Primary caustics and critical points behind a Kerr black hole

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The primary optical caustic surface behind a Kerr black hole is a four-cusped tube displaced from the line of sight. We compute that in the near asymptotic region through a Taylor expansion of the lightlike geodesics up to and including fourth-order terms in m/b and a/b, where m is the black hole mass, a the spin and b the impact parameter. The corresponding critical locus is elliptical and a point-like source inside the caustics will be imaged as an Einstein cross. With regard to lensing near critical points, a Kerr lens is analogous to a circular lens perturbed by a dipole and a quadrupole potential. The caustic structure of the supermassive black hole in the Galactic center could be probed by lensing of low mass X-ray binaries in the Galactic inner regions or by hot spots in the accretion disk.

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Black holes (BHs) are among the most fascinating predictions of the general theory of relativity. Recent progresses in mass measurements of compact objects have supplied compelling evidence for their existence. Nearly every galaxy is supposed to contain a very massive BH at its center in the mass range from 10^6 to 10^9 $M_\odot$ [1]. Despite strong hints that massive BHs mostly rotate rapidly, their spins are still unmeasured so that a full characterization of their properties is not possible. Analyses of luminosities and spectra of accretion disks around spinning BH are very promising tool, but the interpretation can be sometimes unclear due to uncertainties in the physics of the inflowing gas. The classical test of gravitational deflection of light rays passing near compact bodies can provide an alternative probe. In fact, the theoretical bases of lensing are well understood and, on the observational side, the supermassive BH supposed to be hosted in the radio source Sgr A* in the Galactic center, with a mass of $\sim 3.6\times 10^6 M_\odot$ and at a distance of 7.6 Kpc from the Earth [2], offers a very appealing target for future space- and ground-based experiments.

Lensing by rotating BHs has been studied from the very beginning of the Kerr spacetime [3]. Angular momentum first appears in gravitational lensing through terms $\sim ma/b^2$ in the deflection angle. Up to this order, light deflection is well understood. Very different approaches can be undertaken [4–15] to show the degeneracy between a Kerr BH and a suitably displaced Schwarzschild lens [16, 17]. Such analyses can be easily extended to general spinning mass distributions [18–22] and can show many interesting features in the magnification pattern and in the image properties [16, 20, 21] .

Investigations up to higher orders require the full consideration of the lightlike geodesics [23, 24]. The optical appearance of both stars [25] and accretion disk [26, 27] orbiting a Kerr BH together with the optical structure of the primary caustic surface [28] were detailed through numerical investigations. A clear analytical picture of the relativistic caustics in the strong deflection limit has also emerged [29, 30]. The missing part is an analytical treatment of map singularities and caustics in the weak deflection limit. This is the prerequisite for the study of critical points and lensing map inversion near them.

The lightlike geodesic equation have been expanded as a Taylor series up to and including second [13] and third-order terms in m/b and a/b [31], with the last analysis showing that the primary optical caustic is still point-like at that order. Here, we take the further step. The null geodesics for a light ray can be expressed in terms of the first integrals of motion [23, 24]. We consider photon trajectories from the source S, with Boyer-Lindquist coordinates $\{r_\mathrm{s},\vartheta_\mathrm{s},\phi_\mathrm{s}\}$, to the observer O in $\{r_\mathrm{o},\vartheta_\mathrm{o},\phi_\mathrm{o}=0\}$. Both S and O are located in the near flat region of the spacetime. The geodesics are approximated through an expansion of integrand functions and integrals. Radial and angular integrals are treated introducing the variables $\rho\equiv 1/r$ and $\mu\equiv\cos\vartheta$, respectively. We expand quantities of interest in both $\epsilon\sim m/b$ and a/b. In the near asymptotic region $b/r_\mathrm{o}\sim b/r_\mathrm{s}\sim\epsilon$. Mixed terms like $\mathcal{O}(m/b)^i\mathcal{O}(a/b)^j$ are referred to as terms of order $\mathcal{O}(\epsilon^{i+j})$ and we produce our results up to a given formal order in ϵ. Calculations are performed strictly following the techniques developed in Sereno and De Luca [31] up to including terms $\sim\epsilon^4$. Once performed the expansion, the geodesics equations, which are our lens equations, can be written as

$$\phi_\mathrm{s} = \pm\pi+\delta\phi_\mathrm{s}(b_1,b_2)+\mathcal{O}(\epsilon^5), \quad (1)$$
$$\mu_\mathrm{s} = -\mu_\mathrm{o}+\delta\mu_\mathrm{s}(b_1,b_2)+\mathcal{O}(\epsilon^5), \quad (2)$$

where the b_i parameters are function of the constant of motion [31],

$$b_1\equiv-\frac{J}{\sqrt{1-\mu_\mathrm{o}^2}},\ \ b_2\equiv-(-1)^k\sqrt{Q-J^2\frac{\mu_\mathrm{o}^2}{1-\mu_\mathrm{o}^2}}.$$

The parameter k is an even (odd) integer for photons coming from below (above).

The critical points of the lens mapping form the locus of formally infinite magnification. They can be found where the Jacobian nulls out,

$$\left[\frac{\partial\mu_\mathrm{s}\partial\phi_\mathrm{s}}{\partial b_1\partial b_2}\right]=0. \quad (3)$$

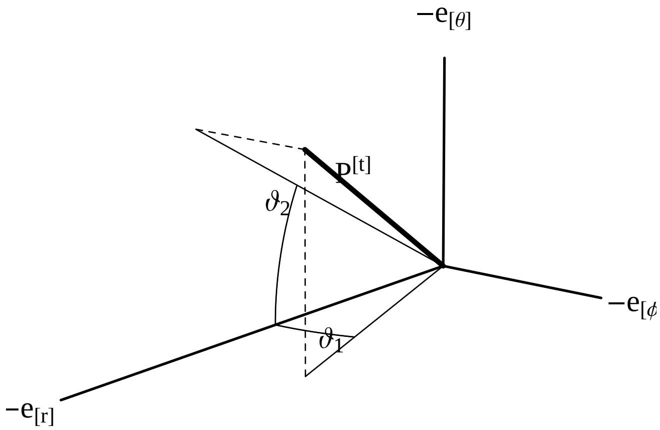

FIG. 1: The angles ϑ_1 and ϑ_2 in the locally flat observer's frame.

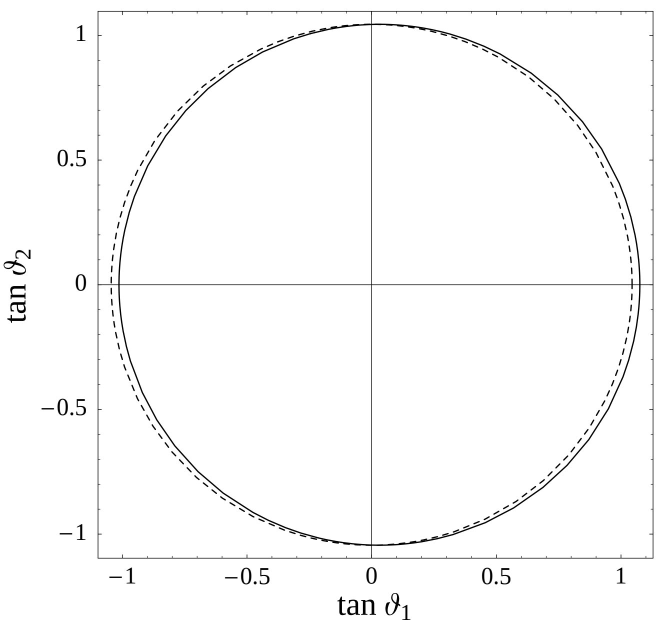

FIG. 2: Critical curve in the $\tan\vartheta_1$-$\tan\vartheta_2$ plane for a source at $r_\mathrm{s} = 10$ AU behind a Sgr A*-like BH. We consider an extreme Kerr, $a = 1$, (full line) and a Schwarzschild lens (dashed line). The observer is equatorial $\mu_\mathrm{o} = 0$. Axis-lengths are in units of the tangent of the Einstein angle, $\theta_\mathrm{E} \equiv R_\mathrm{E}/r_\mathrm{o} \simeq 162$ μarcsec.

Distances can be conveniently rescaled in terms of the Einstein ring $R_\mathrm{E} \equiv \sqrt{4Dr_\mathrm{o}r_\mathrm{g}}$, where $r_\mathrm{g}(\equiv GM/c^2)$ is the gravitational radius and $D \equiv r_\mathrm{s}/(r_\mathrm{o}+r_\mathrm{s})$. A convenient new series expansion parameter is then $\varepsilon \equiv R_\mathrm{E}/(4Dr_\mathrm{o})$. We can find solution to Eq. (3) in the form

$$b_1 \simeq R_\mathrm{E}\cos\varphi\left\{1 + {}^{(1)}b_\mathrm{E}\varepsilon + {}^{(2)}b_\mathrm{E}\varepsilon^2 + {}^{(3)}b_\mathrm{E}\varepsilon^3\right\}, \quad (4)$$

$$b_2 \simeq R_\mathrm{E}\sin\varphi\left\{1 + {}^{(1)}b_\mathrm{E}\varepsilon + {}^{(2)}b_\mathrm{E}\varepsilon^2 + {}^{(3)}b_\mathrm{E}\varepsilon^3\right\}, \quad (5)$$

with $0 \leq \varphi \leq 2\pi$. Solving order by order Eq. (3), we get

$$ {}^{(1)}b_\mathrm{E} = \frac{15}{32}\pi + a\sqrt{1-\mu_\mathrm{o}^2}\cos\varphi, \quad (6)$$

$$ {}^{(2)}b_\mathrm{E} = 4\left(1+D-D^2\right) - \frac{675\pi^2}{2048} + \frac{15\pi}{32}a\sqrt{1-\mu_\mathrm{o}^2}\cos\varphi + \frac{1}{2}\left[\sin^2\varphi\left(\mu_\mathrm{o}^2-1\right) - \mu_\mathrm{o}^2\right]a^2, \quad (7)$$

$$ {}^{(3)}b_\mathrm{E} = \frac{15}{512}\pi\left(-\mu_\mathrm{o}^2 - 3\cos 2\varphi\left(1-\mu_\mathrm{o}^2\right) - 5\right)a^2 + \left[8\left(1+D-D^2\right) - \frac{225}{256}\pi^2\right]\cos\varphi\sqrt{1-\mu_\mathrm{o}^2}\,a + \frac{15\pi}{4}\left(\frac{225\pi^2}{2048} - \frac{153}{128} + D - D^2\right). \quad (8)$$

We are measuring distances in units of r_g. Putting back these solutions in the lens equations, we obtain the caustic surface at r_s in Boyer-Lindquist coordinates,

$$\phi_\mathrm{s} = -\frac{b_1}{|b_1|}\pi - 4a\varepsilon^2 - \frac{5}{4}\pi a\varepsilon^3 + \left[\left(\frac{225}{128}\pi^2 - 16\right)a - \frac{15}{16}\pi a^2\sqrt{1-\mu_\mathrm{o}^2}\cos^3\varphi\right]\varepsilon^4 \quad (9)$$

$$\mu_\mathrm{s} = -\mu_\mathrm{o} - \frac{15}{16}\pi a^2(1-\mu_\mathrm{o}^2)^{3/2}\sin^3\varphi\varepsilon^4. \quad (10)$$

The terms proportional to $\cos^3\varphi$ and $\sin^3\varphi$ trace the astroid caustic surface.

We have now to translate our results in the observer's frame. The local frame of reference can be oriented parallely to the local flat three-space of the locally nonrotating frame (LNRF) and the angles in the observer's sky can be expressed in terms of the tetrad components of the four momentum P [26, 32]. We take the basis vector $-e_{[\theta]}$ as polar axis; the azimuth is zero on the line of sight, i.e. the axis $-e_{[r]}$, and counted positive over $-e_{[\phi]}$, see Fig. 1. The angles ϑ_1 and ϑ_2 are then defined such that $\tan\vartheta_1 = -P^{[\phi]}/P^{[r]}$, i.e. ϑ_1 is the opposite of the azimuthal angle, and $\tan\vartheta_2 = P^{[\theta]}/P^{[r]}$. In a plane orthogonal to the line of sight spanned by ϑ_1 and ϑ_2, the critical curve is an ellipse, see Fig. 2, with equation

$$(\tan\vartheta_1 - \Delta_1)^2 + (1-e)^2\tan^2\vartheta_2 = \vartheta_\mathrm{R}^2; \quad (11)$$

the shift, the radius and the ellipticity are, respectively,

$$e = \frac{105}{256}\pi a^2\left(1-\mu_\mathrm{o}^2\right)\varepsilon^3,$$

$$\Delta_1 = \frac{a\sqrt{1-\mu_\mathrm{o}^2}}{r_\mathrm{o}}\left\{1 + \frac{15\pi}{32}\varepsilon + \left[4\left(2 \pm D + 4D^2\right) - \frac{225\pi^2}{256}\right]\varepsilon^2\right\},$$

$$\vartheta_\mathrm{R} = \theta_\mathrm{E}\left\{1 + \frac{15\pi}{32}\varepsilon + \left[4\left(1+D^2\right) - \frac{675\pi^2}{2048}\right]\varepsilon^2 + \frac{15\pi}{8}\varepsilon^3 \times \left[D + 4D^2 - \frac{9\left(272-25\pi^2\right)}{1024} - \frac{a^2}{8}\left(1+\frac{3}{4}\mu_\mathrm{o}^2\right)\right]\right\},$$

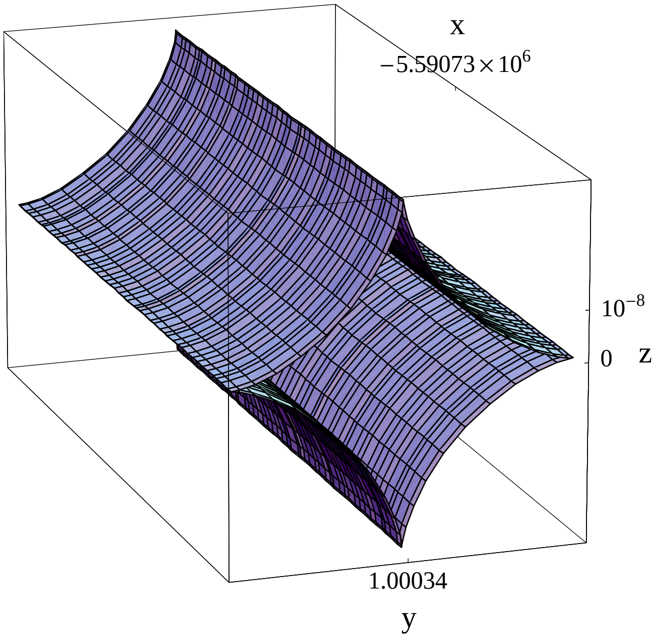

FIG. 3: Primary caustic surface around $r_{\rm s}=1$ pc for a Sgr A*-like lens. We consider extreme Kerr ($a=1$) and an equatorial observer ($\mu_{\rm o}=0$). Axis-lengths are in units of the gravitational radius.

where the upper sign holds for a LNRF and the lower sign for a static observer, respectively. The unperturbed Einstein ring for a Sgr A*-like lens and a source at $r_{\rm s}\sim 100{\rm AU}$ (1pc) is ~ 510 μarcsec (2.3×10^{-2}arcsec). The center shift is dominated by the correction $\propto a$ and is nearly independent of $r_{\rm s}$, $\Delta x_1 \lesssim 4.9(4.8)$ μarcsec. The ellipticity is $\lesssim 1.1\times10^{-6}$ (1.2×10^{-11}) which yields a difference in the axis lengths $\lesssim 5.4\times10^{-4}$ (2.8×10^{-7}) μarcsec. Finally the contribution $\propto a^2$ to the radius is $\lesssim 3.1\times10^{-4}$ (1.6×10^{-7}) μarcsec.

We can plot the caustic surface in a pseudo-Euclidean elliptical coordinate system related to the Boyer-Lindquist coordinates by $(x',y',z')=(\sqrt{r^2+a^2}\sin\theta\cos\phi,\sqrt{r^2+a^2}\sin\theta\sin\phi,r\cos\theta)$. After such a transformation, for $m=0$ the Kerr metric reduces to the Minkowski one [33]. In a rotated system (x,y,z) in which the x-axis lies along the line of sight, we get

$$x_{\rm s} = -r_{\rm s}+\frac{a^2\left(1-\mu_{\rm o}^2\right)}{1-D}R_{\rm E}\varepsilon^3\left[2D(2-D)+\frac{5\pi}{4}\varepsilon\right], \quad (12)$$

$$y_{\rm s} = R_{\rm E}\left\{\frac{a\sqrt{1-\mu_{\rm o}^2}\varepsilon}{1-D}\left[1+\frac{5\pi}{16}\varepsilon+\left(4-\frac{225\pi^2}{512}\right)\varepsilon^2\right] + \frac{15\pi}{64}\frac{a^2\left(1-\mu_{\rm o}^2\right)\cos^3\varphi}{1-D}\varepsilon^3\right\}, \quad (13)$$

$$z_{\rm s} = -R_{\rm E}\frac{a^2\left(1-\mu_{\rm o}^2\right)\varepsilon^3}{1-D}\left[\frac{4D\mu_{\rm o}}{\sqrt{1-\mu_{\rm o}^2}}+\frac{15\pi}{64}\sin^3\varphi\right] \quad (14)$$

The caustic surface is a four-cusped astroid tube, see Fig. 3. The displacement form the line of sight is $\propto a$ in the y-direction, $\Delta y_{\rm s}\sim a\sqrt{1-\mu_{\rm o}^2}/(1-D)$, and $\propto a^2$ along the z-axis. The astroid is symmetric with total width

$$\Delta_c=\frac{15\pi}{128}\frac{a^2\left(1-\mu_{\rm o}^2\right)}{D(1-D)r_{\rm o}}. \quad (15)$$

The width decreases for increasing observer's distance $r_{\rm o}$ and decreases (increases) for increasing source distance if $r_{\rm o}>(<)r_{\rm s}$. For an asymptotic observer ($r_0\rightarrow\infty$), Eq. (15) agrees with the numeric interpolation in eq. 6 in [28], apart from a small difference in the overall numerical factor (0.34 instead of our $15\pi/128\sim0.37$). The caustic width is really small. At $r_{\rm s}\sim 100{\rm AU}$ (1pc) behind a Sgr A*-like BH, $\Delta_c\lesssim 350$ km (170 m), a very tiny fraction of the gravitational radius of the BH, $r_g\sim5.3\times10^6$km.

As well known, the number of images of a source crossing the caustics changes by two. Let us consider a source displaced by $(\delta\phi_{\rm s}\varepsilon^4,\delta\mu_{\rm s}\varepsilon^4)$ with respect to the center of the caustic astroid. The solution for the image positions will be in the form of Eqs. (4), with ${}^{(3)}b_{\rm E}$ differing by ${}^{(3)}\delta b_{\rm E}$ from a point of the critical locus, Eq. (8). We get

$$\frac{15\pi}{16}(1-\mu_{\rm o}^2)^{3/2}a^2\sin\varphi+\delta\mu_{\rm s}-\delta\phi_{\rm s}(1-\mu_{\rm o}^2)\tan\varphi=0, \quad (16)$$

and

$${}^{(3)}\delta b_{\rm E}=\frac{15\pi}{128}\left(1-\mu_{\rm o}^2\right)a^2\cos^2\varphi-\frac{\delta\phi_{\rm s}\sqrt{1-\mu_{\rm o}^2}}{8}\sec\varphi. \quad (17)$$

As can be seen from Eq. (16), a source has four images if inside the caustics and two if outside. The spin breaks the spherical symmetry. The Einstein ring does not form anymore and is replaced by an Einstein cross in the case of point-like source inside the caustics.

The above results on critical points and on lensing near them show how there is a tight connection between a Kerr BH and a Schwarzschild lens slightly perturbed, as already envisaged in [28]. In fact, as far as the caustic structure is concerned, only the potential at the Einstein ring is important [34]. A first order dipole harmonic perturbation is enough to shift the point-like central caustics of a circular lens, whereas the second harmonic yields the four-cusped asteroid. Such a quadrupole perturbation appears at order $(ma)^2/b^4$, breaking the degeneracy between Kerr and displaced Schwarzschild BH.Then, lensing distortion and magnification and formation of arcs by extended sources behind a Kerr lens follows the well known features of nearly circular gravitational lenses.

Our analytical derivation corroborates knowledge about the primary caustics which, in a earlier work [28], have been investigated only numerically and provides an original study of critical points and lens mapping near them. Our approach is the natural complement to qualitative methods which give some information on lensing properties without actually solving the equation for lightlike geodesics, such as the Morse theory [35].

Together with the theoretical motivation, an equally compelling reason for investigating gravitational lensing in a Kerr spacetime comes from lensing observations towards Sgr A* which are coming into the reach of observability and for which the weak-deflection limit approximation at the lowest order is not applicable. The stars surrounding the Galactic center have been considered as suitable targets for detection of lensing effects. Sgr A* is expected to be lensing nearly ten sources at any given time for observations down to $K \sim 21$ [36–38] . Considering the only few stars whose orbital parameters have already been accurately determined, detectable lensing events are expected to occur in a temporal span of ~ 30 years [39]. The typical radius of these sources ($\sim 10^8 - 10^9$ m) is much larger than the caustic width, so that the effects of the finite astroid size are washed out. Nevertheless, the astrometric shift of the caustic center, together with the precession orbital effects related to the spinning central body, must be accounted for when considering future experiments.

Other appealing sources for lensing by Sgr A* are low mass X-ray binaries (LMXBs), whose emission mostly originates in a region a few tens of kilometers across, consisting of the inner accretion disk around the BH accreting from the companion. Tens of thousands of stellar-masses BHs and neutron stars are likely to have settled dynamically into the central parsec of the Galaxy and perhaps few hundreds of them might have stellar companions [40]. Several tens of LMXBs have been already detected in the very inner regions down to a minimum projected distance of only 0.1 pc [41], with a detected overabundance of transients X-ray binaries within 1 pc [40]. These sources have been considered for detection of relativistic images [29] and could as well, since their small radius, probe the primary caustic surface.

Compact emission regions in the clumpy and unsteady accretion flow near Sgr A* are other interesting lensing sources [42, 43]. Relativistic images of orbiting bright-spots could be detected in the near future with observations achieving submilliarcsecond resolution at infrared and submillimetre wavelengths . However, we remark that in our approximation the source distance must be large and close inner orbits can not be considered.

The finite size of the primary caustics behind Sgr A* could then be probed by lensing of either LMXBs or compact hot spots in the accretion flow whereas the effects of angular momentum, such as the shift in the caustic position, affect even lensing of main sequence stars in the Galactic center cluster.

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