - 1| \leq |x|e^{|x|}$, the claimed estimate then follows indeed from Lemma 2.3. Going through the arguments of the proof one sees that (2.49) holds locally uniformly on W and uniformly on bounded subsets of L_0^2. □

Proof of Proposition 2.3. By Corollary 2.2 and Lemma 2.3, $u_n^{\pm}$ and $v_n^{\pm}$ are defined and continuous on W for any $n \geq 1$. Hence the identities (2.27) extend to all of W,

$$z_n^+ = u_n^+ v_n^+ \quad \text{and} \quad z_n^- = u_n^- v_n^-. \tag{2.50}$$

By Corollary 2.1 and Corollary 2.3, it follows that for q in W_N,

$$\begin{aligned} z_n^{\pm} &= \left(2\langle q, \cos 2\pi nx\rangle \pm 2i\langle q, \sin 2\pi nx\rangle + \frac{1}{n^{N+1}}\ell_n^2\right)\left(1 + \frac{1}{n}\ell_n^2\right) \\ &= 2\hat{q}_{\mp n} + \frac{1}{n^{N+1}}\ell_n^2. \end{aligned}$$

Going through the arguments of the proof one sees that the estimate holds locally uniformly on W_N and uniformly on bounded subsets of H_0^N. This proves Proposition 2.3. □

2.3 Asymptotics of Φ

In this section we prove the asymptotics of the Birkhoff map claimed in Theorem 0.4 and use them to derive further properties of this map.

Proof of Theorem 0.4. For any $N \in \mathbb{Z}_{\geq 0}$, let $W_N \subseteq W \cap H^N_{0,\mathbb{C}}$ be the neighbourhood of H^N_0 given by Theorem 0.8. For any $q \in W$, $\Phi(q)$ is given by $\Phi(q) = (z_n(q))_{n\neq 0}$, where for any $n \geq 1$,

$$z_{-n} = x_n + iy_n = \xi_n \frac{z_n^+}{2} e^{i\beta_n}, \quad \text{and} \quad z_n = x_n - iy_n = \xi_n \frac{z_n^-}{2} e^{-i\beta_n}. \qquad (2.51)$$

By Proposition 2.1 and Lemma 2.1

$$\xi_n = \frac{1}{\sqrt{\pi n}} \left(1 + \frac{1}{n}\ell^2_n\right) \quad \text{and} \quad e^{i\beta_n} = 1 + O\left(\frac{1}{n}\right) \qquad (2.52)$$

whereas by Proposition 2.3, for $q \in W_N$,

$$\frac{z_n^+}{2} = \hat{q}_{-n} + \frac{1}{n^{N+1}}\ell^2_n. \qquad (2.53)$$

Hence

$$z_{-n} = \frac{1}{\sqrt{n\pi}} \left(\hat{q}_{-n} + \frac{1}{n^{N+1}}\ell^2_n\right).$$

A similar estimate holds for z_n. Note that all the asymptotic estimates referred to hold locally uniformly on W_N. As by Theorem 0.3, $z_{\pm n}$ are analytic on W for any $n \geq 1$ it follows from Theorem A.5 in [12] that $\Phi - \Phi_0 : W_N \to \mathfrak{h}^{N+3/2}_{\mathbb{C}}$ is analytic. This proves the first part of Theorem 0.4.

Going through the above arguments and using that by Proposition 2.1, Lemma 2.1, and Proposition 2.3, the estimates (2.52) and (2.53) hold uniformly on bounded sets of H^N_0, it follows that $A := (\Phi - \Phi_0) : H^N_0 \to \mathfrak{h}^{N+3/2}$ is bounded. ☐

Proposition 2.5. *Let $N \in \mathbb{Z}_{\geq} 0$. The restriction of Φ^{-1} to $\mathfrak{h}^{N+1/2}$ is of the form $\Phi^{-1} = \Phi_0^{-1} + B$, with $B = -\Phi_0^{-1} \circ A \circ \Phi^{-1}$. B is a map from $\mathfrak{h}^{N+1/2}$ to H^{N+1}_0. It is bounded and real analytic.*

Proof. First let us verify the formula for B

$$Id_{\mathfrak{h}^{N+1/2}} = (\Phi_0 + A) \circ \Phi^{-1} = Id_{\mathfrak{h}^{N+1/2}} + \Phi_0 \circ B + A \circ \Phi^{-1}.$$

Hence

$$B = -\Phi_0^{-1} \circ A \circ \Phi^{-1}.$$

As B is given by the composition of real analytic maps,

$$\mathfrak{h}^{N+1/2} \xrightarrow{\Phi^{-1}} H^N_0 \xrightarrow{A} \mathfrak{h}^{N+3/2} \xrightarrow{-\Phi_0^{-1}} H^{N+1}_0,$$

it is itself real analytic. It remains to prove that for any $N \in \mathbb{Z}_{\geq 0}$, $B : \mathfrak{h}^{N+1/2} \to H^{N+1}$ is bounded. First note that for any $N \in \mathbb{Z}_{\geq 0}$, the inverse of the weighted Fourier transform $\Phi_0^{-1} : \mathfrak{h}^{N+1/2} \to H_0^N$ and, by Theorem 0.4, the nonlinear map $A : H_0^N \to \mathfrak{h}^{N+3/2}$ are bounded for any $N \in \mathbb{Z}_{\geq 0}$. Furthermore, the boundedness of $\Phi^{-1} : \mathfrak{h}^{N+1/2} \to H_0^N$ follows, in the case $N = 0$, from the identity $\sum_{n\geq 1} 2\pi n I_n = \frac{1}{2}\|q\|_{L^2}$, established in [12], Theorem E.1, and, in the case $N \in \mathbb{Z}_{\geq 1}$, from [18], Theorem 2.4 and Theorem 2.6. More precisely, in [18] it is shown that for any $q \in H^N$ with $\lambda_0(q) = 0$ and $N \in \mathbb{Z}_{\geq 1}$, $\|\partial_x^N q\|_{L^2}$ can be bounded in terms of $\|J_n(q)\|_{\mathfrak{h}^{N+1/2}}$. Note that for any $p \in H_0^N$, $q = p - \lambda_0(p)$ satisfies $\lambda_0(q) = 0$ and hence, as $N \geq 1$, $\|\partial_x^N q\|_{L^2} = \|\partial_x^N p\|_{L^2}$. Furthermore, the quantities $(J_n(q))_{n\geq 1}$, introduced in [18], formula (1.2), can be shown to coincide with the action variables $(I_n(p))_{n\geq 1}$, introduced in [12] (cf formula (7.2)). Combining these results it follows that $B = -\Phi_0^{-1} \circ A \circ \Phi^{-1} : \mathfrak{h}^{N+1/2} \to H_0^{N+1}$ is bounded for any $N \in \mathbb{Z}_{\geq 0}$. □

By Cauchy's estimate, Theorem 0.4 yields the following asymptotics of the differential of Φ.

Corollary 2.4. *For any $q \in W_N$ with $N \geq 0$, the differential of the Birkhoff map*

$$d_q\Phi : H_{0,\mathbb{C}}^N \to \mathfrak{h}_{\mathbb{C}}^{N+1/2}$$

satisfies the asymptotic estimate

$$d_q\Phi(f) = \left(\frac{1}{\sqrt{n\pi}}\hat{f}_n + \mathcal{R}_n(f)\right)_{n\neq 0}$$

where

$$(\mathcal{R}_n(f))_{n\neq 0} \in \mathfrak{h}^{N+3/2}.$$

As an immediate application of Corollary 2.4 we obtain

Corollary 2.5. *For any $q \in W_N$ with $N \geq 0$,*

$$d_q\Phi : H_{0,\mathbb{C}}^N \to \mathfrak{h}_{\mathbb{C}}^{N+1/2}$$

is a compact perturbation of the (weighted) Fourier transform. In particular, it satisfies the Fredholm alternative.

Corollary 2.5 is already proved in [12]. The proof argues by approximation and uses quite complicated computations. It is based on the fact that the set of finite gap potentials is dense in H_0^N. In the set-up presented here, its proof is straightforward given the asymptotic estimates stated in Section 2.

Chapter 3

Asymptotics of the Birkhoff map II

It has been discovered by Magri [22] that KdV is a bihamiltonian system. Alternatively, one can write KdV as a Hamiltonian system with respect to the so-called Magri bracket

$$\{F,G\}_M = \int_0^1 \partial_q F(-\partial_x^3 + 2q\circ\partial_x + 2\partial_x\circ q)\partial_q G dx \tag{3.1}$$

where F and G, are differentiable functionals on L^2 with L^2-gradients in H^3. It was proved in [9] that KdV admits Birkhoff coordinates also with respect to the Magri bracket. We denote the corresponding map by $\Phi^{(M)}$. The purpose of this chapter is to prove a result for the map $\Phi^{(M)}$ analogous to Theorem 0.4. To state the result more precisely we need to introduce further notation. The Magri bracket is degenerate and admits the functional

$$L^2 \to \mathbb{R},\ q \mapsto \Delta(0,q)$$

as a Casimir. It leads to a foliation by connected symplectic leaves some of which are topologically nontrivial. Introduce the following connected components of level sets:

$$\begin{aligned}
\mathcal{L}_d :=& \{q\in L^2 | \Delta(0,q) = d,\ \lambda_0(q)\geq 0\}, \quad d\in\mathbb{R}_{\geq 2};\\
\mathcal{L}_{d,k} :=& \{q\in L^2 | \Delta(0,q) = d,\ \lambda_{k-1}(q) < 0 < \lambda_k(q)\}, \quad d\in\mathbb{R},\ k\in\mathbb{Z}_{\geq 1};\\
\mathcal{J}_k^{\pm} :=& \{q\in L^2 |\ \lambda_{k-1/2\pm 1/2}(q) = 0,\ \lambda_{k-1}(q)\neq\lambda_k(q)\}, \quad k\in\mathbb{Z}_{\geq 1};\\
\mathcal{I}_k :=& \{q\in L^2 |\ \lambda_{k-1}(q) = \lambda_k(q) = 0\}, \quad k\in\mathbb{Z}_{\geq 1}.
\end{aligned}$$

Note that L^2 can be written as a disjoint union of connected symplectic leaves in the following way

$$L^2 = \bigcup_{d\geq 2}\mathcal{L}_d \bigcup_{k\in\mathbb{Z}_{\geq 1}}\left(\mathcal{J}_{2k}^+ \cup \mathcal{J}_{2k}^- \cup \mathcal{I}_{2k} \bigcup_{|d|<2}\mathcal{L}_{d,2k-1} \bigcup_{(-1)^k d>2}\mathcal{L}_{d,2k}\right). \tag{3.2}$$

An arbitrary leaf of this foliation is denoted by $\mathcal{L}_\star$. Define the Hamiltonian

$$\mathcal{H}^{(M)}(q) := \frac{1}{2}\int_0^1 q^2(x)dx.$$

Then the Hamiltonian flow of $\mathcal{H}^{(M)}$ with respect to the Magri bracket is the KdV flow. In [9], Birkhoff coordinates with respect to the Magri bracket are constructed. To state this result we have to introduce some further notation. Define

$$\mathcal{Z} := \mathbb{R} \times S^1,\ \mathcal{Z}^+ := \mathbb{R}_{>0} \times S^1,\ \mathcal{Z}^- := \mathbb{R}_{<0} \times S^1.$$

and for $k \in \mathbb{Z}_{>0}, \alpha \in \mathbb{R}$

$$\mathfrak{h}_k^\alpha := \ell^{2,\alpha}(\mathbb{Z}_0 \setminus \{k, -k\}, \mathbb{C}),$$

endowed with the standard symplectic structure. Now we formulate Theorem 2 in [9] in complex coordinates $z_{\pm n}^{(M)} = x_n \mp iy_n,\ n > 1$.

Theorem 3.1. *There is a map $\Phi^{(M)}$ on L^2 (defined leafwise using (3.2)) with the property that for each $N \in \mathbb{Z}_{\geq 0}$ the following restrictions are real analytic symplectic diffeomorphisms.*

$$\begin{array}{rlll}
\Phi^{(M)}_{|H^N \cap \mathcal{L}_d} : H^N \cap \mathcal{L}_d & \to \mathfrak{h}^{N+3/2} & & d \geq 2 \\
\Phi^{(M)}_{|H^N \cap \mathcal{L}_{d,2k-1}} : H^N \cap \mathcal{L}_{d,2k-1} & \to \mathfrak{h}^{N+3/2} & & |d| < 2,\ k \in \mathbb{Z}_{>0} \\
\Phi^{(M)}_{|H^N \cap \mathcal{L}_{d,2k}} : H^N \cap \mathcal{L}_{d,2k} & \to \mathcal{Z} \times \mathfrak{h}_{2k}^{N+3/2} & & (-1)^k d > 2,\ k \in \mathbb{Z}_{>0} \\
\Phi^{(M)}_{|H^N \cap \mathcal{J}^\pm_{2k}} : H^N \cap \mathcal{J}^\pm_{2k} & \to \mathcal{Z}^\pm \times \mathfrak{h}_{2k}^{N+3/2} & & k \in \mathbb{Z}_{>0} \\
\Phi^{(M)}_{|H^N \cap \mathcal{I}_{2k}} : H^N \cap \mathcal{I}_{2k} & \to \mathfrak{h}_{2k}^{N+3/2} & & k \in \mathbb{Z}_{>0}.
\end{array}$$

When expressed in the new coordinates, the KdV-Hamiltonian $\mathcal{H}^{(M)} \circ (\Phi^{(M)}_{|L_\star})^{-1}$, is a real analytic function of the action variables.

To state the next result it is convenient to write $\mathfrak{h}_\star^\alpha$ for a space of the form $\mathfrak{h}^\alpha$, $\mathcal{Z} \times \mathfrak{h}_k^\alpha$, $\mathcal{Z}^\pm \times \mathfrak{h}_k^\alpha$ or $\mathfrak{h}_k^\alpha$; $k \in \mathbb{Z}_{\geq 1}$, $\alpha \in \mathbb{R}$.

In Section 3.1 we prove for $\Phi^{(M)}$ the analogous to Theorem 0.4.

Theorem 3.2. *Let $\Phi_\star^{(M)} := \Phi^{(M)}_{|\mathcal{L}_\star} : \mathcal{L}_\star \to \mathfrak{h}_\star^{3/2}$ be the restriction of $\Phi^{(M)}$ to a symplectic leaf $\mathcal{L}_\star$ and corresponding target $\mathfrak{h}_\star^{3/2}$ according to Theorem 3.1. Then $\Phi_\star^{(M)}$ has the property that for any $N \geq 0$, $\Phi_\star^{(M)} - \Phi_0^{(M)}$ maps $H^N \cap \mathcal{L}_\star$ into $\mathfrak{h}_\star^{N+5/2}$, where*

$$\Phi_0^{(M)}(h) := \left(\frac{1}{(|n|\pi)^{3/2}}\hat{h}_n\right)_{n \neq 0}.$$

The restriction $(\Phi_\star^{(M)} - \Phi_0^{(M)})_{|H^N \cap \mathcal{L}_\star} : H^N \cap \mathcal{L}_\star \to \mathfrak{h}_\star^{N+5/2}$ is bounded on bounded subsets of $H^N \cap \mathcal{L}_\star$.

In other words, we have the formula for $(z^{(M)}_{\pm n})_{n\in\mathbb{Z}_{>0}} = \Phi^{(M)}(q)$

$$z^{(M)}_{\pm n} = \frac{1}{2(n\pi)^{3/2}}\left(\hat{q}_{\pm n} + \frac{1}{n^{N+1}}\ell^2_n\right),\ n\in\mathbb{Z}_{>0},$$

where the error is bounded on bounded subsets of $\mathcal{L}_\star$.

In Section 3.2 we consider the modified Korteweg-de Vries equation (mKdV)

$$\partial_t r + \partial_x^3 r - 6r^2\partial_x r = 0, \tag{3.3}$$

introduced by Miura [24]. It is a model for acoustic waves in a certain anharmonic lattice – see e.g. [28]. It has played an important rôle in the discovery of conserved quantities of the KdV equation. Both equations are known to have many features in common. Actually, the two equations are related by a Bäcklund transformation. It is given by the so called Miura transformation

$$B : r \mapsto \partial_x r + r^2$$

and maps solutions of mKdV to solutions of KdV.
Miura [24] observed that mKdV is a Hamiltonian PDE on the phase space $(H^1, \{\cdot,\cdot\})$ where $\{\cdot,\cdot\}$ denotes the Gardner bracket, introduced in the Introduction. Then mKdV takes the form

$$r_t = \partial_x(\partial_r \mathcal{H}^{mKdV})$$

where the mKdV-Hamiltonian is given by

$$\mathcal{H}^{mKdV}(r) = \frac{1}{2}\int_0^1 ((\partial_x r)^2 + r^4)dx.$$

Note that the mean value functional $r \mapsto [r] = \int_0^1 r(x)dx$ is a Casimir of the Gardner bracket and its level sets

$$H^N_c := \{r \in H^N | [r] = c\}$$

describe a symplectic foliation of H^N.
Notice now that the Miura map, viewed as a map between Poisson spaces,

$$B : (H^1, \{\cdot,\cdot\}) \to (L^2, \{\cdot,\cdot\}_M)$$

is canonical. Its image is

$$B(H^1) = \bigcup_{d\geq 2}\mathcal{L}_d.$$

Birkhoff coordinates are then obtained by pulling back the Birkhoff map $\Phi^{(M)}$ of Theorem 3.1 on $(L^2, \{\cdot,\cdot\}_M)$ – cf [14]. This Birkhoff map is denoted by Ω. To be more precise, we formulate Theorem 1 in [14] in complex coordinates $z^{(mKdV)}_{\pm n} = x_n \mp iy_n,\ n > 1$.

Theorem 3.3. *There exist a real analytic map $\Omega : H^1 \to \mathfrak{h}^{3/2} \times \mathbb{R}$ with the following properties:*

(BNF1) For any $N \in \mathbb{Z}_{>0}$, the restriction of Ω to H^N is a canonical, real analytic diffeomorphism onto $\mathfrak{h}^{N+1/2} \times \mathbb{R}$.

(BNF2) When expressed in the new coordinates, the mKdV-Hamiltonian $\mathcal{H}^{mKdV} \circ \Omega^{-1}$, defined on $\mathfrak{h}^{3/2}$, is a real analytic function of the action variables $I_k = (x_k^2 + y_k^2)/2$, $k \geq 1$, alone.

In Section 3.2 we prove for the map $\Omega : H^1 \to \mathfrak{h}_{\mathbb{C}}^{3/2} \times \mathbb{R}$ of Theorem 3.3 an analogous result to Theorem 0.4.

Theorem 3.4. *For any $N \in \mathbb{Z}_{\geq 1}$, $\Omega - \Omega_0$ maps H^N into $\mathfrak{h}^{N+5/2}$ where*

$$\Omega_0(h) = \left(\left(\frac{\pi i \operatorname{sign}(n)}{(|n|\pi)^{1/2}} \hat{h}_n \right)_{n \neq 0}, [h] \right),$$

and $[h] = \int_0^1 h(x)dx$. The restriction $(\Omega - \Omega_0)_{|H^N} : H^N \to \mathfrak{h}^{N+5/2}$ is bounded on bounded subsets of H^N.

3.1 Asymptotics of $\Phi^{(M)}$

In this section we prove Theorem 3.2. First we improve on estimates of the action variables obtained in [9]. Recall that W is a (sufficiently small) neighbourhood of L^2 in $L^2_{\mathbb{C}}$. For q in W, the n'th action variable with respect to the Magri bracket is given by

$$I_n^{(M)} = \frac{1}{4\pi} \int_{\Gamma_n} \log(|\lambda|) \frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} d\lambda \qquad (n \geq 1),$$

where $\Delta(\lambda)$ is the discriminant, $\dot{\Delta}(\lambda) = \partial_\lambda \Delta(\lambda)$ and Γ_n is a contour around the interval $G_n := \{t\lambda_{2n-1} + (1-t)\lambda_{2n} | \ 0 \leq t \leq 1\}$ in the isolating neighbourhood U_n of G_n – see Section 7 of [12] for more details. The main result of this section is an improved estimate of the quotient $I_n^{(M)}/\gamma_n^2$, given in [9], Lemma 15.

Proposition 3.1. *Locally uniformly on W, the quotient $I_n^{(M)}/\gamma_n^2$ satisfies*

$$32(\pi n)^3 \frac{I_n^{(M)}}{\gamma_n^2} = 1 + \frac{1}{n} \ell_n^2 \tag{3.4}$$

Moreover

$$\xi_n^{(M)} = \sqrt[+]{32 I_n^{(M)}/\gamma_n^2} = \frac{1}{(\pi n)^{3/2}} (1 + \frac{1}{n} \ell_n^2) \tag{3.5}$$

is well-defined as a real analytic, non-vanishing function on W. In particular, at $q = 0$, we have $\xi_n^{(M)} = \frac{1}{(\pi n)^{3/2}}$ for all $n \geq 1$. On L_0^2, the estimates hold uniformly on bounded subsets of L_0^2.

Proof. We refer to [12], Section 7, for all notions, notations (and results) not explained here. In view of Lemma 15 in [9] it only remains to be shown the improved asymptotics (2.2). Recall the product expansions

$$\Delta^2(\lambda) - 4 \quad = \quad 4(\lambda_0 - \lambda) \prod_{n\geq 1} \frac{(\lambda_{2n} - \lambda)(\lambda_{2n-1} - \lambda)}{\pi_n^4}$$

and

$$\dot{\Delta}(\lambda) \quad = \quad - \prod_{n\geq 1} \frac{\dot{\lambda}_n - \lambda}{\pi_n^2}.$$

For λ on Γ_n write

$$\frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} = \frac{1}{2\pi n} \frac{\lambda - \dot{\lambda}_n}{\sqrt[s]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} \chi_n(\lambda) \tag{3.6}$$

where for $\lambda \in U_n$,

$$\chi_n(\lambda) = (-1)^{n-1} \frac{n\pi}{\sqrt[+]{\lambda - \lambda_0}} \prod_{m\neq n} \frac{\dot{\lambda}_m - \lambda}{\sqrt[+]{(\lambda_{2m} - \lambda)(\lambda_{2m-1} - \lambda)}}, \tag{3.7}$$

and where the canonical root $\sqrt[c]{\Delta^2(\lambda) - 4}$ has been introduced earlier. (For questions of signs, see Section 6 in [12].) Note that for $\lambda \in U_n$,

$$\frac{n\pi}{\sqrt[+]{\lambda - \lambda_0}} = 1 + \frac{1}{n}\ell_n^2. \tag{3.8}$$

Furthermore, by Proposition 1.4, the roots $\dot{\lambda}_n$ of $\dot{\Delta}$ satisfy $\dot{\lambda}_n = \tau_n + \frac{\gamma_n^2}{n}\ell_n^2$ and hence, by Proposition 1.1, one has for $\lambda \in U_n$,

$$\begin{aligned} \left(\prod_{m\neq n} \frac{\dot{\lambda}_m - \lambda}{\sqrt[+]{(\lambda_{2m} - \lambda)(\lambda_{2m-1} - \lambda)}} \right)^2 &= \prod_{m\neq n} \frac{\dot{\lambda}_m - \lambda}{\lambda_{2m} - \lambda} \prod_{m\neq n} \frac{\dot{\lambda}_m - \lambda}{\lambda_{2m-1} - \lambda} \\ &= 1 + \frac{1}{n}\ell_n^2 \end{aligned}$$

or

$$(-1)^{n-1} \prod_{m\neq n} \frac{\dot{\lambda}_m - \lambda}{\sqrt[+]{(\lambda_{2m} - \lambda)(\lambda_{2m-1} - \lambda)}} = 1 + \frac{1}{n}\ell_n^2. \tag{3.9}$$

Combining (3.8) and (3.9) leads to the estimate

$$\chi_n(\lambda) = 1 + \frac{1}{n}\ell_n^2 \tag{3.10}$$

uniformly for $\lambda \in U_n$. Introduce

$$Z_n = \{q \in W \mid \lambda_{2n}(q) = \lambda_{2n-1}(q)\}.$$

Arguing as in Section 7 of [12] one gets for q in $W \setminus Z_n$

$$\begin{aligned} I_n^{(M)} &= \frac{1}{4\pi} \int_{\Gamma_n} (\log(\lambda) - \log(\dot{\lambda}_n)) \frac{\dot{\Delta}(\lambda)}{\sqrt[c]{\Delta^2(\lambda) - 4}} d\lambda \\ &= \frac{1}{8\pi^2 n} \int_{\Gamma_n} \log\left(\frac{\lambda}{\dot{\lambda}_n}\right) \frac{(\lambda - \dot{\lambda}_n)}{\sqrt[s]{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} \chi_n(\lambda) d\lambda. \end{aligned}$$

Substituting $\lambda(\zeta) = \tau_n + \zeta\gamma_n/2$ and setting $\delta_n = 2(\dot{\lambda}_n - \tau_n)/\gamma_n$ yields

$$I_n^{(M)} = \frac{\gamma_n}{16\pi^2 n} \int_{\Gamma'_n} \log\left(\frac{\lambda(\zeta)}{\dot{\lambda}_n}\right) (\zeta - \delta_n)\chi_n(\tau_n + \zeta\frac{\gamma_n}{2}) \frac{d\zeta}{\sqrt[s]{1 - \zeta^2}}$$

where Γ'_n is some circuit in $\mathbb{C}$ around $[-1, 1]$. Thus on $W \setminus Z_n$,

$$\begin{aligned} 16\pi n \frac{I_n^{(M)}}{\gamma_n} &= \frac{1}{\pi} \int_{\Gamma'_n} \log\left(\frac{\lambda(\zeta)}{\dot{\lambda}_n}\right) (\zeta - \delta_n)\chi_n\left(\tau_n + \zeta\frac{\gamma_n}{2}\right) \frac{d\zeta}{\sqrt[s]{1 - \zeta^2}} \\ &= \frac{1}{\pi} \int_{\Gamma'_n} \log\left(\frac{\lambda(\zeta)}{\dot{\lambda}_n}\right) (\zeta - \delta_n)\chi_n(\tau_n) \frac{d\zeta}{\sqrt[s]{1 - \zeta^2}} \\ &+ \frac{1}{\pi} \int_{\Gamma'_n} \log\left(\frac{\lambda(\zeta)}{\dot{\lambda}_n}\right) (\zeta - \delta_n) \left(\chi_n\left(\tau_n + \zeta\frac{\gamma_n}{2}\right) - \chi_n(\tau_n)\right) \frac{d\zeta}{\sqrt[s]{1 - \zeta^2}}. \end{aligned}$$

By Proposition 1.4, δ_n is well defined on all of W, hence so is the r.h.s. of the latter identity. In order to investigate the integrand we make a Taylor expansion of $\log\left(\frac{\lambda(\zeta)}{\dot{\lambda}_n}\right)$ – see [9]

$$\begin{aligned} \log\left(\frac{\lambda(\zeta)}{\dot{\lambda}_n}\right) &= \log\left(1 + s\left(\frac{\lambda(\zeta)}{\dot{\lambda}_n} - 1\right)\right)\Big|_{s=1} \\ &= \left(\frac{\lambda(\zeta)}{\dot{\lambda}_n} - 1\right) \int_0^1 \frac{ds}{1 + s(\frac{\lambda(\zeta)}{\dot{\lambda}_n} - 1)} = \frac{\gamma_n/2}{\dot{\lambda}_n}(\zeta - \delta_n) \left(\int_0^1 \frac{ds}{1 + s(\frac{\lambda(\zeta)}{\dot{\lambda}_n} - 1)}\right) \end{aligned}$$

inserting this gives

$$\begin{aligned} 32\pi^3 n^3 \frac{I_n^{(M)}}{\gamma_n^2} =& \frac{1}{\pi} \int_{\Gamma'_n} \frac{n^2\pi^2}{\dot{\lambda}_n} \left(\int_0^1 \frac{ds}{1 + s(\frac{\lambda(\zeta)}{\dot{\lambda}_n} - 1)}\right) (\zeta - \delta_n)^2 \chi_n(\tau_n) \frac{d\zeta}{\sqrt[s]{1 - \zeta^2}} \\ &+ \frac{1}{\pi} \int_{\Gamma'_n} \frac{n^2\pi^2}{\dot{\lambda}_n} \left(\int_0^1 \frac{ds}{1 + s(\frac{\lambda(\zeta)}{\dot{\lambda}_n} - 1)}\right) (\zeta - \delta_n)^2 \\ &\cdot \left(\chi_n\left(\tau_n + \zeta\frac{\gamma_n}{2}\right) - \chi_n(\tau_n)\right) \frac{d\zeta}{\sqrt[s]{1 - \zeta^2}}. \end{aligned}$$

thus we get, using $\delta_n = \frac{\gamma_n}{n}\ell_n^2$ and $\dot\lambda_n = n^2\pi^2 + [q] + \ell_n^2$ we obtain

$$\left(\int_0^1 \frac{ds}{1+s(\frac{\lambda(\zeta)}{\dot\lambda_n}-1)}\right) = 1 + \frac{\ell_n^2}{n^2}$$
$$\frac{n^2\pi^2}{\dot\lambda_n} = 1 + \frac{\ell_n^2}{n^2\pi^2}$$
$$(\zeta-\delta_n)^2 = \zeta^2 + \frac{\gamma_n}{n}\ell_n^2$$

Lemma M.1 in [12] and (3.10) then imply that

$$\begin{aligned}&\frac{1}{\pi}\int_{\Gamma_n'} \frac{n^2\pi^2}{\dot\lambda_n}\left(\int_0^1 \frac{ds}{1+s(\frac{\lambda(\zeta)}{\dot\lambda_n}-1)}\right)(\zeta-\delta_n)^2\chi_n(\tau_n)\frac{d\zeta}{\sqrt[s]{1-\zeta^2}}\\ =&\frac{1}{\pi}\int_{\Gamma_n'}\frac{\zeta^2 d\zeta}{\sqrt[s]{1-\zeta^2}} + \frac{1}{n}\ell_n^2\\ =&1+\frac{1}{n}\ell_n^2.\end{aligned}$$

By the Taylor expansion of order 0 of χ_n at $\lambda = \tau_n$, Cauchy's estimate, and (3.10) to bound $\dot\chi_n(\lambda)$, one gets

$$\begin{aligned}&\frac{1}{\pi}\int_{\Gamma_n'}\frac{1}{\dot\lambda_n}\left(\int_0^1 \frac{ds}{1+s(\lambda/\dot\lambda_n-1)}\right)(\zeta-\delta_n)^2\\ &\cdot\left(\chi_n\left(\tau_n+\zeta\frac{\gamma_n}{2}\right)-\chi_n(\tau_n)\right)\frac{d\zeta}{\sqrt[s]{1-\zeta^2}}\\ =&\frac{1}{n}\ell_n^2\end{aligned}$$

Altogether,

$$32\pi^3 n^3 I_n^{(M)}/\gamma_n^2 = 1 + \frac{1}{n}\ell_n^2$$

and thus

$$\xi_n^{(M)} = \frac{1}{(\pi n)^{3/2}}(1+\frac{1}{n}\ell_n^2).$$

Going through the arguments of the proof one sees that the estimate holds locally uniformly on W and uniformly on bounded subsets of L^2. □

Proposition 2.1 leads to the following asymptotics of the action variables.

Proposition 3.2. *Locally uniformly on* $W \cap H_{\mathbb{C}}^N$

$$2I_n^{(M)} = \frac{1}{4\pi^3 n^3}\left(\hat q_n \hat q_{-n} + \frac{1}{n^{2N+1}}\ell_n^1\right).$$

On H^N*, the error is uniformly bounded on bounded subsets of* H^N*.*

Proof. By Theorem 1.1 one has, uniformly on bounded subsets of $W \cap H^N_{0,\mathbb{C}}$

$$\gamma_n^2 = 4\langle q, e^{2\pi inx}\rangle\langle q, e^{-2\pi inx}\rangle + \frac{1}{n^{2N+1}}\ell^1_n.$$

Since $\gamma_n(q) = \gamma_n(q - [q])$ this estimate hold also uniformly on bounded subsets of W. Combined with the asymptotics (3.4) we then get,

$$\begin{aligned} 2I_n^{(M)} =& \frac{1}{16n^3\pi^3}\gamma_n^2\left(1 + \frac{1}{n}\ell^2_n\right) \\ =& \frac{1}{4\pi^3 n^3}(1 + \frac{1}{n}\ell^2_n)\left(\langle q, e^{2\pi inx}\rangle\langle q, e^{-2\pi inx}\rangle + \frac{1}{n^{2N+1}}\ell^1_n\right) \end{aligned}$$

or, again uniformly on bounded subsets of W,

$$2I_n^{(M)} = \frac{1}{4\pi^3 n^3}\left(\langle q, e^{2\pi inx}\rangle\langle q, e^{-2\pi inx}\rangle + \frac{1}{n^{2N+1}}\ell^1_n\right).$$

Going through the arguments of the proof one sees that the estimates hold locally uniformly on $W \cap H^N_{\mathbb{C}}$ and uniformly on bounded subsets of H^N. □

Notice that the definition of the actions with respect to the Gardner bracket, defined in (2.1) makes sense for any $q \in L^2_{\mathbb{C}}$ sufficiently close to L^2. For $q \in E_n := \{q \in L^2 | \, 0 \notin [\lambda_{2n-1}(q), \lambda_{2n}(q)]\}$ we define

$$\zeta_n(q) := \begin{cases} \left(\frac{I_n^{(M)}(q)}{I_n(q)}\right)^{1/2} & \text{if } 0 < \lambda_{2n-1}(q) < \lambda_{2n}(q) \\ -\left(\frac{-I_n^{(M)}(q)}{I_n(q)}\right)^{1/2} & \text{if } \lambda_{2n-1}(q) < \lambda_{2n}(q) < 0 \\ 0 & \text{if } \lambda_{2n-1}(q) = \lambda_{2n}(q). \end{cases} \quad (3.11)$$

In [9] is shown that ζ_n extend to real analytic functions on a neighbourhood of E_n in $L^2_{\mathbb{C}}$. Note that for each $q \in L^2$ there is $N \in \mathbb{Z}_{\geq 1}$ such that $q \in E_n$, $\forall n > N$, thus it makes sense to state asymptotics of ζ_n as $n \to \infty$. Combining Proposition 2.2 and Proposition 3.2 yields the following asymptotics.

Corollary 3.1. *Locally uniformly on* W

$$\zeta_n(q) = \frac{1}{2n\pi}\left(1 + \frac{1}{n}\ell^2_n\right). \quad (3.12)$$

On L^2*, the error in* (2.9) *is uniformly bounded on bounded subsets of* L^2*.*

With the help of this we can now prove Theorem 3.2.

Proof of Theorem 3.2. Note that the Birkhoff map with respect to the Gardner bracket Φ can analytically be extended to an neighbourhood of L^2 in $L^2_{\mathbb{C}}$ by

$$\Phi(q) = \Phi(q - [q]).$$

Fix now a symplectic leaf $\mathcal{L}_\star$ of the union (3.2). Then there is a $N \in \mathbb{Z}_{\geq 1}$ such that for all $q \in \mathcal{L}_\star$

$$\lambda_{2n-1}(q) > 0, \quad \forall n > N,$$

and thus we have (see [9])

$$(\Phi^{(M)})_{\pm n} = \zeta_n(q)(\Phi(q))_{\pm n}, \quad \forall n > N.$$

Let $(z_{\pm n})_{n\geq 1}$ be the components of $\Phi(q)$, and $(z^{(M)}_{\pm n})_{n\geq 1}$ the components of $\Phi^{(M)}(q)$. In the proof of Theorem 0.4 we have shown that

$$z_{\pm n} = \frac{1}{\sqrt{n\pi}}\left(\hat{q}_{\pm n} + \frac{1}{n^{N+1}}\ell^2_n\right).$$

Combining this with (3.12) leads for $z^{(M)}_{\pm n} = \zeta_n z_{\pm n}$ to the asymptotics

$$z^{(M)}_{\pm n} = \zeta_n z_{\pm n} = \frac{1}{2(n\pi)^{3/2}}\left(\hat{q}_{\pm n} + \frac{1}{n^{N+1}}\ell^2_n\right).$$

Going through the arguments of the proof one sees that the estimate holds uniformly on bounded subsets of H^N. □

3.2 Asymptotics of the Birkhoff map of mKdV

Proof of Theorem 3.4. The Birkhoff map of mKdV established in [14] is given for each $N \geq 1$ as follows

$$\begin{aligned}\Omega : H^N &\to h^{N+1/2} \times \mathbb{R}\\ r &\mapsto \left(\Phi^{(M)}\left(B(r) - \lambda_0(B(r))\right), [r]\right),\end{aligned}$$

where

$$\begin{aligned}B : H^N &\to H^{N-1}\\ r &\mapsto r' + r^2\end{aligned}$$

is the Miura map. Denote the Fourier transform by $\mathcal{F}$, then we get using Theorem 3.2

$$\begin{aligned}\Omega(r) &= \left(\Phi^{(M)}\left(r' + r^2 - \lambda_0(B(r))\right), [r]\right)\\ &= \left(\left(\frac{1}{2(|n|\pi)^{3/2}}\left(\mathcal{F}(r' + r^2 - \lambda_0(B(r)))_n + \frac{1}{|n|^N}\ell^2_n\right)\right)_{n\neq 0}, [r]\right)\\ &= \left(\left(\frac{1}{2(|n|\pi)^{3/2}}\left(\mathcal{F}(r')_n + \mathcal{F}(r^2 - \lambda_0(B(r)))_n + \frac{1}{|n|^N}\ell^2_n\right)\right)_{n\neq 0}, [r]\right)\end{aligned}$$

Since $r^2 - \lambda_0(B(r)) \in H^N$ we get

$$\begin{aligned}\Omega(r) &= \left(\left(\frac{1}{2(|n|\pi)^{3/2}} \left(\mathcal{F}(r')_n + \frac{1}{|n|^N} \ell^2_n \right) \right)_{n\neq 0}, [r] \right) \\ &= \left(\left(\frac{\pi i \operatorname{sign}(n)}{(|n|\pi)^{1/2}} \left(\hat{r}_n + \frac{1}{|n|^{N+1}} \ell^2_n \right) \right)_{n\neq 0}, [r] \right).\end{aligned}$$

Going through the arguments of the proof one sees that the estimates hold uniformly on bounded subsets of H^N. □

Chapter 4

Qualitative features of KdV and mKdV

In the first section of this chapter we prove the qualitative features of solutions for KdV stated in Theorem 0.1 and Theorem 0.2.

In Section 4.2 we prove analogous results for the mKdV equation – see Theorem 4.1 and Theorem 4.2.

4.1 Qualitative properties of the KdV flow

In this section we prove Theorem 0.1 and Theorem 0.2. First we need to derive some auxiliary results. Let $N \in \mathbb{Z}_{\geq 0}$ and $c \in \mathbb{R}$ be given. For any q in H^N_c, write $q = p + c$ where $p \in H^N_0$ and $\int_0^1 q(x)dx = \hat{q}_0 = c$. Then

$$\mathcal{H}(p + c) = \mathcal{H}(p) + 3c \int_0^1 p^2 dx + c^3.$$

Note that $\frac{1}{2}\int_0^1 p^2 dx$ is the second Hamiltonian in the KdV hierarchy. By Parseval's identity for KdV (cf [12], Appendix E)

$$\frac{1}{2}\int_0^1 p^2 dx = \sum_{n\geq 1} 2\pi n I_n$$

and thus the KdV frequencies satisfy

$$\omega_n(q) = \omega_n(p) + 12cn\pi \quad \forall n \geq 1, \tag{4.1}$$

and the KdV flow is given by

$$u(t) \equiv S^t(q) = S^t_c(p) + c \tag{4.2}$$

where S_c^t denotes the flow on H_0^N corresponding to the Hamiltonian

$$\mathcal{H}_c(p) := \mathcal{H}(p) + 3c \int_0^1 p^2 dx.$$

The equations of motion corresponding to $\mathcal{H}_c$ read, when expressed in Birkhoff coordinates $(z_n)_{n\neq 0}$,

$$\dot{z}_n = i\omega_n^c z_n \quad \text{and} \quad \dot{z}_{-n} = -i\omega_n^c z_{-n}$$

where

$$\omega_n^c \equiv \omega_n^c(p) = \omega_n(p) + 12cn\pi. \tag{4.3}$$

Define $\omega_{-n}^c := -\omega_n^c$ $(n \geq 1)$. Then

$$v(t) \equiv \Omega_c^t(z) = \left(e^{i\omega_n^c t} z_n\right)_{n\neq 0} \tag{4.4}$$

is the flow of $\mathcal{H}_c$ expressed in (complex) Birkhoff coordinates. Here, by a slight abuse of terminology, ω_n^c is viewed as a function of z. Note that Φ conjugates the flow maps S_c^t and Ω_c^t,

$$S_c^t = \Phi^{-1} \circ \Omega_c^t \circ \Phi.$$

With $B := \Phi^{-1} - \Phi_0^{-1}$, we get

$$S_c^t = \Phi_0^{-1} \circ \Omega_c^t \circ \Phi + B \circ \Omega_c^t \circ \Phi. \tag{4.5}$$

We now analyse the map $\Phi_0^{-1} \circ \Omega_c^t \circ \Phi$ in more detail. First note that for any $z = (z_n)_{n\neq 0} \in \mathfrak{h}^{N+1/2}$,

$$\Phi_0^{-1}(z)(x) = \sum_{n\neq 0} \sqrt{|n|\pi} z_n e^{2\pi i n x}.$$

Hence for any $p \in H_0^N$ and $t, c \in \mathbb{R}$,

$$\Phi_0^{-1} \circ \Omega_c^t \circ \Phi(p) = \sum_{n\neq 0} e^{i\omega_n^c t} \sqrt{|n|\pi} z_n(p) e^{2\pi i n x} \tag{4.6}$$

where $(z_n(p))_{n\neq 0} = \Phi(p)$. For the proof of item (i) of Theorem 0.1 we need to consider the KdV flow on all of H^N. For any $t \in \mathbb{R}$, introduce

$$E^t : H^N \times \mathfrak{h}^{N+1/2} \to \mathfrak{h}^{N+1/2},\ (q, z) \mapsto (e^{i\omega_n(q)t} z_n)_{n\neq 0}.$$

Denoting by Π the projection $\Pi : H^N \to H_0^N$, $q \mapsto q - \hat{q}_0$ it follows that for any $q \in H^N$,

$$E^t(q, \Phi \circ \Pi(q)) = (e^{i\omega_n(q)t} z_n(\Pi(q)))_{n\neq 0} = \Omega_{\hat{q}_0}^t \circ \Phi(\Pi(q)). \tag{4.7}$$

Lemma 4.1. *For any given $N \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{R}$, the map $E^t : H^N \times \mathfrak{h}^{N+1/2} \to \mathfrak{h}^{N+1/2}$ is continuous.*

Proof. For any $q, p \in H^N$, $z, w \in \mathfrak{h}^{N+1/2}$, and for any $K > 0$ one has

$$\|E^t(q,z) - E^t(p,w)\|^2_{N+1/2} = \sum_{n \neq 0} |n|^{2N+1} |e^{i\omega_n(q)t} z_n - e^{i\omega_n(p)t} w_n|^2.$$

As the KdV frequencies are real valued functions on H^N

$$\begin{aligned}|e^{i\omega_n(q)t} z_n - e^{i\omega_n(p)t} w_n|^2 =& |z_n - e^{i(\omega_n(p)-\omega_n(q))t} w_n|^2 \\ \leq& 2|z_n - w_n|^2 + 2|1 - e^{i(\omega_n(p)-\omega_n(q))t}|^2 |w_n|^2\end{aligned}$$

and hence

$$\begin{aligned}\|E^t(q,z) - E^t(p,w)\|^2_{N+1/2} \leq 2\|z-w\|^2_{N+1/2} + 8 \sum_{|n| \geq K} |n|^{2K+1} |w_n|^2 \\ +2 \sum_{0<|n|<K} |n|^{2N+1} |e^{i(\omega_n(p)-\omega_n(q))t} - 1|^2 |w_n|^2. \quad (4.8)\end{aligned}$$

By (4.1),

$$\omega_n(p) - \omega_n(q) = 12n\pi \widehat{(p-q)}_0 + O(1)$$

locally uniformly on $H^N \times H^N$. Moreover, it follows from (4.1) and [17], Theorem 1.9 that for any $n \neq 0$, the n'th frequency $\omega_n : H^N \to \mathbb{R}$ is continuous. This combined with (4.8) implies the statement of the lemma. ☐

Proof of Theorem 0.1. Let $N \in \mathbb{Z}_{\geq 0}$. For any $q \in H^N$ let $p = \Pi(q) = q - \hat{q}_0$. Then by (4.2) and (4.5), with $c = \hat{q}_0$,

$$S^t(q) = S^t_c(p) + c = \Phi_0^{-1} \circ \Omega^t_c \circ \Phi(p) + B \circ \Omega^t_c \circ \Phi(p) + c.$$

Substituting $\Phi = \Phi_0 + A$ into $\Phi_0^{-1} \circ \Omega^t_c \circ \Phi(p)$ then yields

$$\Phi_0^{-1} \circ \Omega^t_c \circ \Phi(p) = \sum_{n \neq 0} e^{i\omega^c_n t} \hat{p}_n e^{2\pi i n x} + \Phi_0^{-1} \circ \Omega^t_c \circ A(p)$$

where by a slight abuse of terminology we denote by $\Omega^t_c \circ A(p)$ the element $\Omega^t_c \circ A(p) = \left(e^{i\omega^c_n(p)t} a_n(p)\right)_{n \neq 0}$ and $A(p) = (a_n(p))_{n \neq 0}$. Taking into account that for any $n \neq 0$, $\hat{q}_n = \hat{p}_n$ and $\omega^c_n(p) = \omega_n(q)$ we conclude

$$\begin{aligned}R^t(q) =& S^t(q) - \sum_n e^{i\omega_n t} \hat{q}_n e^{2\pi i n x} \\ =& B \circ \Omega^t_c \circ \Phi(p) + \Phi_0^{-1} \circ \Omega^t_c \circ A(p). \quad (4.9)\end{aligned}$$

Statements (ii) and (iii) of Theorem 0.1 then follow from Theorem 0.4, Proposition 2.5, and the boundedness of $\Phi = \Phi_0 + A$. To prove item (i) note that by (4.7) and (4.9),

$$R^t(q) = (B \circ E^t)(q, \Phi \circ \Pi(q)) + (\Phi_0^{-1} \circ E^t)(q, A \circ \Pi(q)). \quad (4.10)$$

It then follows from Lemma 4.1, Theorem 0.4, and Proposition 2.5 that R^t is continuous.
To prove statement (iv) write for $n \in \mathbb{Z}$ arbitrary,

$$\hat{u}_n(t) = e^{i\omega_n t}(\hat{q}_n + \tilde{\rho}_n(t)),$$

where $\omega_0 := 0$ and

$$\tilde{\rho}_n(t) := \widehat{R}^t_n(q) e^{-i\omega_n t}.$$

By the definition of R^t, $\hat{R}^t_0(q) = 0$ implying that $\tilde{\rho}_0(t)$ vanishes identically. Moreover $\tilde{\rho}_n(0) = 0$ for any $n \in \mathbb{Z}$ as $R^0(q) = 0$. Clearly $\partial_t \hat{u}_n(t) = i\omega_n \hat{u}_n(t) + e^{i\omega_n t}\partial_t \tilde{\rho}_n(t)$ and when substituted into the KdV equation

$$-\partial_t \hat{u}_n(t) + 8i\pi^3 n^3 \hat{u}_n(t) + 6i\pi n \sum_l \hat{u}_l(t)\hat{u}_{n-l}(t) = 0,$$

one gets

$$-i\omega_n \hat{u}_n(t) - e^{i\omega_n t}\partial_t \tilde{\rho}_n(t) + 8i\pi^3 n^3 \hat{u}_n(t) + 6i\pi n \sum_l \hat{u}_l(t)\hat{u}_{n-l}(t) = 0$$

or

$$ie^{i\omega_n t}\partial_t \tilde{\rho}_n(t) = (\omega_n - 8\pi^3 n^3)\hat{u}_n(t) - 6\pi n \sum_l \hat{u}_l(t)\hat{u}_{n-l}(t). \tag{4.11}$$

Recall from [12], p 229

$$\omega_n(q) = 8\pi n(\tau_n + \lambda_0/2 - \sum_{m\geq 1}(\sigma^n_m - \tau_m)), \tag{4.12}$$

where λ_0, τ_n, and σ^n_m have been introduced either in the Introduction or Section 1.4. For $N \geq 1$, $H^N \hookrightarrow L^2$ is compact. As $L^2 \to \mathbb{R}, q \mapsto \lambda_0(q)$ is continuous, it then follows that $H^N \to \mathbb{R} : q \mapsto \lambda_0(q)$ is compact. By Theorem 0.8

$$\tau_n = \hat{q}_0 + n^2\pi^2 + \frac{1}{n}\ell^2_n$$

whereas by Proposition 1.5 and Theorem 1.1 , uniformly in n

$$\sigma^n_m - \tau_m = \frac{\gamma^2_m}{m}\ell^2_m = \frac{1}{m^{2N+1}}\ell^1_m.$$

Both asymptotic estimates hold uniformly on bounded subsets of H^N. Hence formula (4.12) leads to the asymptotics

$$\omega_n(q) = 8\pi^3 n^3 + O(n) \tag{4.13}$$

uniformly on bounded subsets of H^N and statement (iv) follows. □

Remark 4.1. *As pointed out in Remark 0.1, the restrictions of R^t and $\partial_t R^t$ to H^N_c are real analytic. To formulate this result more precisely, for any $c \in \mathbb{R}$, denote by $H^N_{c,\mathbb{C}}$ the complexification of H^N_c,*

$$H^N_{c,\mathbb{C}} := \{q \in H^N_{\mathbb{C}} \,|\, \int_0^1 q(x)dx = c\} = c + H^N_{0,\mathbb{C}}.$$

Note that $H^N_{c,\mathbb{C}}$ is not *an open complex neighbourhood of H^N_c in $H^N_{\mathbb{C}}$ as for q in $H^N_{c,\mathbb{C}}$ the value of $\hat{q}_0$ is kept fixed. Let E be a real Banach space and denote by $E_{\mathbb{C}}$ its complexification. A map $f : H^N_c \to E$ is said to be real analytic if f extends to an analytic map $f : W \to E_{\mathbb{C}}$ where W is an open neigbourhood of H_c in $H^N_{c,\mathbb{C}}$. In view of the fact mentioned above that $\hat{R}^t_0 = 0$ for any $t \in \mathbb{R}$, R^t maps H^N into H^{N+1}_0 and $\partial_t R^t$ maps H^N into H^{N-1}_0 . Theorem 0.1 can then be amended as follows:*

(v) for any $N \in \mathbb{Z}_{\geq 0}$ and $c \in \mathbb{R}$, $R^t_{|H^N_c} : H^N_c \to H^{N+1}_0$ is real analytic;

(vi) for any $N \in \mathbb{Z}_{\geq 1}$ and $c \in \mathbb{R}$, $\partial_t R^t_{|H^N_c} : H^N_c \to H^{N-1}_0$ is real analytic.

To see that $R^t_{|H^N_c}$ is real analytic, note that it follows from [3], Theorem 1 and formula (4.1) *that for any $q \in H^N_c$,*

$$\omega_n(q) = 8\pi^3 n^3 + 12cn\pi + O(1), \tag{4.14}$$

locally uniformly in a complex neighbourhood V of H^N_c in $H^N_{c,\mathbb{C}}$ and that for any $n \geq 1$, ω_n is real analytic on V. One then concludes that for any $t \in \mathbb{R}$

$$E^t : V \times \mathfrak{h}^{N+1/2}_{\mathbb{C}} \to \mathfrak{h}^{N+1/2}_{\mathbb{C}},\ (q, (z_n)_{n\neq 0}) \mapsto (e^{i\omega_n(q)t} z_n)_{n\neq 0}$$

is analytic. Together with the analyticity of Φ as well as Φ^{-1} and hence of $B = \Phi^{-1} - \Phi_0^{-1}$, it then follows from (4.10) *that $R^t_{|H^N_c}$ is real analytic. The analyticity of $\partial_t R^t_{|H^N_c}$ stated in item (vi) is proved in a similar fashion.*

We now turn to the proof of Theorem 0.2.

Proof of Theorem 0.2. Let $\epsilon, M > 0$ be given and let $q \in H^N$ satisfy $\|q\|_{H^N} \leq M$. Apply $(Id - P_L)$ to $S^t(q) = \sum_{n\in\mathbb{Z}} e^{i\omega_n t}\hat{q}_n e^{2\pi inx} + R^t(q)$,

$$(Id - P_L)S^t(q) = \sum_{|n|>L} e^{i\omega_n t}\hat{q}_n e^{2\pi inx} + (Id - P_L)R^t(q).$$

Note that

$$\| \sum_{|n|>L} e^{i\omega_n t}\hat{q}_n e^{2\pi inx} \|_{H^N} = \|(Id - P_L)q\|_{H^N}. \tag{4.15}$$

By Theorem 0.4 (iii), $B_M := \{R^t(p) | \, t \in \mathbb{R}, \, p \in H^N, \|p\|_{H^N} \leq M\}$ is bounded in H^{N+1} and thus by the Sobolev embedding theorem relatively compact in H^N. Hence there exists $L_\star \in \mathbb{N}$ such that for any $L \geq L_\star$

$$\|(Id - P_L)r\|_{H^N} < \epsilon \quad \forall r \in B_M.$$

Thus

$$\begin{aligned} \|(Id - P_L)S^t(q)\|_{H^N} \leq & \|(Id - P_L)q\|_{H^N} + \|(Id - P_L)R^t(q)\|_{H^N} \\ \leq & \|(Id - P_L)q\|_{H^N} + \epsilon \end{aligned}$$

and

$$\|(Id - P_L)S^t(q)\|_{H^N} \geq \|(Id - P_L)q\|_{H^N} - \epsilon.$$

□

There are other ways of approximating the KdV flow than the one considered in Theorem 0.2. As an alternative to the projection of the solution of KdV onto the space of trigonometric polynomials of order L one could involve the orthogonal projection $Q_L : \mathfrak{h}^{1/2} \to \mathfrak{h}^{1/2}$ onto the $2L$ dimensional $\mathbb{R}$-vector space

$$\{(z_k)_{k \neq 0} | \, z_k = 0 \; \forall |k| > L\}$$

and study

$$\Phi^{-1} \circ Q_L \circ \Phi \circ S_c^t(p) = \Phi^{-1} \circ Q_L \circ \Omega_c^t \circ \Phi(p).$$

Results similar to the ones of Theorem 0.2 can be obtained for such a type of approximation.

4.2 Qualitative properties of the mKdV flow

Recall that the Miura map $B : H^1 \to L^2$, $r \mapsto r' + r^2$ maps solutions of mKdV to solutions of KdV. And furthermore $B_c := B_{|H_c^1} : H_c^1 \to \mathcal{L}_d$ is a real analytic diffeomorphism, where $d = 2\cosh c$ and $\mathcal{L}_d := \{q \in L^2 | \Delta(0, q) = d, \; \lambda_0(q) \geq 0\}$ – see [14]. We can write

$$B_c = \partial_x + Q \tag{4.16}$$

where $Q : H^1 \to L^2$, $r \mapsto r^2$. To describe $B_c^{-1} : \mathcal{L}_d \to H_c^1$ introduce

$$\partial_x^{-1} : L^2 \to H_0^1, \quad q \mapsto \sum_{n \neq 0} \frac{1}{2\pi i n} \hat{q}_n e^{2\pi i n x};$$

as well as

$$\mathfrak{c} : L^2 \to L^2, \quad \text{where } \mathfrak{c}(q)(x) = c \; \forall x \in [0, 1]$$

and

$$K_c := -\partial_x^{-1} \circ Q \circ B_c^{-1}. \tag{4.17}$$

We claim now

Lemma 4.2.

$$B_c^{-1} = \partial_x^{-1} + \mathfrak{c} + K_c, \tag{4.18}$$

Proof. First note that $(\partial_x^{-1} + \mathfrak{c}) \circ \partial_x = Id_{H_c^1}$ and $\partial_x = (B_c - Q)$ Then we get

$$\begin{aligned} B_c^{-1} =&(\partial_x^{-1} + \mathfrak{c}) \circ \partial_x \circ B_c^{-1} \\ =&(\partial_x^{-1} + \mathfrak{c}) \circ (B_c - Q) \circ B_c^{-1} \\ =&(\partial_x^{-1} + \mathfrak{c}) \circ (Id_{\mathcal{L}_d} - Q \circ B_c^{-1}) \\ =&\partial_x^{-1} - \partial_x^{-1} \circ Q \circ B_c^{-1} + \mathfrak{c}, \end{aligned}$$

which concludes the proof. □

Special notation for this section In this section we will use the norm on H^N given by $\|r\|_{H^N} = |\int_0^1 r(x)dx| + (\sum_{n\neq 0} |n|^{2N} |\hat{r}_n|^2)^{1/2}$. (This norm is equivalent to the norm given in the Introduction of this thesis.) Furthermore in this section we mean by a bounded map a map which is bounded on bounded sets.

Lemma 4.3. *For $N \in \mathbb{Z}_{>0}$, $c \in \mathbb{R}$ and $d = 2\cosh(c)$, then holds*

(i) B is a bounded map from H^N to H^{N-1}.

(ii) B_c^{-1} is a bounded map from $H^{N-1} \cap \mathcal{L}_d$ to H_c^N.

(iii) For all $N \in \mathbb{Z}_{>0}$ the restrictions $K_c : H^{N-1} \cap \mathcal{L}_d \to H^N$ and $Q : H^N \to H^{N-1}$ are compact operators.

Proof. (i) Let $r \in H^N$ then

$$\|B(r)\|_{H^{N-1}} = \|r' + r^2\|_{H^{N-1}} \leq \|r'\|_{H^{N-1}} + \|r^2\|_{H^{N-1}}. \tag{4.19}$$

We have $\|r\|_{H^{N-1}} \leq \|r\|_{H^N}$ and H^N is a Banach algebra as $N \geq 1$, so there is a constant $C_1 > 0$ such that

$$\|r^2\|_{H^{N-1}} \leq \|r^2\|_{H^N} \leq C_1 \|r\|_{H^N}^2. \tag{4.20}$$

Thus we get with (4.19)

$$\|B(r)\|_{H^{N-1}} = \|r' + r^2\|_{H^{N-1}} \leq \|r\|_{H^N} + C_1 \|r\|_{H^N}^2.$$

(ii) We show that for any given $M > 0$

$$\sup_{r \in H_c^N,\, \|B(r)\|_{H^{N-1}} < M} \|r\|_{H^k} < \infty, \quad \forall 1 \leq k \leq N. \tag{4.21}$$

We prove (4.21) by induction in k. Let $k = 1$ and $r \in H^N_c$, then

$$\begin{aligned} &M^2 \geq (\|B(r)\|_{H^{N-1}})^2 \geq (\|B(r)\|_{H^0})^2 \\ \geq &\int_0^1 (r')^2 + 2r'r^2 + r^4 dx = \|r'\|^2_{L^2} + \|r^2\|^2_{L^2} \geq (\|r\|_{H^1} - |c|)^2 \end{aligned}$$

which proves (4.21) in the case $k = 1$. Assume now (4.21) holds for a $k < N$. Recall that H^k is a Banach algebra for $k \geq 1$, i.e. there is $C_1 > 0$ such that

$$\|r^2\|_{H^k} \leq C_1 \|r\|^2_{H^k}$$

and by the induction assumption there is a $C_2 > 0$ such that

$$\sup_{r \in H^N_c,\, \|B(r)\|_{H^{N-1}} < M} \|r\|_{H^k} \leq C_2$$

thus we have for $r \in H^N_c$ with $\|B(r)\|_{H^{N-1}} < M$

$$\|r^2\|_{H^k} \leq C_1 C_2^2 \quad \text{and} \quad \|r' + r^2\|_{H^k} \leq \|r' + r^2\|_{H^{N-1}} < M.$$

Combining these two bounds one gets with another constant $C_3 > 0$

$$\begin{aligned} &\|r\|_{H^{k+1}} \leq C_3 \|(r' + r^2) - r^2\|_{H^k} + |c| \\ \leq &C_3 \|r' + r^2\|_{H^k} + C_3 \|r^2\|_{H^k} + |c| \leq C_3 M + C_1 C_2^2 C_3 + |c| \end{aligned}$$

and (4.21) follows also for $k + 1$, thus (ii) is proven.

(iii) By (4.20) and Sobolev embedding theorem $Q : H^N \to H^{N-1}$ is compact. By item (ii) of this lemma and (4.17) follows then that $K_c : H^{N-1} \cap \mathcal{L}_d \to H^N$ is compact as well. □

Note that the mKdV equation is globally well-posed on H^N, $N \in \mathbb{Z}_{>0}$, Denote the mKdV flow by S^t_m, i.e. $S^t_m(u(0)) = u(t)$ for any solution curve $u(t)$ in H^N. In addition, let

$$R^t_m(u(0)) := S^t_m(u(0)) - \sum_{n \in \mathbb{Z}} e^{i\omega_n t} \hat{u}_n(0) e^{2\pi i n x}, \tag{4.22}$$

where for $r \in H^1$ and $n \in \mathbb{Z}_{\neq 0}$ the mKdV frequencies are given by

$$\omega_n \equiv \omega_n^{mKdV}(r) = \omega_n^{KdV}(B(r)) \tag{4.23}$$

– for details see [14].

Theorem 4.1. *For $r = u(0) \in H^N$, $N \in \mathbb{Z}_{\geq 1}$, the error $R^t_m(r)$ of the approximation $\sum_{n \in \mathbb{Z}} e^{i\omega_n t} \hat{r}_n e^{2\pi i n x}$ of the flow $S^t_m(u(0))$ has the following properties*

(i) $R^t_m : H^N \to H^{N+1}$ is continuous;

(ii) for any $r \in H^N$, $t \mapsto R^t_m(r)$ is a relatively compact curve in H^{N+1};

(iii) *for any* $M > 0$*, the set* $\{R^t_m(r)|\, t \in \mathbb{R},\, r \in H^N, \|r\|_{H^N} \leq M\}$ *is bounded in* H^{N+1}*;*

(iv) $\partial_t R^t_m : H^N \to H^{N-1}$ *is continuous and for any* $q \in H^N$*, the orbit* $\{\partial_t R^t_m(r)|\, t \in \mathbb{R}\}$ *is relatively compact in* H^{N-1}*. Moreover for any* $M > 0$ *the set of orbits* $\{\partial_t R^t_m(r)|\, t \in \mathbb{R},\, r \in H^N, \|r\|_{H^N} \leq M\}$ *is bounded in* H^{N-1}*.*

Remark 4.2. *Actually, one can prove that for any* $c \in \mathbb{R}$*, the restrictions of* R^t_m *and* $\partial_t R^t_m$ *to the affine subspace* $H^N_c = \{r \in H^N |\, \int_0^1 r(x)dx = c\}$ *are real analytic.*

Proof. By Theorem 0.1 we have

$$S^t(q) = \sum_{n\in\mathbb{Z}} e^{i\omega_n t}\hat{q}_n e^{2\pi i n x} + R^t(q) \tag{4.24}$$

where $R^t(q)$ is a relatively compact curve in H^{N+1} (given $q \in H^N$). The mKdV evolution, denoted by S^t_m, is then given by

$$S^t_m(r) = B_c^{-1} S^t B_c(r),$$

where $r \in H^N$, $N \geq 1$, $c := \int_0^1 r(x)dx$ – see [14]. First we use the decomposition (4.18) to get

$$S^t_m(r) = B_c^{-1} S^t B_c(r) = (\partial_x^{-1} + \mathfrak{c}) \circ S^t \circ B_c(r) + K_c \circ S^t \circ B_c(r).$$

By Theorem 0.1 $\{S^t(B_c(r))|\, t \in \mathbb{R}\}$ is relatively compact in H^N, then by Lemma 4.2, $K_c \circ S^t \circ B_c(r)$ is a relatively compact curve in H^{N+1} fulfilling also the boundedness statement of item (iii). Thus we have just to concentrate on $(\partial_x^{-1} + c) \circ S^t \circ B_c(r)$. Applying the formulas (4.16), (4.24) and recalling $\omega_n \equiv \omega_n^{mKdV}(r) = \omega_n^{KdV}(B(r))$ yields

$$\begin{aligned}(\partial_x^{-1} + \mathfrak{c}) \circ S^t \circ B_c(r) =& (\partial_x^{-1} + \mathfrak{c}) S^t (\partial_x + Q)(r)\\ =& (\partial_x^{-1} + \mathfrak{c}) \left(\sum_{n\in\mathbb{Z}} e^{i\omega_n t} (\widehat{r' + r^2})_n e^{2\pi i n x} + R^t(B_c(r)) \right)\\ =& c + \sum_{n\neq 0} e^{i\omega_n t} \hat{r}_n e^{2\pi i n x} + \sum_{n \neq 0} e^{i\omega_n t} \frac{1}{2\pi i n} (\widehat{r^2})_n e^{2\pi i n x}\\ &+ \partial_x^{-1}(R^t(B_c(r)))\end{aligned}$$

Since $\hat{r}_0 = c$ we get the formula

$$\begin{aligned}R^t_m(r) =& \sum_{n\neq 0} e^{i\omega_n t} \frac{1}{2\pi i n} (\widehat{r^2})_n e^{2\pi i n x} + \partial_x^{-1} \circ R^t \circ B_c(r)\\ &+ K_c \circ S^t \circ B_c(r)\end{aligned} \tag{4.25}$$

Observe now that $\sum_{n\neq 0} e^{i\omega_n t}\frac{1}{2\pi i n}(\widehat{r^2})_n e^{2\pi i n x}$ and $\partial_x^{-1}\circ R^t\circ B_c(r)$ are relatively compact curves in H^{N+1}. Thus (i) and (ii) follow. Using that B is bounded, and that the boundedness statement holds for $\sum_{n\neq 0} e^{i\omega_n t}\frac{1}{2\pi i n}(\widehat{r^2})_n e^{2\pi i n x}$ and $R^t(q)$ we conclude item (iii).
To prove statement (iv) write for $n\in\mathbb{Z}$ arbitrary,

$$\hat{u}_n(t) = e^{i\omega_n t}(\hat{r}_n + \tilde{\rho}_n(t)),$$

where $\omega_0 := 0$ and

$$\tilde{\rho}_n(t) := \widehat{(R^t_m)}_n(q)e^{-i\omega_n t}.$$

By the definition of R^t_m, $\widehat{(R^t_m)}_0(q) = 0$ implying that $\tilde{\rho}_0(t)$ vanishes identically. Moreover $\tilde{\rho}_n(0) = 0$ for any $n\in\mathbb{Z}$ as $R^0_m(q) = 0$. Clearly $\partial_t\hat{u}_n(t) = i\omega_n\hat{u}_n(t) + e^{i\omega_n t}\partial_t\tilde{\rho}_n(t)$ and when substituted into the mKdV equation

$$-\partial_t\hat{u}_n(t) + 8i\pi^3 n^3\hat{u}_n(t) + 4i\pi n\widehat{(u^3)}_n(t) = 0,$$

one gets

$$-i\omega_n\hat{u}_n(t) - e^{i\omega_n t}\partial_t\tilde{\rho}_n(t) + 8i\pi^3 n^3\hat{u}_n(t) + 4i\pi n\widehat{(u^3)}_n(t) = 0$$

or

$$ie^{i\omega_n t}\partial_t\tilde{\rho}_n(t) = (\omega_n - 8\pi^3 n^3)\hat{u}_n(t) - 4\pi n\widehat{(u^3)}_n(t). \tag{4.26}$$

For $q := B(r)$ we have by (4.23) $\omega_n \equiv \omega_n(r) \equiv \omega_n^{mKdV}(r) = \omega_n^{KdV}(q)$ and by (4.12)

$$\omega_n(r) = 8\pi n(\tau_n(q) + \lambda_0/2(q) - \sum_{m\geq 1}(\sigma^n_m(q) - \tau_m(q))), \tag{4.27}$$

where λ_0, τ_n, and σ^n_m have been introduced either in the Introduction or Section 1.4. Arguing as in the proof of item (iv) of Theorem 0.1 and using that by Lemma 4.3 $B : H^N \to H^{N-1}$ is bounded on bounded subsets of H^N, formula (4.12) leads to the asymptotics

$$\omega_n(r) = 8\pi^3 n^3 + O(n) \tag{4.28}$$

uniformly on bounded subsets of H^N and statement (iv) follows.

□

The proof of the following Theorem is similar to the proof of Theorem 0.2. Let $u(t) = S^t_m(r)$ then we have the following result.

Theorem 4.2. *Let $N\in\mathbb{Z}_{\geq 1}$ be arbitrary. Then for any $M > 0$ and $\epsilon > 0$ there exists $L_{\epsilon,M}\geq 1$ such that for any $u(0)\in H^N$, with $\|u(0)\|_{H^N}\leq M$, $L\geq L_{\epsilon,M}$ and any $t\in\mathbb{R}$*

$$\|(Id - P_L)u(0)\|_{H^N} - \epsilon \leq \|(Id - P_L)u(t)\|_{H^N} \leq \|(Id - P_L)u(0)\|_{H^N} + \epsilon.$$

In particular, if $u(0)$ with $\|u(0)\|_{H^N}\leq M$ is a trigonometric polynomial of order $L_\star$, then for any $L\geq\max(L_\star, L_{\epsilon,M})$, $P_L u(t)$ approximates $u(t)$ uniformly in $t\in\mathbb{R}$ up an error of size ϵ.

Proof. Let $\epsilon, M > 0$. For $r = u(0)$, we have by (4.22)

$$u(t) = S_m^t(r) = \sum_n e^{i\omega_n t}\hat{r}_n e^{2\pi i n x} + R_m^t(r).$$

Applying $Id - P_L$ yields

$$(Id - P_L)(u(t)) = \sum_{|n|>L} e^{i\omega_n t}\hat{r}_n e^{2\pi i n x} + (Id - P_L)(R_m^t(r)).$$

By Theorem 4.1 (iii) $\{R_m^t(r) | \, \|r\|_{H^N} \leq M\}$ is bounded in H^{N+1}. By Sobolev embedding theorem $H^{N+1} \hookrightarrow H^N$ is compact, thus we can, in view the of the characterization of compact subsets of H^N, choose $L_{\epsilon,M}$ such that $\forall L > L_{\epsilon,M}$

$$\|(Id - P_L)(R_m^t(r))\|_{H^N} \leq \epsilon \quad \text{whenever } \|r\|_{H^N} \leq M.$$

On the other hand we have

$$\|(Id - P_L)(r)\|_{H^N} = \|\sum_{|n|>L} e^{i\omega_n t}(\hat{r}_n)e^{2\pi i n x}\|_{H^N}.$$

Thus the Theorem follows by triangle inequality. □

Bibliography

[1] D. Bambusi, B. Grébert: *Birkhoff normal form for PDE's with tame modulus*, Duke Math. J. 135 (2005), 507-567.

[2] M. Erdogan, N. Tzirakis, V. Zharnitsky : *Near-linear dynamics in KdV with periodic boundary conditions*, Nonlinearity 23 (2010), 1675-1694.

[3] D. Bättig, T. Kappeler, B. Mityagin : *On the Korteweg-de Vries equation: frequencies and initial value problem*, Pacific J. of Math. 181 (1997), 1-55.

[4] H. Flaschka, D. McLaughlin: *Canonically conjugate variables for the Korteweg-de Vries equation and Toda lattices with periodic boundary conditions*, Progress Theor.Phys. 55(1976), 438-456.

[5] B. Grébert, T. Kappeler, J. Pöschel: *Normal form theory of the NLS equation*, preliminary version, arxiv.org/abs/0907.3938v1 2009. To appear in *EMS Series of Lectures in Mathematics*, EMS Publishing House.

[6] G. Hardy, J. Littlewood, G. Pólya: *Inequalities*, Cambridge University Press, paperback edition, reprinted 1994.

[7] A. Its, V. Matveev : *A class of solutions of the Korteweg-de Vries equation*(in Russian), Problems in Math. Physics N08, Izdat. Leningrad Univ., Leningrad, 1976 , 79-92.

[8] T. Kappeler: *Fibration of the phase-space for the Korteweg-de Vries equation*, Ann. Inst. Fourier 41 (1991), 539-575.

[9] T. Kappeler, M. Makarov: *On action-angle variables for the second Poisson bracket*, Comm. in Math. Phys., 214, 3 (2000), 651-677.

[10] T. Kappeler, M. Makarov: *On Birkhoff coordinates for KdV*, Ann. H. Poincaré 2 (2001), 807 - 856.

[11] T. Kappeler, B. Mityagin: *Estimates for periodic and Dirichlet eigenvalues of the Schrödinger operator*, SIAM J. Math Anal. 33 (2001), p 113-152.

[12] T. Kappeler, J. Pöschel: *KdV & KAM*, Ergebnisse Math. u. Grenzgebiete, Springer, Berlin, 2003.

[13] T. Kappeler, B. Schaad, P. Topalov: *Asymptotic estimates of spectral quantities of Schrödinger operators*, preprint.

[14] T. Kappeler, B. Schaad, P. Topalov: *mKdV and its Birkhoff coordinates*, Physica D: Nonlinear Phenomena, 237, 10-12 (2008), 1655-1662.

[15] T. Kappeler, B. Schaad, P. Topalov: *Qualitative features of periodic solutions of KdV*, preprint.

[16] T. Kappeler, P. Topalov: *Global fold structure of the Miura map on* $L^2(\mathbb{T}, \mathbb{R})$, Int. Math. Research Notices (2004), 2039-2068.

[17] T. Kappeler, P. Topalov: *Global wellposedness of KdV in* $H^{-1}(\mathbb{T}, \mathbb{R})$, Duke Math. J. 135 (2006), 327-360.

[18] E. Korotyaev: *Estimates for the Hill operator, II*, J. Diff. Eq. 223 , 2006, 229-260.

[19] S. Kuksin: *Damped driven KdV and effective equation for long-time behaviour of its solution*, preliminary version, 2009.

[20] S. Kuksin, G. Perelman: *Vey theorem in infinite dimensions and its application to KdV*, Discrete and Continuous Dynamical Systems Ser. A, 27, no 1, (2010), 1-24.

[21] S. Kuksin, A. Piatnitski: *Khasminskii - Whitham averaging for randomly perturbated KdV equation*, J. Math. Pures Appl. 89 (2008), 400-428.

[22] F. Magri: *A simple model of the integrable Hamiltonian equation.* J. Math. Phys. 19 (5) (1975), 1156 - 1162.

[23] V. Marchenko: *Sturm-Liouville operators and applications*, Birkhäuser, Basel, 1986.

[24] R.M. Miura: *Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation.* J. Math. Phys. 9 (1968), 1202 - 1204.

[25] H. McKean, E. Vaninsky: *Action-angle variables for the cubic Schroedinger equation*, Comm. Pure Appl. Math. 50 (1997), 489-562.

[26] J. Pöschel, E. Trubowitz: *Inverse spectral theory*, Academic Press, Boston, 1987.

[27] A. Savchuk, A. Shkalikov: *On the eigenvalues of the Sturm-Liouville operator with potentials from Sobolev spaces*, Math. Notes 80, no 6 (2006), 864-884.

[28] M. Wadati: *The modified Korteweg-de Vries equation.* J. Phys. Soc. Japan 34 (1973), 1289 - 1296.

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