Trace Functor and Categorified Quantum $\mathfrak{sl}_n$

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Abstract

M. Khovanov and A. Lauda [35] define a 2-category $\mathcal{U}$ such that the split Grothendieck group $K_0(\mathcal{U})$ is isomorphic to an integral version of the universal enveloping algebra $\mathbf{U}(\mathfrak{sl}_n)$, $n \geq 2$. The trace Tr is a decategorification functor that is an alternative to the usual decategorification given by K_0. We compute the trace $\mathrm{Tr}\,\mathcal{U}$ using the diagrammatic presentation of $\mathcal{U}$. We find a graded algebra homomorphism between $\mathrm{Tr}\,\mathcal{U}$ and the current algebra $\mathbf{U}(\mathfrak{sl}_n[t])$, which is the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_n \otimes \mathbb{C}[t]$. More recently, this homomorphism is proven to be an isomorphism in [5]. Thus, $\mathrm{Tr}\,\mathcal{U}$ has a richer structure than $K_0(\mathcal{U})= \mathbf{U}(\mathfrak{sl}_n)$.

A 2-representation is a categorification of the notion of a representation. In particular, a 2-representation of $\mathcal{U}$ is a 2-functor from $\mathcal{U}$ to a linear, additive 2-category. Our next result provides a framework on how to get a current algebra module from any 2-representation of $\mathcal{U}$. We are interested in the 2-representation, defined by Khovanov-Lauda using bimodules over cohomology rings of flag varieties. This 2-representation induces an action of the current algebra $\mathbf{U}(\mathfrak{sl}_n[t])$ on the cohomology rings. We explicitly compute the action of $\mathbf{U}(\mathfrak{sl}_n[t])$ generators using the trace functor. It turns out that the obtained current algebra module is related to another family of $\mathbf{U}(\mathfrak{sl}_n[t])$-modules, called local Weyl modules. Using known results about the cohomology rings, we are able to provide a new proof of the character formula for the local Weyl modules.

Finally, we apply the trace functor to the Bao-Shan-Wang-Webster [2] categorification $\mathcal{U}^\jmath$ of the coideal subalgebra of quantized $\mathfrak{sl}_{2n+1}$. We compute the trace of the 2-category $\mathcal{U}^\jmath$ in this thesis. We define a new algebra, called the current coideal algebra, and we show that $\mathrm{Tr}\,\mathcal{U}^\jmath$ is homomorphic to the current coideal algebra.

Zusammenfassung

M. Khovanov and A. Lauda [35] definieren eine 2-Kategorie $\mathcal{U}$, so dass die geteilte Grothendieck-Gruppe $K_0(\mathcal{U})$ zu einer integralen Version der universellen einhüllenden Algebra $\mathbf{U}(\mathfrak{sl}_n)$, $n \geq 2$ isomorph ist. Die Spur Tr ist eine Alternative zu dem De-kategorisierungsfunktor, normalerweise gegeben durch K_0. Wir berechnen Tr$\mathcal{U}$ mithilfe der diagrammatischen Präsentation von $\mathcal{U}$. Wir finden einen graduierten Algebra-Homomorphismus zwischen Tr$\mathcal{U}$ und der Stromalgebra $\mathbf{U}(\mathfrak{sl}_n[t])$, welche die universelle einhüllende Algebra der Lie-Algebra $\mathfrak{sl}_n \otimes \mathbb{C}[t]$ ist. Kürzlich wurde in [5] bewiesen, dass dieser Homomorphismus sogar einen Isomorphismus darstellt. Somit hat Tr$\mathcal{U}$ eine reichere Struktur als $K_0(\mathcal{U})= \mathbf{U}(\mathfrak{sl}_n)$.

Eine 2-Darstellung ist eine Kategorisierung des Begriffs der Darstellung. Insbesondere ist eine 2-Darstellung von $\mathcal{U}$ ein 2-Funktor aus $\mathcal{U}$ in eine lineare, additive 2-Kategorie. Unser nächstes Ergebnis stellt einen Rahmen dar, um aus einer beliebigen 2-Darstellung ein Stromalgebra-Modul zu erhalten. Wir interessieren uns für die 2-Darstellung, definierte von Khovanov-Lauda durch Bimoduln über Kohomologien der Fahnenmannigfaltigkeiten. Diese 2-Darstellung induziert eine Wirking der Stromalgebra $\mathbf{U}(\mathfrak{sl}_n[t])$ auf die Kohomologieringe. Wir berechnen die explizite Stromalgebrawirkung durch den Spurfunktor. Es wird festgestellt, dass der erhaltene Stromalgebramodul sich auf eine andere Familie von $\mathbf{U}(\mathfrak{sl}_n[t])$-Moduln bezieht, bekannt als lokale Weyl-Moduln. Mithilfe bekannter Ergebnisse geben wir einen neuen Beweis der Charakterformeln von lokalen Weyl-Moduln.

Schliesslich wenden wir den Spurfunktor auf die Bao-Shan-Wang-Webster Kategorisierung [2] $\mathcal{U}^\jmath$ der koidealen Unteralgebra der quantisierten Lie-Algebra $\mathfrak{sl}_{2n+1}$ an. Wir berechnen die Spur der 2-Kategorie $\mathcal{U}^\jmath$ in dieser Arbeit. Wir definieren eine neue Algebra, als koideale Stromalgebra genannt, und wir zeigen, dass Tr$\mathcal{U}^\jmath$ homomorph zu der koidealen Stromalgebra ist.

Contents

Introduction

The special linear Lie algebra $\mathfrak{sl}_n$, $n \geq 2$ is the Lie algebra of $n \times n$ complex matrices with zero trace. The quantum group $\mathbf{U}_q(\mathfrak{sl}_n)$ is a q-deformation of the universal enveloping algebra $\mathbf{U}(\mathfrak{sl}_n)$. A reason why quantum groups are interesting is that they can be used to obtain invariants of links in $\mathbb{R}^3$. For example, the Jones polynomial is a classical link invariant defined using the vector representation of $\mathbf{U}_q(\mathfrak{sl}_2)$.

L. Crane and I. Frenkel [27] conjectured that quantum groups might be shadows of higher categorical structures – the *categorified quantum groups*. This conjecture was supported by Lusztig's [42] discovery of canonical bases, which have certain integrality and positivity properties. M. Khovanov and A. Lauda [35] categorified $\mathbf{U}_q(\mathfrak{sl}_n)$. Namely, they defined a diagrammatic 2-category $\mathcal{U}$ and showed that *split Grothendieck group* decategorification of $\mathcal{U}$ is isomorphic to an integral version of $\mathbf{U}_q(\mathfrak{sl}_n)$. Categorified quantum groups are used to obtain categorified link invariants, which are usually more subtle than the classical ones, and their additional structure gives more information about the link.

Trace is an alternative decategorification functor to the split Grothendieck group. In this thesis we apply the *trace functor* on the 2-category $\mathcal{U}$ and its representations. We show that the trace of $\mathcal{U}$ is algebraically homomorphic to the *current algebra* $\mathbf{U}(\mathfrak{sl}_n \otimes \mathbb{C}[t])$. There is a 2-category $\mathbf{Pol}_N$ that admits an action of $\mathcal{U}$. The 2-category $\mathbf{Pol}_N$ is constructed by Khovanov-Lauda [35] using bimodules over cohomology rings of flag varieties to prove the non-degeneracy of $\mathcal{U}$. We present this 2-category in terms of symmetric polynomials. We define a current algebra module using the objects of $\mathbf{Pol}_N$ and explicitly compute the current algebra action. We are able to relate this module to well-studied current algebra modules, called local Weyl modules, and we give a new proof of character formula for the local Weyl modules. We go over each of these steps in this introductory chapter.

Categorification and K_0 decategorification

Categorification is a process of finding category-theoretic analogs of set-like mathematical structures by replacing sets with categories, functions with functors and equations between functions by natural isomorphisms between functors. *Decategorification* is the inverse process which 'forgets' the morphisms in the category and identifies the isomorphic objects.

For example, let $\mathbf{Vect}_{\Bbbk}$ be the category of finite-dimensional vector spaces over a field $\Bbbk$. Since finite-dimensional vector spaces are isomorphic if and only if their dimensions are equal, the set of isomorphism classes of objects in $\mathbf{Vect}_{\Bbbk}$ are in bijection with the natural numbers $\mathbb{N}$. Hence, we can view $\mathbf{Vect}_{\Bbbk}$ as a categorification of $\mathbb{N}$.

One of the most prominent instances of categorification is the *Khovanov homology.* To any link Khovanov assigns a chain complex of graded vector spaces. He then proves that homology of this chain complex is a link invariant. The Euler characteristic of the Khovanov homology gives the Jones polynomial. Hence, Khovanov homology categorifies the Jones polynomial [3]. Khovanov homology is proven to be a strictly stronger link invariant than the Jones polynomial.

Decategorification is usually understood as a functor that returns the split Grothendieck group $K_0(\mathcal{C})$ of an additive category $\mathcal{C}$. The split Grothendieck group $K_0(\mathcal{C})$ is the abelian group generated by isomorphisms classes $\{[x]_{\cong}\}_{x\in \mathrm{Ob}(\mathcal{C})}$, modulo the relation

$$[x \oplus y]_{\cong} = [x]_{\cong} + [y]_{\cong}.$$

In our example of $\mathbf{Vect}_{\Bbbk}$, by choosing a basis, every vector space $V \in \mathrm{Ob}(\mathbf{Vect}_{\Bbbk})$ is isomorphic to a direct sum of copies of the ground field: $[V]_{\cong} = [\Bbbk^{\oplus \dim(V)}]_{\cong} = \dim(V)[\Bbbk]_{\cong}$. Hence, the Grothendieck group $K_0(\mathbf{Vect}_{\Bbbk})$ can be identified with $\mathbb{N}$ by sending $[\Bbbk]_{\cong} \mapsto 1$.

A 2-category is a category with an additional structure – the hom-sets are also categories. That is, we have 2-morphisms between 1-morphisms. A 2-category can be thought of as a categorification of the usual notion of a category. For example, **Cat** is the 2-category whose objects are small categories, morphisms are functors, and 2-morphisms are natural transformations between functors. The split Grothendieck group of a 2-category can be thought of as a procedure for turning the 2-categorical structure of an additive category into a 1-categorical structure of an abelian group. We get a 1-category if we forget 2-morphisms in a 2-category and identify all isomorphic 1-morphisms.

Trace decategorification

The trace of a square matrix is the sum of its diagonal elements. The notion of a trace can be generalized to a $\Bbbk$-algebra A. We define the trace $\operatorname{Tr} A$ to be $A/[A, A]$, which is also known as the zeroth Hochschild homology of A.

We say that a category is $\Bbbk$-linear if the set of morphisms form a $\Bbbk$-linear vector space, and compositions are bilinear over $\Bbbk$. The trace of a $\Bbbk$-linear category $\mathcal{C}$ is defined by

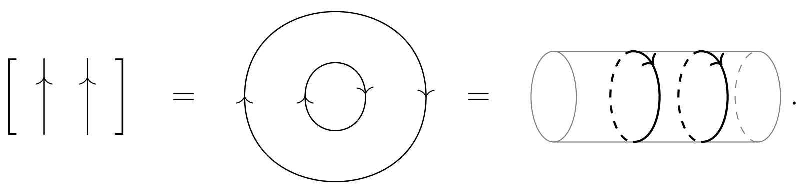
$$\operatorname{Tr}\mathcal{C} = \left(\bigoplus_{x\in \mathrm{Ob}(\mathcal{C})} \mathrm{End}_{\mathcal{C}}(x) \right) / \, \mathrm{Span}_{\Bbbk}\{f \circ g - g \circ f\},$$

where f and g run through all pairs of morphisms $f\colon x \to y$ and $g\colon y \to x$ with $x, y \in \mathrm{Ob}(\mathcal{C})$. This is a generalization of the trace of an algebra, since a $\Bbbk$-algebra is a $\Bbbk$-linear category with one object. The trace of a $\Bbbk$-linear category $\mathcal{C}$ is the same as its *zeroth Hochschild-Mitchell homology.* The trace is another procedure to turn $\mathcal{C}$ into an abelian group. For example, in the category $\mathbf{Vect}_{\Bbbk}$, the trace class $[\varphi]$ of a $\Bbbk$-linear endomorphism $\varphi\colon V \to V$ is equal to the class $\mathrm{tr}(\varphi)[1_{\Bbbk}]$, where $\mathrm{tr}(\varphi)$ is the usual trace of the $\Bbbk$-linear endomorphism φ. Since $\dim_V = \mathrm{tr}(1_V)$, the trace can be seen as a generalization of the dimension decategorification map. There is an abelian group homomorphism

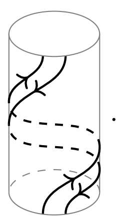
$$h_{\mathcal{C}}\colon K_0(\mathcal{C}) \to \mathrm{Tr}\,\mathcal{C}, \qquad h_{\mathcal{C}}[x]_{\cong} = [1_x].$$

The map $h_{\mathcal{C}}$ is called *Chern character map*, and it is generally neither injective nor surjective. That is to say, K_0 and Tr functors may produce different results. As we will see in this thesis, Tr is often richer than K_0.

The trace $\mathrm{Tr}\,\mathcal{C}$ of a 2-category $\mathcal{C}$ is a category who has the same object set as $\mathcal{C}$, and morphisms of $\mathrm{Tr}\,\mathcal{C}$ are the trace classes of 2-endomorphisms of $\mathcal{C}$. If 2-morphisms of $\mathcal{C}$ admit a diagrammatic presentation, then the trace class of a 2-endomorphism can be described either by closing the loose ends of a diagram to right on a plane or by closing them vertically around an horizontal cylinder, *e.g.*



There is another notion of a trace $\mathrm{Tr}^h\,\mathcal{C}$ of a 2-category $\mathcal{C}$, called the *horizontal trace*, introduced in [7]. The horizontal trace is more complicated to define and to work with in full generality. Diagrammatically, images of 2-morphisms under $\mathrm{Tr}^h\,\mathcal{C}$ can be described by revolving diagrams horizontally around a vertical cylinder, *e.g.*



There is an injective map $\mathrm{Tr}\,\mathcal{C} \to \mathrm{Tr}^h\,\mathcal{C}$ that can be diagrammatically described by turning the cylinder 90 degrees to the right. This suggests that $\mathrm{Tr}^h\,\mathcal{C}$ is richer than $\mathrm{Tr}\,\mathcal{C}$.

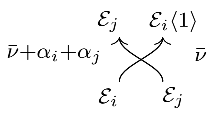
H. Queffelec and D. Rose [50] use horizontal trace functor to construct an $\mathfrak{sl}_n$ homology for colored links in a thickened annulus. We will describe their construction in section 2.2.3 of this thesis.

Another application of the horizontal trace is given by A. Beliakova, K. Putyra and S. Wehrli [9]. Let **Tan** be the 2-category whose objects are points in a thickened stripe $\mathbb{R} \times I$, 1-morphisms are oriented tangles in $\mathbb{R} \times I \times I$, and 2-morphisms are tangle cobordisms in $\mathbb{R} \times I \times I \times I$. Beliakova-Putyra-Wehrli prove that the horizontal trace of **Tan** is equivalent to the category of oriented links in $S^1 \times \mathbb{R}^2$ and cobordisms between them. They actually prove this equivalence in a more general setting, where tangles in any thickened surface are considered, and the notion of the horizontal trace is quantized. Beliakova-Putyra-Wehrli use it to introduce the *quantum annular link homology* for annular links.

Trace of categorified quantum $\mathfrak{sl}_n$.

Another milestone in categorification was achieved independently by M. Khovanov and A. Lauda [35] and by R. Rouquier [53]. They define a 2-category $\mathcal{U}_Q(\mathfrak{sl}_n)$ whose split Grothendieck group gives the integral version of the idempotent completion $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ of the associated quantum group $\mathbf{U}_q(\mathfrak{sl}_n)$. In this thesis we will mainly work with the diagrammatic 2-category defined by Khovanov-Lauda [35]. They categorify Beilinson-Lusztig-MacPherson [4] idempotent version $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$, in which the unit in $\mathbf{U}_q(\mathfrak{sl}_n)$ is replaced by a collection of idempotents $\{1_{\bar{\nu}}\}$, indexed by elements $\bar{\nu}$ of the integral $\mathfrak{sl}_n$ weight lattice X. The $\mathbb{Q}(q)$-module $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ has a natural category structure with the object set X, and hom-sets from $\bar{\nu} \in X$ to $\bar{\mu} \in X$ are given by $1_{\bar{\mu}}\dot{\mathbf{U}}_q(\mathfrak{sl}_n)1_{\bar{\nu}}$. The algebra $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ is generated over $\mathbb{Q}(q)$ by elements $1_{\bar{\nu}}$, $1_{\bar{\nu}+\alpha_i}E_i1_{\bar{\nu}}$ and $1_{\bar{\nu}-\alpha_i}F_i1_{\bar{\nu}}$, modulo certain relations, where $\bar{\nu} \in X$, $i \in I = \{1, \ldots, n-1\}$ and α_i is the i-th simple root.

Khovanov and Lauda define $\mathcal{U}_Q(\mathfrak{sl}_n)$ to be the 2-category with the object set X and 1-morphisms generated by the identity 1-morphism $\mathbf{1}_{\bar{\nu}}$ and the morphisms $\mathcal{E}_i\mathbf{1}_{\bar{\nu}} \colon \bar{\nu} \to \bar{\nu} + \alpha_i$, $\mathcal{F}_i\mathbf{1}_{\bar{\nu}} \colon \bar{\nu} \to \bar{\nu} - \alpha_i$ for all $i \in I$. For each 1-morphism f there is also a degree shifted 1-morphism $f\langle t\rangle$ in $\mathcal{U}_Q(\mathfrak{sl}_n)$ for each $t \in \mathbb{Z}$. The 2-morphisms are $\Bbbk$-vector spaces spanned by certain oriented planar diagrams, modulo isotopies and local relations. For example,



is a 2-morphism $\mathcal{E}_i\mathcal{E}_j\mathbf{1}_{\bar{\nu}} \to \mathcal{E}_j\mathcal{E}_i\mathbf{1}_{\bar{\nu}}\langle 1\rangle$. Precise set of generators and relations of 2-morphisms in $\mathcal{U}_Q(\mathfrak{sl}_n)$ is given in section 1.2.3 of this thesis. Let $\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)$ be the smallest 2-category which contains $\mathcal{U}_Q(\mathfrak{sl}_n)$ and has splitting idempotents. The split Grothendieck group $K_0(\dot{\mathcal{U}}_Q(\mathfrak{sl}_n))$ can be regarded as a $\mathbb{Z}[q, q^{-1}]$-module if we forget the 2-morphisms, identify isomorphic 1-morphisms and let the degree shift $\langle 1\rangle$ of a 1-morphism correspond to a multiplication by q. Khovanov and Lauda prove that there is a $\mathbb{Z}[q,q^{-1}]$-module isomorphism between an integral form of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ and the split Grothendieck group $K_0(\dot{\mathcal{U}}_Q(\mathfrak{sl}_n))$ which maps $q^tE_i1_{\bar{\nu}}$ to $[\mathcal{E}_i\mathbf{1}_{\bar{\nu}}\langle t\rangle]_{\cong}$, $q^tF_i1_{\bar{\nu}}$ to $[\mathcal{F}_i\mathbf{1}_{\bar{\nu}}\langle t\rangle]_{\cong}$ for all $i\in I$ and $q^t1_{\bar{\nu}}$ to $[\mathbf{1}_{\bar{\nu}}\langle t\rangle]_{\cong}$.

Let $\mathcal{U}$ be the modification of $\mathcal{U}_Q(\mathfrak{sl}_n)$, where degree shifts of each 1-morphism are identified, and 2-morphisms are not required to preserve the degree. The split Grothendieck group $K_0(\mathcal{U})$ is a $\mathbb{Z}$-module rather than a $\mathbb{Z}[q,q^{-1}]$-module since identifying morphisms that differ by a degree shifts will correspond to setting $q=1$ on the K_0 level. The trace $\mathrm{Tr}\,\mathcal{U}$ turns out to be a larger algebra with a $\mathbb{Z}$-grading. Let $\mathfrak{sl}_n[t]=\mathfrak{sl}_n\otimes\Bbbk[t]$ be the extension of $\mathfrak{sl}_n$ by the polynomials in parameter t. The Lie bracket $[a\otimes t^r,b\otimes t^s]=[a,b]\otimes t^{r+s}$ defines a graded Lie algebra structure on $\mathfrak{sl}_n\otimes\Bbbk[t]$ with $\deg(a\otimes t^r)=2r$. We call the universal enveloping algebra $\mathbf{U}(\mathfrak{sl}_n[t])$ the *current algebra*. Current algebra is generated over $\Bbbk$ by $E_{i,r}=E_i\otimes t^r, F_{i,r}=F_i\otimes t^r, H_{i,r}=H_i\otimes t^r$ for all $r\geq 0$ and $i\in I$, where E_i,F_i,H_i are the Chevalley basis of $\mathfrak{sl}_n$. We denote by $\dot{\mathbf{U}}(\mathfrak{sl}_n[t])$ the idempotented version of the current algebra. The first result of this thesis is the following theorem.

Theorem 1. *There is a well-defined graded algebra homomorphism* $\rho\colon\dot{\mathbf{U}}(\mathfrak{sl}_n[t])\to\mathrm{Tr}\,\mathcal{U}$.

The definition of the map ρ is given in section 1.2.4 of this thesis. We prove Theorem 1 by using explicit generators of $\mathrm{Tr}\,\mathcal{U}$ as suggested in [7] and checking the current algebra relations. Our computation utilizes diagrammatic presentation of generators and relations of 2-morphisms in $\mathcal{U}$. We describe the trace classes of 2-endomorphisms by closing the loose ends of the diagrams to the right. The case $n=2$ is first proven by Beliakova-Habiro-Lauda-Zivkovic [7]. They obtain a stronger statement which says that ρ is an isomorphism. Their proof is based on decompositions of arbitrary 1-morphisms into indecomposables, which is not possible for general n. The homomorphism ρ has been proven by Beliakova-Habiro-Lauda-Webster [5] to be an isomorphism for any $n\geq 2$. They use properties of local Weyl modules of current algebra to prove their result.

2-representations

Representations of $\mathbf{U}_q(\mathfrak{sl}_n)$ are used to obtain invariants of links and 3-manifolds. A motivation for categorifying the representations is to achieve even stronger invariants. A *2-representation* or *categorical representation* of $\mathbf{U}_q(\mathfrak{sl}_n)$ is a graded, additive 2-functor $\mathcal{U}_Q(\mathfrak{sl}_n)\to\mathcal{K}$ for some graded, additive 2-category $\mathcal{K}$. Categorical representations of $\mathbf{U}_q(\mathfrak{sl}_n)$ have been studied even before the categorification of the quantum group $\mathbf{U}_q(\mathfrak{sl}_n)$ itself. A categorification of the tensor products of the fundamental $\mathfrak{sl}_2$ representations was first algebraically formulated by Bernstein-Frenkel-Khovanov [10], later extended to the quantum group $\mathbf{U}_q(\mathfrak{sl}_2)$ by Frenkel-Khovanov-Stroppel [28], Stroppel [57], Chuang-Rouquier [26], and to $\mathbf{U}_q(\mathfrak{sl}_n)$ by Mazorchuk-Stroppel [48], Webster [61]. A big role here has always played the graded, parabolic BGG category $\mathcal{O}$ of certain finitely generated $\mathfrak{gl}_m$ weight modules.

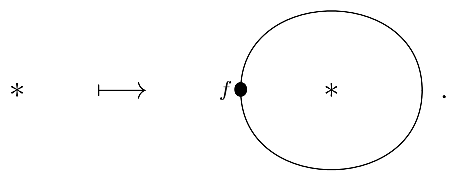
A modification of $\mathcal{K}$ by allowing 2-morphisms of any degree lets us to define a 2-representation $\mathcal{U} \to \mathcal{K}$. We define the center $Z(x)$ of an object $x \in \mathrm{Ob}(\mathcal{K})$ to be the endomorphism ring $\mathrm{End}(1_x)$ and *the center of objects* $Z(\mathcal{K})$ to be $Z(\mathcal{K}) = \bigoplus_{x\in\mathrm{Ob}(\mathcal{K})} Z(x)$. We are able to prove the following statement by using Theorem 1 and a certain cyclicity property of $\mathcal{U}$.

Theorem 2. *Any 2-representation $\mathcal{U} \to \mathcal{K}$ induces an action of $\mathbf{U}(\mathfrak{sl}_n[t])$ on $\mathrm{Tr}\,\mathcal{K}$ and on $Z(\mathcal{K})$.*

The action on $\mathrm{Tr}\,\mathcal{K}$ is defined by applying the trace functor on $\mathcal{U} \to \mathcal{K}$ and composing with the homomorphism ρ in Theorem 1:

$$\dot{\mathbf{U}}(\mathfrak{sl}_n[t]) \to \mathrm{Tr}\,\mathcal{U} \to \mathrm{Tr}\,\mathcal{K}.$$

The precise action on $Z(\mathcal{K})$ will be given in section 1.2.4 of this thesis. Intuitively, it can be understood as follows. In terms of diagrams, an element in $Z(\mathcal{K})$ is a closed diagram $*$. The action of $f \in \mathbf{U}(\mathfrak{sl}_n[t])$ corresponds to the closure of diagram f around $*$, and this gives us another closed diagram:



Local Weyl modules and the trace of cyclotomic KLR algebras

Here we describe the 2-representation of $\mathcal{U}$ over cyclotomic KLR algebras, and we give a brief overview how the trace and the center of cyclotomic KLR algebras are related to local Weyl modules.

A local Weyl module $W(\bar{\lambda})$ of the current algebra $\mathbf{U}(\mathfrak{sl}_n[t])$ assigned to a dominant weight $\bar{\lambda} \in X$ is defined to be the graded module generated by $v_{\bar{\lambda}}$ over $\mathbf{U}(\mathfrak{sl}_n[t])$, subject to the following relations for all $i \in I$ and $j \geq 0$:

$$E_{i,j}v_{\bar{\lambda}} = F_{i,0}^{\bar{\lambda}_i+1}v_{\bar{\lambda}} = 0, \qquad H_{i,j}v_{\bar{\lambda}} = \delta_{0,j}\,\bar{\lambda}_i v_{\bar{\lambda}},$$

where δ is the Kronecker symbol. $W(\bar{\lambda})$ is finite-dimensional, inherits degree from the current algebra, and each homogeneous piece $W(\bar{\lambda})\{2r\}$ is an $\mathfrak{sl}_n$-module.

Local Weyl modules play an important role in the study of finite-dimensional representations of affine Lie algebras. Chari-Pressley [25] were first to introduce them, where they prove certain universality properties. Since then local Weyl modules and their infinite-dimensional analogues, called *global Weyl modules*, have attracted a vast amount of interest. Chari-Loktev [24] show that local Weyl modules are isomorphic to another families of current algebra modules, called Demazure modules and fusion modules. Fusion modules are certain graded tensor products of fundamental $\mathfrak{sl}_n$ representations, and by setting $t = 1$ we recover the usual tensor products. Kodera-Naoi [37] prove that radical and socle series of local Weyl modules coincide with their grading filtration. They also relate local Weyl modules to the homology groups of Nakajima's quiver varieties of type A.

The *Khovanov-Lauda-Rouquier (KLR) algebra* R, a generalization of nilHecke algebra, was introduced independently by Khovanov-Lauda [33] and by Rouquier [53] to categorify $\mathbf{U}_q(\mathfrak{sl}_n)$. R is a graded algebra and admits a rich diagrammatic presentation.

For each dominant weight $\bar{\lambda} \in X$, R has a graded, finite-dimensional quotient $R^{\bar{\lambda}}$ which is called the *cyclotomic KLR algebra*. Brundan-Kleshchev [15] prove the isomorphism between $R^{\bar{\lambda}}$ and cyclotomic Hecke algebras. Khovanov-Lauda [33] show that the category $R^{\bar{\lambda}}$-pMod of graded, finitely generated, projective $R^{\bar{\lambda}}$-modules is a 2-representation of $\mathcal{U}_Q(\mathfrak{sl}_n)$, and they conjecture that $R^{\bar{\lambda}}$-pMod categorifies the finite-dimensional, simple $\mathbf{U}_q(\mathfrak{sl}_n)$-module with the highest weight $\bar{\lambda}$. This conjecture was proven independently by Kang-Kashiwara [32] and Webster [61]. Webster defines a flat deformation $\check{R}^{\bar{\lambda}}$ of the algebra $R^{\bar{\lambda}}$ and shows that the category $\check{R}^{\bar{\lambda}}$-pMod is Rouquier's [54] *universal categorification* $\mathcal{V}(\bar{\lambda})$.

It is well-known that for a graded algebra A, we have $\operatorname{Tr} A = \operatorname{Tr}(A\text{-pMod})$ (see Shan-Varagnolo-Vasserot [56], Proposition 2.1). Hence, by applying the trace to the 2-category $R^{\bar{\lambda}}$-pMod and using Theorem 2, we obtain a current algebra module structure on the trace $\operatorname{Tr} R^{\bar{\lambda}}$ of the cyclotomic KLR algebra $R^{\bar{\lambda}}$.

Let $Z(R^{\bar{\lambda}})$ be the center of $R^{\bar{\lambda}}$. $\operatorname{Tr} R^{\bar{\lambda}}$ and $Z(R^{\bar{\lambda}})$ can be understood as a zeroth Hochschild homology and cohomology respectively. There is a *symmetrizing form* $R^{\bar{\lambda}} \to \Bbbk$ (also known as a *Frobenius trace*, see Webster [61], Theorem 3.18), and this form induces a duality between $\operatorname{Tr} R^{\bar{\lambda}}$ and $Z(R^{\bar{\lambda}})$ by Shan-Varagnolo-Vasserot [56] (Propositions 2.7 and 3.10). Using this duality, one can translate the current algebra action on $\operatorname{Tr} R^{\bar{\lambda}}$ to an action on $Z(R^{\bar{\lambda}})$.

It turns out that $W(\bar{\lambda})$ and $\operatorname{Tr} R^{\bar{\lambda}}$ are isomorphic as graded current algebra modules. This result is proven independently by Beliakova-Habiro-Lauda-Webster [5] and by Shan-Varagnolo-Vasserot [56]. They also prove that this isomorphism induces an isomorphism between $Z(R^{\bar{\lambda}})$ and an appropriate dualization of the local Weyl module $W(\bar{\lambda})$.

The 2-representation in equivariant cohomology of flag varieties

Here we apply Theorem 2 to a 2-representation defined by Khovanov-Lauda [35] to obtain a current algebra module. We first present their 2-representation in terms of symmetric polynomials and then compute the current algebra action on the center of objects.

A *n-composition of N* is an ordered n-tuple of non-negative integers which sum up to a fixed integer N. Let $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ be a *n-composition of N* and P_ν be the subalgebra of the polynomial ring in variables $X_1, X_2, \dots, X_N$, which is symmetric in the first ν_1-tuple of variables, in the second ν_2-tuple of variables and so forth. This ring is generated by complete symmetric polynomials $h_r(\nu; i)$ in i-th tuple of ν_i variables over all $r \geq 0$, $1 \leq i \leq n$, as well as by elementary symmetric polynomials $e_r(\nu; i)$ or by power sum symmetric polynomials $p_r(\nu; i)$.

We will construct a 2-category $\mathbf{Pol}_N$ together with a 2-representation $\Theta_N \colon \mathcal{U} \to \mathbf{Pol}_N$. Objects in $\mathbf{Pol}_N$ are the graded rings P_ν for each n-composition of N, 1-morphisms are bimodules over P_ν, and 2-morphisms are bimodule maps. $\mathbf{Pol}_N$ is equivalent to the 2-category $\mathbf{EqFlag}_N$, which is defined by Khovanov-Lauda to show the non-degeneracy of $\mathcal{U}$. $\mathbf{EqFlag}_N$ is constructed using bimodules over equivariant cohomology rings of partial flag varieties, and mapping Chern classes in $\mathbf{EqFlag}_N$ to elementary symmetric polynomials in $\mathbf{Pol}_N$ defines an equivalence of 2-categories.

The split Grothendieck group of $\mathbf{Pol}_N$ gives the finite-dimensional, simple highest weight $\mathfrak{sl}_n$ module $V(\bar{\lambda}_0)$ of the highest weight $\bar{\lambda}_0 = N\omega_1$ for the first fundamental weight ω_1. This proven by Frenkel-Khovanov-Stroppel [28] for the case $n = 2$ and by Khovanov-Lauda [35] for general n.

The 2-functor Θ_N factors through Rouquier's universal categorification $\mathcal{V}(\bar{\lambda}_0)$:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\Theta_N} & \mathbf{Pol}_N \\ & \searrow & \uparrow \Theta'_N \\ & & \mathcal{V}(\bar{\lambda}_0) \end{array} ,$$

where Θ'_N is a strongly equivariant 2-functor in the sense of Webster [61].

The 2-representation Θ_N induces a current algebra action on the center of objects $Z(\mathbf{Pol}_N) = \bigoplus_\nu P_\nu$ by Theorem 2. We identify the current algebra module structure in $\bigoplus_\nu P_\nu$ explicitly in our next result.

Theorem 3. *Let ν be an n-composition of N, $i \in I$, $\alpha_i = (0, \dots, 0, 1, -1, 0, \dots 0)$, $k_i = \sum_{l=1}^{i} \nu_l$ and $m, j \geq 0$. The action of the current algebra $\mathbf{U}(\mathfrak{sl}_n[t])$ on $\bigoplus_\nu P_\nu$ is defined as follows.*

1. *The map* $F_{i,j}\colon P_\nu \to P_{\nu-\alpha_i}$ *is the* $P_{\nu-\alpha_i}$*-module homomorphism such that*

$$F_{i,j}(X_{k_i}^m) = \sum_{l=0}^{\nu_i-1}(-1)^l e_l(\nu-\alpha_i;i)h_{m+j+\nu_i-\nu_{i+1}-1-l}(\nu-\alpha_i;i+1).$$

2. *The map* $E_{i,j}\colon P_\nu \to P_{\nu+\alpha_i}$ *is the* $P_{\nu+\alpha_i}$*-module homomorphism such that*

$$E_{i,j}(X_{k_i+1}^m) = \sum_{l=0}^{\nu_{i+1}-1}(-1)^l e_l(\nu+\alpha_i;i+1)h_{m+j+\nu_{i+1}-\nu_i-1-l}(\nu+\alpha_i;i).$$

3. *The map* $H_{i,j}\colon P_\nu \to P_\nu$ *is the multiplication by* $(-1)^j(p_j(\nu;i+1)-p_j(\nu;i))$ *if* $j>0$, *and by* $\nu_i-\nu_{i+1}$ *if* $j=0$.

The proof of Theorem 3 follows from a direct computation using diagrammatic description of 2-morphisms in $\mathcal{U}$ and some relations in the polynomial algebra. The action of $E_{i,j}$ and $F_{i,j}$ for $j=0$ are already given by Brundan [13] using Schur-Weyl duality, Theorem 3 generalizes it for a current algebra parameter j in the context of 2-representation theory.

Current algebra action on the cohomology of flag varieties

Brundan [13] defines a finite-dimensional quotient C_ν^λ of the algebra P_ν for a given n-composition ν and n*-partition* λ of N. We will show that the current algebra action defined in Theorem 3 descends to the quotient ring $\bigoplus_\nu C_\nu^\lambda$, following a similar proof for $\mathfrak{sl}_n$ action given by Brundan [13].

The rings C_ν^λ appear naturally in geometry and 2-representation theory. Brundan and Ostrik [17] prove that C_ν^λ is isomorphic to the cohomology ring of a Spaltenstein variety – a quiver variety of type A. Mackaay [45] shows that $\bigoplus_\nu C_\nu^\lambda$ is isomorphic to the center of the $\mathfrak{sl}_m$ web algebra and to the center of the foam category. Brundan [13] proves an algebra isomorphism between $\bigoplus_\nu C_\nu^\lambda$ and the center of parabolic category $\mathcal{O}^\lambda$. Varagnolo-Vasserot-Shan [56] and Webster [60] prove a graded algebra isomorphism between the center of the cyclotomic KLR algebra $R^{\bar\lambda}$ and $\bigoplus_\nu C_\nu^\lambda$.

The graded character and dimension formulas for $\bigoplus_\nu C_\nu^\lambda$ are given by Brundan [13]. This result allows us to compute graded characters and dimensions of the local Weyl modules in terms of the *Kostka-Foulkes polynomials* (see [43]). The Kostka-Foulkes polynomials $K_{\lambda,\nu}(t)$, indexed by a partition λ and a composition ν play major role in the representation theory of affine Lie algebras and symmetric groups.

Theorem 4. *Let* λ *be a* n*-partition of a positive integer* N. *Let* $\bar\lambda = \sum_{i=1}^{n-1}(\lambda_i-\lambda_{i+1})\omega_i$, *where* ω_i, $i \in I$ *are fundamental* $\mathfrak{sl}_n$ *weights. The graded character of the local Weyl module*

$W(\bar{\lambda})$ *is given by the formula*

$$ch_t\, W(\bar{\lambda}) = \sum_{\tau} K_{\tau^T,\lambda^T}(t)\; ch\, V(\bar{\tau}),$$

and the graded dimension of the weight space $\bar{\nu}$ *is given by*

$$\sum_{r\geq 0} \dim_{\mathbb{C}}\, W_{\bar{\nu}}(\bar{\lambda})\{2r\}\, t^r = \sum_{\tau} K_{\tau,\nu}(1)\, K_{\tau^T,\lambda^T}(t),$$

where the summations on the right hand sides are over all n*-compositions of* N*,* $V(\bar{\tau})$ *is the finite-dimensional, simple* $\mathfrak{sl}_n$ *module with the highest weight* $\bar{\tau} = \sum_{i=1}^{n-1}(\tau_i - \tau_{i+1})\omega_i$*, and* $W_{\bar{\nu}}(\bar{\lambda})$ *is the weight space* $W_{\bar{\nu}}(\bar{\lambda}) = \{w \in W(\bar{\lambda}) \mid H_{i,0}w = (\nu_i - \nu_{i+1})w\}$*.*

The proof of Theorem 4 follows easily from the work of Shan-Varagnolo-Vasserot [56] (also see Webster [60]). They show that $\bigoplus_\nu C_\nu^\lambda$ is isomorphic to the dual Weyl module $W^*(\bar{\lambda})$. We only need to adjust the degree, and this will gives a character formula. This formula can be understood as a *character decategorification* of $W(\bar{\lambda})$. The graded dimension is obtained from the character formula.

Theorem 4 is probably already known to experts. We, however, give a new proof with a different approach and a simpler formula. Sanderson [55] computes characters of Demazure modules, and this can be combined with results of Chari-Loktev [24] to derive the characters for local Weyl modules. Chari-Ion ([23], Theorem 4.1) also give a character formula, however they use an entirely different approach, combinatorics and proof of which is beyond our understanding.

Trace of categorified quantum symmetric pairs

Bao-Wang [1] use the *coideal algebra* $\mathbf{U}^\jmath$ to develop a new Kazhdan-Lusztig theory for the category $\mathcal{O}$. The $\mathbb{Q}(q)$-algebra $\mathbf{U}^\jmath$ is isomorphic to a subalgebra of the quantum group $\mathbf{U}_q(\mathfrak{sl}_{2n+1})$, $n \geq 1$. The pair $(\mathbf{U}_q(\mathfrak{sl}_{2n+1}), \mathbf{U}^\jmath)$ forms a quantum symmetric pair in the sense of Letzter [41].

Let $\dot{\mathbf{U}}^\jmath$ be the idempotent completion of the coideal algebra $\mathbf{U}^\jmath$. The algebra $\dot{\mathbf{U}}^\jmath$ can be viewed as a category with objects indexed by a quotient weight lattice $X_\jmath$ of $\mathfrak{sl}_{2n+1}$ weight lattice, and morphisms generated by 1_μ, $1_{\mu+\alpha_i}\mathrm{E}_i 1_\mu$, $1_{\mu-\alpha_i}\mathrm{F}_i 1_\mu$ for $i \in \mathbb{I}^\jmath = \{\diamond, \diamond + 1, \ldots, r - \diamond\}$, $\diamond = \frac{1}{2}$, $\mu \in X_\jmath$. For all $i \neq \diamond$, the generators E_i, F_i satisfy the same relations as standard generators of the usual idempotented quantum group and special relations for $i = \diamond$.

Bao-Shan-Wang-Webster [2] construct a graded, additive 2-category $\dot{\mathfrak{U}}^\jmath$ which categorifies $\dot{\mathbf{U}}^\jmath$. Namely, the split Grothendieck group of the Karoubi envelope of $\dot{\mathfrak{U}}^\jmath$ is isomorphic to the integral form of $\dot{\mathbf{U}}^\jmath$. The 2-morphisms in $\mathfrak{U}^\jmath$ are also generated by planar diagrams as in $\mathcal{U}$, but there is a special set of relations between $\diamond$-labeled strands.

We define a new $\Bbbk$-algebra $\mathbf{U}_t^\jmath$, called *current coideal algebra*, in terms of generators and relations similar to those of current algebra. In fact, $\mathbf{U}_t^\jmath$ is isomorphic to a certain subalgebra of $\mathbf{U}(\mathfrak{sl}_{2n+1}[t])$. Let $\mathrm{Tr}\,\mathcal{U}^\jmath$ be the trace of the graded version $\mathcal{U}^\jmath$ of the 2-category $\mathfrak{U}^\jmath$. We have the following result.

Theorem 5. *There is a graded algebra homomorphism between* $\mathbf{U}_t^\jmath$ *and* $\mathrm{Tr}\,\mathcal{U}^\jmath$*.*

We prove Theorem 5 by checking the relations in current coideal algebra for the generators of $\mathrm{Tr}\,\mathcal{U}^\jmath$. We only need to check the relation between $\diamond$-labeled generators, and the rest of the proof follows from Theorem 1.

We are also able to obtain the trace decategorification of Bao-Shan-Wang-Webster 2-category $\mathfrak{U}^\jmath$. The proof uses a similar argument as the trace decategorification of $\mathcal{U}_Q(\mathfrak{sl}_n)$ by Beliakova-Habiro-Lauda-Webster [5], Theorem 8.1.

Theorem 6. *There is a an algebra isomorphism between* $\dot{\mathbf{U}}^\jmath \cong \mathrm{Tr}\,\mathfrak{U}^\jmath$*.*

Organization of thesis

The thesis is organized as follows. In the first chapter we construct a homomorphism between the current algebras and the trace of categorified quantum groups. We start by giving the relevant definitions concerning traces and 2-categories. We define the Khovanov-Lauda 2-category using the diagrammatic presentation. We then construct a homomorphism between the current algebras and the trace of categorified quantum groups. This part is based on a joint work [8] with A. Beliakova, K. Habiro and A. Lauda.

In second chapter, we concern ourselves with 2-representations of categorified quantum $\mathfrak{sl}_n$ and current algebra modules. The most part of this chapter is devoted to the polynomial construction of the 2-category $\mathbf{Pol}_N$. We then explicitly describe the current algebra action on the center of objects of $\mathbf{Pol}_N$ and use it to derive the character formula of local Weyl modules.

The third chapter is the computation of the trace of categorified coideal algebra $\mathcal{U}^\jmath$. We prove that $\mathrm{Tr}\mathcal{U}^\jmath$ homomorphic to an extension of the coideal algebra. This chapter mainly consists of diagrammatic computations in $\mathcal{U}^\jmath$.

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Chapter I

Trace of categorified quantum groups

This chapter is divided into two sections. The first section introduces the notion of trace for algebras, categories and 2-categories. The material given here is well-known (see, *e.g.*, [8]). We present Khovanov-Lauda's diagrammatic definition of categorified quantum $\mathfrak{sl}_n$ in the second section. The remaining part of the chapter is devoted to our first main result.

1 Trace functor

1.1 Trace of a ring

Let $\Bbbk$ be a field of zero characteristics. The trace of a square matrix with elements in $\Bbbk$ is the sum of its diagonal entries. Given two square matrices A, B, the trace satisfies the property

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)\,. \tag{1.1}$$

More generally, for any finite-dimensional $\Bbbk$-vector space V, a trace is a linear map

$$\mathrm{tr}\colon \mathrm{End}_{\Bbbk}(V) \to \Bbbk$$

such that the *trace relation* holds:

$$\mathrm{tr}(fg) = \mathrm{tr}(gf) \quad \text{for any} \quad f, g \in \mathrm{End}_{\Bbbk}(V)\,.$$

As a simple invariant of a linear endomorphism, the trace has many important applications. For example, given a representation $\varphi\colon G \to \mathrm{Aut}_{\Bbbk}(V)$ of a group G, the character $\chi_\varphi\colon G \to \Bbbk$ of the representation φ is the function sending each group element $g \in G$ to the trace of $\varphi(g)$. The character of a representation carries much of the essential information about a representation.

We now generalize the notion of the trace from the ring of endomorphisms of V to any $\Bbbk$-algebra. For a $\Bbbk$-algebra A, the *Hochschild homology* $\mathrm{HH}_*(A)$ of A can be defined as the homology of the Hochschild chain complex

$$C_\bullet = C_\bullet(A) \colon \qquad \dots \longrightarrow C_n(A) \xrightarrow{d_n} C_{n-1}(A) \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} C_1(A) \xrightarrow{d_1} A \longrightarrow 0, \tag{1.2}$$

where $C_n(A) = A^{\otimes n+1}$ and

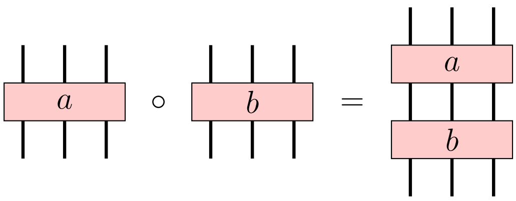
$$d_n(a_0 \otimes \dots \otimes a_n) := \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \dots \otimes a_{n-1}$$

for $a_0, \dots, a_{n-1} \in A$. We define *the trace* or *the cocenter* of A to be the zeroth Hochschild homology $\mathrm{HH}_0(A)$ of A.

Definition 1.1.1. *The trace of a $\Bbbk$-algebra A defined as*

$$\operatorname{Tr} A = \mathrm{HH}_0(A) = A/[A,A] = A/\,\mathit{Span}_{\Bbbk}\{ab - ba \mid a, b \in A\}.$$

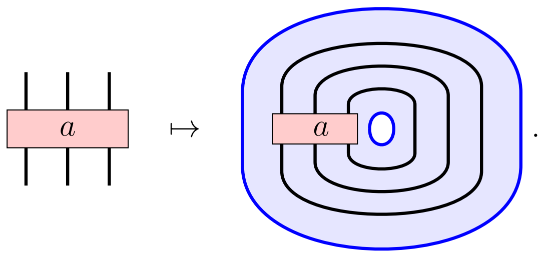
Many interesting algebras can be presented by diagrams in the plane modulo some local relations. Multiplication in A is represented by stacking diagrams on top of each other:



for $a, b \in A$. Then the trace $\operatorname{Tr} A$ can be described by diagrams on the annulus, modulo the same local relations as those for A and the map

$$A \longrightarrow A/[A,A]$$

has a diagrammatic description



Notice that by sliding elements of A around the annulus we see that $ab = ba$ in the quotient $\mathrm{HH}_0(A)$.

The Hochschild cochain complex is defined as

$$C^\bullet = C^\bullet(A)\colon \quad 0 \to A \xrightarrow{\delta_0} C^0(A) \xrightarrow{\delta_1} C^1(A) \xrightarrow{\delta_2} \dots \xrightarrow{\delta_{n-1}} C^{n-1}(A) \xrightarrow{\delta_n} C^n(A) \longrightarrow \dots,$$

where $C^n(A) = \mathrm{Hom}_{\mathbf{k}}(A^{\otimes n+1}, A)$ and

$$\delta_n(f)(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i f(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) + (-1)^n f(a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}),$$

$$\delta_0(a_0)(a_1) = a_0 a_1 - a_1 a_0$$

for $a_0, \dots, a_{n-1} \in A$, $f \in \mathrm{Hom}_{\mathbf{k}}(A^{\otimes n}, A)$. The *Hochschild cohomology* is

$$HH^n(A) = \ker \delta_n / \mathrm{im}\, \delta_{n-1}$$

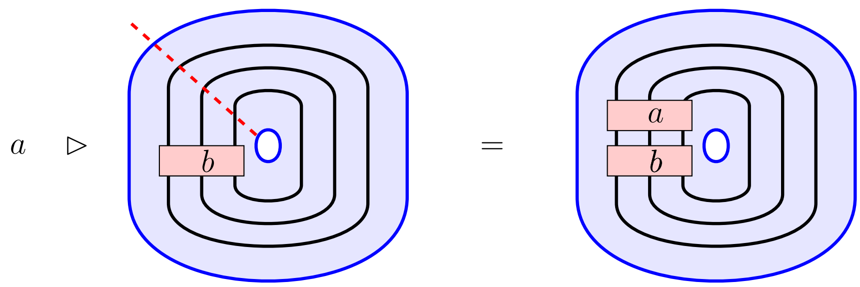
for $n \geq 0$. In particular,

$$HH^0(A) = \ker \delta_0 = \{a \in A \,|\, \delta_0(a) = 0\} = \{a \in A \,|\, a_0 a - a a_0 = 0 \quad \text{for all} \quad a_0 \in A\}.$$

The zeroth Hochschild cohomology $\mathrm{HH}^0(A)$ of a ring A can be identified with the center of A:

$$\mathrm{HH}^0(A) \cong Z(A).$$

The center $Z(A)$ acts naturally on the trace $\mathrm{Tr}\, A$. This action can be graphically understood by cutting the diagram of an element $b \in \mathrm{Tr}\, A$ and inserting the diagram of $a \in Z(A)$:



1.2 Trace of a linear category

A *small category* $\mathcal{C}$ consists of a set of objects $\mathrm{Ob}(\mathcal{C})$ and a set of morphisms $\mathcal{C}(x, y)$ for each pair $x, y \in \mathrm{Ob}(\mathcal{C})$. We assume that categories are small for the rest of this section. A $\Bbbk$-*linear category* is a category in which hom-sets are $\Bbbk$-linear vector spaces, and compositions of morphisms are bilinear over $\Bbbk$. A $\Bbbk$-algebra can be understood as a $\Bbbk$-linear category with one object. We generalize the notion of trace to $\Bbbk$-linear categories.

Definition 1.2.1. *The trace* $\mathrm{Tr}\,\mathcal{C}$ *of the category* $\mathcal{C}$ *is defined as*

$$\mathrm{Tr}\,\mathcal{C} = \left(\bigoplus_{x\in\mathrm{Ob}(\mathcal{C})} \mathrm{End}_{\mathcal{C}}(x)\right) / \,\mathrm{Span}_{\Bbbk}\{f\circ g - g\circ f\}, \tag{1.3}$$

where f *and* g *run through all pairs of morphisms* $f\colon x\to y$, $g\colon y\to x$ *and* $x, y\in Ob(\mathcal{C})$.

We denote the trace class of a morphism f by $[f]$. A $\Bbbk$-*linear functor* between two $\Bbbk$-linear categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $F\colon \mathcal{C}\to\mathcal{D}$ such that the map $F\colon \mathcal{C}(x, y)\to \mathcal{D}(F(x), F(y))$ is an abelian group homomorphism for $x, y\in\mathrm{Ob}(\mathcal{C})$. The trace Tr is a $\Bbbk$-linear functor.

An *additive* category is a $\Bbbk$-linear category equipped with a zero object and biproducts, also called direct sums.

Lemma 1.2.2. *If* $\mathcal{C}$ *is an additive category, then for* $f\colon x\to x$ *and* $g\colon y\to y$ *we have*

$$[f\oplus g] = [f] + [g]$$

in $\mathrm{Tr}\,\mathcal{C}$.

Proof. Since $f\oplus g = (f\oplus 0) + (0\oplus g)\colon x\oplus y\to x\oplus y$, we have

$$[f\oplus g] = [f\oplus 0] + [0\oplus g].$$

Now we have $[f\oplus 0] = [ifp] = [pif] = [f]$, where $p\colon x\oplus y\to x$ and $i\colon x\to x\oplus y$ are the projection and the inclusion maps. Similarly, $[0\oplus g] = [g]$ holds. □

Example 1.2.3. *Let* $\mathcal{C} = \mathbf{Vect}_{\Bbbk}$ *be the category of finite-dimensional* $\Bbbk$*-vector spaces. Since any finite-dimensional vector space is isomorphic to a finite direct sum of copies of* $\Bbbk$*, by Lemma 1.2.2 we have* $\mathrm{Tr}\,\mathbf{Vect}_{\Bbbk} = \Bbbk$.

Example 1.2.4. *Let* $\mathcal{C} = \mathbf{grVect}_{\Bbbk}$ *be the category of finite-dimensional* $\mathbb{Z}$*-graded vector spaces, i.e. any* $V\in\mathrm{Ob}(\mathcal{C})$ *decomposes as* $V = \bigoplus_{n\in\mathbb{Z}} V_n$ *with* $\deg(x) = n$ *for any* $x\in V_n$, *and morphisms in* $\mathbf{grVect}_{\Bbbk}$ *are degree preserving. Then* $\operatorname{Tr}\mathbf{grVect}_{\Bbbk} = \bigoplus_{n\in\mathbb{Z}}\Bbbk = \Bbbk[q,q^{-1}]$. *The multiplication by* q *is interpreted as a shift of the degree by one.*

The previous examples can be further generalized. For a linear category $\mathcal{C}$ there is a universal additive category generated by $\mathcal{C}$, called the *additive closure* $\mathcal{C}^{\oplus}$, in which the objects are formal finite direct sums of objects in $\mathcal{C}$, and the morphisms are matrices of morphisms in $\mathcal{C}$. There is a canonical fully faithful functor $\mathcal{C} \to \mathcal{C}^{\oplus}$. Every linear functor $F\colon \mathcal{C} \to \mathcal{D}$ to an additive category $\mathcal{D}$ factors through $\mathcal{C} \to \mathcal{C}^{\oplus}$ uniquely up to natural isomorphism. We consider the trace of the additive closure $\mathcal{C}^{\oplus}$ of a linear category $\mathcal{C}$. The homomorphism

$$\operatorname{Tr} i\colon\ \operatorname{Tr}\mathcal{C} \to \operatorname{Tr}\mathcal{C}^{\oplus}$$

induced by the canonical functor $i\colon \mathcal{C} \to \mathcal{C}^{\oplus}$ is an isomorphism. The inverse $\operatorname{tr}\colon\ \operatorname{Tr}\mathcal{C}^{\oplus} \to \operatorname{Tr}\mathcal{C}$ is defined by

$$\operatorname{tr}[(f_{k,l})_{k,l}] = \sum_k [f_{k,k}]$$

for an endomorphism in $\mathcal{C}^{\oplus}$

$$(f_{k,l})_{k,l\in\{1,\dots,n\}}\colon x_1 \oplus \cdots \oplus x_n \to x_1 \oplus \cdots \oplus x_n$$

with $f_{k,l}\colon x_l \to x_k$ in $\mathcal{C}$.

1.3 Idempotent completions and split Grothendieck group

A *projection* or *idempotent* in $\mathbf{Vect}_{\Bbbk}$ is an endomorphism $p\colon V \to V$ satisfying the relation $p^2 = p$. In this case $(\mathrm{Id}_V - p)\colon V \to V$ is also an idempotent, and together these two idempotents decompose the space V into a direct sum of the image of the projection p and the image of the projection $\mathrm{Id}_V - p$.

More generally, an idempotent in a $\Bbbk$-linear category $\mathcal{C}$ is an endomorphism $e\colon x \to x$ satisfying $e^2 = e$. An idempotent $e\colon x \to x$ in $\mathcal{C}$ is said to *split* if there is an object y and morphisms $g\colon x \to y$, $h\colon y \to x$ such that $hg = e$ and $gh = 1_y$. The Karoubi envelope $\mathrm{Kar}(\mathcal{C})$ is the category whose objects are pairs (x, e) of objects $x \in \mathrm{Ob}(\mathcal{C})$ and an idempotent endomorphism $e\colon x \to x$ in $\mathcal{C}$. The morphisms

$$f\colon (x,e) \to (y,e')$$

are morphisms $f\colon x \to y$ in $\mathcal{C}$ such that $f = e'fe$. Composition is induced by the composition in $\mathcal{C}$, and the identity morphism is $e\colon (x,e) \to (x,e)$. $\mathrm{Kar}(\mathcal{C})$ is equipped with a $\Bbbk$-linear category structure. The Karoubi envelope can be thought of as a minimal enlargement of the category $\mathcal{C}$ in which all idempotents split.

We can identify $(x, 1_x)$ with the object x of $\mathcal{C}$. This identification gives rise to a natural embedding functor $\iota\colon \mathcal{C} \to \mathrm{Kar}(\mathcal{C})$ such that $\iota(x) = (x, 1_x)$ for $x \in \mathrm{Ob}(\mathcal{C})$ and $\iota(f\colon x \to y) = f$. The Karoubi envelope $\mathrm{Kar}(\mathcal{C})$ has the universality property that if $F\colon \mathcal{C} \to \mathcal{D}$ is a $\Bbbk$-linear functor to a $\Bbbk$-linear category $\mathcal{D}$ with split idempotents, then F extends to a functor from $\mathrm{Kar}(\mathcal{C})$ to $\mathcal{D}$ uniquely up to natural isomorphism. The following proposition illustrates one of the key advantages of the trace, namely its invariance under passing to the Karoubi envelope.

Proposition 1.3.1. *The map* $\mathrm{Tr}\,\iota\colon \mathrm{Tr}\,\mathcal{C} \longrightarrow \mathrm{Tr}\,\mathrm{Kar}(\mathcal{C})$ *induced by* ι *is bijective.*

Proof. Recall that an endomorphism $f\colon (x, e) \to (x, e)$ in $\mathrm{Kar}(\mathcal{C})$ is just a morphism $f\colon x \to x$ in $\mathcal{C}$ satisfying the condition that $f = efe$. Define a map $u\colon \mathrm{Tr}\,\mathrm{Kar}(\mathcal{C}) \longrightarrow \mathrm{Tr}\,\mathcal{C}$ sending $[f] \in \mathrm{Tr}\,\mathrm{Kar}(\mathcal{C})$ to $[f] \in \mathrm{Tr}\,\mathcal{C}$. Then one can check that u is an inverse to $\mathrm{Tr}\,\iota$. □

The split Grothendieck group $K_0(\mathcal{C})$ of an additive category $\mathcal{C}$ is an abelian group generated by isomorphism classes $[x]_{\simeq}$ of objects $x \in \mathrm{Ob}(\mathcal{C})$ modulo the relation $[x \oplus y]_{\simeq} = [x]_{\simeq} + [y]_{\simeq}$. The split Grothendieck group K_0 defines a functor from additive categories to abelian groups.

Example 1.3.2. $K_0(\mathbf{Vect}_{\Bbbk}) = \mathbb{Z}$ *is generated by* $[\Bbbk]_{\cong}$.

Define a homomorphism

$$h_{\mathcal{C}}\colon K_0(\mathcal{C}) \longrightarrow \mathrm{Tr}\,\mathcal{C}$$

by

$$h_{\mathcal{C}}([x]_{\cong}) = [1_x]$$

for $x \in \mathrm{Ob}(\mathcal{C})$. Indeed, one can easily check that

$$h_{\mathcal{C}}([x \oplus y]_{\cong}) = [1_x] + [1_y]\,,$$

since $1_{f \oplus g} = 1_f \oplus 1_g$. The map $h_{\mathcal{C}}$ defines a natural transformation

$$h\colon K_0 \Rightarrow \mathrm{Tr}\colon \mathbf{AdCat} \to \mathbf{Ab},$$

where **AdCat** denotes the category of additive small categories. The map $h_{\mathcal{C}}$ is called *Chern character map*. It is neither injective nor surjective in general.

A *graded* $\Bbbk$-*linear category* is a $\Bbbk$-linear category equipped with an auto-equivalence $\langle 1\rangle\colon \mathcal{C} \to \mathcal{C}$. In a graded $\Bbbk$-linear category, for objects $x, y \in \mathrm{Ob}(\mathcal{C})$ and a morphism $f\colon x \to y$, there exist objects $x\langle t\rangle, y\langle t\rangle \in \mathrm{Ob}(\mathcal{C})$ and a morphism $f\langle t\rangle\colon x\langle t\rangle \to y\langle t\rangle$ for each $t \in \mathbb{Z}$, where $\langle t\rangle$ is $\langle 1\rangle$ applied $t \in \mathbb{Z}$ times. The trace and the split Grothendieck groups of a graded linear category $\mathcal{C}$ are $\mathbb{Z}[q, q^{-1}]$-modules such that $[x\langle t\rangle]_{\simeq} = q^t[x]_{\simeq}$, and $[f\langle t\rangle] = q^t[f]$ for $x \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{End}(x)$.

A *translation* in $\mathcal{C}$ is a family of natural isomorphisms $x \simeq x\langle t\rangle$ for all $x \in \mathrm{Ob}(\mathcal{C})$ and $t \in \mathbb{Z}$. Given a graded $\Bbbk$-linear category $\mathcal{C}$, we can form a category $\mathcal{C}^\star$ with the same objects as $\mathcal{C}$ and morphisms

$$\mathcal{C}^\star(x, y) = \bigoplus_{t\in\mathbb{Z}} \mathcal{C}(x, y\langle t\rangle). \tag{1.4}$$

The category $\mathcal{C}^\star$ admits a translation, since for any $x \in \mathrm{Ob}(\mathcal{C})$ and $t \in \mathbb{Z}$ the natural isomorphism $x \simeq x\langle t\rangle$ is given by $1_x \in \mathcal{C}(x, x) = \mathcal{C}(x, x\langle t\rangle\langle -t\rangle) \subset \mathcal{C}^\star(x, x\langle t\rangle)$ together with the inverse map $1_x\langle t\rangle \in \mathcal{C}(x\langle t\rangle, x\langle t\rangle) = \mathcal{C}(x\langle t\rangle, x\langle 0\rangle\langle t\rangle) \subset \mathcal{C}^\star(x\langle t\rangle, x)$.

The isomorphism $x \simeq x\langle t\rangle$ forgets the q-grading by setting $q = 1$ on the split Grothendieck group and makes $K_0(\mathcal{C}^\star)$ a $\mathbb{Z}$-module. The same is also true for $\mathrm{Tr}\,\mathcal{C}^\star$ since $[f\langle t\rangle] = [f]$ for any endomorphism $f\colon x \to x$.

However, hom-spaces of $\mathcal{C}^\star$ contain degree t morphisms $\mathcal{C}^\star(x, y\langle t\rangle)$ for all $t \in \mathbb{Z}$. Hence $\mathrm{Tr}\,\mathcal{C}^\star$ decomposes into equivalence classes of morphisms of degree t:

$$\mathrm{Tr}\,\mathcal{C}^\star(x, y) = \bigoplus_{t\in\mathbb{Z}} \mathrm{Tr}\,\mathcal{C}(x, y\langle t\rangle). \tag{1.5}$$

Thus, $\mathrm{Tr}\,\mathcal{C}^\star(x, y)$ is a $\mathbb{Z}$-graded abelian group.

1.4 Pivotal categories

Definition 1.4.1. *A (strict) monoidal category (or tensor category) is a category $\mathcal{C}$ equipped with a bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called tensor product, such that*

- *the tensor product is associative: for any $\alpha, \beta, \gamma \in \mathcal{C}$ we have*

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma),$$

- *there exists an object $\mathbf{1}$, called the unit object, such that for each $A \in \mathrm{Ob}(\mathcal{C})$,*

$$A = A \otimes \mathbf{1} = \mathbf{1} \otimes A.$$

For example, the category $\mathbf{Vect}_{\Bbbk}$ is monoidal with the usual tensor product of vector spaces and the unit object $\Bbbk$. *A pivotal category* is a monoidal category such that for each object $A \in \mathcal{C}$ there is an associated dual object $A^* \in \mathcal{C}$, and there are four morphisms

$$\mathrm{ev}_A \colon A^* \otimes A \to \mathbf{1}, \quad \mathrm{coev}_A \colon \mathbf{1} \to A^* \otimes A,$$

$$\widetilde{\mathrm{ev}}_A \colon A \otimes A^* \to \mathbf{1}, \quad \widetilde{\mathrm{coev}}_A \colon \mathbf{1} \to A \otimes A^*,$$

such that the following biadjointness relations hold:

$$(\widetilde{\mathrm{ev}}_A \otimes 1_A)\,(1_A \otimes \widetilde{\mathrm{coev}}_A) = (1_A \otimes \mathrm{ev}_A)\,(\mathrm{coev}_A \otimes 1_A) = 1_A,$$

$$(\mathrm{ev}_A \otimes 1_{A^*})\,(1_{A^*} \otimes \mathrm{coev}_A) = (1_{A^*} \otimes \widetilde{\mathrm{ev}}_A)\,(\widetilde{\mathrm{coev}}_A \otimes 1_{A^*}) = 1_{A^*},$$

where 1_A is the identity morphism of A. For a morphism $f \colon A \to B$ in a pivotal category $\mathcal{C}$, the morphisms

$$f^* = (\mathrm{ev}_B \otimes 1_{A^*})\,(1_B \otimes f \otimes 1_{A^*})\,(1_{B^*} \otimes \mathrm{coev}_A) : B^* \to A^*,$$

$${}^*f = (1_{A^*} \otimes \widetilde{\mathrm{ev}}_B)\,(1_{A^*} \otimes f \otimes 1_{B^*})\,(\widetilde{\mathrm{coev}}_A \otimes 1_{B^*}) : B^* \to A^*$$

are called left and right duals of f, and the monoidal natural transformation

$$\varphi = \{\varphi_A = (\widetilde{\mathrm{ev}_A} \otimes 1_{A^{**}})\,(1_A \otimes \mathrm{coev}_{A^*}) : A \to A^{**}\}_{A \in \mathrm{Ob}(\mathcal{C})}$$

is called the *pivotal structure*. In what follows, we shall see that a pivotal category and its trace admit well-defined diagrammatic descriptions when the left and right duals coincide.

1.5 2-categories and trace

Definition 1.5.1. *A 2-category $\mathcal{C}$ consists of the following data.*

- *A collection of objects.*
- *For each pair of objects x, y, a category $\mathcal{C}(x, y)$. The objects of $\mathcal{C}(x, y)$ are called 1-morphisms from x to y. For any $u, u' \in Ob(\mathcal{C}(x, y))$, morphisms $f \colon u \to u'$ are called 2-morphisms.*
- *There is a composition of 1-morphisms: if $u \in Ob(\mathcal{C}(x, y))$ and $v \in Ob(\mathcal{C}(y, z))$, then $vu \in Ob(\mathcal{C}(x, z))$. This composition is associative: for any object ζ and $w \in Ob(\mathcal{C}(z, \zeta))$ we have*
$$(wv)u = w(vu).$$

- *There is a horizontal composition of 2-morphisms: if* $u, u' \in Ob(\mathcal{C}(x,y))$*,* $f\colon u \to u'$ *and* $v, v' \in Ob(\mathcal{C}(y,z))$*,* $g\colon v \to v'$*, then there exists a 2-morphism* $g * f\colon vu \to v'u'$*. The horizontal composition is associative.*

- *2-morphisms can also be composed vertically: if* $u, u', u'' \in Ob(\mathcal{C}(x,y))$*,* $f\colon u \to u'$ *and,* $f'\colon u' \to u''$*, then there is a 2-morphism* $f'f\colon u \to u''$*. This composition is associative. Moreover, let* $v, v', v'' \in Ob(\mathcal{C}(y,z))$*,* $g\colon v \to v'$ *and,* $g'\colon v' \to v''$*. The following rule holds between horizontal and vertical compositions of 2-morphisms:*

$$(g'g) * (f'f) = (g' * f')(g * f).$$

- *For every object* x *there is an identity 1-morphism* $1_x \in Ob(\mathcal{C}(x,x))$ *such that for any* $u \in Ob(\mathcal{C}(x,y))$*, we have* $1_y u = u 1_x = u$*.*

- *For every 1-morphism* $u \in Ob(\mathcal{C}(x,y))$ *there exists an identity 2-morphism* 1_u *such that for all* $u' \in Ob(\mathcal{C}(x,y))$ *and* $f\colon u \to u'$*, we have* $1_{u'} f = f 1_u = f$ *and* $1_{1_y} * f = f * 1_{1_x} = f$*. Additionally, the identity 2-morphisms must be compatible with composition of 1-morphisms:* $1_v * 1_u = 1_{vu}$ *for any* $v \in Ob(\mathcal{C}(y,z))$*.*

In the literature, this definition usually belongs to a strict 2-category in order to distinguish it from a weaker one, called bicategory. In this thesis, we will only discuss strict 2-categories. For example, a monoidal category $\mathcal{C}$ is a 2-category with one object $*$ and $\mathrm{End}(*) = \mathcal{C}$. **Cat**, the category of small categories, is the 2-category whose objects are (small) categories, morphisms are functors, and 2-morphisms are natural transformations between functors.

Definition 1.5.2. *A 2-functor* $F\colon \mathcal{C} \to \mathcal{D}$ *from a 2-category* $\mathcal{C}$ *to a 2-category* $\mathcal{D}$ *consists of*

- *a function* $F\colon Ob(\mathcal{C}) \to Ob(\mathcal{D})$*, and*

- *for each pair of objects* x, y *in* $\mathcal{C}$*, a functor* $F\colon \mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy)$*.*

1.6 Kar, K_0 and Tr for 2-categories

A 2-category $\mathcal{C}$ is $\Bbbk$-linear if the categories $\mathcal{C}(x,y)$ are $\Bbbk$-linear for all objects x, y in $\mathcal{C}$, and the composition functor preserves the $\Bbbk$-linear structure. Similarly, an additive $\Bbbk$-linear 2-category is a $\Bbbk$-linear 2-category in which the categories $\mathcal{C}(x,y)$ are additive.

A *graded linear 2-category* is a 2-category whose hom-spaces form graded linear categories, and the composition map preserves degree.

The following definitions extend notions of trace, split Grothendieck group and Karoubi envelope to the 2-categorical setting.

- Given an additive $\Bbbk$-linear 2-category $\mathcal{C}$, define the split Grothendieck group $K_0(\mathcal{C})$ of $\mathcal{C}$ to be the $\Bbbk$-linear category with $\mathrm{Ob}(K_0(\mathcal{C})) = \mathrm{Ob}(\mathcal{C})$ and with $K_0(\mathcal{C})(x,y) := K_0(\mathcal{C}(x,y))$ for any two objects $x, y \in \mathrm{Ob}(\mathcal{C})$. For $[f]_{\cong} \in K_0(\mathcal{C})(x,y)$ and $[g]_{\cong} \in K_0(\mathcal{C})(y,z)$ the composition in $K_0(\mathcal{C})$ is defined by $[g]_{\cong} \circ [f]_{\cong} := [g \circ f]_{\cong}$.
- The trace $\mathrm{Tr}\,\mathcal{C}$ of a $\Bbbk$-linear 2-category is the $\Bbbk$-linear category with $\mathrm{Ob}(\mathrm{Tr}\,\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$ and with $\mathrm{Tr}\,\mathcal{C}(x,y) := \mathrm{Tr}\,\mathcal{C}(x,y)$ for any two objects $x, y \in \mathrm{Ob}(\mathcal{C})$. For a 2-endomorphism σ in $\mathcal{C}(x,y)$ and a 2-endomorphism τ in $\mathcal{C}(y,z)$, we have $[\tau] \circ [\sigma] = [\tau \circ \sigma]$. The identity morphism for $x \in \mathrm{Ob}(\mathrm{Tr}\,\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$ is given by $[1_{1_x}]$.
- The Karoubi envelope $\mathrm{Kar}(\mathcal{C})$ of a $\Bbbk$-linear 2-category $\mathcal{C}$ is the $\Bbbk$-linear 2-category with $\mathrm{Ob}(\mathrm{Kar}(\mathcal{C})) = \mathrm{Ob}(\mathcal{C})$ and with hom-categories $\mathrm{Kar}(\mathcal{C})(x,y) := \mathrm{Kar}(\mathcal{C}(x,y))$. The composition functor $\mathrm{Kar}(\mathcal{C})(y,z) \times \mathrm{Kar}(\mathcal{C})(x,y) \to \mathrm{Kar}(\mathcal{C})(x,z)$ is induced by the universal property of the Karoubi envelope from the composition functor in $\mathcal{C}$. The fully-faithful additive functors $\mathcal{C}(x,y) \to \mathrm{Kar}(\mathcal{C}(x,y))$ form an additive 2-functor $\mathcal{C} \to \mathrm{Kar}(\mathcal{C})$ that is universal with respect to splitting idempotents in the hom-categories $\mathcal{C}(x,y)$.

A 2-functor $F\colon \mathcal{C} \to \mathcal{D}$ between $\Bbbk$-linear 2-categories $\mathcal{C}$ and $\mathcal{D}$ is a $\Bbbk$-linear 2-functor if for objects x, y in $\mathcal{C}$ the functor $F\colon \mathcal{C}(x,y) \to \mathcal{D}(x,y)$ is $\Bbbk$-linear. In this case F induces a $\Bbbk$-linear functor

$$\mathrm{Tr}\,F\colon\ \mathrm{Tr}\,\mathcal{C} \to \mathrm{Tr}\,\mathcal{D},$$

such that

$$\mathrm{Tr}\,F = F\colon \mathrm{Ob}(\mathrm{Tr}\,\mathcal{C}) \to \mathrm{Ob}(\mathrm{Tr}\,\mathcal{D})$$

on the objects, and for $x, y \in \mathrm{Ob}(\mathcal{C})$

$$(\mathrm{Tr}\,F)_{x,y} = \mathrm{Tr}(F_{x,y})\colon\ \mathrm{Tr}\,\mathcal{C}(x,y) \to \mathrm{Tr}\,\mathcal{D}(F(x), F(y)).$$

Let $\mathcal{C}$ be a $\Bbbk$-linear 2-category. The *center* $Z(x)$ of an object $x \in \mathrm{Ob}(\mathcal{C})$ is the commutative ring of endomorphisms $\mathcal{C}(1_x, 1_x)$. We call $Z(\mathcal{C}) = \bigoplus_{x \in \mathrm{Ob}(\mathcal{C})} Z(x)$ the *center of objects* of $\mathcal{C}$.

We call $\mathcal{C}$ *a pivotal 2-category* if for every 1-morphism $u\colon x \to y$ there exists a biadjoint morphism $u^*\colon y \to x$, together with 2-morphisms

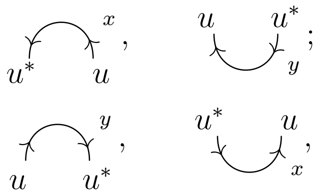
$$\mathrm{ev}_u\colon u^*u \to 1_x, \quad \mathrm{coev}_u\colon 1_y \to uu^*, \tag{1.6}$$
$$\widetilde{\mathrm{ev}}_u\colon uu^* \to 1_y, \quad \widetilde{\mathrm{coev}}_u\colon 1_x \to u^*u \tag{1.7}$$

such that biadjointness relations

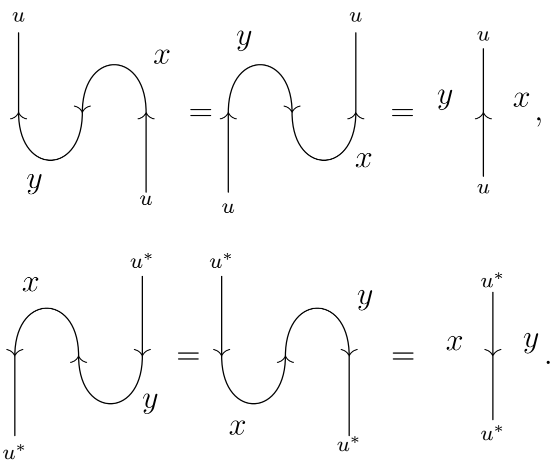
$$(\widetilde{\mathrm{ev}}_u \otimes 1_u)\,(1_u \otimes \widetilde{\mathrm{coev}}_u) = (1_u \otimes \mathrm{ev}_u)\,(\mathrm{coev}_u \otimes 1_u) = 1_u, \tag{1.8}$$

$$(\mathrm{ev}_u \otimes 1_{u^*})\,(1_{u^*} \otimes \mathrm{coev}_u) = (1_{u^*} \otimes \widetilde{\mathrm{ev}}_u)\,(\widetilde{\mathrm{coev}}_u \otimes 1_{u^*}) = 1_{u^*} \tag{1.9}$$

hold. Pivotal 2-categories admit rich a diagrammatic presentation in the following way. We denote the objects by areas labeled with the objects. Then 1-morphisms are described by the vertices of the diagrams separating two regions, and 2-morphisms are the edges of the diagrams. The horizontal composition of two diagrams $A \circ B$ places A to the right of B in the plane. The vertical composition AB of two diagrams means the stacking of A on top of B. The 2-morphisms (1.6) and (1.7) can be depicted as



and the relations (1.8) and (1.9) are



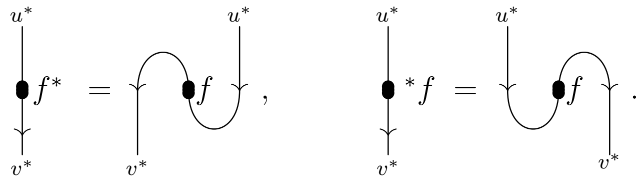
In a pivotal 2-category $\mathcal{C}$ we can define the left and right duals of a given 2-morphism $f \colon u \to v$ as follows:

$$f^* := (\mathrm{ev}_v \otimes 1_{u^*})\,(1_v \otimes f \otimes 1_{u^*})\,(1_{v^*} \otimes \mathrm{coev}_u) : v^* \to u^*,$$

$${}^*f := (1_{u^*} \otimes \widetilde{\mathrm{ev}}_v)\,(1_{u^*} \otimes f \otimes 1_{v^*})\,(\widetilde{\mathrm{coev}}_u \otimes 1_{v^*}) : v^* \to u^*.$$

Diagrammatically, the right and left duals of the 2-morphism

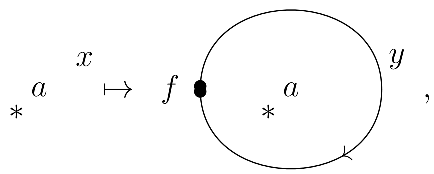
can be depicted as



A pivotal 2-category $\mathcal{C}$ is called *cyclic* if the left and right duals of all 2-morphisms are equal. In a cyclic category $\mathcal{C}$ there is a well-defined action of the trace $\operatorname{Tr}\mathcal{C}$ on the center of objects $Z(\mathcal{C})$ as follows. Let $u\colon x \to y$ be a 1-morphism, $[f]$ denote the trace class of a 2-morphism $f\colon u \to u$, and $a \in Z(x)$. Then we define the map $[f]\colon Z(x) \to Z(y)$,

$$[f](a) = \widetilde{\mathrm{ev}}_u \circ (f \otimes a \otimes 1_{u^*}) \circ \mathrm{coev}_u.$$

In terms of diagrams an element in $Z(x)$ is a closed diagram at a region labeled by x. The action of the trace corresponds to the closure of diagram f around a, and this will give us a closed diagram at region y:



(1.10)

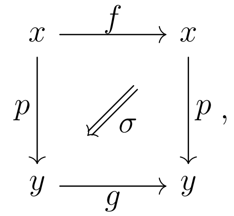
1.7 Horizontal trace of a 2-category

A more general notion of trace of a 2-category $\mathcal{C}$ is introduced in [7]. It is called the *horizontal trace*, denoted by $\operatorname{Tr}^h \mathcal{C}$.

Definition 1.7.1. *The horizontal trace* $\operatorname{Tr}^h \mathcal{C}$ *of a 2-category* $\mathcal{C}$ *is a* $\Bbbk$*-linear category in which objects are* 1*-endomorphisms* $f\colon x \to x$, $x \in \mathrm{Ob}(\mathcal{C})$. *Morphisms from* $f\colon x \to x$ *to* $g\colon y \to y$ *are equivalence classes* $[p, \sigma]$ *of pairs* (p, σ) *of a morphism* $p\colon x \to y$ *in* $\mathcal{C}$ *and a* 2*-morphism*

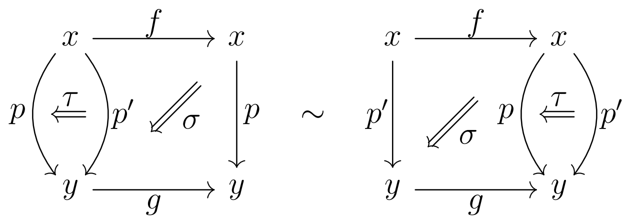
$$\sigma\colon p \circ f \Rightarrow g \circ p\colon x \to y,$$

depicted by the commuting diagram



where the equivalence relation on such pairs is generated by

$$(p, (g \circ \tau)\sigma) \sim (p', \sigma(\tau \circ f)),$$



for $p, p' \colon x \to y$, $\sigma \colon p \circ f \Rightarrow g \circ p' \colon x \to y$, $\tau \colon p' \Rightarrow p \colon x \to y$. *The composition of morphisms* $[p, \sigma] \colon (f \colon x \to x) \to (g \colon y \to y)$ *and* $[q, \tau] \colon (g \colon y \to y) \to (h \colon z \to z)$ *is defined by*

$$[q, \tau][p, \sigma] = [qp, (\tau \circ p)(q \circ \sigma)].$$

The identity morphism is given by

$$1_f = [1_x, 1_f]$$

for $f \colon x \to x$.

There is a canonical functor $\mathrm{i} \colon \operatorname{Tr} \mathcal{C} \to \operatorname{Tr}^h \mathcal{C}$ defined as follows. For an object $x \in \mathrm{Ob}(\mathcal{C})$ in $\operatorname{Tr} \mathcal{C}$, set

$$\mathrm{i}(x) = (1_x \colon x \to x).$$

For a morphism $[\sigma \colon f \Rightarrow f \colon x \to y] \colon x \to y$ in $\operatorname{Tr} \mathcal{C}$, set

$$\mathrm{i}([\sigma]) = [f, \sigma \colon f \circ 1_x \Rightarrow 1_y \circ f].$$

The functor i is full and faithful. Thus, $\operatorname{Tr}^h \mathcal{C}$ has more information about $\mathcal{C}$ than $\operatorname{Tr} \mathcal{C}$.

2 The categorified quantum $\mathfrak{sl}_n$

In this section we will set our notation for $\mathfrak{sl}_n$ weight lattice. We will then introduce the quantum $\mathfrak{sl}_n$ and its diagrammatic categorification $\mathcal{U}_Q(\mathfrak{sl}_n)$ by Khovanov and Lauda. There is an alternative categorification of the quantum $\mathfrak{sl}_n$, which is given by Rouquier [53]. Brundan [12] proves the isomorphism between Rouquier's 2-category and $\mathcal{U}_Q(\mathfrak{sl}_n)$.

2.1 Cartan datum for $\mathfrak{sl}_n$

Let $I = \{1, 2, \ldots, n-1\}$ be the set of vertices of the Dynkin diagram of type A_{n-1}, $n \geq 2$:

$$\overset{1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{2}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{3}{\circ}\!\!-\!\!\!-\!\!\!-\quad \cdots \quad -\!\!\!-\!\!\!-\!\!\overset{n-1}{\circ}. \tag{2.1}$$

Simple roots are defined as $\alpha_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0) \in \mathbb{Z}^n$, $i \in I$, where 1 occurs in the i-th component. Let $(\cdot, \cdot)$ be the standard scalar product on $\mathbb{Z}^n$. Then

$$a_{ij} = (\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{if } |i-j| > 1 \end{cases}$$

are the entries of the Cartan matrix associated to $\mathfrak{sl}_n$.

We will call the elements $\omega_i = (1, \ldots 1, 0, \ldots, 0) \in \mathbb{Z}^n_+$, $i \in I$ with i number of 1's the *fundamental weights*. Notice that we have $(\omega_i, \alpha_j) = \delta_{ij}$. Fundamental weights generate the integral $\mathfrak{sl}_n$-weight lattice X over $\mathbb{Z}$.

By $\mathcal{P}(n, N)$ we denote n-compositions of N – the set of n-tuples of non-negative integers which sum up to a positive integer N, and by $\mathcal{P}^+(n, N)$ the set of n-partitions of N, that is, elements $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{P}(n, N)$ such that $\lambda_i \geq \lambda_{i+1}$ for all $i \in I$. For each composition $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathcal{P}(n, N)$ there is a corresponding $\mathfrak{sl}_n$ weight $\bar{\nu} = \sum_{i \in I}(\nu_i - \nu_{i+1})\omega_i$. Note that $\bar{\nu}$ does not define ν uniquely unless we fix N. We will sometimes abuse the notation and write $\bar{\nu} = (\bar{\nu}_1, \bar{\nu}_2, \ldots, \bar{\nu}_{n-1}) \in \mathbb{Z}^{n-1}$, $\bar{\nu}_i = \nu_i - \nu_{i+1}$ although $\bar{\nu}$ is not a composition. If $\bar{\nu}_i \geq 0$ for all $i \in I$, we call $\bar{\nu}$ a dominant integral weight. If $\lambda \in \mathcal{P}^+(n, N)$, then $\bar{\lambda}$ is a dominant weight.

Define the composition $\nu + \alpha_i \in \mathcal{P}(n, N)$ as

$$\nu + \alpha_i = (\nu_1, \ldots, \nu_{i-1}, \nu_i + 1, \nu_{i+1} - 1, \nu_{i+2}, \ldots, \nu_n)$$

if $\nu_{i+1} > 0$ and $\varnothing$ otherwise. Similarly,

$$\nu - \alpha_i = (\nu_1, \ldots, \nu_{i-1}, \nu_i - 1, \nu_{i+1} + 1, \nu_{i+2}, \ldots, \nu_n)$$

for $\nu_i > 0$ and $\varnothing$ if $\nu_i = 0$. In this case we have

$$\overline{\nu+\alpha_i} = \begin{cases} (\bar{\nu}_1+2, \bar{\nu}_2-1, \bar{\nu}_3, \dots, \bar{\nu}_{n-2}, \bar{\nu}_{n-1}) & \text{if } i=1, \\ (\bar{\nu}_1, \bar{\nu}_2, \dots, \bar{\nu}_{n-2}, \bar{\nu}_{n-1}-1, \bar{\nu}_{n-1}+2) & \text{if } i=n-1, \\ (\bar{\nu}_1, \dots, \bar{\nu}_{i-1}-1, \bar{\nu}_i+2, \bar{\nu}_{i+1}-1, \dots, \bar{\nu}_{n-1}) & \text{if } 1<i<n-1, \end{cases}$$

$$\overline{\nu-\alpha_i} = \begin{cases} (\bar{\nu}_1-2, \bar{\nu}_2+1, \bar{\nu}_3, \dots, \bar{\nu}_{n-2}, \bar{\nu}_{n-1}) & \text{if } i=1, \\ (\bar{\nu}_1, \bar{\nu}_2, \dots, \bar{\nu}_{n-2}, \bar{\nu}_{n-1}+1, \bar{\nu}_{n-1}-2) & \text{if } i=n-1, \\ (\bar{\nu}_1, \dots, \bar{\nu}_{i-1}+1, \bar{\nu}_i-2, \bar{\nu}_{i+1}+1, \dots, \bar{\nu}_{n-1}) & \text{if } 1<i<n-1. \end{cases}$$

2.2 The quantum group $\mathbf{U}_q(\mathfrak{sl}_n)$

The algebra $\mathbf{U}_q(\mathfrak{sl}_n)$ is the unital $\mathbb{Q}(q)$-algebra generated by the elements E_i, F_i and K_i, K_i^- for $i \in I$ together with the following relations:

$$K_i^{-1}K_i = K_iK_i^{-1} = 1, \quad K_iK_j = K_jK_i,$$

$$K_iE_jK_i^{-1} = q^{a_{ij}}E_j, \quad K_iF_jK_i^{-1} = q^{-a_{ij}}F_j,$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q-q^{-1}},$$

$$E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0 \quad \text{if } a_{ij} = -1,$$

$$F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 = 0 \quad \text{if } a_{ij} = -1,$$

$$E_iE_j = E_jE_i, \quad F_iF_j = F_jF_i \quad \text{if } a_{ij} = 0.$$

Let $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ be Lusztig's idempotented version of $\mathbf{U}_q(\mathfrak{sl}_n)$, where the unit is replaced by a collection of orthogonal idempotents $1_{\bar{\nu}}$ indexed by the weight lattice X of $\mathfrak{sl}_n$,

$$1_{\bar{\nu}}1_{\bar{\nu}'} = \delta_{\bar{\nu}\bar{\nu}'}1_{\bar{\nu}}, \tag{2.2}$$

such that if $\bar{\nu} = (\bar{\nu}_1, \bar{\nu}_2, \dots, \bar{\nu}_{n-1})$, then

$$K_i1_{\bar{\nu}} = 1_{\bar{\nu}}K_i = q^{\bar{\nu}_i}1_{\bar{\nu}}, \quad E_i1_{\bar{\nu}} = 1_{\overline{\nu+\alpha_i}}E_i, \quad F_i1_{\bar{\nu}} = 1_{\overline{\nu-\alpha_i}}F_i. \tag{2.3}$$

$\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ can be viewed as a category with objects $\bar{\nu} \in X$ and morphisms given by compositions of E_i and F_i with $1 \leq i < n$ modulo the above relations.

For $\mathcal{A} := \mathbb{Z}[q, q^{-1}]$, the $\mathcal{A}$-algebra ${}_{\mathcal{A}}\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$, the integral form of $\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$ is generated by products of divided powers $E_i^{(a)}1_{\bar{\nu}} := \frac{E_i^a}{[a]!}1_{\bar{\nu}}$, $F_i^{(a)}1_{\bar{\nu}} := \frac{F_i^a}{[a]!}1_{\bar{\nu}}$ for $\bar{\nu} \in X$ and $i \in I$. The subalgebra ${}_{\mathcal{A}}\dot{\mathbf{U}}_q^{+}(\mathfrak{sl}_n)$ generated by $E_i^{(a)}1_{\bar{\nu}}$, $\bar{\nu} \in X$ is called the *positive half* of ${}_{\mathcal{A}}\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$.

2.3 The 2-category $\mathcal{U}_Q(\mathfrak{sl}_n)$

Here we describe a categorification of $\mathbf{U}_q(\mathfrak{sl}_n)$ by M. Khovanov and A. Lauda. From now on we assume that the underlying field is complex field. Fix the following choice of scalars Q consisting of $\{t_{ij}\}_{i,j\in I}$ such that

- $t_{ii} = 1$ for all $i \in I$ and $t_{ij} \in \mathbb{C}^*$ for $i \neq j$,
- $t_{ij} = t_{ji}$ if $a_{ij} = 0$,

and a choice of bubble parameters $c_{i,\bar{\nu}} \in \mathbb{C}^*$ for $i \in I$ and a weight $\bar{\nu}$ such that

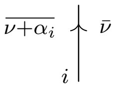
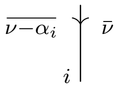
$$c_{j,\overline{\nu+\alpha_j}}/c_{i,\bar{\nu}} = t_{ij}.$$

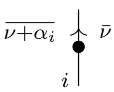
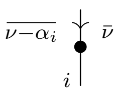
Definition 2.3.1. *The 2-category $\mathcal{U}_Q(\mathfrak{sl}_n)$ is the graded linear 2-category consisting of:*

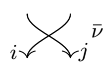
- *objects are $\mathfrak{sl}_n$ weights $\bar{\nu}$.*
- *1-morphisms are formal direct sums of (degree shifts of) compositions of*

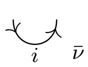
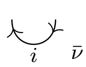
$$\mathbf{1}_{\bar{\nu}}, \quad \mathbf{1}_{\overline{\nu+\alpha_i}}\mathcal{E}_i = \mathbf{1}_{\overline{\nu+\alpha_i}}\mathcal{E}_i\mathbf{1}_{\bar{\nu}} = \mathcal{E}_i\mathbf{1}_{\bar{\nu}}, \quad \text{and} \quad \mathbf{1}_{\overline{\nu-\alpha_i}}\mathcal{F}_i = \mathbf{1}_{\overline{\nu-\alpha_i}}\mathcal{F}_i\mathbf{1}_{\bar{\nu}} = \mathcal{F}_i\mathbf{1}_{\bar{\nu}}$$

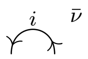
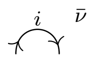
  for $i \in I$ and a weight $\bar{\nu}$.
- *2-morphisms are $\mathbb{C}$-vector spaces spanned by compositions of the following tangle-like diagrams label by $i \in I$:*


$: \mathcal{E}_i\mathbf{1}_{\bar{\nu}} \to \mathcal{E}_i\mathbf{1}_{\bar{\nu}}$,

$: \mathcal{F}_i\mathbf{1}_{\bar{\nu}} \to \mathcal{F}_i\mathbf{1}_{\bar{\nu}}$, (2.4)


$: \mathcal{E}_i\mathbf{1}_{\bar{\nu}} \to \mathcal{E}_i\mathbf{1}_{\bar{\nu}}\langle 2\rangle$,

$: \mathcal{F}_i\mathbf{1}_{\bar{\nu}} \to \mathcal{F}_i\mathbf{1}_{\bar{\nu}}\langle 2\rangle$, (2.5)


$: \mathcal{E}_i\mathcal{E}_j\mathbf{1}_{\bar{\nu}} \to \mathcal{E}_j\mathcal{E}_i\mathbf{1}_{\bar{\nu}}\langle -a_{ij}\rangle$,

$: \mathcal{F}_i\mathcal{F}_j\mathbf{1}_{\bar{\nu}} \to \mathcal{F}_j\mathcal{F}_i\mathbf{1}_{\bar{\nu}}\langle -a_{ij}\rangle$, (2.6)

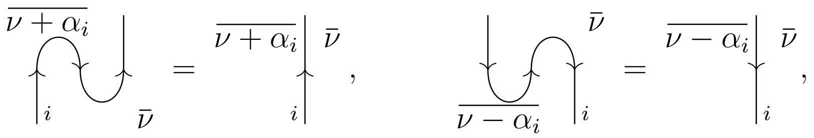

$: \mathbf{1}_{\bar{\nu}} \to \mathcal{F}_i\mathcal{E}_i\mathbf{1}_{\bar{\nu}}\langle 1 + \bar{\nu}_i\rangle$,

$: \mathbf{1}_{\bar{\nu}} \to \mathcal{E}_i\mathcal{F}_i\mathbf{1}_{\bar{\nu}}\langle 1 - \bar{\nu}_i\rangle$, (2.7)

 $: \mathcal{F}_i\mathcal{E}_i\mathbf{1}_{\bar\nu} \to \mathbf{1}_{\bar\nu}\langle 1+\bar\nu_i\rangle,$  $: \mathcal{E}_i\mathcal{F}_i\mathbf{1}_{\bar\nu} \to \mathbf{1}_{\bar\nu}\langle 1-\bar\nu_i\rangle.$ (2.8)

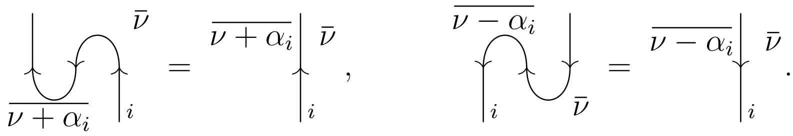
The two 2-morphisms (2.4) are identity 2-morphisms. We read diagrams from right to left and from bottom to top. That is, the horizontal composition of the 2-morphisms corresponds to drawing the respective diagrams side by side from right to left, and vertical composition means stacking diagrams on top of each other. If the labels of the strands do not match in the vertical composition, then we set the composition to zero. Isotopies are allowed as long as they do not change the combinatorial type of a diagram.

The 2-morphisms satisfy the following relations:

1. The identity morphisms of $\mathcal{E}_i\mathbf{1}_{\bar\nu}$ and $\mathcal{F}_i\mathbf{1}_{\bar\nu}$ are biadjoint up to a degree shift:

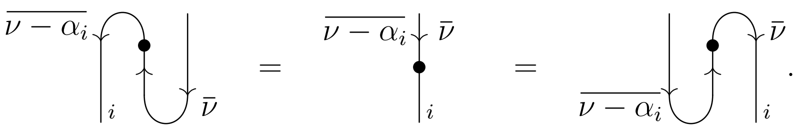


(2.9)



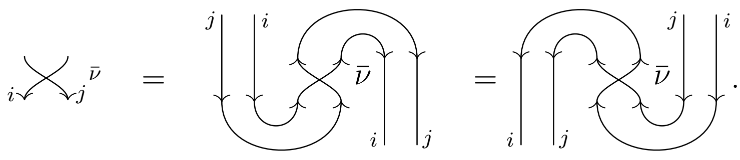
(2.10)

2. The 2-morphisms are cyclic with respect to this biadjoint structure:



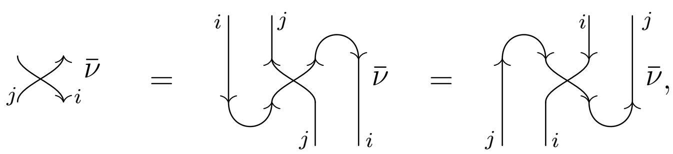
(2.11)

The cyclicity for crossings are given by

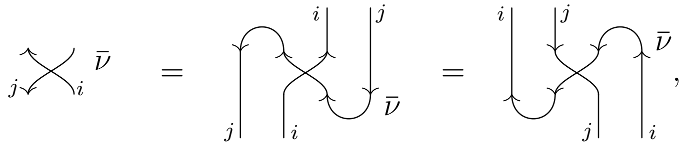


(2.12)

Sideways crossings are defined by the equalities:



(2.13)

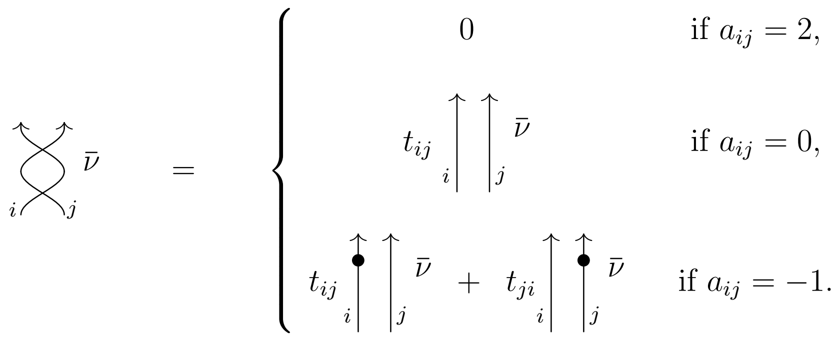


(2.14)

where the second equalities in (2.13) and (2.14) follow from (2.12).

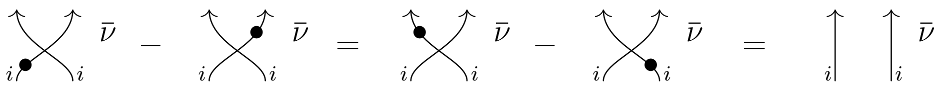
3. The following local relations hold for upwards oriented strands:

   i)



(2.15)

   ii) The nilHecke dot sliding relations



(2.16)

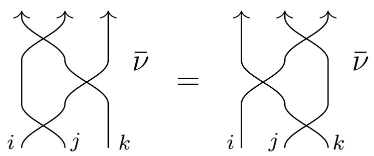
   hold.

   iii) For $i \neq j$ the dot sliding relations

(2.17)

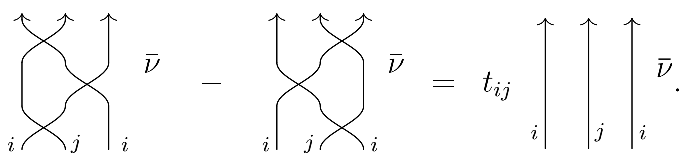
   hold.

   iv) If $i \neq k$ and $a_{ij} \geq 0$, the relation



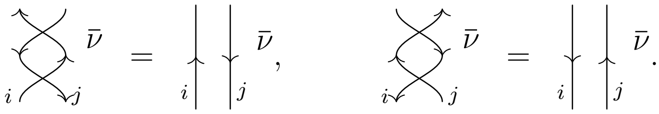
(2.18)

   holds. Otherwise if $a_{ij} = -1$, we have



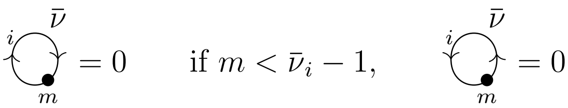
(2.19)

4. When $i \neq j$ one has the mixed relations



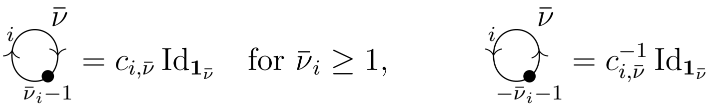
(2.20)

5. Negative degree bubbles are zero. That is, for all $m \in \mathbb{Z}_+$ one has



$= 0 \quad$ if $m < \bar{\nu}_i - 1$, $\qquad = 0 \quad$ if $m < -\bar{\nu}_i - 1$. (2.21)

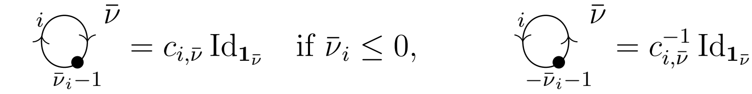
A dotted bubble of degree zero is just the scaled identity 2-morphism:



$= c_{i,\bar{\nu}} \,\mathrm{Id}_{\mathbf{1}_{\bar{\nu}}}$ for $\bar{\nu}_i \geq 1$, $\qquad = c_{i,\bar{\nu}}^{-1} \,\mathrm{Id}_{\mathbf{1}_{\bar{\nu}}}$ for $\bar{\nu}_i \leq -1$.

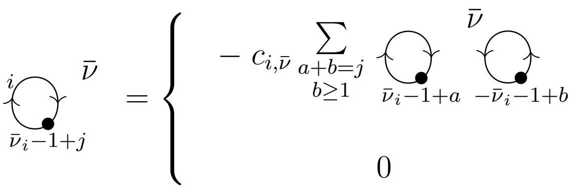
6. It is convenient to introduce so called fake bubbles. These diagrams are dotted bubbles where the number of dots is negative, but the total degree of the dotted bubble taken with these negative dots is still positive.

   - Degree zero fake bubbles are equal to the identity 2-morphisms



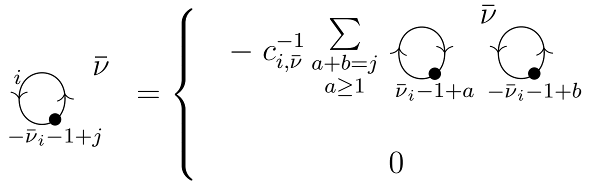
$= c_{i,\bar{\nu}} \,\mathrm{Id}_{\mathbf{1}_{\bar{\nu}}}$ if $\bar{\nu}_i \leq 0$, $\qquad = c_{i,\bar{\nu}}^{-1} \,\mathrm{Id}_{\mathbf{1}_{\bar{\nu}}}$ if $\bar{\nu}_i \geq 0$.

   - Higher degree fake bubbles for $\bar{\nu}_i < 0$ are defined inductively as



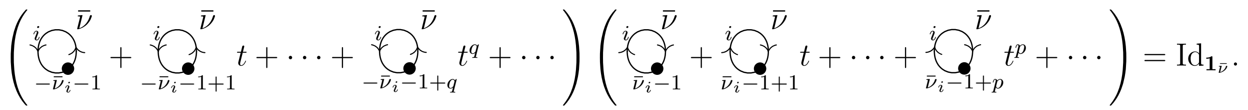
if $0 \leq j < -\bar{\nu}_i + 1$; $\quad 0$ if $j < 0$. (2.22)

   - Higher degree fake bubbles for $\bar{\nu}_i > 0$ are defined inductively as



if $0 \leq j < \bar{\nu}_i + 1$; $\quad 0$ if $j < 0$. (2.23)

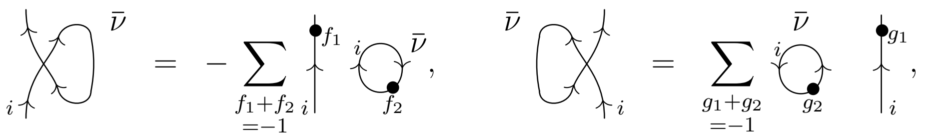
These equations arise from the homogeneous terms in t of the Grassmannian equation



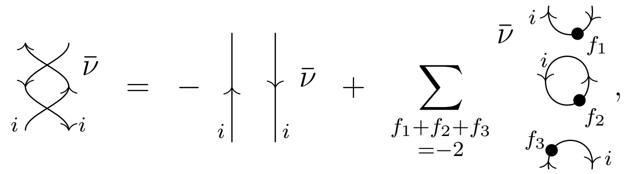
(2.24)

Using the Grassmannian relation, we can express clockwise bubbles in terms of fake and real counterclockwise bubbles. We will use the following notation in order to emphasize the degree of bubble:

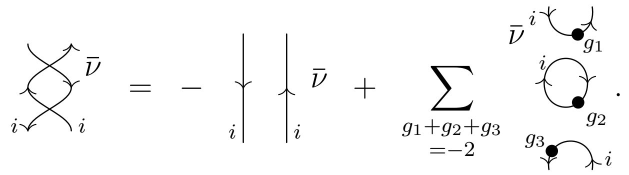
7. The following set of relations are sometimes called extended $\mathfrak{sl}_2$ relations:



(2.25)



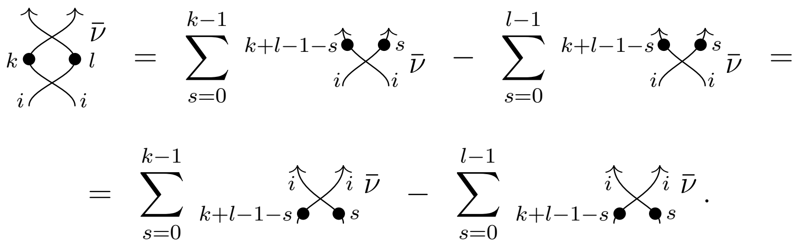
(2.26)



(2.27)

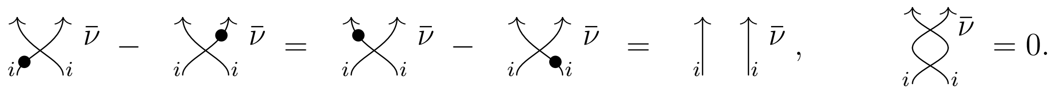
We prove the following relation which we will need for our main result.

Proposition 2.3.2. *Let $k, l \geq 0$ be integers such that $k + l > 0$. We have*

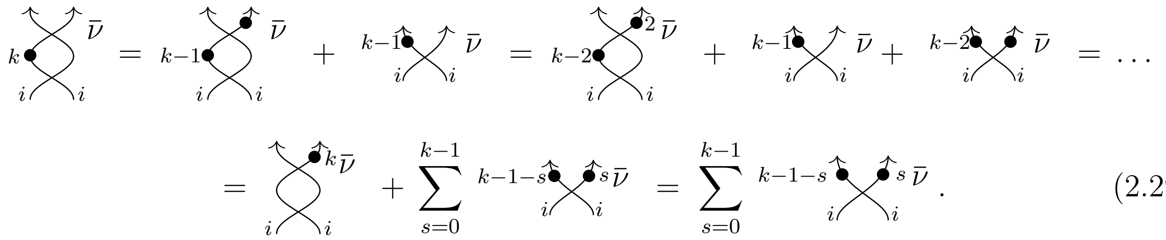


(2.28)

Proof. We apply the following two nilHecke relations for the proof of the statement:

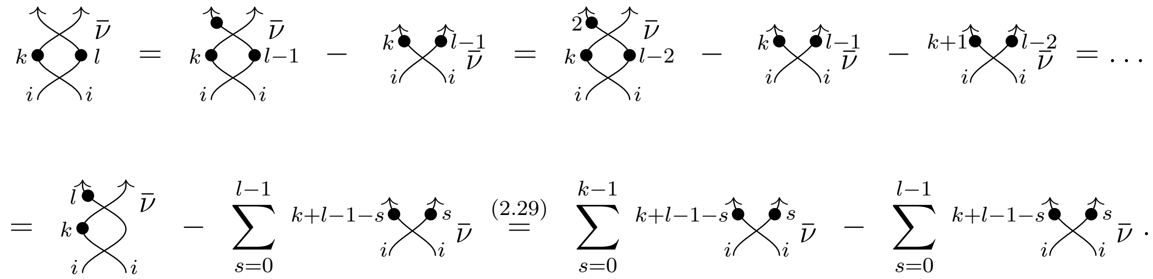


First assume $l = 0$. Then by sliding the k dots upwards through the crossing, we get



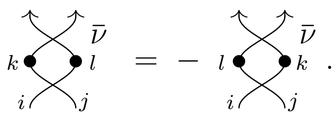
(2.29)

We now let $l \geq 0$ and slide the l dots upwards:



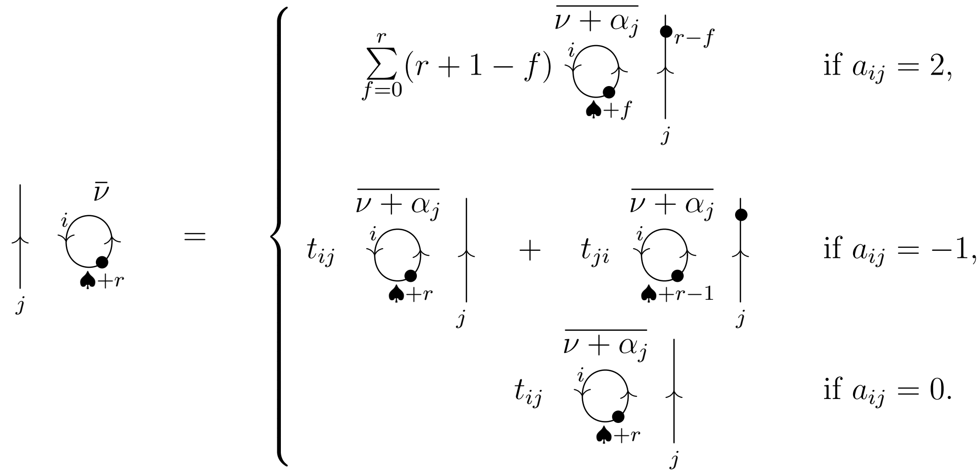
The proof of the second equality is entirely similar, we slide the dots downwards through the crossing. □

Proposition 2.3.2 implies the following relation:

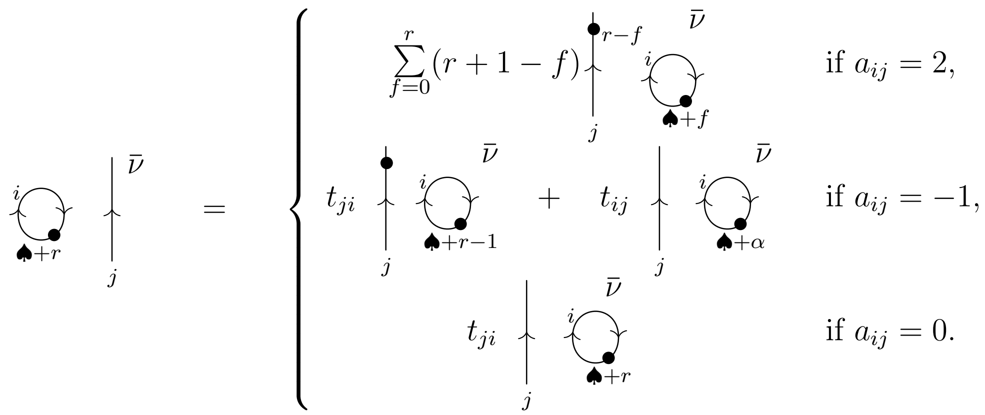


(2.30)

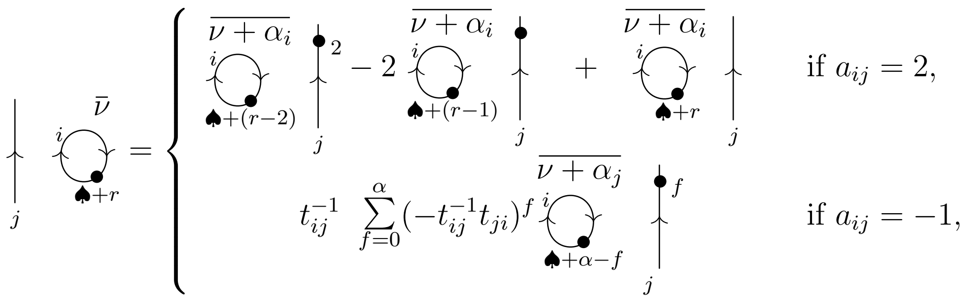
The relations between 2-morphisms can be used to obtain the following bubble slide equations:



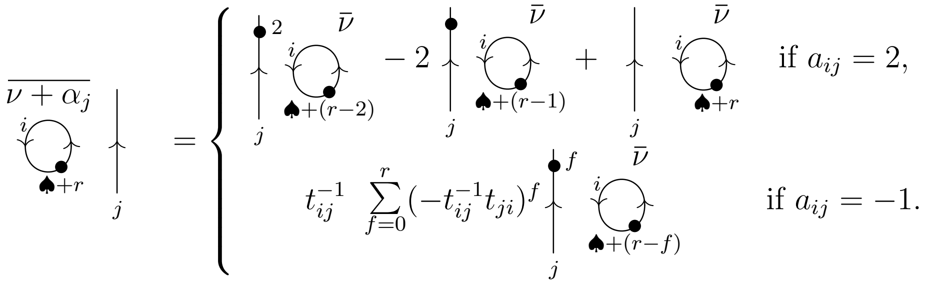
(2.31)



(2.32)



(2.33)



(2.34)

Let $\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)$ be the Karoubi envelope of $\mathcal{U}_Q(\mathfrak{sl}_n)$, the smallest 2-category which contains $\mathcal{U}_Q(\mathfrak{sl}_n)$ and has splitting idempotents. The diagrammatic description of $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$ is given in [36], but has not been worked out yet in full generality for any n. The split Grothendieck group $K_0(\dot{\mathcal{U}}_Q(\mathfrak{sl}_n))$ is the additive category with objects $\bar{\nu} \in X$, and the abelian group of morphisms $\bar{\nu} \to \bar{\mu}$, $\bar{\mu} \in X$ is the split Grothendieck group $K_0\left(\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)(\bar{\nu},\bar{\mu})\right)$ of the additive category $\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)(\bar{\nu},\bar{\mu})$. We can view $K_0(\dot{\mathcal{U}}_Q(\mathfrak{sl}_n))$ as a $\mathcal{A} = \mathbb{Z}[q,q^{-1}]$ module, since the degree shift functor on $\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)$ induces a multiplication by q in $K_0(\dot{\mathcal{U}}_Q(\mathfrak{sl}_n))$. The following theorem is the categorification of quantum $\mathfrak{sl}_n$ by Khovanov and Lauda.

Theorem 2.3.3. *[39] There is $\mathcal{A}$-module isomorphism*

$$\gamma : {}_{\mathcal{A}}\dot{\mathbf{U}}_q(\mathfrak{sl}_n) \to K_0(\dot{\mathcal{U}}_Q(\mathfrak{sl}_n)), \tag{2.35}$$

such that $\gamma(1_{\bar{\nu}}) = [\mathbf{1}_{\bar{\nu}}]_{\cong}$, $\gamma(E_i 1_{\bar{\nu}}) = [\mathcal{E}_i \mathbf{1}_{\bar{\nu}}]_{\cong}$, *and* $\gamma(F_i 1_{\bar{\nu}}) = [\mathcal{F}_i \mathbf{1}_{\bar{\nu}}]_{\cong}$.

We define $\mathcal{U} = \mathcal{U}_Q^{\star}(\mathfrak{sl}_n)$ to be the 2-category with the same objects and 1-morphisms as $\mathcal{U}_Q(\mathfrak{sl}_n)$ and with 2-morphisms

$$\mathcal{U}(\bar{\nu},\bar{\mu})(f,g) = \bigoplus_{t\in\mathbb{Z}} \mathcal{U}_Q(\mathfrak{sl}_n)(\bar{\nu},\bar{\mu})(f,g\langle t\rangle)$$

for all $\bar{\nu},\bar{\mu} \in \mathrm{Ob}(\mathcal{U}))$ and $f,g\colon \bar{\nu} \to \bar{\mu}$. Horizontal composition in $\mathcal{U}$ is induced from the horizontal composition in $\mathcal{U}_Q(\mathfrak{sl}_n)$. We will work mainly with the 2-category $\mathcal{U}$ from now on. K_0 decategorification of $\mathcal{U}$ is the same as that of $\mathcal{U}_Q(\mathfrak{sl}_n)$, however, it has more interesting trace decategorification as we will see next.

2.4 The current algebra and the trace of $\mathcal{U}$

Here we will compute the trace of the 2-category $\mathcal{U}$ and compare it to the split Grothendieck group. Let E_i, F_i, H_i, $i \in I$ be the Chevalley generators of $\mathfrak{sl}_n$. By $\mathfrak{sl}_n[t] = \mathfrak{sl}_n \otimes \Bbbk[t]$, we denote the extension of $\mathfrak{sl}_n$ by polynomials in t. Then $\mathfrak{sl}_n[t]$ is a Lie algebra with the Lie bracket is defined as $[a\otimes t^r, b\otimes t^s] = [a,b]\otimes t^{r+s}$, for $a,b \in \mathfrak{sl}_n$ and $r,s \geq 0$. We call $\mathbf{U}(\mathfrak{sl}_n[t])$ the current algebra. Current algebra is graded with $\deg(a\otimes t^k) = 2k$. This algebra is generated over $\Bbbk$ by $E_{i,r} = E_i \otimes t^r, F_{i,r} = F_i \otimes t^r, H_{i,r} = H_i \otimes t^r$ for $r \geq 0$ and $i \in I$ modulo the following relations:

C1 For $i, j \in I$ and $r, s \geq 0$

$$[H_{i,r}, H_{j,s}] = 0,$$

C2 For any $i, j \in I$ and $r, s \geq 0$,

$$[H_{i,r}, E_{j,s}] = a_{ij} E_{j,r+s}, \quad [H_{i,r}, F_{j,s}] = -a_{ij} F_{j,r+s},$$

C3 For any $i, j \in I$ and $r, s \geq 0$,

$$[E_{i,r+1}, E_{j,s}] = [E_{i,r}, E_{j,s+1}], \quad [F_{i,r+1}, F_{j,s}] = [F_{i,r}, F_{j,s+1}],$$

C4 For any $i, j \in I$ and $r, s \geq 0$

$$[E_{i,r}, F_{j,s}] = \delta_{i,j} H_{i,r+s},$$

C5 Let $i \neq j$ and set $m = 1 - a_{ij}$. For every $r_1, \ldots, r_m, s \geq 0$

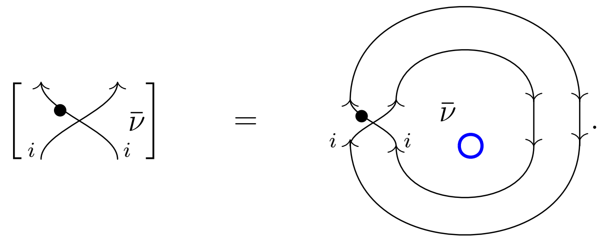
$$\sum_{\pi \in S_m} \sum_{l=0}^{m} (-1)^l \binom{m}{l} E_{i,k_{\pi(1)}} \ldots E_{i,k_{\pi(l)}} E_{j,s} E_{i,k_{\pi(l+1)}} \ldots E_{i,k_{\pi(m)}} = 0,$$

$$\sum_{\pi \in S_m} \sum_{l=0}^{m} (-1)^l \binom{m}{l} F_{i,k_{\pi(1)}} \ldots F_{i,k_{\pi(l)}} F_{j,s} F_{i,k_{\pi(l+1)}} \ldots F_{i,k_{\pi(m)}} = 0.$$

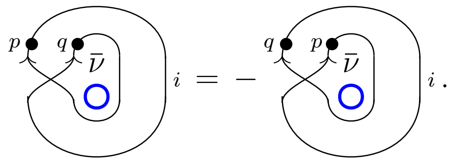
We identify $\mathbf{U}(\mathfrak{sl}_n)$ as a subalgebra of $\mathbf{U}(\mathfrak{sl}_n[t])$ generated by degree zero elements $E_{i,0}, F_{i,0}, H_{i,0}$, $i \in I$. Let $\dot{\mathbf{U}}(\mathfrak{sl}_n[t])$ be the idempotented version of the current algebra $\mathbf{U}(\mathfrak{sl}_n[t])$, where the unit is replaced by the collection of mutually orthogonal idempotents $1_{\bar{\nu}}$ for each $\mathfrak{sl}_n$-weight $\bar{\nu}$, such that

$$1_{\bar{\nu}} 1_{\bar{\nu}'} = \delta_{\bar{\nu}.\bar{\nu}'} 1_{\bar{\nu}}, \quad 1_{\bar{\nu}} H_{i,0} = H_{i,0} 1_{\bar{\nu}} = \bar{\nu}_i 1_{\bar{\nu}}, \quad 1_{\overline{\nu+\alpha_i}} E_{i,j} = E_{i,j} 1_{\bar{\nu}}, \quad 1_{\overline{\nu-\alpha_i}} F_{i,j} = F_{i,j} 1_{\bar{\nu}}.$$

Now we describe the trace Tr$\mathcal{U}$ of the 2-category $\mathcal{U}$. The trace class of a 2-morphisms in $\mathcal{U}$ can be depicted by closing a 2-morphism diagrams to the right:

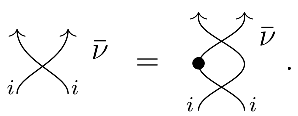


Proposition 2.4.1. *The following relation holds in* Tr$\mathcal{U}$ *:*



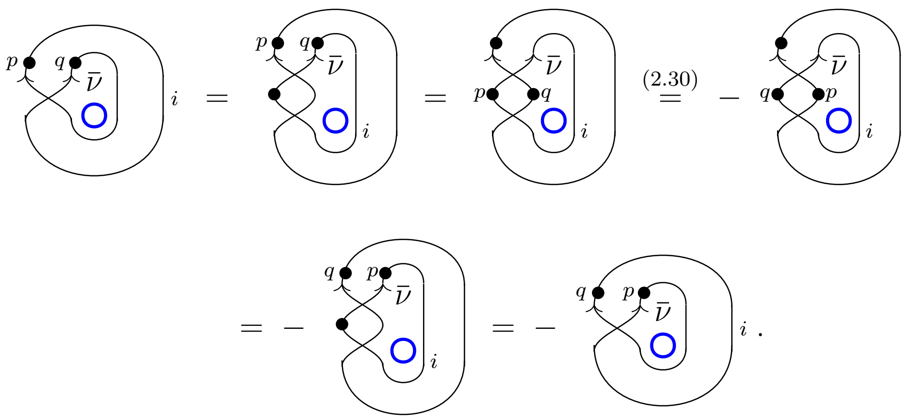
(2.36)

Proof. By putting $k = 1$, $l = 0$ in the Proposition 2.3.2, we get



(2.37)

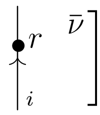
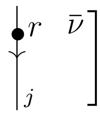
Then we have



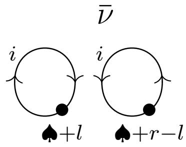
(2.38)

Hence, the statement indeed holds. ◻

Let $\mathsf{E}_{i,r}1_{\bar{\nu}}, \mathsf{F}_{j,r}1_{\bar{\nu}}, \mathsf{H}_{i,r}1_{\bar{\nu}}$ denote the following trace classes:

$\mathsf{E}_{i,r}1_{\bar{\nu}} = \Bigg[$  $\Bigg]$, $\quad \mathsf{F}_{j,r}1_{\bar{\nu}} = \Bigg[$  $\Bigg]$, $\quad \mathsf{H}_{i,r}1_{\bar{\nu}} = [\pi_{i,r}(\bar{\nu})\, \mathrm{Id}_{\mathbf{1}_{\bar{\nu}}}]$,

where

$$\pi_{i,r}(\bar{\nu}) = \sum_{l=0}^{r}(l+1)$$ 

for $r > 0$ and $\pi_{i,0}(\bar{\nu}) = \bar{\nu}_i$.

Theorem 2.4.2. *There is an algebra homomorphism*

$$\rho\colon \dot{\mathbf{U}}(\mathfrak{sl}_n[t]) \longrightarrow \mathrm{Tr}\,\mathcal{U}, \tag{2.39}$$

given by

$$E_{i,r}1_{\bar{\nu}} \mapsto \mathsf{E}_{i,r}1_{\bar{\nu}}, \qquad F_{i,r}1_{\bar{\nu}} \mapsto \mathsf{F}_{i,r}1_{\bar{\nu}}, \qquad H_{i,r}1_{\bar{\nu}} \mapsto \mathsf{H}_{i,r}1_{\bar{\nu}}.$$

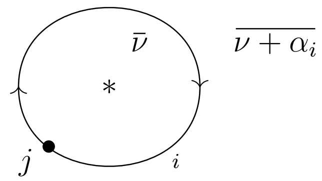
The proof of Theorem 2.4.2 is given in the next subsection.

Let $Z(\bar{\nu})$, the center of an object $\bar{\nu} \in \mathrm{Ob}(\mathcal{U})$, to be the endomorphism ring $\mathrm{End}(1_{\bar{\nu}})$. The ring $Z(\bar{\nu})$ is a commutative ring, diagrammatically given by closed diagrams label by $\bar{\nu}$ on the outside of the diagram. By [35], $Z(\bar{\nu})$ is freely generated by counterclockwise oriented fake and real bubbles. We will define the center of objects $Z(\mathcal{U})$ of $\mathcal{U}$ as

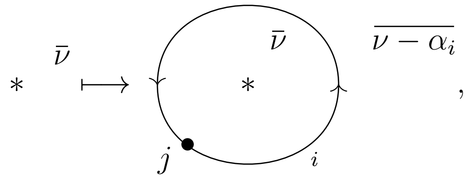
$$Z(\mathcal{U}) = \bigoplus_{\bar{\nu}\in \mathrm{Ob}(\mathcal{U})} Z(\bar{\nu}).$$

Cyclicity relations show that $\mathcal{U}$ is cyclic as a 2-category. Hence, we can define an action of $\mathrm{Tr}\,\mathcal{U}$ on the center of objects $Z(\mathcal{U})$, and thus an action of the current algebra.

Corollary 2.4.3. *The vector space $Z(\mathcal{U})$ is a $\mathbf{U}(\mathfrak{sl}_n[t])$-module with*

$$E_{i,j}\colon Z(\bar{\nu}) \to Z(\overline{\nu+\alpha_i}), \qquad * \ \ \bar{\nu} \ \longmapsto$$  , (2.40)

$F_{i,j}\colon Z(\bar{\nu}) \to Z(\overline{\nu - \alpha_i}),$

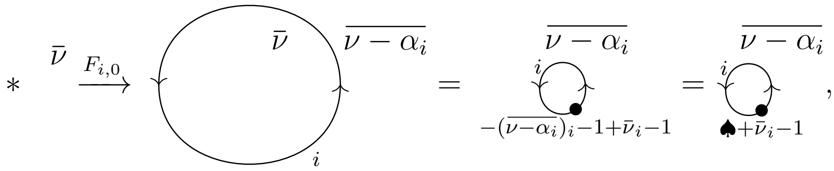


, (2.41)

$$H_{i,j}\colon Z(\bar{\nu}) \to Z(\bar{\nu}), \qquad *^{\,\bar{\nu}} \longmapsto \pi_{i,j}(\bar{\nu}) \; *^{\,\bar{\nu}}. \tag{2.42}$$

Proof. This theorem follows immediately from Theorem 2.4.2. In particular, the fact that the current algebra relations hold in the trace $\mathrm{Tr}\,\mathcal{U}$ imply that these relations hold under the action described above. □

Notice that the maps $F_{i,j}$ and $E_{i,j}$ send an element of degree d to an element of degree $d + 2(j + \bar{\nu}_i - 1)$ and $d + 2(j - \bar{\nu}_i - 1)$, respectively. In order to see this for $F_{i,0}$, let $*$ the empty diagram and $j = 0$. Then $F_{i,0}$ sends it to a counterclockwise bubble of degree $2(\bar{\nu}_i - 1)$:



, (2.43)

where we use the identity $(\overline{\nu - \alpha_i})_i = \bar{\nu}_i - 2$.

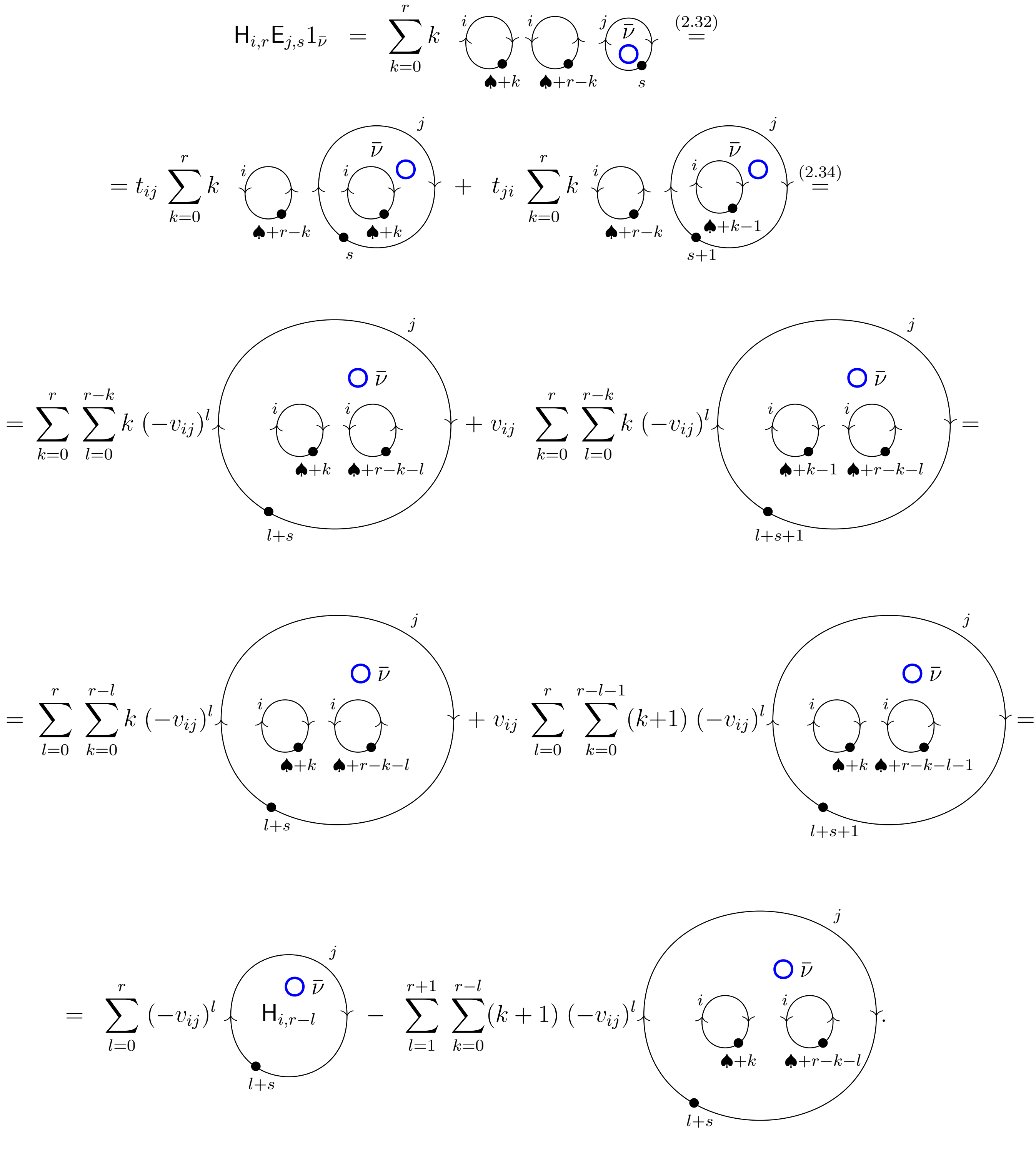
Theorem 2.4.2 was later improved in the following theorem by A. Beliakova, K. Habiro, A. Lauda, B. Webster in [5], Theorem 8.3.

Theorem 2.4.4. *The homomorphism* $\rho\colon \dot{\mathbf{U}}(\mathfrak{sl}_n[t]) \longrightarrow \mathrm{Tr}\,\mathcal{U}$ *is an isomorphism.*

2.5 Proof of the homomorphism theorem

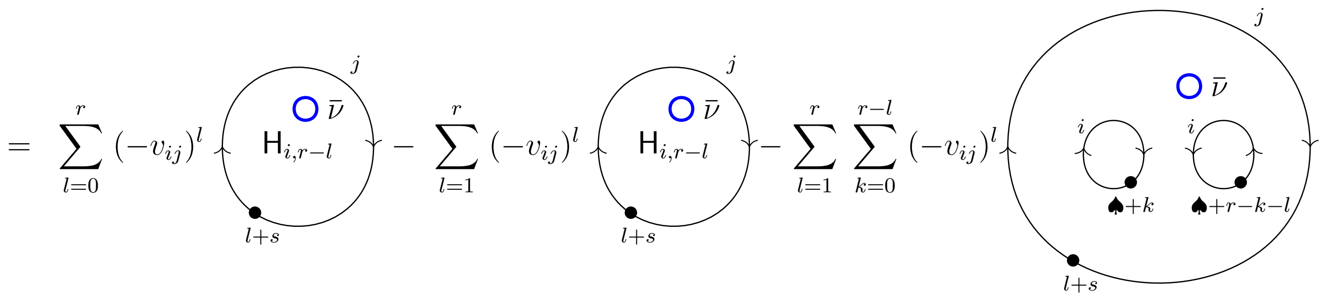
Here we give the proof of the Theorem 2.4.2 using diagrammatic algebra of 2-morphism.

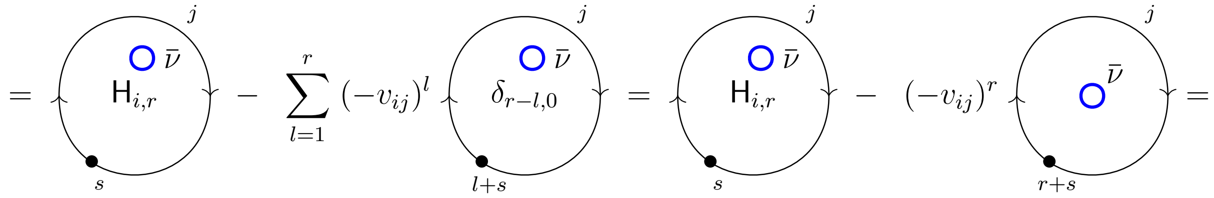
Proof. To prove this theorem, we verify the current algebra relations using the relations in the 2-category $\mathcal{U}$. We only need to consider the case $i \neq j$, since the $\mathfrak{sl}_2$ case has been proven in [7]. **C1** is clear, since multiplication by a bubble is a commutative operation. We prove the equality **C2** for the case $j = i \pm 1$. For convenience, we will only depict the interior of the annulus for diagrams on the annulus in our calculation below, where $v_{ij} := t_{ij}^{-1} t_{ji}$.



H_{i,r}E_{j,s}1_{ν̄} = Σ_{k=0}^{r} k
(2.32)
= t_{ij} Σ_{k=0}^{r} k
+ t_{ji} Σ_{k=0}^{r} k
(2.34)
= Σ_{k=0}^{r} Σ_{l=0}^{r−k} k (−v_{ij})^l
+ v_{ij} Σ_{k=0}^{r} Σ_{l=0}^{r−k} k (−v_{ij})^l
= Σ_{l=0}^{r} Σ_{k=0}^{r−l} k (−v_{ij})^l
+ v_{ij} Σ_{l=0}^{r} Σ_{k=0}^{r−l−1} (k+1) (−v_{ij})^l
= Σ_{l=0}^{r} (−v_{ij})^l
H_{i,r−l}
− Σ_{l=1}^{r+1} Σ_{k=0}^{r−l} (k+1) (−v_{ij})^l
♠+k
♠+r−k
♠+k−1
♠+r−k−l
♠+r−k−l−1
s
s+1
l+s
l+s+1

Pulling off k of the summands in the second term and observing that the $l = r+1$ term vanishes, we have





$$= \mathsf{E}_{j,s}\mathsf{H}_{i,r}1_{\bar{\nu}} - (-v_{ij})^r\, \mathsf{E}_{j,r+s}1_{\bar{\nu}}.$$

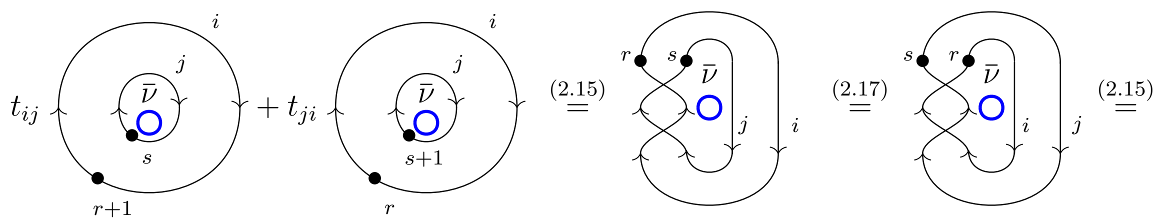
By choosing $v_{ij} = -1$, we get

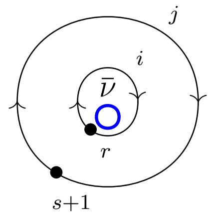
$$[\mathsf{H}_{i,r}, \mathsf{E}_{j,s}]1_{\bar{\nu}} = -\mathsf{E}_{j,r+s}1_{\bar{\nu}}.$$

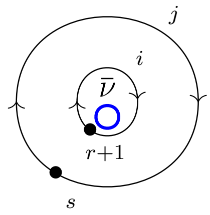
The relation

$$[\mathsf{H}_{i,r}, \mathsf{F}_{j,s}]1_{\bar{\nu}} = \mathsf{F}_{j,r+s}1_{\bar{\nu}}.$$

can be proven in a similar way. **C3** follows from the relations (2.15) and (2.17):



$=t_{ji}$ 

$+\, t_{ij}$ 

.

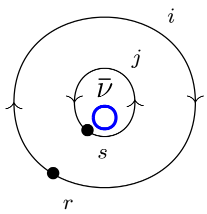
Dividing both sides of this equation by t_{ij}, and by choosing $v_{ij} = -1$, we get

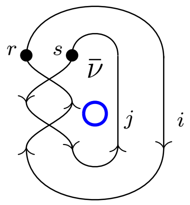
$$[\mathsf{E}_{i,r+1}, \mathsf{E}_{j,s}]1_{\bar{\nu}} = [\mathsf{E}_{i,r}, \mathsf{E}_{j,s+1}]1_{\bar{\nu}}.$$

If we reverse the arrows in the preceding equation, they still hold:

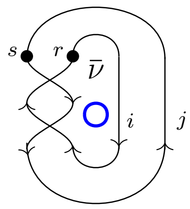
$$[\mathsf{F}_{i,r+1}, \mathsf{F}_{j,s}]1_{\bar{\nu}} = [\mathsf{F}_{i,r}, \mathsf{F}_{j,s+1}]1_{\bar{\nu}}.$$

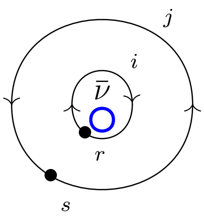
The relation **C4** for the case $i \neq j$ can be checked the following way:

$\mathsf{E}_{i,r}\mathsf{F}_{j,s}1_{\bar{\nu}} \;=$ 

$\overset{(2.20)}{=}$ 

$\overset{(2.17)}{=}$

$=$ 

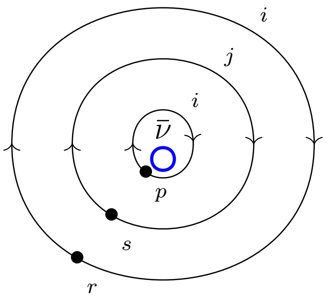
$\overset{(2.20)}{=}$ 

$= \mathsf{F}_{j,s}\mathsf{E}_{i,r}1_{\bar{\nu}}.$

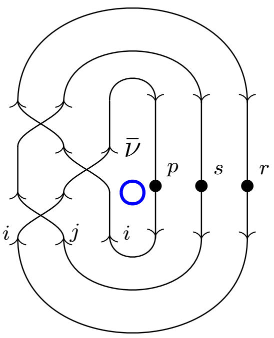
We now prove the Serre relation **C5** in the case when $|i-j| = 1$. Let us prove the identity

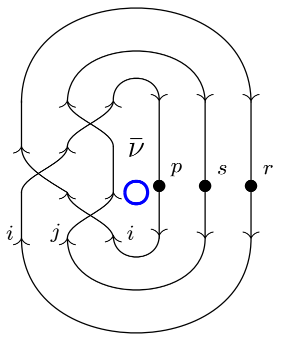
$$\mathsf{E}_{i,r}\mathsf{E}_{j,s}\mathsf{E}_{i,p}1_{\bar{\nu}} + \mathsf{E}_{i,p}\mathsf{E}_{j,s}\mathsf{E}_{i,r}1_{\bar{\nu}} = \mathsf{E}_{i,r}\mathsf{E}_{i,p}\mathsf{E}_{j,s}1_{\bar{\nu}} + \mathsf{E}_{j,s}\mathsf{E}_{i,r}\mathsf{E}_{i,p}1_{\bar{\nu}}.$$

We have

$$\mathsf{E}_{i,r}\mathsf{E}_{j,s}\mathsf{E}_{i,p}1_{\bar{\nu}} =$$ 

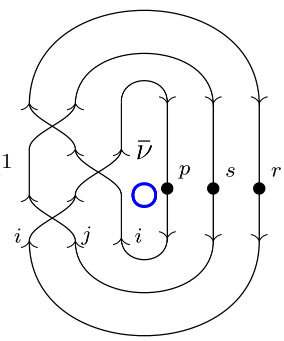
$$\overset{(2.19)}{=} \qquad (2.44)$$

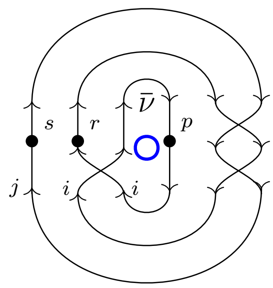
$$= t_{ij}^{-1}$$ 

$$- \; t_{ij}^{-1}$$ 

.

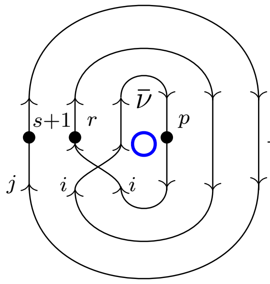
The first term on the right hand side of (2.44) can be simplified

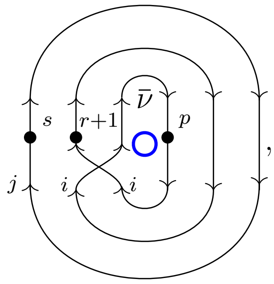
$$t_{ij}^{-1}$$ 

$$= \; t_{ij}^{-1}$$ 

$$. \qquad (2.45)$$

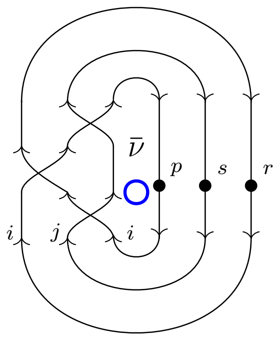
using the biadjoint relations and equations (2.13) and (2.14) to slide crossings around the annulus. This diagram can be further simplified using the relation (2.15):

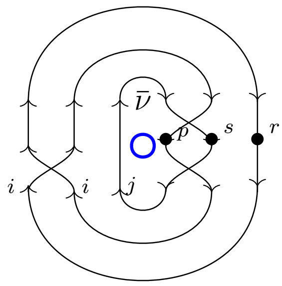
$\overset{(2.15)}{=}\ v_{ji}$ 

$+$ 

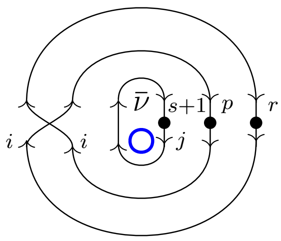
, (2.46)

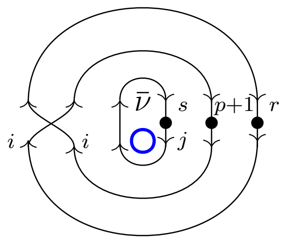
where we freely slide dots through caps and cups using the dot cyclicity relation (2.11). The second term on the right-hand-side of (2.44) can also be simplified

t_{ij}^{-1} 

$\overset{(2.12),(2.17)}{=}\ t_{ij}^{-1}$ 

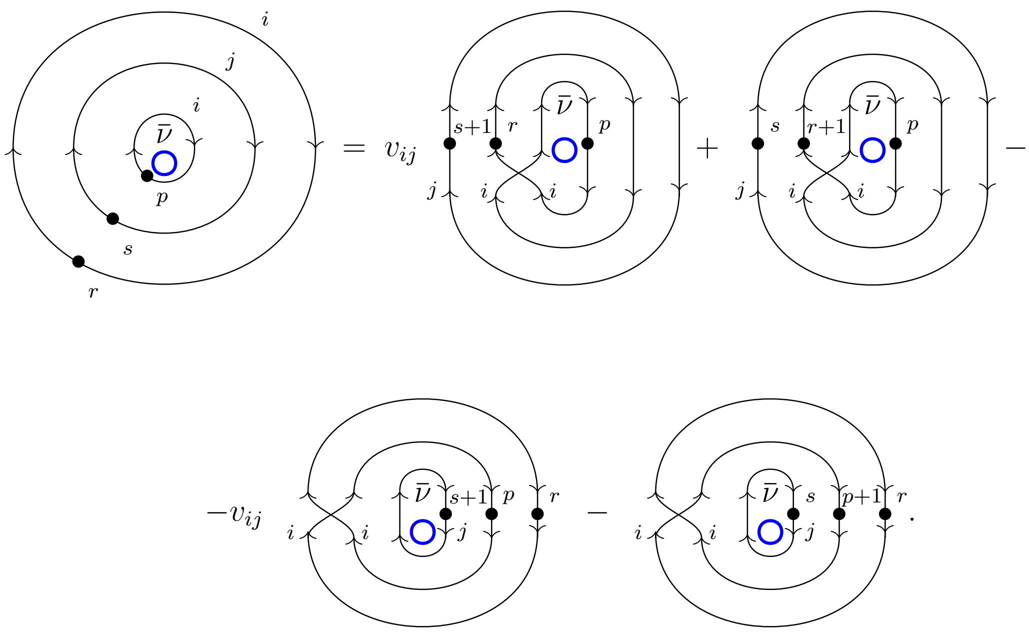
$=$

$\overset{(2.15)}{=}\ v_{ij}$ 

$+$ 

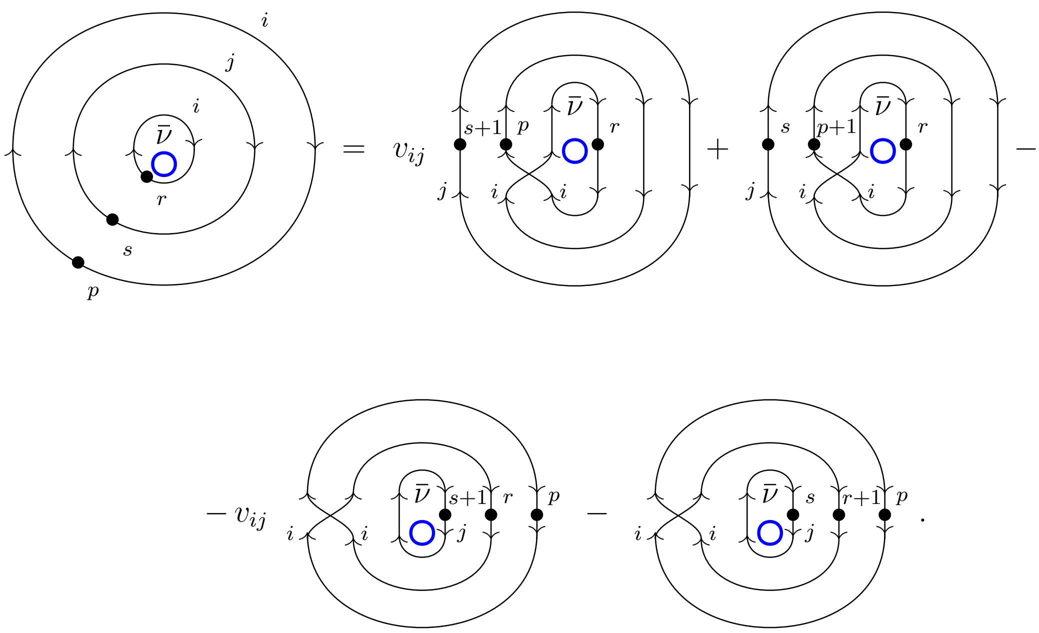
.

By combining terms and simplifying, we get



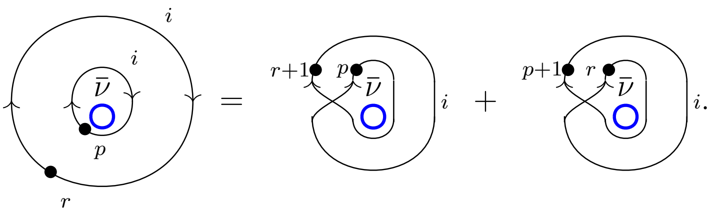
(2.47)

By interchanging r and p in the equation (2.47), we get



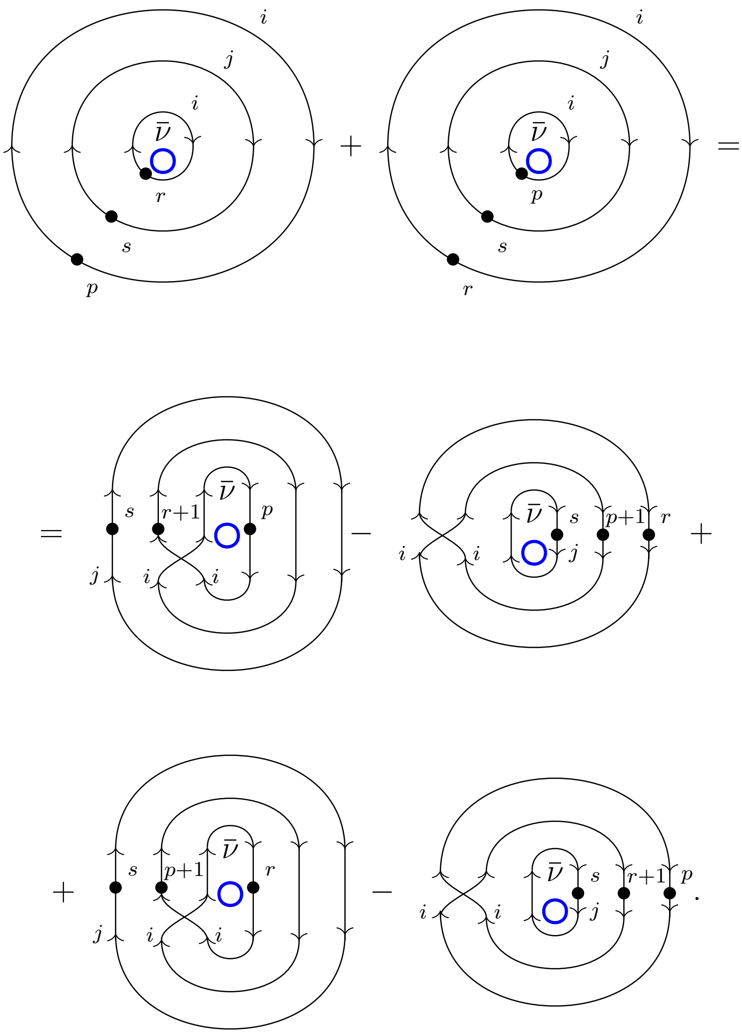
(2.48)

The following equation directly follows from (2.16) and Proposition 2.4.1:



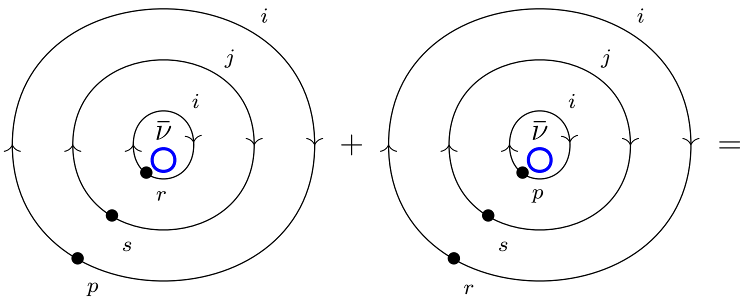
(2.49)

We add the right and left hand sides of the equations (2.47) and (2.48), and after eliminating terms using (2.36), we get

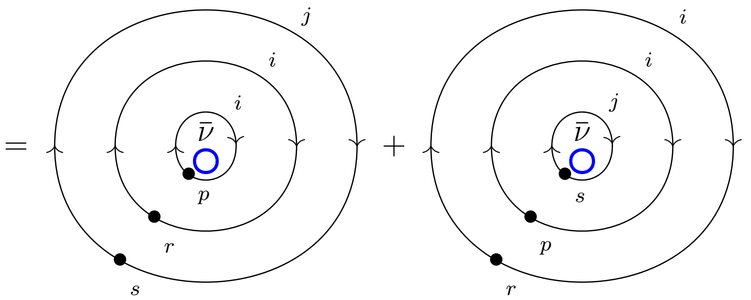


(2.50)

Combining the respective terms using the relation (2.49) in the equation (2.50), we have



(2.51)



$= \mathsf{E}_{j,s}\mathsf{E}_{i,r}\mathsf{E}_{i,p}1_{\bar{\nu}} + \mathsf{E}_{i,r}\mathsf{E}_{i,p}\mathsf{E}_{j,s}1_{\bar{\nu}}.$

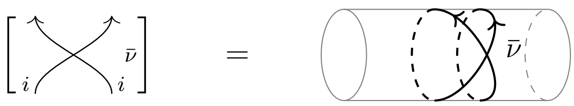
By reversing the orientation of the arrows, we get the identity

$$\mathsf{F}_{i,r}\mathsf{F}_{j,s}\mathsf{F}_{i,p}1_{\bar{\nu}} + \mathsf{F}_{i,p}\mathsf{F}_{j,s}\mathsf{F}_{i,r}1_{\bar{\nu}} = \mathsf{F}_{i,r}\mathsf{F}_{i,p}\mathsf{F}_{j,s}1_{\bar{\nu}} + \mathsf{F}_{j,s}\mathsf{F}_{i,r}\mathsf{F}_{i,p}1_{\bar{\nu}}.$$

The case $|i-j|>1$ is very easy to prove since all i and j labeled terms commute. □

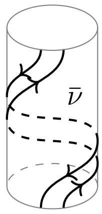
2.6 Horizontal trace of $\mathcal{U}$

The trace of a 2-endomorphism in $\mathcal{U}$ is obtained by closing the matching loose ends of the strands to the right on a plane. Alternatively, it can be described by closing it around a cylinder in the space, for example,



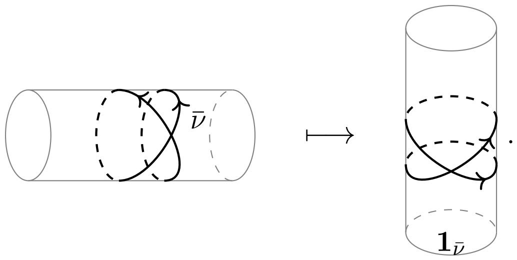
. (2.52)

We defined the horizontal trace of a linear 2-category in a general setting. The horizontal trace of $\mathcal{U}$ is a linear category with object set of 1-endomorphisms of $\mathcal{U}$ and hom-sets are 2-morphisms of $\mathcal{U}$ drawn on a cylinder in the horizontal direction. For example, the following is a morphism in $\mathrm{Tr}^h\,\mathcal{U}$:



$$. \tag{2.53}$$

Sometimes the usual trace is called the *vertical trace.* Recall that the vertical trace can be embedded into the horizontal trace in a canonical way. This embedding sends on object in $\mathrm{Tr}\,\mathcal{U}$ to the identity 1-morphism of that object. There is a nice diagrammatic description of an injection $\mathrm{Tr}\,\mathcal{U} \hookrightarrow \mathrm{Tr}^h\,\mathcal{U}$ on the morphisms. It is given by turning cylinders by 90 degrees to the right:



Chapter II

Trace and 2-representations

In this chapter, 2-representations of categorified quantum $\mathfrak{sl}_n$ and their traces will be discussed. The main object of study is the 2-category of bimodules over cohomology of flag varieties and construction of current algebra modules. We also mention Rouquier's universal categorification, projective module category of KLR algebras and category of foams. We start by defining local Weyl modules and their duals, which will play a central role for the main results in the following sections.

3 Local Weyl modules of current algebra

3.1 Local Weyl modules

Let X be the integral $\mathfrak{sl}_n$ weight lattice and $\bar{\lambda} \in X$ be a dominant integral weight. There is a partial order on X, defined as $\bar{\lambda} \geq \bar{\nu}$ if $\bar{\lambda} - \bar{\nu}$ is a positive linear combination of simple roots. A $\mathfrak{sl}_n$-module is called a *weight module* if it is a direct sum of its weight spaces, and an *integrable module* if E_i and F_i act nilpotently for all $i \in I$.

We define $V(\bar{\lambda})$ to be the $\mathbf{U}(\mathfrak{sl}_n)$-module generated by a vector $v_{\bar{\lambda}}$ over $\mathbf{U}(\mathfrak{sl}_n)$, subject to the following relations for all $i \in I$:

$$E_i v_{\bar{\lambda}} = F_i^{\bar{\lambda}_i+1} v_{\bar{\lambda}} = 0, \qquad H_i v_{\bar{\lambda}} = \bar{\lambda}_i v_{\bar{\lambda}}.$$

$V(\bar{\lambda})$ is the unique, up to isomorphism, finite-dimensional irreducible $\mathbf{U}(\mathfrak{sl}_n)$-module with the highest weight $\bar{\lambda}$. We can also view $V(\bar{\lambda})$ as a $\mathbf{U}(\mathfrak{sl}_n[t])$-module, where the action of positive degree elements of $\mathbf{U}(\mathfrak{sl}_n[t])$ are trivial. Let $\mathbb{Z}[X]$ be the integral group ring spanned by elements $e(\bar\nu)$, $\bar\nu \in X$. We call the element

$$\operatorname{ch} V(\bar\lambda) = \sum_{\bar\lambda \geq \bar\nu} \dim_{\mathbb{C}} V_{\bar\nu}(\bar\lambda)\, e(\bar\nu)$$

in $\mathbb{Z}[X]$ the *character* of $V(\bar\lambda)$, where $V_{\bar\nu}(\bar\lambda) = \{v \in V(\bar\lambda) \mid H_i v = \bar\nu_i v\}$.

In this section we give a short introduction to $\mathbf{U}(\mathfrak{sl}_n[t])$-modules, called local Weyl modules. The simple $\mathfrak{sl}_n$-modules $V(\bar\lambda)$ are sometimes also called Weyl modules. In this work, however, we apply the term only to the following current algebra modules.

Definition 3.1.1. *The local Weyl module $W(\bar\lambda)$ is the $\mathbf{U}(\mathfrak{sl}_n[t])$-module generated by the element $v_{\bar\lambda}$ over $\mathbf{U}(\mathfrak{sl}_n[t])$ together with the following relations for all $i \in I$ and $j \geq 0$:*

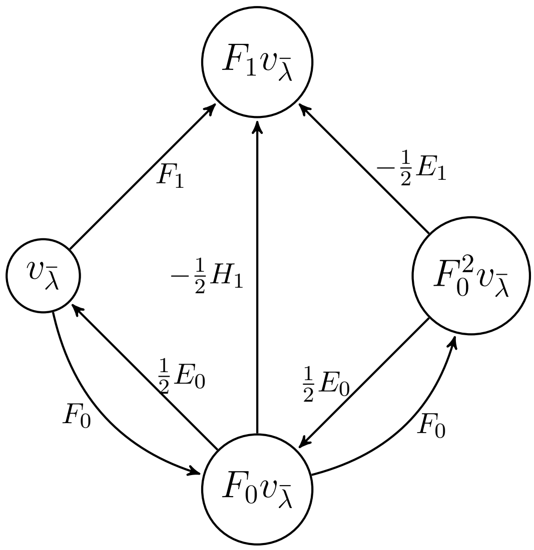
$$E_{i,j} v_{\bar\lambda} = F_{i,0}^{\bar\lambda_i+1} v_{\bar\lambda} = 0, \qquad H_{i,0} v_{\bar\lambda} = \delta_{j,0}\, \bar\lambda_i v_{\bar\lambda}. \tag{3.1}$$

$W(\bar\lambda)$ is a weight module, that is, it decomposes into a direct sum of weight spaces:

$$W(\bar\lambda) = \bigoplus_{\bar\lambda \geq \bar\nu} W_{\bar\nu}(\bar\lambda), \tag{3.2}$$

where $W_{\bar\nu}(\bar\lambda) = \{v \in W(\bar\lambda) \mid H_{i,0} v = \bar\nu_i v\}$.

In the simplest case $n = 2$ and $\bar\lambda = 2\omega_1$, the local Weyl module and the current algebra action can be illustrated by the following graph. The upward oriented edge describes the action of $H_1 := H_{1,1}$, the left and right oriented edges shows the actions of $E_j := E_{1,j}$ and $F_j := F_{1,j}$, respectively.



As a module generated by a single element $v_{\bar{\lambda}}$ over a graded algebra, $W(\bar{\lambda})$ inherits the degree from the current algebra, where we set the degree of $v_{\bar{\lambda}}$ to zero. Since the degree of t-parameter of the current algebra is 2, we can write $W(\bar{\lambda})$ as

$$W(\bar{\lambda}) = \bigoplus_{r\geq 0} W(\bar{\lambda})\{2r\},$$

where each homogeneous $2r$-graded piece $W(\bar{\lambda})\{2r\} = \{v \in W(\bar{\lambda}), \deg(v) = 2r\}$ is a $\mathfrak{sl}_n$-module. Then the character of a local Weyl module $W(\bar{\lambda})$ is

$$\operatorname{ch}_t W(\bar{\lambda}) = \sum_{r\geq 0} \operatorname{ch} W(\bar{\lambda})\{2r\}\, t^r. \tag{3.3}$$

The local Weyl modules have the following universal property.

Theorem 3.1.2. *[25] Any finite-dimensional current algebra module generated by an element $v_{\bar{\lambda}}$ satisfying the relations* (3.1) *is a quotient of the local Weyl module $W(\bar{\lambda})$.*

3.2 Duals of local Weyl modules

Let $\mathcal{P}(n, N)$ be the set of n-compositions of N. We define a partial order, called the *dominance order*, on $\mathcal{P}(n, N)$ by setting $\mu \geq \nu$ for $\nu, \mu \in \mathcal{P}(n, N)$ if $\mu - \nu$ is a positive linear composition of simple roots.

For any compositions $\nu, \mu \in \mathcal{P}(n, N)$ define the integer

$$d_\nu^\mu = \max\{(\mu, \mu) - (\nu, \nu), 0\}. \tag{3.4}$$

Notice that d_ν^μ is an even non-negative integer, and it depends on $\bar{\mu}$ and $\bar{\nu}$ rather than μ and ν. To see this, take the partition $\mu' = \mu + (m, m, \dots, m) \in \mathcal{P} + (n, N + nm)$ and $\nu' = \nu + (m, m, \dots, m) \in \mathcal{P} + (n, N + nm)$ for any $m \in \mathbb{Z}_+$. Then we have $d_{\nu'}^{\mu'} = d_\nu^\mu$.

We now define the duals of $V(\bar{\lambda})$ and $W(\bar{\lambda})$ as follows. Let $V^*_{\bar{\nu}}(\bar{\lambda})$ be the dual vector space of the weight space $V_{\bar{\nu}}(\bar{\lambda})$. Namely, for each $w \in V_{\bar{\nu}}(\bar{\lambda})$, $V^*_{\bar{\nu}}(\bar{\lambda})$ is spanned by linear maps $\delta_w \colon V_{\bar{\nu}}(\bar{\lambda}) \to \mathbb{C}$ such that $\delta_w(v)$ equals the Kronecker delta function $\delta_{w,v}$.

The dual of $V(\bar{\lambda})$ is the $\mathbf{U}(\mathfrak{sl}_n)$ -module

$$V^*(\bar{\lambda}) = \bigoplus_{\bar{\lambda}\geq\bar{\nu}} V^*_{\bar{\nu}}(\bar{\lambda})$$

in which the action is defined as

$$(x \cdot \delta_w)v = \delta_w(\omega(x)v). \tag{3.5}$$

In the equation (3.7), $v, w \in V(\bar{\lambda})$, $\delta_w \in V^*(\bar{\lambda})$, $x \in \mathbf{U}(\mathfrak{sl}_n)$ and ω is the anti-involution on $\mathbf{U}(\mathfrak{sl}_n)$ which sends F_i to E_i and fixes H_i. Under this definition the dual $V^*(\bar{\lambda})$ is isomorphic to $V(\bar{\lambda})$ as a $\mathbf{U}(\mathfrak{sl}_n)$-module.

As we will see next, the dual local Weyl module $W^*(\bar{\lambda})$ is defined similarly, by dualizing each weight space and taking the direct sum. However, we have to consider the fact that the weight spaces $W_{\bar{\nu}}(\bar{\lambda})$, $\bar{\nu} \in X$ are graded. We will need the following statement which is proven in Proposition 4.2 in [52].

Proposition 3.2.1. *The top degree elements of the weight space $W_{\bar{\nu}}(\bar{\lambda})$ have degree d_ν^λ.*

Definition 3.2.2. *Let $W^*_{\bar{\nu}}(\bar{\lambda})$ be the dual space of the vector space $W_{\bar{\nu}}(\bar{\lambda})$, spanned by elements $\{\delta_w\}_{w \in W_{\bar{\nu}}(\bar{\lambda})}$. We define the grading on $W^*_{\bar{\nu}}(\bar{\lambda})$ as follows:*

$$W^*_{\bar{\nu}}(\bar{\lambda})\{2r\} := \left(W_{\bar{\nu}}(\bar{\lambda})\{d_\nu^\lambda - 2r\}\right)^*,$$

and let $W^*_{\bar{\nu}}(\bar{\lambda}) = \bigoplus_{r \geq 0} W^*_{\bar{\nu}}(\bar{\lambda})\{2r\}$. Then

$$W^*(\bar{\lambda}) := \bigoplus_{\bar{\lambda} \geq \bar{\nu}} W^*_{\bar{\nu}}(\bar{\lambda}), \tag{3.6}$$

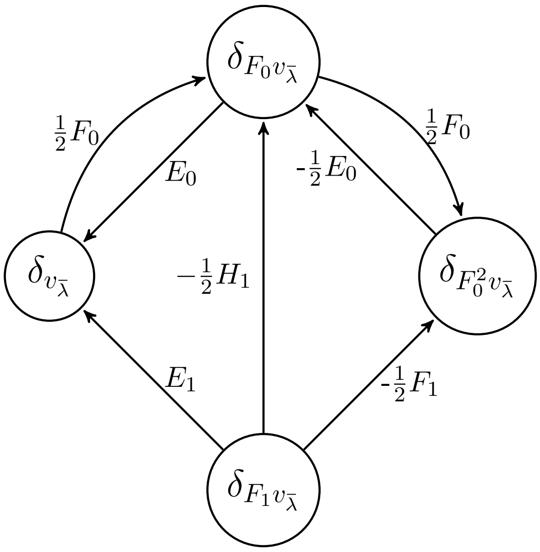
is called the dual Weyl module of highest weight $\bar{\lambda}$, together with a current algebra action

$$(x \cdot \delta_w)v = \delta_w(\omega(x)v), \tag{3.7}$$

where $v, w \in W(\bar{\lambda})$, $x \in \mathbf{U}(\mathfrak{sl}_n[t])$, and ω is the anti-involution on $\mathbf{U}(\mathfrak{sl}_n[t])$ which sends $F_{i,j}$ to $E_{i,j}$ and fixes $H_{i,j}$.

There is a non-degenerate bilinear pairing $(\delta_w, v) \mapsto \delta_w(v)$ of degree d_ν^λ between $W^*_{\bar{\nu}}(\bar{\lambda})$ and $W_{\bar{\nu}}(\bar{\lambda})$. Note that current algebra action on local Weyl modules preserves the degree, but the generators $E_{i,j}$ and $F_{i,j}$ does not preserve the degree on dual Weyl module. The action of $H_{i,j} \colon W^*_{\bar{\nu}}(\bar{\lambda}) \to W^*_{\bar{\nu}}(\bar{\lambda})$ increases the degree by $2j$, while $E_{i,j} \colon W^*_{\bar{\nu}}(\bar{\lambda}) \to W^*_{\overline{\nu+\alpha_i}}(\bar{\lambda})$ increases the degree by $d_\nu^\lambda - d_{\nu+\alpha_i}^\lambda + 2j = 2(-\bar{\nu}_i - 1 + j)$, and $F_{i,j} \colon W^*_{\bar{\nu}}(\bar{\lambda}) \to W^*_{\overline{\nu-\alpha_i}}(\bar{\lambda})$ increases the degree by $d_\nu^\lambda - d_{\nu-\alpha_i}^\lambda + 2j = 2(\bar{\nu}_i - 1 + j)$.

For the case $n = 2$ and $\bar{\lambda} = 2\omega$, the dual Weyl module $W^*(\bar{\lambda})$ and the current algebra action has the following graphical description:



4 2-representations

While classical representation theory studies actions on vector spaces, 2-representation theory concerns with actions on $\Bbbk$-linear categories. Instead of linear maps, 2-representation theory studies linear functors and natural transformations between them. There has been significant progress in understanding 2-representations of $\mathcal{U}$. In this section we define some important examples of 2-representations of $\mathcal{U}$.

4.1 Integrable 2-representations

Definition 4.1.1. *A 2-representation of $\mathcal{U}$ is a graded, additive 2-functor $\Phi\colon \mathcal{U} \to \mathcal{K}$ for some graded, additive 2-category $\mathcal{K}$.*

We will denote the image of an object $\bar{\nu} \in \mathrm{Ob}(\mathcal{U})$ under the 2-functor by $\mathcal{K}_{\bar{\nu}}$. A 2-representation $\Phi\colon \mathcal{U} \to \mathcal{K}$ is called *integrable* if the 1-morphisms $\Phi(\mathcal{E}_i)$ and $\Phi(\mathcal{F}_i)$ are locally nilpotent for all $i \in I$. Integrability condition implies that only finite number of objects can be non-zero in $\mathcal{K}$.

Definition 4.1.2. *Let $\Phi_1\colon \mathcal{U} \to \mathcal{K}_1$ and $\Phi_2\colon \mathcal{U} \to \mathcal{K}_2$ be two representations. A strongly equivariant functor $\gamma\colon \mathcal{K}_1 \to \mathcal{K}_2$ is a collection of functors $\gamma(\bar{\nu})\colon \Phi_1(\bar{\nu}) \to \Phi_2(\bar{\nu})$ for each $\bar{\nu} \in \mathrm{Ob}(\mathcal{U})$ together with natural isomorphisms of functors $c_u\colon \gamma \circ \Phi_1(u) \cong \Phi_2(u) \circ \gamma$ for every 1-morphism $u \in \mathcal{U}$ such that*

$$c_v(\mathrm{Id}_\gamma \circ \Phi_1(\alpha)) \cong (\Phi_2(\alpha) \circ \mathrm{Id}_\gamma)c_u \tag{4.1}$$

for every 2-morphism $\alpha\colon u \to v$ in $\mathcal{U}$. In (4.1), we use $\circ$ for horizontal composition, and multiplication for vertical composition of 2-morphisms. We call γ a strongly equivariant equivalence if $\gamma(\bar{\nu})$ is an equivalence for each $\bar{\nu} \in \mathrm{Ob}(\mathcal{U})$.

Fix a highest weight $\bar{\lambda}$. We define Rouquier's universal categorification $\mathcal{L}(\bar{\lambda})$ of the simple $\mathfrak{sl}_n$-module $V(\bar{\lambda})$ following Brundan-Davidson [14]. Let $\Phi_{\bar{\lambda}}\colon \mathcal{U} \to \mathcal{R}(\bar{\lambda})$ be the 2-representation defined by $\mathcal{R}(\bar{\lambda})_{\bar{\nu}} = \mathrm{Hom}_{\mathcal{U}}(\bar{\lambda}, \bar{\nu})$, in words, $\Phi_{\bar{\lambda}}$ sends each $\bar{\nu} \in \mathrm{Ob}(\mathcal{U})$ to a the graded category of 1-morphisms $\mathrm{Hom}_{\mathcal{U}}(\bar{\lambda}, \bar{\nu})$ in $\mathcal{U}$. The 1-morphisms $\Phi_{\bar{\lambda}}(\mathcal{E}_i)$ and $\Phi_{\bar{\lambda}}(\mathcal{F}_i)$ are composing 1-morphisms on the left by $\mathcal{E}_i$ and $\mathcal{F}_i$, and 2-morphisms horizontally on the left by $\uparrow_i^{\bar{\nu}}$ and $\downarrow_i^{\bar{\nu}}$ respectively. The 2-morphisms are generated by the 2-morphisms in $\mathcal{U}$.

Now let $\mathcal{I}(\bar{\lambda})$ be the full, additive 2-subcategory of $\mathcal{R}(\bar{\lambda})$ in which the objects are generated by objects of the form $R\,\mathcal{E}_i$, $i \in I$ for a 1-morphism R in $\mathcal{U}$ from $\overline{\lambda + \alpha_i}$ to $\bar{\nu}$. We define the 2-category $\mathcal{V}(\bar{\lambda}) = \mathcal{R}(\bar{\lambda})/\mathcal{I}(\bar{\lambda})$ as a quotient of additive 2-categories. $\mathcal{V}(\bar{\lambda})$ has the same objects as in $\mathcal{R}(\bar{\lambda})$ and $\mathrm{Hom}_{\mathcal{V}(\bar{\lambda})}(a,b) = \mathrm{Hom}_{\mathcal{R}(\bar{\lambda})}(a,b)/\mathrm{Hom}_{\mathcal{I}(\bar{\lambda})}(a,b)$. $\Phi_{\bar{\lambda}}\colon \mathcal{U} \to \mathcal{R}(\bar{\lambda})$ induces a well-defined 2-representation $\Phi_{\bar{\lambda}}\colon \mathcal{U} \to \mathcal{V}(\bar{\lambda})$.

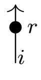
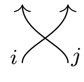
There is another construction of a 2-representation called the *minimal categorification* $\Phi^{\min}_{\bar{\lambda}}\colon \mathcal{U} \to \mathcal{V}_{\min}(\bar{\lambda})$ of the irreducible $\mathfrak{sl}_n$-module $V(\bar{\lambda})$. The image $\Phi^{\min}_{\bar{\lambda}}(\bar{\lambda})$ of the object $\bar{\lambda}$ is isomorphic to the ground field $\Bbbk$, and the image of 2-morphisms of $\mathcal{U}$ in $\mathcal{V}_{\min}(\bar{\lambda})$ is finite-dimensional. Actually, Rouquier [53, 54] proves that any 2-representation Φ which categorifies $V(\bar{\lambda})$ such that $\Phi(\bar{\lambda}) \simeq \Bbbk$ is strongly equivariantly equivalent to $\Phi^{\min}_{\bar{\lambda}}$.

The 2-representations $\Phi_{\bar{\lambda}}$ and $\Phi^{\min}_{\bar{\lambda}}$ are integrable. They are related to module categories of cyclotomic KLR algebras as we will see next.

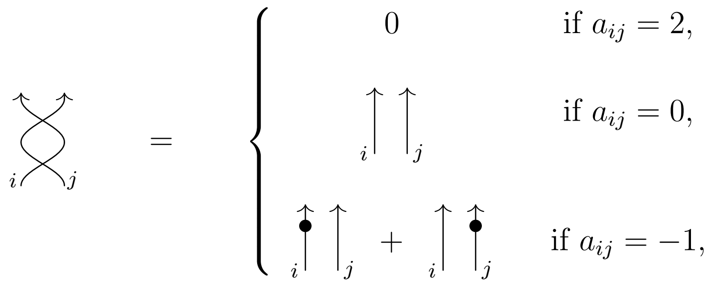
4.2 Cyclotomic KLR algebras

In this subsection we define the cyclotomic quotients of KLR algebras. Their module category is a 2-representation and has an interesting trace decategorification. We start by defining KLR algebras using the diagrammatics that is given in [33].

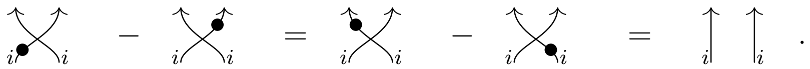
Let $\beta = \sum_{i \in I} \beta_i \alpha_i$ be a linear combination of simple roots over $\mathbb{Z}^+$. Let $R(\beta)$ be the graded $\Bbbk$-algebra generated by diagrams

x_i^r =  , $i \in I, r \geq 0$ $\qquad$ τ_{ij} =  , $i, j \in I$ $\qquad$ (4.2)

with β_i number of i labeled strands subject to the relations (2.15)-(2.19). The multiplication is given by stacking two diagrams on top of each other from bottom to top if the labels of strands at the conjunction match, and zero otherwise. The diagrams are subject to the following relations:



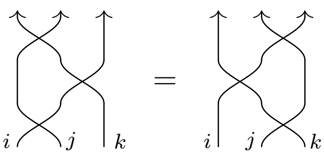
(4.3)


(4.4)

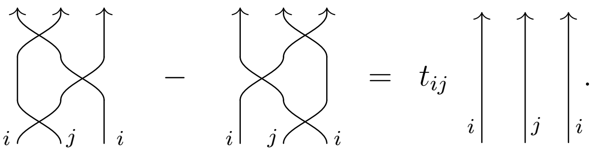
For $i \neq j$ we also have


(4.5)

If $i \neq k$ and $a_{ij} \geq 0$, the relation


(4.6)

holds. Otherwise if $a_{ij} = -1$, we have

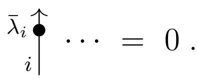

(4.7)

The multiplication is zero if the labels at the conjunction of diagrams do not match. We will call both $R(\beta)$ and $R = \bigoplus_\beta R(\beta)$ KLR algebra when there is no confusion. Let $R(\beta)$-pMod be the category of graded, finitely-generated, projective $R(\beta)$-modules. In [33] Proposition 3.18, Khovanov and Lauda showed that

$$R\text{-pMod} = \bigoplus_\beta R(\beta)\text{-pMod}$$

categorifies the positive half ${}_{\mathcal{A}}\dot{\mathbf{U}}_q^+(\mathfrak{sl}_n)$ of ${}_{\mathcal{A}}\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$.

Definition 4.2.1. *For a dominant $\mathfrak{sl}_n$ weight $\bar{\lambda} \in X$, the cyclotomic quotient $R^{\bar{\lambda}}(\beta)$ of the KLR algebra $R(\beta)$ is the quotient, in which leftmost strand with color i and $\bar{\lambda}_i$ dots is set to be zero for all $i \in I$:*


(4.8)

The algebra

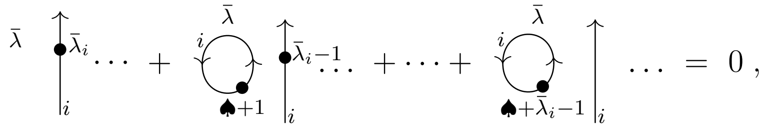
$$R^{\bar{\lambda}} := \bigoplus_{\beta \in \mathbb{N}[I]} R^{\bar{\lambda}}(\beta). \tag{4.9}$$

is called cyclotomic KLR algebra and its summand $R^{\bar{\lambda}}(\beta)$ is called cyclotomic KLR algebra of rank β.

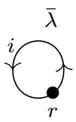
$R^{\bar{\lambda}}$ is a graded, unital and finite-dimensional algebra. The algebra $R^{\bar{\lambda}}$ is interesting for the following reason. There is a 2-representation $\mathcal{U} \to R^{\bar{\lambda}}$-pMod (see Theorem 3.17 in [61] for the details). $R^{\bar{\lambda}}$-pMod categorifies the irreducible finite-dimensional ${}_{\mathcal{A}}\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$-module with the highest weight $\bar{\lambda}$. This result was first conjectured by Khovanov and Lauda in [33], and proved later independently by Brundan and Kleshchev [16], Kang and Kashiwara [32], Webster [61].

We now define a deformation of $R^{\bar{\lambda}}$ according to Webster [61]. Let $\mathrm{End}_{\mathcal{U}}(\oplus_{\mathbf{i}} 1_{\bar{\lambda}}\mathcal{E}_{\mathbf{i}})$ be the endomorphism algebra of directs sums of 1-morphisms in $\mathcal{U}$ of form $1_{\bar{\lambda}}\mathcal{E}_{i_1}^{a_1} \dots \mathcal{E}_{i_l}^{a_l}$, $l \geq 0$, $a_1, \dots, a_l \geq 0$, $i_1, \dots i_l \in I$.

Definition 4.2.2. *Let $\check{R}^{\bar{\lambda}}$ be the quotient of* $\mathrm{End}_{\mathcal{U}}(\oplus_{\mathbf{i}} 1_{\bar{\lambda}}\mathcal{E}_{\mathbf{i}})$ *by the relations*



(4.10)



$= 0$ *for all $r \geq 0$.* (4.11)

for all $i \in I$.

The following result is due to Webster [61], section 3.5:

Theorem 4.2.3. *The algebra $\check{R}^{\bar{\lambda}}$ is a flat deformation of $R^{\bar{\lambda}}$.*

Unlike $R^{\bar{\lambda}}$, $\check{R}^{\bar{\lambda}}$ is infinite-dimensional. The category $\check{R}^{\bar{\lambda}}$-pMod also admits an action of $\mathcal{U}$, and this action gives another categorification of $V(\bar{\lambda})$. The next theorem, proven by Rouquier (see Theorem 4.19 in [13]), relates universal categorification $\mathcal{V}(\bar{\lambda})$ and $\check{R}^{\bar{\lambda}}$-*pMod.*

Theorem 4.2.4. *There is a strongly equivariant equivalence between* $\check{R}^{\bar{\lambda}}$-pMod *and the Karoubi envelope of $\mathcal{V}(\bar{\lambda})$.*

The next theorem shows the universality property of $\check{R}^{\bar{\lambda}}$-pMod (see Corollary 5.7 in Rouquier [53]). It emphasizes the importance of $\check{R}^{\bar{\lambda}}$-pMod in higher representation theory and its relation to other integrable 2-representations.

Theorem 4.2.5. *Given a 2-representation $\Phi\colon \mathcal{U} \to \mathcal{K}$ for some additive, idempotent complete 2-category such that $\Phi(\mathcal{E}_i)(\mathcal{K}_{\bar{\lambda}}) = 0$ for all $i \in I$ there is a unique strongly equivariant functor $\check{R}^{\bar{\lambda}}$-pMod $\to \mathcal{K}$.*

Let $\Phi\colon \mathcal{U} \to \mathcal{K}$ be a 2-representation. By functoriality of the trace, Φ induces a $\mathbb{C}$-algebra homomorphism

$$\operatorname{Tr}\mathcal{U} \xrightarrow{\operatorname{Tr}\Phi} \operatorname{Tr}\mathcal{K}. \tag{4.12}$$

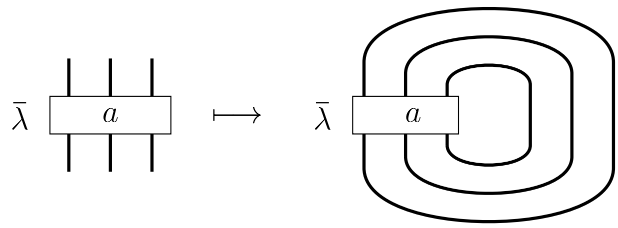
We compose $\operatorname{Tr}\Phi$ with $\dot{\mathbf{U}}(\mathfrak{sl}_n[t]) \xrightarrow{\rho} \operatorname{Tr}\mathcal{U}$ to obtain a $\mathbb{C}$-algebra homomorphism

$$\dot{\mathbf{U}}(\mathfrak{sl}_n[t]) \xrightarrow{\operatorname{Tr}\Phi} \operatorname{Tr}\mathcal{K}. \tag{4.13}$$

Since $\mathcal{K}$ is a cyclic 2-category, $\operatorname{Tr}\mathcal{K}$ acts on the center of objects $Z(\mathcal{K})$. Thus, combining this action with the algebra homomorphism (4.13), we deduce that $Z(\mathcal{K})$ is a current algebra module.

Let $Z(R^{\bar{\lambda}}\text{-pMod}) = \bigoplus_{\bar{\nu}\leq\bar{\lambda}} Z(R^{\bar{\lambda}}(\bar{\lambda}-\bar{\nu}))$-pMod be the center of objects of $R^{\bar{\lambda}}$-pMod, and it is the same as the center $Z(R^{\bar{\lambda}})$ of the cyclotomic KLR algebra $R^{\bar{\lambda}}$. Hence, by the homomorphism (4.13), both $\operatorname{Tr} R^{\bar{\lambda}}$ and $Z(R^{\bar{\lambda}})$ becomes a current algebra module.

There is a map $t\colon R^{\bar{\lambda}} \to \mathbb{C}$ which is a Frobenius trace (see Theorem 3.18, [61]). This means that the kernel of the map $t\colon R^{\bar{\lambda}} \to \mathbb{C}$ contains no nonzero left ideal of $R^{\bar{\lambda}}$. The map can be realized diagrammatically by closing the diagram representing an element in $R^{\bar{\lambda}}$ to the right. We obtain an element in $R^{\bar{\lambda}}(0) \simeq \mathbb{C}$:



(4.14)

The map $t\colon R^{\bar{\lambda}} \to \mathbb{C}$ is symmetric, that is, $t(aa') = t(a'a)$ for all $a, a' \in R^{\bar{\lambda}}$. This map induces a non-degenerate $Z(R^{\bar{\lambda}}(\beta))$-bilinear form $t\colon Z(R^{\bar{\lambda}}(\beta)) \times \operatorname{Tr} R^{\bar{\lambda}}(\beta) \to \mathbb{C}$ of degree $-d^{\bar{\lambda}}_{\bar{\lambda}-\beta}$(see [56], Prop. 3.10).

We will need the following crucial result, which is proven by independently Beliakova-Habiro-Lauda-Webster [5], Theorem 7.3 and Shan-Varagnolo-Vasserot [56], Theorem 3.34.

Theorem 4.2.6. *As a current algebra module, $\operatorname{Tr} R^{\bar{\lambda}}$ is isomorphic to the local Weyl module $W(\bar{\lambda})$. Dually, $Z(R^{\bar{\lambda}})$ is isomorphic to the dual local Weyl module $W^*(\bar{\lambda})$.*

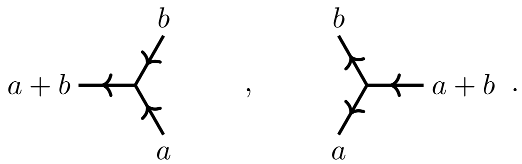
4.3 Web algebra and foamation 2-representation

In this subsection we give a short overview of quantum skew duality, foamation 2-functor and Queffelec-Rose annular $\mathfrak{sl}_m$ link homology. Foamation 2-representation is interesting for it has a direct application in Khovanov-Rozansky homology theory.

Quantum skew Howe duality

Fix an integer $m \geq 2$ and let $\mathcal{P}_m(n, N)$ be the set of n-compositions of N with entries in the set $\{0, 1, \dots, m\}$. The action of the quantum group $\mathbf{U}_q(\mathfrak{sl}_m)$ on the vector representation $\mathbb{C}_q^m$ induces action on the quantized wedge product $\wedge_q^a \mathbb{C}_q^m$. The $\mathbf{U}_q(\mathfrak{sl}_m)$-modules $\wedge_q^a \mathbb{C}_q^m$, $1 \leq a \leq m$ are called fundamental representations. Cautis-Kamnitzer-Morrison [19] use the $\mathfrak{sl}_m$ web category to describe the pivotal category $\mathbf{grRep}(\mathfrak{sl}_m)$ of $\mathbf{U}_q(\mathfrak{sl}_m)$-modules generated by the fundamental representations. The category $m\mathbf{Web}_n(N)$ is defined as follows.

- Objects are the elements $\nu \in \mathcal{P}_m(n, N)$ labeling n points in the interval $[0, 1]$, together with a zero object.
- *Webs* are directed, labeled trivalent graphs with boundary, for example,



(4.15)

  The composition of webs is joining the webs from right to left if the labels at the adjunction match, and zero otherwise. Morphisms in $m\mathbf{Web}_n(N)$ are given by $\mathbb{C}(q)$-linear combinations of webs, generated by two webs (4.15), modulo planar isotopies and relations. We read webs from right to left. Edges labeled zero are erased. If a label lies outside the range $\{0, \dots, n\}$, then the morphism is zero.

The object $\nu \in \mathcal{P}_m(n, N)$ corresponds to the $\mathbf{U}_q(\mathfrak{sl}_m)$-module

$$\wedge_q^{\nu} \mathbb{C}_q^m := \wedge_q^{\nu_1} \mathbb{C}_q^m \otimes \cdots \otimes \wedge_q^{\nu_n} \mathbb{C}_q^m,$$

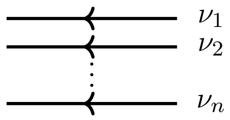
and the webs (4.15) are the unique (up to a scalar multiple) canonical maps $\wedge_q^{a+b} \mathbb{C}_q^m \to \wedge_q^a \mathbb{C}_q^m \otimes \wedge_q^b \mathbb{C}_q^m$ and $\wedge_q^a \mathbb{C}_q^m \otimes \wedge_q^b \mathbb{C}_q^m \to \wedge_q^{a+b} \mathbb{C}_q^m$.

Cautis-Kamnitzer-Morrison [19] prove the existence of the functor

$$\varphi_{m,N} \colon \dot{\mathbf{U}}_q(\mathfrak{sl}_n) \to m\mathbf{Web}_n(N), \tag{4.16}$$

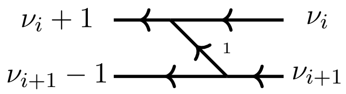
called the *quantum skew Howe duality*. On the level of objects, $\varphi_{m,N}$ sends the $\mathfrak{sl}_n$ weight μ to the composition $\nu \in \mathcal{P}_m(n, N)$ if there is ν that satisfies $\mu = \bar{\nu}$, and to the zero object otherwise. On the level of objects, $\varphi_{m,N}$ is defined as follows:

$$\varphi_{m,N}(\mathbf{1}_{\bar{\nu}}) \quad := \tag{4.17}$$

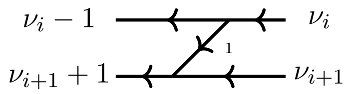


,

$\varphi_{m,N}(E_i\mathbf{1}_{\bar{\nu}}) \;:=$



, $\quad \varphi_{m,N}(F_i\mathbf{1}_{\bar{\nu}}) \;:=$



.

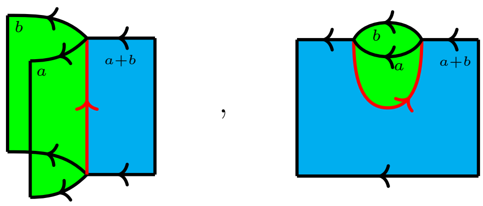
Cautis-Kamnitzer-Morrison [19] build on the previous work on the geometric quantum skew Howe duality by Cautis-Kamnitzer-Licata [20] and give the list of relations between morphisms in $m\mathbf{Web}_n(N)$.

The foamation 2-functor

Here we give Lauda-Queffelec-Rose (see [40] for $m = 2, 3$ and [51] for any m) combinatorial categorification of the quantum skew Howe duality. This is another example of a 2-representation of $\mathcal{U}$. This 2-representation is particularly interesting for $\mathfrak{sl}_m$ link homology theory. We only give an overview here and refer the reader to the original papers for the details.

Let $m\mathbf{Foam}_n(N)$ be the 2-category whose

- objects are the elements $\nu \in \mathcal{P}_m(n, N)$ labeling n points in the interval $[0, 1]$, together with a zero object.
- 1-morphisms are formal directs sums of the compositions of $\mathfrak{sl}_m$ web diagrams together with their formal degree shifts.
- 2-morphisms are matrices of $\Bbbk$-linear combinations of $\mathfrak{sl}_m$ *foams* – labeled, decorated singular surfaces with oriented seams whose horizontal slices are $\mathfrak{sl}_m$ webs, for example,

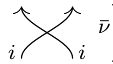


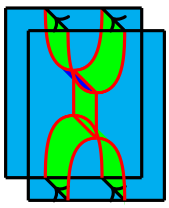
subject to local relations. Vertical and horizontal compositions of 2-morphisms are given by attaching along the relevant boundaries. Each foam is assigned a degree and the local relations preserve this degree.

We refer to [51], Section 3.1 for the precise set of generators and relations since we will not use them here. Lauda-Queffelec-Rose [40, 51] construct 2-representations

$$\Phi_{m,N} \colon \mathcal{U}_Q(\mathfrak{sl}_n) \to m\mathbf{Foam}_n(N) \tag{4.18}$$

for each $m \geq 2, N > 0$, called *foamation functors* (the term first used in [44]). On the level of objects the foamation functor $\Phi_{m,N}$ sends $\mu \in \mathrm{Ob}(\mathcal{U}_Q(\mathfrak{sl}_n))$ to the composition $\nu \in \mathcal{P}_m(n, N)$ such that $\mu = \bar{\nu}$ if such solution exists and to the zero object otherwise. On the level of 1-morphisms $\Phi_{m,N}$ maps the 1-morphisms of $\mathcal{U}_Q(\mathfrak{sl}_n)$ to the 1-morphisms (4.17) with the same name. The complete description of the images of 2-morphisms will be skipped here. For example, we have

$\Phi_{m,N}\Big($ 

$\Big) =$  .

The foamation functor $\Phi_{m,N}$ is an integrable 2-representation, and if $m \geq N$, $\Phi_{m,N}$ factors through $\check{R}^{\bar{\lambda}_0}$-*pMod* for the highest weight $\bar{\lambda}_0 = (N, 0, \dots, 0)$.

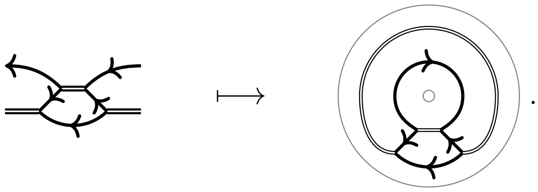
The 2-category $m\mathbf{Foam}_n(N)$ is interesting for the following reason: it categorifies the $\mathfrak{sl}_m$ web category $m\mathbf{Web}_n(N)$. Moreover, $\Phi_{m,N}$ gives a combinatorial categorification of the quantum skew Howe duality functor $\varphi_{m,N}$. In the case $m = 2$, $2\mathbf{Foam}_n(N)$ is a modification of the Bar-Natan's skein category of 2-dimensional cobordisms, originally developed by Blanchet to fix functoriality of the Khovanov homology. For general m, the 2-representation $\Phi_{m,N}$ is used in [51] to construct Khovanov-Rozansky homology combinatorially.

Let $m\mathbf{Foam}^{\star}_n(N)$ be the graded version of the 2-category $m\mathbf{Foam}_n(N)$. Then the foamation functor $\Phi_{m,N}$ extends to a 2-representation $\Phi^{\star}_{m,N} \colon \mathcal{U} \to m\mathbf{Foam}^{\star}_n(N)$. Note that since we have killed the gradings on 1-morphisms in $m\mathbf{Foam}^{\star}_n(N)$, this will translate into $q = 1$ in the split Grothendieck group and $K_0(m\mathbf{Foam}^{\star}_n(N))$ will be equivalent to the monoidal category of finite-dimensional $\mathfrak{sl}_m$ representations.

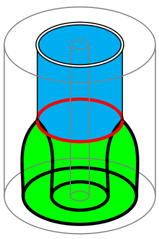
Annular $\mathfrak{sl}_n$ link homology

Queffelec and Rose [50] compose the foamation and the trace functor to construct an invariant *sutured annular Khovanov-Rozansky homology* $\mathsf{saKhR}(\mathcal{L})$ for a given colored, framed, annular link $\mathcal{L}$. We briefly discuss their construction for the rest of this section. Readers who are not familiar with the basics of link homologies may skip to the next section.

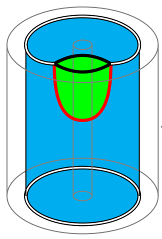
We consider the horizontal trace $\mathrm{Tr}^h\, m\mathbf{Foam}_n(N)$ of the foam 2-category. Let $\mathcal{A} := S^1 \times [0,1]$ be the annulus. The objects in $\mathrm{Tr}^h\, m\mathbf{Foam}_n(N)$ can be described as graded formal direct sums of annular closures of 1-endomorphisms in $m\mathbf{Foam}_n(N)$, for example,



The morphisms in $\mathrm{Tr}^h\, m\mathbf{Foam}_n(N)$ are matrices of $\Bbbk$-linear combinations of degree zero foams embedded in $\mathcal{A} \times [0,1]$, for example,



or



, (4.19)

modulo isotopy and local relations.

In [20] Theorem 4.3, Cautis-Kamnitzer-Licata prove that composing the quantum skew Howe duality functor with the action of the quantum Weyl group action gives the braiding in $\mathbf{grRep}(\mathfrak{sl}_m)$. In [18], Cautis categorifies the quantum Weyl group to Rickard complexes in $\mathcal{U}_Q(\mathfrak{sl}_n)$, which can be utilized to obtain categorical braid group actions on 2-representations. Lauda-Queffelec-Rose [40, 51] show that the images of Rickards complexes give the categorified skein relations to formulate Khovanov-Rozansky homology. The categorified skein relations assign a complex of webs in $m\mathbf{Foam}_n(N)$ to any framed tangle. This complex, viewed in the homotopy category of complexes in $m\mathbf{Foam}_n(N)$ is an invariant of the given framed tangle.

Since any framed link $\mathcal{L}$ in the thickened annulus $\mathcal{A} \times [0,1]$ can be presented as the annular closure of a framed tangle, we can assign to $\mathcal{L}$ a complex $C(\mathcal{L})$ in $\mathrm{Tr}^h\, m\mathbf{Foam}_n(N)$ using the functoriality of the horizontal trace.

Queffelec and Rose modify $\mathrm{Tr}\,\mathcal{U}$ and $\mathrm{Tr}\, m\mathbf{Foam}_n^\star(N)$ to $\widetilde{\mathrm{Tr}}\,\mathcal{U}_Q(\mathfrak{sl}_n)$ and $\widetilde{\mathrm{Tr}}\, m\mathbf{Foam}_n(N)$ so that the objects admit formal q-shifts and only degree zero morphisms are allowed. Since we have an algebra isomorphism $\mathrm{Tr}\,\mathcal{U} \cong \dot{\mathbf{U}}(\mathfrak{sl}_n[t])$, the morphisms in $\widetilde{\mathrm{Tr}}\mathcal{U}$ are q-shifts of elements of current algebra. Composing $\widetilde{\mathrm{Tr}}$ with the foamation functor $\Phi_{m,N}$ and setting $t = 0$ gives the map $\widetilde{\mathrm{Tr}}\,\mathcal{U} \to \mathbf{grRep}(\mathfrak{sl}_m)$ which factors through $\widetilde{\mathrm{Tr}}\, m\mathbf{Foam}_n(N)$. This induces a functor

$$\widetilde{\varphi}_{m,N} \colon \widetilde{\operatorname{Tr}}\, m\mathbf{Foam}_n(N) \to \mathbf{grRep}(\mathfrak{sl}_m). \tag{4.20}$$

Then Queffelec and Rose go on to prove that the homotopy category of complexes of in $\widetilde{\operatorname{Tr}}\, m\mathbf{Foam}_n(N)$ and in $\operatorname{Tr}^h m\mathbf{Foam}_n(N)$ are equivalent:

$$Kom(\widetilde{\operatorname{Tr}}\, m\mathbf{Foam}_n(N)) \cong Kom(\operatorname{Tr}^h m\mathbf{Foam}_n(N)).$$

Hence, for the complex $C(\mathcal{L})$ in $\operatorname{Tr}^h m\mathbf{Foam}_n(N)$ assigned to a colored annular link $\mathcal{L}$ there is a a homotopy equivalent complex $\widetilde{C}(\mathcal{L})$ in $\widetilde{\operatorname{Tr}}\, m\mathbf{Foam}_n(N)$.

We now apply the functor (4.20) on the complex $\widetilde{C}(\mathcal{L})$. The image $\widetilde{\varphi}_{m,N}\left(\widetilde{C}(\mathcal{L})\right)$ is a complex in the homotopy category of complexes in $\mathbf{grRep}(\mathfrak{sl}_m)$. The homology of this complex is the $\mathsf{saKhR}(\mathcal{L})$ homology, which is an invariant of $\mathcal{L}$. In the case $n = 2$ this homology is called *sutured annular Khovanov homology* ($\mathsf{saKh}(\mathcal{L})$).

5 The 2-category $\mathbf{Pol}_N$

In this section we will construct an integrable 2-representation of $\mathcal{U}$ using symmetric polynomials. This is very similar to Khovanov-Lauda's construction of the 2-representation using cohomology ring of flag varieties. We will compare this 2-representation to Rouquier's universal categorification.

5.1 Symmetric polynomials

We denote the set of n-compositions and n-partitions of N by $\mathcal{P}(n,N)$ and $\mathcal{P}^+(n,N)$, respectively. The algebra of polynomials $P = \mathbb{C}[X_1, X_2, \dots, X_N]$ in N commuting variables admits an action of the symmetric group S_N. The action of $\sigma \in S_N$ on a polynomial $q(X_1, X_2, \dots, X_N) \in P$ is defined as

$$\sigma q(X_1, X_2, \dots, X_N) = q(X_{\sigma(1)}, X_{\sigma(2)}, \dots, X_{\sigma(N)}).$$

We will assume that all variables are of degree 2. By $Sym_N = P^{S_N}$, we denote the subalgebra of symmetric polynomials. Consider a n-composition $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in \mathcal{P}(n,N)$ and the parabolic subgroup $S_\nu = S_{\nu_1} \times S_{\nu_2} \times \cdots \times S_{\nu_n}$. To be clear, there are many subgroups of S_N isomorphic to S_ν, but all of them are conjugate. Let $P_\nu = P^{S_\nu}$ be the subalgebra of P which is symmetric separately in the first ν_1 variables, in the second ν_2 variables and so on. We have $P_{(1,1,\dots,1)} = P$ and $P_{(N,0,\dots,0)} = Sym_N$.

For $r \geq 0$ and $1 \leq i \leq n$, let $e_r(\nu; i)$, $h_r(\nu; i)$ and $p_r(\nu; i)$ be the r-th elementary, complete and power sum symmetric polynomials respectively, in variables

$$\Omega_i = \{X_k \,|\, \nu_1 + \cdots + \nu_{i-1} + 1 \leq k \leq \nu_1 + \cdots + \nu_i\}\,.$$

For example, if $N = 6$, $n = 3$, and $\nu = (2,1,3)$, then $e_r(\nu;1) = e_r(X_1, X_2)$, $e_1(\nu;2) = e_1(X_3) = X_3$, and $e_r(\nu;3) = e_r(X_4, X_5, X_6)$. The algebra P_ν is generated by $\{e_r(\nu;i)\}_{r\geq 0,\, 1\leq i\leq n}$, as well as by $\{h_r(\nu;i)\}_{r\geq 0,\, 1\leq i\leq n}$ or by $\{p_r(\nu;i)\}_{r\geq 0,\, 1\leq i\leq n}$.

We extend our notation by defining the following symmetric polynomials:

$$h_r(\nu; i_1, \ldots, i_m) = \sum_{r_1+\cdots+r_m=r} h_{r_1}(\nu; i_1) h_{r_2}(\nu; i_2) \cdots h_{r_m}(\nu; i_m), \tag{5.1}$$

$$e_r(\nu; i_1, \ldots, i_m) = \sum_{r_1+\cdots+r_m=r} e_{r_1}(\nu; i_1) e_{r_2}(\nu; i_2) \cdots e_{r_m}(\nu; i_m), \tag{5.2}$$

where $1 \leq i_1 < \cdots < i_m \leq n$, $m > 0$. It is not difficult to see that $e_r(\nu; i_1, i_2, \ldots, i_m)$ and $h_r(\nu; i_1, i_2, \ldots, i_m)$ are the r-th elementary and complete symmetric polynomials, respectively, in variables $\Omega_{i_1} \cup \cdots \cup \Omega_{i_m}$. This follows from the following well-known identities about symmetric polynomials, which can be found in [43]. We have

$$\sum_{s=0}^{r} (-1)^s e_s(x_1, \ldots, x_n) h_{r-s}(x_1, \ldots, x_n) = 0 \quad \text{for all } r \geq 1, \tag{5.3}$$

and for $r \geq 0$ we have

$$h_r(X_1, \ldots, X_p, Y_1, \ldots, Y_q) = \sum_{s=0}^{r} h_s(X_1, \ldots, X_p) h_{r-s}(Y_1, \ldots, Y_q), \tag{5.4}$$

$$e_r(X_1, \ldots, X_p, Y_1, \ldots, Y_q) = \sum_{s=0}^{r} e_s(X_1, \ldots, X_p) e_{r-s}(Y_1, \ldots, Y_q), \tag{5.5}$$

$$h_r(Y_1, \ldots, Y_q) = \sum_{s=0}^{r} (-1)^s e_s(X_1, \ldots, X_p) h_{r-s}(X_1, \ldots, X_p, Y_1, \ldots, Y_q), \tag{5.6}$$

$$e_r(Y_1, \ldots, Y_q) = \sum_{s=0}^{r} (-1)^s h_s(X_1, \ldots, X_p) e_{r-s}(X_1, \ldots, X_p, Y_1, \ldots, Y_q). \tag{5.7}$$

By convention, $e_r(X_1, \ldots, X_p) = h_r(X_1, \ldots, X_p) = p_r(X_1, \ldots, X_p) = 0$ for $r < 0$, and $e_0(X_1, \ldots, X_p) = h_0(X_1, \ldots, X_p) = p_0(X_1, \ldots, X_p) = 1$. Moreover, $e_r(X_1, \ldots, X_p) = 0$ holds for all $r > p$.

Let $\nu = (\nu_1, \dots, \nu_n) \in \mathcal{P}(n, N)$ be a n-composition of a fixed positive integer N as before. By $\nu - \alpha_i \in \mathcal{P}(n, N)$ and $\nu + \alpha_i \in \mathcal{P}(n, N)$, we mean the linear sum

$$\nu - \alpha_i = (\nu_1, \dots, \nu_{i-1}, \nu_i - 1, \nu_{i+1} + 1, \nu_{i+2}, \dots, \nu_n),$$

$$\nu + \alpha_i = (\nu_1, \dots, \nu_{i-1}, \nu_i + 1, \nu_{i+1} - 1, \nu_{i+2}, \dots, \nu_n),$$

if all components are non-negative, otherwise we set them to $\varnothing$. We will also need the following compositions. Define the compositions $(\nu - \alpha_i, \nu) \in \mathcal{P}(n+1, N)$ and $(\nu + \alpha_i, \nu) \in \mathcal{P}(n+1, N)$

$$(\nu - \alpha_i, \nu) = (\nu_1, \dots, \nu_{i-1}, \nu_i - 1, 1, \nu_{i+1}, \nu_{i+2}, \dots, \nu_n),$$
$$(\nu + \alpha_i, \nu) = (\nu_1, \dots, \nu_{i-1}, \nu_i, 1, \nu_{i+1} - 1, \nu_{i+2}, \dots, \nu_n),$$

if all components are non-negative, and $\varnothing$ otherwise. Recall the definition of the ring $P_\nu = P^{S_\nu}$. We can also define the rings $P_{\nu-\alpha_i,\nu}$ and $P_{\nu+\alpha_i,\nu}$ corresponding to the compositions $(\nu - \alpha_i, \nu)$ and $(\nu + \alpha_i, \nu)$ as well. Notice that $P_{\nu-\alpha_i,\nu}$ is a free P_ν-module with basis $\{1, X_{k_i}, \dots, X_{k_i}^{\nu_i - 1}\}$, where $k_i = \sum_{j=1}^{i} \nu_j$, and a free $P_{\nu-\alpha_i}$-module with basis $\{1, X_{k_i}, \dots, X_{k_i}^{\nu_{i+1}}\}$. This is due to the fact that $P^{S_{l-1}}$ is a free P^{S_l} -module with basis $\{1, X_l, X_l^2, \dots, X^{l-1}\}$.

Example 5.1.1. *If $\nu = (2, 1, 3)$ and $i = 1$, then $\nu - \alpha_1 = (1, 2, 3)$, $k_1 = 2$ and $(\nu - \alpha_1, \nu) = (1, 1, 1, 3)$ and we have*

$$P_\nu = \langle e_r(X_1, X_2), X_3, e_r(X_4, X_5, X_6) \rangle,$$

$$P_{\nu-\alpha_1} = \langle X_1, e_r(X_2, X_3), e_r(X_4, X_5, X_6) \rangle,$$
$$P_{\nu-\alpha_1,\nu} = \langle X_1, X_2, X_3, e_r(X_4, X_5, X_6) \rangle.$$

It is easy to see that we have $X_1 = e_1(X_1, X_2) - X_2$, and $P_{\nu-\alpha_1,\nu}$ is a free P_ν-module with basis $\{1, X_2\}$.

We agree that $P_{\nu-\alpha_i,\nu}$ is a right P_ν-module and a left $P_{\nu-\alpha_i}$-module, although this is artificial, since $P_{\nu,\nu-\alpha_i}$ is a commutative ring. Similarly, $P_{\nu+\alpha_i,\nu}$ is a free right P_ν-module with basis $\{1, X_{k_i+1}, \dots X_{k_i+1}^{\nu_i}\}$ and a free left $P_{\nu+\alpha_i}$-module with basis $\{1, X_{k_i+1}, \dots X_{k_i+1}^{\nu_{i+1}-1}\}$. The following relations hold in $P_{\nu-\alpha_i,\nu}$:

$$e_r(\nu; i+1) = \sum_{s=0}^{r} (-1)^s e_{r-s}(\nu - \alpha_i; i+1) X_{k_i}^s. \tag{5.8}$$

$$e_r(\nu - \alpha_i; i) = \sum_{s=0}^{r} (-1)^s X_{k_i}^s e_{r-s}(\nu; i), \tag{5.9}$$

$$e_r(\nu - \alpha_i; i+1) = X_{k_i} e_{r-1}(\nu; i+1) + e_r(\nu; i+1), \tag{5.10}$$

$$e_r(\nu; i) = e_{r-1}(\nu - \alpha_i; i) X_{k_i} + e_r(\nu - \alpha_i; i), \tag{5.11}$$

$$h_r(\nu; i+1) = h_r(\nu - \alpha_i; i+1) - h_{r-1}(\nu - \alpha_i; i+1) X_{k_i}, . \tag{5.12}$$

$$h_r(\nu - \alpha_i; i) = h_r(\nu; i) - X_{k_i} h_{r-1}(\nu; i), \tag{5.13}$$

$$h_r(\nu - \alpha_i; i+1) = \sum_{s=0}^{r} X_{k_i}^s h_{r-s}(\nu; i+1), \tag{5.14}$$

$$h_r(\nu; i) = \sum_{s=0}^{r} h_{r-s}(\nu - \alpha_i; i) X_{k_i}^s, \tag{5.15}$$

which are easily seen to follow from the properties (5.4), (5.6) of symmetric polynomials. Similarly, in $P_{\nu+\alpha_i,\nu}$, we have the identities

$$e_r(\nu + \alpha_i; i+1) = \sum_{s=0}^{r} (-1)^s X_{k_i+1}^s e_{r-s}(\nu; i+1). \tag{5.16}$$

$$e_r(\nu; i) = \sum_{s=0}^{r} (-1)^s X_{k_i+1}^s e_{r-s}(\nu + \alpha_i; i), \tag{5.17}$$

$$e_r(\nu; i+1) = X_{k_i+1} e_{r-1}(\nu + \alpha_i; i+1) + e_r(\nu + \alpha_i; i+1), \tag{5.18}$$

$$e_r(\nu + \alpha_i; i) = X_{k_i+1} e_{r-1}(\nu; i) + e_r(\nu; i), \tag{5.19}$$

$$h_r(\nu + \alpha_i; i+1) = h_r(\nu; i+1) - X_{k_i+1} h_{r-1}(\nu; i+1), . \tag{5.20}$$

$$h_r(\nu; i) = h_r(\nu + \alpha_i; i) - X_{k_i+1} h_{r-1}(\nu + \alpha_i; i), \tag{5.21}$$

$$h_r(\nu; i+1) = \sum_{s=0}^{r} X_{k_i+1}^s h_{r-s}(\nu + \alpha_i; i+1), \tag{5.22}$$

$$h_r(\nu + \alpha_i; i) = \sum_{s=0}^{r} X_{k_i+1}^s h_{r-s}(\nu; i). \tag{5.23}$$

The following identities will also be useful:

$$X_{k_i}^r = \sum_{l=0}^{r} (-1)^l h_{r-l}(\nu - \alpha_i; i+1) e_l(\nu; i+1), \tag{5.24}$$

$$X_{k_i}^r = \sum_{l=0}^{r} (-1)^l e_l(\nu - \alpha_i; i) h_{r-l}(\nu; i), \tag{5.25}$$

$$X_{k_i+1}^r = \sum_{l=0}^{r} (-1)^l e_l(\nu + \alpha_i; i+1) h_{r-l}(\nu; i+1), \tag{5.26}$$

$$X_{k_i+1}^r = \sum_{l=0}^{r} (-1)^{r-l} h_l(\nu + \alpha_i; i) e_{r-l}(\nu; i). \tag{5.27}$$

We can extend the definition of the bimodule $P_{\nu+\alpha_i,\nu}$ by letting

$$P_{\nu+\alpha_i\pm\alpha_j,\nu} = P_{\nu+\alpha_i\pm\alpha_j,\nu+\alpha_i} \otimes_{P_{\nu+\alpha_i}} P_{\nu+\alpha_i,\nu},$$

and hence, inductively define the bimodule $P_{\nu+\sum m_i\alpha_i,\nu}$ for $m_i \in \mathbb{Z}$. Note that these bimodules are graded, and by $P_{\nu+\alpha_i,\nu}\langle s\rangle$ we will mean the degree shift of $P_{\nu+\alpha_i,\nu}$ by $s \in \mathbb{Z}$.

5.2 The 2-functor $\mathcal{U} \to \mathbf{Pol}_N$

Let $\mathbf{Pol}_N$ be the additive 2-category, in which

- objects are P_ν for all $\nu \in \mathcal{P}(n, N)$,
- 1-morphisms between the objects P_ν and $P_{\nu+\sum m_i\alpha_i}$ are the finite direct sums of bimodules $P_{\nu+\sum m_i\alpha_i,\nu}$,
- 2-morphisms are matrices of bimodule homomorphisms.

The next theorem is due to Khovanov and Lauda [35], however, we replace their $\mathbf{EqFlag}_N^*$ 2-category with the 2-category $\mathbf{Pol}_N$. While the 2-category $\mathbf{EqFlag}_N^*$ is presented in terms of $GL_{\mathbb{C}}(n)$-equivariant cohomology rings of partial flag varieties in [35], we can pass to $\mathbf{Pol}_N$ by identifying the Chern classes with elementary symmetric polynomials.

Theorem 5.2.1. *[35] There is a 2-representation $\Theta_N \colon \mathcal{U} \to \boldsymbol{Pol}_N$, which is defined as follows:*

- *On objects*

$$\bar{\nu} \quad \mapsto \begin{cases} P_\nu & \text{if } \nu \in \mathcal{P}(n,N) \text{ and } \bar{\nu}_i = \nu_i - \nu_{i+1} \text{ for each } i \in I, \\ 0 & \text{otherwise.} \end{cases}$$

- *On 1-morphisms*

$$\mathbf{1}_{\bar{\nu}}\langle t\rangle \quad \mapsto \quad P_\nu\langle t\rangle,$$

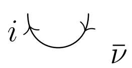
$$\mathcal{E}_i \mathbf{1}_{\bar{\nu}}\langle t\rangle \quad \mapsto P_{\nu+\alpha_i,\nu}\langle t\rangle,$$

$$\mathcal{F}_i \mathbf{1}_{\bar{\nu}}\langle t\rangle \quad \mapsto P_{\nu-\alpha_i,\nu}\langle t\rangle.$$

- *On 2-morphisms*

$\Theta_N\Bigg($  $\Bigg)$

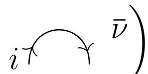
$$: \begin{cases} P_\nu & \longrightarrow \quad (P_{\nu,\nu+\alpha_i} \otimes_{P_{\nu+\alpha_i}} P_{\nu+\alpha_i,\nu})\langle 2\nu_i\rangle, \\ 1 & \mapsto \quad c_{\bar{\nu},i}^{-1} \sum_{r=0}^{\nu_i} (-1)^{\nu_i - r} X_{k_i+1}^r \otimes e_{\nu_i - r}(\nu; i), \end{cases}$$

$\Theta_N\Bigg($  $\Bigg)$

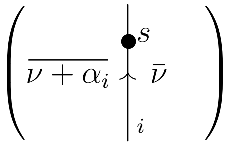
$$: \begin{cases} P_\nu & \longrightarrow \quad P_{\nu,\nu-\alpha_i} \otimes_{P_{\nu-\alpha_i}} P_{\nu-\alpha_i,\nu}\langle 2\nu_{i+1}\rangle, \\ 1 & \mapsto \quad \sum_{r=0}^{\nu_{i+1}} (-1)^{\nu_{i+1} - r} X_{k_i}^r \otimes e_{\nu_{i+1} - r}(\nu; i+1), \end{cases}$$

$\Theta_N\Bigg($  $\Bigg)$

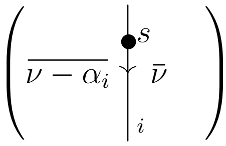
$$: \begin{cases} P_{\nu,\nu+\alpha_i} \otimes_{P_{\nu+\alpha_i}} P_{\nu+\alpha_i,\nu} & \rightarrow \quad P_\nu\langle 2(1-\nu_{i+1})\rangle, \\ X_{k_i+1}^{r_1} \otimes X_{k_i+1}^{r_2} & \mapsto \quad h_{r_1+r_2+1-\nu_{i+1}}(\nu; i+1), \end{cases}$$

$\Theta_N\Bigg($  $\Bigg)$

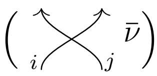
$$: \begin{cases} P_{\nu,\nu-\alpha_i} \otimes_{P_{\nu-\alpha_i}} P_{\nu-\alpha_i,\nu} & \rightarrow \quad P_\nu\langle 2(1-\nu_i)\rangle, \\ X_{k_i}^{r_1} \otimes X_{k_i}^{r_2} & \mapsto \quad c_{i,\bar{\nu}}\, h_{r_1+r_2+1-\nu_i}(\nu; i), \end{cases}$$

Θ_N 

$$:\quad \begin{cases} P_{\nu+\alpha_i,\nu} & \to \quad P_{\nu+\alpha_i,\nu}\langle 2s\rangle, \\ X^r_{k_i+1} & \mapsto \quad X^{r+s}_{k_i+1}, \quad s \geq 0, \end{cases}$$

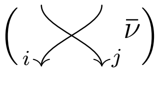
Θ_N 

$$:\quad \begin{cases} P_{\nu-\alpha_i,\nu} & \to \quad P_{\nu-\alpha_i,\nu}\langle 2s\rangle, \\ X^r_{k_i} & \mapsto \quad X^{r+s}_{k_i}, \quad s \geq 0, \end{cases}$$

Θ_N 

$$: \quad P_{\nu+\alpha_i+\alpha_j,\nu+\alpha_j} \otimes_{P_{\nu+\alpha_j}} P_{\nu+\alpha_j,\nu} \to P_{\nu+\alpha_i+\alpha_j,\nu+\alpha_i} \otimes_{P_{\nu+\alpha_i}} P_{\nu+\alpha_i,\nu}\langle -a_{ij}\rangle,$$

$$X^{r_1}_{k_i+1} \otimes X^{r_2}_{k_j+1} \quad \mapsto \quad \begin{cases} \sum_{f=0}^{r_1-1} X^{r_1+r_2-1-f}_{k_i+2} \otimes X^f_{k_i+1} - \sum_{g=0}^{r_2-1} X^{r_1+r_2-1-g}_{k_i+2} \otimes X^g_{k_i+1} & \textit{if } a_{ij}=2, \\ \left(t_{ij} X^{r_2}_{k_j+1} \otimes X^{r_1+1}_{k_i+1} + t_{ji} X^{r_2+1}_{k_j+1} \otimes X^{r_1}_{k_i+1}\right)\langle -1\rangle & \textit{if } i=j+1, \\ \left(X^{r_2}_{k_j+1} \otimes X^{r_1}_{k_i+1}\right)\langle 1\rangle & \textit{if } i=j-1, \\ X^{r_2}_{k_j+1} \otimes X^{r_1}_{k_i+1} & \textit{if } a_{ij}=0, \end{cases}$$

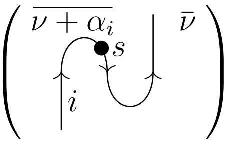
Θ_N 

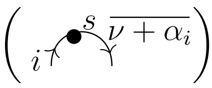
$$: \quad P_{\nu-\alpha_i-\alpha_j,\nu-\alpha_j} \otimes_{P_{\nu-\alpha_j}} P_{\nu-\alpha_j,\nu} \to P_{\nu-\alpha_i-\alpha_j,\nu-\alpha_i} \otimes_{P_{\nu-\alpha_i}} P_{\nu-\alpha_i,\nu}\langle -a_{ij}\rangle,$$

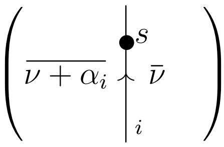
$$X^{r_1}_{k_i} \otimes X^{r_2}_{k_j} \quad \mapsto \quad \begin{cases} \sum_{f=0}^{r_2-1} X^{r_1+r_2-1-f}_{k_i-1} \otimes X^f_{k_i} - \sum_{g=0}^{r_1-1} X^{r_1+r_2-1-g}_{k_i-1} \otimes X^g_{k_i} & \textit{if } a_{ij}=2, \\ t_{ji}\left(X^{r_2}_{k_j} \otimes X^{r_1}_{k_i}\right)\langle -1\rangle & \textit{if } i=j+1, \\ t_{ji}\left(X^{r_2+1}_{k_j} \otimes X^{r_1}_{k_i} + X^{r_2}_{k_j} \otimes X^{r_1+1}_{k_i}\right)\langle 1\rangle & \textit{if } i=j-1, \\ X^{r_2}_{k_j} \otimes X^{r_1}_{k_i} & \textit{if } a_{ij}=0. \end{cases}$$

Proof. Theorem is proved in [35]. We, however, show a way to do it using symmetric polynomials instead of Chern classes. Clearly, the 2-functor Θ_N preserves the degrees of 2-morphisms. We need to check that Φ preserves the relations between 2-morphisms.

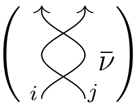
- Biadjointness property of 1-morphisms can be shown the following way:

Θ_N  $(1) =$

$= \Theta_N$  $\left(c_{\bar{\nu},i}^{-1} \sum_{l=0}^{\nu_i} (-1)^{\nu_i - l} X_{k_i+1}^{s} \otimes X_{k_i+1}^{l} \otimes e_{\nu_i - l}(\nu; i) \right) =$

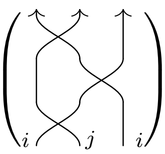
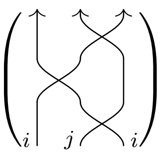
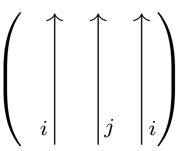
$= X_{k_i+1}^{s} \sum_{l=0}^{\nu_i} (-1)^{\nu_i - l} h_{l-\nu_i}(\nu + \alpha_i; i) e_{\nu_i - l}(\nu; i) = X_{k_i+1}^{s} = \Theta_N$  $(1)\,.$

- The following relations are easy computations obtained by applying the images of crossings twice:

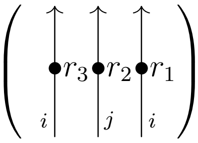
Θ_N  $(X_{k_i+1}^{r_1} \otimes X_{k_j+1}^{r_2}) =$

$$= \begin{cases} 0 & \text{if } a_{ij} = 2, \\ t_{ij}(X_{k_i+1}^{r_1} \otimes X_{k_j+1}^{r_2}) & \text{if } a_{ij} = 0, \\ t_{ij}(X_{k_i+1}^{r_1+1} \otimes X_{k_j+1}^{r_2}) \;+\; t_{ji}(X_{k_i+1}^{r_1} \otimes X_{k_j+1}^{r_2+1}) & \text{if } a_{ij} = -1, \end{cases}$$

- We only show the following relation for $i = j + 1$:

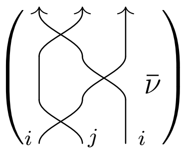
Θ_N  $-\ \Theta_N$  $=\ t_{ij}\Theta_N$  $.$ (5.28)

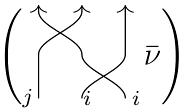
We prove this relation by applying the 2-morphisms on the both sides of the equalities on $X^{r_3}_{k_i+2} \otimes X^{r_2}_{k_j+1} \otimes X^{r_1}_{k_i+1}$. Since

$$X^{r_3}_{k_i+2} \otimes X^{r_2}_{k_j+1} \otimes X^{r_1}_{k_i+1} = \Theta_N$$


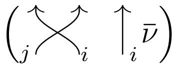
$$(1 \otimes 1 \otimes 1)$$

holds, without loss of generality we can take $r_1 = r_2 = r_3 = 0$:

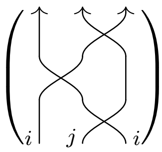
$$\Theta_N$$


$$(1 \otimes 1 \otimes 1) = \Theta_N$$


$$(t_{ij}\, 1 \otimes X_{k_i+2} \otimes 1 + t_{ji}\, X_{k_j+1} \otimes 1 \otimes 1)\langle 1 \rangle =$$

$$= \Theta_N$$


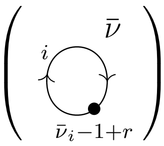
$$(t_{ij}\, 1 \otimes 1 \otimes 1)\langle 1 \rangle = t_{ij}\, 1 \otimes 1 \otimes 1,$$

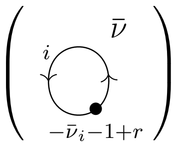
$$\Theta_N$$


$$(1 \otimes 1 \otimes 1) = 0.$$

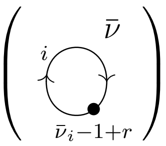
We leave the remaining relations for the readers to check. They all follow from the identities (5.8) to (5.23) via direct computations. □

Under the 2-functor Θ_N, the images of the clockwise and counterclockwise bubbles are:

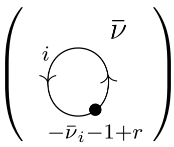
$$\Theta_N$$


$$, \Theta_N$$


$$: P_\nu \to P_\nu,$$

$$\Theta_N$$


$$(1) = c_{i,\bar{\nu}} \sum_{l=0}^{r} (-1)^l e_l(\nu; i+1) h_{r-l}(\nu; i), \tag{5.29}$$

$$\Theta_N$$


$$(1) = c^{-1}_{i,\bar{\nu}} \sum_{l=0}^{r} (-1)^l e_l(\nu; i) h_{r-l}(\nu; i+1). \tag{5.30}$$

If we replace the ring P_ν with ring $C_\nu := P_\nu / Sym_N$ for each $\nu \in \Lambda(n, N)$, we can still define a 2-category $\mathbf{CPol}_N$ with objects C_ν and a 2-functor $\Psi\colon \mathcal{U} \to \mathbf{CPol}_N$ in an entirely similar way. Khovanov-Lauda [35] construct a 2-category $\mathbf{Flag}^*_N$ using the ordinary cohomology of partial flag varieties to show the non-degeneracy of $\mathcal{U}$. The isomorphism between $\mathbf{EqFlag}^*_N$ and $\mathbf{Pol}_N$ descends to a isomorphism between $\mathbf{Flag}^*_N$ and $\mathbf{CPol}_N$. Taking the quotients of the objects in $\mathbf{Pol}_N$ by Sym_N corresponds to imposing Grassmannian relation on the objects of $\mathbf{EqFlag}^*_N$. The 2-morphisms in $\mathbf{CPol}_N$ are finite-dimensional, and we have $C_{\bar{\lambda}_0} := P_{\bar{\lambda}_0}/Sym_N \simeq \mathbb{C}$. The following result is also due to Khovanov-Lauda [35], Theorem 6.14.

Theorem 5.2.2. *The 2-categories* $\boldsymbol{Pol}_N$ *and* $\boldsymbol{CPol}_N$ *categorify the finite-dimensional, simple* $\mathfrak{sl}_n$ *representation* $V(\bar{\lambda}_0)$*.*

It is easy to check that both $\mathbf{Pol}_N$ and $\mathbf{CPol}_N$ are integrable. Recall the two categorifications of $V(\bar{\lambda}_0)$ – the categories of graded, finitely generated, projective modules $\check{R}^{\bar{\lambda}}$-pMod and $R^{\bar{\lambda}}$-pMod over the deformed cyclotomic KLR algebra $\check{R}^{\bar{\lambda}}$ and the cyclotomic KLR algebra $R^{\bar{\lambda}}$ respectively. By Theorem 4.2.5, they carry a universality property which can depicted in the following commuting diagrams:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\Theta_N} & \mathbf{Pol}_N \\ & \searrow & \uparrow{\scriptstyle \Theta'_N} \\ & & \check{R}^{\bar{\lambda}_0}\text{-pMod} \end{array} \quad , \qquad \begin{array}{ccc} \mathcal{U} & \xrightarrow{\Psi_N} & \mathbf{CPol}_N \\ & \searrow & \uparrow{\scriptstyle \Psi'_N} \\ & & R^{\bar{\lambda}_0}\text{-pMod} \end{array} \quad ,$$

where Θ'_N and Ψ'_N are strongly equivariant 2-functors. The 2-category $\mathbf{CPol}_N$ is a minimal categorification since $C_{\bar{\lambda}_0} \simeq \mathbb{C}$, i.e., Ψ'_N is a strongly equivariant equivalence.

5.3 $Z(\mathbf{Pol}_N)$ as a current algebra module

The 2-functor $\Theta_N\colon \mathcal{U} \to \mathbf{Pol}_N$ induces a $\mathbb{C}$-linear functor $\operatorname{Tr}\Theta_N\colon \operatorname{Tr}\mathcal{U} \to \operatorname{Tr}\mathbf{Pol}_N$. The center of the object P_ν of the 2-category $\mathbf{Pol}_N$ is isomorphic to P_ν, since P_ν is a commutative ring and acts on itself by multiplication. Composing the linear isomorphism $\rho\colon \dot{\mathbf{U}}(\mathfrak{sl}_n[t]) \longrightarrow \operatorname{Tr}\mathcal{U}$ with the functor $\operatorname{Tr}\Theta_N\colon \operatorname{Tr}\mathcal{U} \to \operatorname{Tr}\mathbf{Pol}_N$ gives the current algebra module structure on $\bigoplus_{\nu \in \mathcal{P}(n,N)} P_\nu$, where each ring P_ν corresponds to the $\mathfrak{sl}_n$ weight space $\bar{\nu}$. Recall the definition of the generators $\mathsf{F}_{i,j}$, $\mathsf{E}_{i,j}$, $\mathsf{H}_{i,j}$ of $\operatorname{Tr}\mathcal{U}$ in the equation (2.39). In what follows, we will define the action of $\operatorname{Tr}\mathbf{Pol}_N$ on the the center of objects $\bigoplus_{\nu \in \mathcal{P}(n,N)} P_\nu$ to be the images of the trace maps, defined in (2.40) to (2.42), under the functor Θ_N :

$$\mathtt{F_{i,j}} = \operatorname{Tr}\Theta_N(\mathsf{F}_{i,j})\colon P_\nu \to P_{\nu-\alpha_i},$$

$$\mathtt{E_{i,j}} = \operatorname{Tr}\Theta_N(\mathsf{E}_{i,j})\colon P_\nu \to P_{\nu+\alpha_i},$$
$$\mathtt{H_{i,j}} = \operatorname{Tr}\Theta_N(\mathsf{H}_{i,j})\colon P_\nu \to P_\nu,$$

for all $i \in I$ and $j \geq 0$. The next theorem defines the current algebra action and generalizes Brundan's formula ([13], Theorem 3.4) for the action of $\mathfrak{sl}_n$.

Theorem 5.3.1. *1. The map $\mathtt{F_{i,j}}\colon P_\nu \to P_{\nu-\alpha_i}$ is the $P_{\nu-\alpha_i}$-module homomorphism such that*

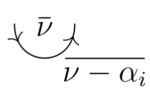
$$\mathtt{F_{i,j}}(X_{k_i}^m) = c_{i,\bar\nu}^{-1} \sum_{l=0}^{\nu_i-1} (-1)^l e_l(\nu-\alpha_i; i) h_{m+j+\bar\nu_i-1-l}(\nu-\alpha_i; i+1).$$

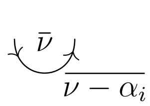
2. The map $\mathtt{E_{i,j}}\colon P_\nu \to P_{\nu+\alpha_i}$ is the $P_{\nu+\alpha_i}$-module homomorphism such that

$$\mathtt{E_{i,j}}(X_{k_i+1}^m) = c_{i,\bar\nu} \sum_{l=0}^{\nu_{i+1}-1} (-1)^l e_l(\nu+\alpha_i; i+1) h_{m+j-\bar\nu_i-1-l}(\nu+\alpha_i; i).$$

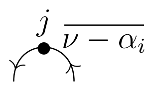
3. The map $\mathtt{H_{i,j}}\colon P_\nu \to P_\nu$ is the multiplication by $(-1)^j(p_j(\nu; i+1) - p_j(\nu; i))$ if $j > 0$, and by $\bar\nu_i$ if $j = 0$.

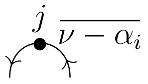
Proof.

1. We compute the action of $\mathtt{F_{i,j}}$ on $p \in P_\nu$. To do this, we first apply the map $\Theta_N\Big($  $\Big)$ on p :

$\Theta_N\Big($  $\Big)$ $$(p) = c_{i,\overline{\nu-\alpha_i}}^{-1} \sum_{r=0}^{\nu_i-1} (-1)^l e_l(\nu-\alpha_i; i) p \otimes X_{k_i}^{\nu_i-1-l} \in P_{\nu-\alpha_i,\nu} \otimes_{P_\nu} P_{\nu,\nu-\alpha_i}.$$

Every element $p \in P_{\nu',\nu}$ can be written as $p = \sum_{m=0}^{\nu_{i+1}} p'_m X_{k_i}^m$ for some elements $p'_m \in P_{\nu-\alpha_i}$.

Now we apply the cap $\Theta_N\Big($  $\Big)$:

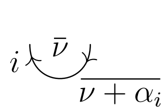
$\Theta_N\Big($  $\Big)$ $$\left(c_{i,\overline{\nu-\alpha_i}}^{-1} \sum_{m=0}^{\nu_{i+1}} p'_m \sum_{l=0}^{\nu_i-1} (-1)^l e_l(\nu-\alpha_i; i) X_{k_i}^m \otimes X_{k_i}^{j+\nu_i-1-l}\right) =$$

$$= c_{i,\bar{\nu}}^{-1} \sum_{m=0}^{\nu_{i+1}} p'_m \sum_{l=0}^{\nu_i - 1} (-1)^l e_l(\nu - \alpha_i; i) h_{m+j+\bar{\nu}_i-1-l}(\nu - \alpha_i; i+1).$$

Thus,

$$\mathtt{F_{i,j}}(X_{k_i}^m) = c_{i,\bar{\nu}}^{-1} \sum_{l=0}^{\nu_i - 1} (-1)^l e_l(\nu - \alpha_i; i) h_{m+j+\bar{\nu}_i-1-l}(\nu - \alpha_i; i+1).$$

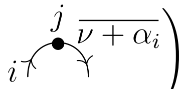
2. Similarly, the action of the $\mathtt{E_{i,j}}$ on $p \in P_\nu$ is defined by closing it with a cup and a cup consecutively, and hence obtaining an element of $P_{\nu+\alpha_i}$. As an element of $P_{\nu,\nu+\alpha_i}$, we can write $p = \sum_{m=0}^{\nu_{i+1}-1} p'_m X_{k_i+1}^m$ for some elements $p'_m \in P_{\nu+\alpha_i}$. Then we have

$\Theta_N \Big($  $\Big)$

$$(p) = \sum_{r=0}^{\nu_{i+1}-1} (-1)^l e_l(\nu + \alpha_i; i+1) p \otimes X_{k_i}^{\nu_{i+1}-1-l} =$$

$$= \sum_{m=0}^{\nu_{i+1}-1} p'_m \sum_{l=0}^{\nu_{i+1}-1} (-1)^l e_l(\nu + \alpha_i; i+1) X_{k_i+1}^m \otimes X_{k_i+1}^{\nu_{i+1}-1-l} \quad \in P_{\nu+\alpha_i,\nu} \otimes_{P_\nu} P_{\nu,\nu+\alpha_i}.$$

We close the diagram by applying the cap:

$\Theta_N \Big($  $\Big)$

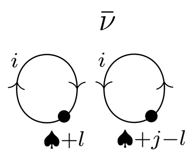
$$\left(\sum_{m=0}^{\nu_{i+1}-1} p'_m \sum_{l=0}^{\nu_{i+1}-1} (-1)^l e_l(\nu + \alpha_i; i+1) X_{k_i+1}^m \otimes X_{k_i+1}^{j+\nu_{i+1}-1-l} \right) =$$

$$= c_{i,\overline{\nu+\alpha_i}} \sum_{m=0}^{\nu_{i+1}-1} p'_m \sum_{l=0}^{\nu_{i+1}-1} (-1)^l e_l(\nu + \alpha_i; i+1) h_{m+j-\bar{\nu}_i-1-l}(\nu + \alpha_i; i).$$

Thus, we get

$$\mathtt{E_{i,j}}(X_{k_i}^m) = c_{i,\bar{\nu}} \sum_{l=0}^{\nu_{i+1}-1} (-1)^l e_l(\nu + \alpha_i; i+1) h_{m+j-\bar{\nu}_i-1-l}(\nu + \alpha_i; i).$$

3. The map $\mathsf{H}_{i,j}$, $j \geq 1$ is a multiplication by $\operatorname{Tr} \Theta_N(\pi_{i,j}(\bar{\nu}))$, where

$$\pi_{i,j}(\bar{\nu}) = \sum_{l=0}^{j} (l+1)$$  . (5.31)

We define the following generating functions:

$$ {}^{i}\overset{\bar{\nu}}{\circlearrowleft}_{\bullet}(t)=\sum_{l=0}^{\infty} {}^{i}\overset{\bar{\nu}}{\circlearrowleft}_{-\bar{\nu}_i-1+l}\, t^{l}, \qquad {}^{i}\overset{\bar{\nu}}{\circlearrowright}_{\bullet}(t)=\sum_{l=0}^{\infty} {}^{i}\overset{\bar{\nu}}{\circlearrowright}_{\bar{\nu}_i-1+l}\, t^{l}, \qquad E(\nu;i)(t)=\sum_{l=0}^{\nu_i} e_l(\nu;i)t^{l}, $$

$$ H(\nu;i)(t)=\sum_{l=0}^{\infty} h_l(\nu;i)t^{l}, \qquad p(\nu;i)(t)=\sum_{l=0}^{\infty} p_l(\nu;i)t^{l-1}, \qquad \pi_i(\bar{\nu})(t)=\sum_{l=0}^{\infty} \pi_{i,l}(\bar{\nu})t^{l-1}. $$

Then the equation 5.31 can be written as

$$ \pi_i(\bar{\nu})(t)=\left({}^{i}\overset{\bar{\nu}}{\circlearrowright}_{\bullet}(t)\right)' {}^{i}\overset{\bar{\nu}}{\circlearrowright}_{\bullet}(t)=\frac{\left({}^{i}\overset{\bar{\nu}}{\circlearrowright}_{\bullet}(t)\right)'}{{}^{i}\overset{\bar{\nu}}{\circlearrowright}_{\bullet}(t)}=\left(\log {}^{i}\overset{\bar{\nu}}{\circlearrowright}_{\bullet}(t)\right)'. \tag{5.32} $$

The equation (5.29) implies

$$ \mathrm{Tr}\,\Theta_N\left({}^{i}\overset{\bar{\nu}}{\circlearrowright}_{\bullet}(t)\right)=H(\nu;i)(-t)E(\nu;i+1)(t)=\frac{E(\nu;i+1)(t)}{E(\nu;i)(t)}, $$

so we have

$$ \mathrm{Tr}\,\Theta_N\left(\pi_i(\bar{\nu})(t)\right)=\left(log\frac{E(\nu;i)(t)}{E(\nu;i+1)(t)}\right)'= \tag{5.33} $$

$$ =(log\,E(\nu;i+1)(t))'-(log\,E(\nu;i)(t))'=p(\nu;i+1)(-t)-p(\nu;i)(-t). $$

The last equality in the equation (5.33) is given on page 23 in [43]. Thus, $\mathrm{Tr}\,\Theta_N\left(\pi_{i,j}(\bar{\nu})\right)=(-1)^j(p_j(\nu;i+1)-p_j(\nu;i))$. □

5.4 The coinvariant algebra C_ν^λ

Fix a partition $\lambda\in\mathcal{P}^+(n,N)$. Let $\nu\in\mathcal{P}(n,N)$ and I_ν^λ be the ideal of P_ν generated by

$$ \left\{ h_r(\nu;i_1,\dots,i_m) \;\middle|\; \begin{array}{l} 1\le m\le n,\ 1\le i_1<\cdots<i_m\le n, \\ r>\lambda_1+\cdots+\lambda_m-\nu_{i_1}-\cdots-\nu_{i_m} \end{array} \right\}. \tag{5.34} $$

We will need the following statement for the special case $\lambda_0=(N,0,\dots,0)\in\mathcal{P}^+(n,N)$.

Proposition 5.4.1. *If $\lambda_0=(N,0,\dots,0)$, then $I_\nu^{\lambda_0}=Sym_N$.*

Proof. We have

$$I_\nu^{\lambda_0} = \left\langle h_r(\nu; i_1, \dots, i_m) \,\middle|\, \begin{array}{l} 1 \le m \le n, \ 1 \le i_1 < \cdots < i_m \le n, \\ r > N - (\nu_{i_1} + \cdots + \nu_{i_m}) \end{array} \right\rangle.$$

Since $Sym_N = \langle h_r(\nu; 1, 2, \dots, n) \,|\, r > 0 \rangle$, we have $Sym_N \subseteq I_\nu^{\lambda_0}$. To prove $I_\nu^{\lambda_0} \subseteq Sym_N$, let $h_r(\nu; i_1, \dots, i_m)$ be a generator of I_ν^N with $r > N - (\nu_{i_1} + \cdots + \nu_{i_m})$. Define

$$\{j_1, j_2, \dots, j_{n-m}\} = \{1, 2, \dots n\} \setminus \{i_1, i_2, \dots i_m\}.$$

to be the ordered set. Then according to (5.6), we have

$$h_r(\nu; i_1, \dots, i_m) = \sum_{s=0}^{r} (-1)^{r-s} e_{r-s}(\nu; j_1, \dots, j_{n-m}) h_s(\nu; 1, 2, \dots, n) =$$

$$= (-1)^r e_r(\nu; j_1, \dots, j_{n-m}) + \sum_{s=1}^{r} (-1)^{r-s} e_{r-s}(\nu; j_1, \dots, j_{n-m}) h_s(\nu; 1, 2, \dots, n).$$

since $e_r(\nu; j_1, \dots, j_{n-m})$ is an elementary symmetric polynomial in $N - (\nu_{i_1} + \cdots + \nu_{i_m})$ variables, it is zero when $r > N - (\nu_{i_1} + \cdots + \nu_{i_m})$. Thus, $I_\nu^{\lambda_0} \subseteq Sym_N$ holds. □

Let $C_\nu^\lambda = P_\nu / I_\nu^\lambda$. Notice that $Sym_N \subseteq I_\nu^\lambda$ holds, thus we can regard the ring C_ν^λ as a quotient of $C_\nu = C_\nu^{\lambda_0}$. Since P_ν is a finitely generated free module over Sym_N, C_ν is finite-dimensional and hence so is C_ν^λ. Notice that $C_\nu^\lambda \neq 0$ if and only if $\lambda \geq \nu$ with respect to the dominance order.

Proposition 5.4.2. *The ideal* $\bigoplus_\nu I_\nu^\lambda$ *of* $\bigoplus_\nu P_\nu$ *is invariant under the action of generators* $E_{i,j}$ *and* $F_{i,j}$ *for all* $i \in I$ *and* $j \geq 0$.

Proof. We will only show $E_{i,j}\left(I_\nu^\lambda\right) \subseteq I_{\nu+\alpha_i}^\lambda$, and the proof of $F_{i,j}(I_\nu^\lambda) \subseteq I_{\nu-\alpha_i}^\lambda$ will be similar. We have to show that images of all $h_r(\nu; i_1, \dots, i_m)$ under $E_{i,j}$ are in $I_{\nu+\alpha_i}^\lambda$ for all $j \geq 0$, $m \geq 1$, $1 \leq i_1 < \cdots < i_m \leq n$ and $r > \lambda_1 + \cdots + \lambda_m - \nu_{i_1} - \cdots - \nu_{i_m}$. We proceed with the case distinction.

1. If $i \in \{i_1, \dots, i_m\}$ and $i+1 \in \{i_1, \dots, i_m\}$, then $h_r(\nu; i_1, \dots, i_m) = h_r(\nu+\alpha_i; i_1, \dots, i_m)$ holds. In this case, we have

$$E_{i,j}(h_r(\nu; i_1, \dots, i_m)) = h_r(\nu + \alpha_i; i_1, \dots, i_m) E_{i,j}(1),$$

which belongs to $I_{\nu+\alpha_i}^\lambda$. The same is true when $i \notin \{i_1, \dots, i_m\}$ and $i+1 \notin \{i_1, \dots, i_m\}$ hold simultaneously .

2. We consider the case $i \in \{i_1, \ldots, i_m\}$ and $i+1 \notin \{i_1, \ldots, i_m\}$.

   We have the identity (5.21):

$$h_r(\nu; i_1, \ldots, i_m) = h_r(\nu + \alpha_i; i_1, \ldots, i_m) - X_{k_i+1} h_{r-1}(\nu + \alpha_i; i_1, \ldots, i_m).$$

   Since $r-1 > \lambda_1 + \cdots + \lambda_m - (\nu+\alpha_i)_{i_1} - \cdots - (\nu+\alpha_i)_{i_m} = \lambda_1 + \cdots + \lambda_m - \nu_{i_1} - \cdots - \nu_{i_m} - 1$, the element

$$h_r(\nu + \alpha_i; i_1, \ldots, i_m)\, E_{i,j}(1) - h_{r-1}(\nu + \alpha_i; i_1, \ldots, i_m)\, E_{i,j}(X_{k_i+1})$$

   belongs to $I^{\lambda}_{\nu+\alpha_i}$.

3. Finally suppose that $i \notin \{i_1, \ldots, i_m\}$ and $i+1 = i_m$. By (5.22), we have that

$$h_r(\nu; i_1, \ldots, i+1) = \sum_{l=0}^{r} h_l(\nu + \alpha_i; i_1, \ldots, i+1) X_{k_i+1}^{r-l}$$

   .

   Applying $E_{i,j}$ on $h_r(\nu; i_1, \ldots, i+1)$, we get the element

$$c_{i,\bar{\nu}} \sum_{s=0}^{\nu_{i+1}-1} (-1)^s e_s(\nu+\alpha_i; i+1) \sum_{l=0}^{r} h_{r-l+j-\bar{\nu}_i-1-s}(\nu+\alpha_i; i) h_l(\nu+\alpha_i; i_1, \ldots, i+1). \quad (5.35)$$

   Notice that since $r > \lambda_1 + \cdots + \lambda_m - \nu_{i_1} - \cdots - \nu_{i_m} = \lambda_1 + \cdots + \lambda_m - (\nu + \alpha_i)_{i_1} - \cdots - (\nu + \alpha_i)_{i_m} - 1$, the second summand in the identity

$$h_{r+j-\bar{\nu}_i-1-s}(\nu+\alpha_i; i_1, \ldots, i+1, i) = \sum_{l=0}^{r} h_{r-l+j-\bar{\nu}_i-1-s}(\nu+\alpha_i; i) h_l(\nu+\alpha_i; i_1, \ldots, i+1)$$

$$+ \sum_{l=r+1}^{r+j-\bar{\nu}_i-1-s} h_{r-l+j-\bar{\nu}_i-1-s}(\nu+\alpha_i; i) h_l(\nu+\alpha_i; i_1, \ldots, i+1)$$

   is in $I^{\lambda}_{\nu+\alpha_i}$. Therefore,

$$h_{r+j-\bar{\nu}_i-1-s}(\nu + \alpha_i; i_1, \ldots, i+1, i) \equiv$$

$$\equiv \sum_{l=0}^{r} h_{r-l+j-\bar{\nu}_i-1-s}(\nu + \alpha_i; i) h_l(\nu + \alpha_i; i_1, \ldots, i+1) \bmod I^{\lambda}_{\nu+\alpha_i}.$$

We can insert this equivalence in the element (5.4.4). To finish the proof, we need to check that the element

$$c_{i,\bar{\nu}} \sum_{s=0}^{\nu_{i+1}-1} (-1)^s e_s(\nu + \alpha_i; i+1) h_{r+j-\bar{\nu}_i-1-s}(\nu + \alpha_i; i_1, \ldots, i+1, i)$$

also belongs to $I^\lambda_{\nu+\alpha_i}$. Another polynomial identity shows that it is equal to $c_{i,\bar{\nu}} h_{r+j-\bar{\nu}_i-1}(\nu + \alpha_i; i_1, \ldots, i_{m-1}, i)$, which is an element of $I^\lambda_{\nu+\alpha_i}$, since $r + j - \bar{\nu}_i - 1 > \lambda_1 + \cdots + \lambda_m - (\nu + \alpha_i)_{i_1} - \cdots - (\nu + \alpha_i)_{i_{m-1}} - (\nu + \alpha_i)_{i+1}$. □

The proof of Proposition 5.4.2 is a generalization of Lemma 4.1 of Brundan [13] for the j parameter. This proposition implies the following statement.

Corollary 5.4.3. *The ring* $\bigoplus_\nu C^\lambda_\nu$ *is a finite-dimensional graded current algebra module with the highest weight* $\bar{\lambda} = (\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n)$.

The current algebra action on $\bigoplus_\nu P_\nu$ descends to $\bigoplus_\nu C^\lambda_\nu$. In other words, the following diagrams commute:

$$\begin{array}{ccc} P_\nu & \xrightarrow{E_{i,j}} & P_{\nu+\alpha_i} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ C^\lambda_\nu & \xrightarrow[E_{i,j}]{} & C^\lambda_{\nu+\alpha_i}, \end{array} \qquad \begin{array}{ccc} P_\nu & \xrightarrow{F_{i,j}} & P_{\nu-\alpha_i} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p''} \\ C^\lambda_\nu & \xrightarrow[F_{i,j}]{} & C^\lambda_{\nu-\alpha_i}, \end{array}$$

where p, p' and p'' are projections with kernels I^λ_ν, $I^\lambda_{\nu+\alpha_i}$ and $I^\lambda_{\nu-\alpha_i}$ respectively.

Recall that we defined an integer

$$d^\lambda_\nu = \max\{(\lambda, \lambda) - (\nu, \nu), 0\}. \tag{5.36}$$

It is easy to show that if $\lambda > \nu$ with respect to the dominance order, we have $d^\lambda_\nu > 0$. Brundan-Ostrik [17] prove that the ring C^λ_ν is isomorphic to the cohomology ring of Spaltenstein variety – the variety of partial flags of type ν which are annihilated by matrices of Jordan type λ^T.

For a graded, finite-dimensional current algebra module M, *graded composition multiplicity* of a simple $\mathfrak{sl}_n$-module V is the polynomial

$$\sum_{s \geq 0} [M\{2s\} : V]\ t^s,$$

where $[M\{2s\} : V]$ is the number of copies of V in $M\{2s\}$. Given a partition τ, the graded composition multiplicity of $V(\bar{\tau})$ in $\bigoplus_\nu C^\lambda_\nu$ is given by Brundan [13] via the formula

$$\sum_{r\geq 0} \left[\bigoplus\nolimits_\nu C^\lambda_\nu\{d^\lambda_\nu - 2r\} : V(\bar{\tau})\right] t^r = K_{\tau^T,\lambda^T}(t), \tag{5.37}$$

where τ^T and λ^T are the transposes of the partitions τ, λ, and $K_{\tau^T,\lambda^T}(t)$ is the *Kostka-Foulkes polynomial*. We refer to [43], section III.6 for the definition of the Kostka-Foulkes polynomial. We will use their following property.

Proposition 5.4.4. *Let $\lambda, \mu \in \mathcal{P}^+(n, N)$ be partitions, and let $\lambda' = \lambda + (m, m, \ldots, m)$, $\mu' = \mu + (m, m, \ldots, m)$ for some non-negative integer m. Then the following equations hold:*

$$K_{\lambda,\mu}(t) = K_{\lambda',\mu'}(t), \tag{5.38}$$

$$K_{\mu^T,\lambda^T}(t) = K_{(\mu')^T,(\lambda')^T}(t).$$

Proof. $K_{\lambda,\mu}(t)$ is nonzero if and only if $\lambda \geq \mu$ with respect to the dominance order. Let $\alpha_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$ be the i-th simple root as before, and let

$$R^+ = \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j \,|\, 1 \leq i \leq j \leq n-1\}$$

be the set of positive roots. For any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{Z}$ such that $\sum_i \xi_i = 0$, we define the polynomial

$$P(\xi; t) = \sum_{\{m_\alpha\}_{\alpha\in R^+}} t^{\sum m_\alpha},$$

where the sum is over all families $\{m_\alpha\}_{\alpha\in R^+}$ of non-negative integers such that $\xi = \sum m_\alpha \alpha$. The polynomial $P(\xi; t)$ is nonzero if and only if $\xi = \sum_i \eta_i \alpha_i$ for some non-negative integers η_i, $1 \leq i \leq n-1$.

By example III.6.4 in [43],

$$K_{\lambda,\mu}(t) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, P(\sigma(\lambda + \delta) - (\mu + \delta); t), \tag{5.39}$$

where $\delta = (n-1, n-2, \ldots, 1, 0)$. The first equality in (5.38) immediately follows from (5.39).

To prove the second equality, notice that $(\lambda')^T = (\underbrace{n, \ldots, n}_{m\,\text{times}}, \lambda^T_1, \ldots, \lambda^T_N)$, $(\mu')^T = (\underbrace{n, \ldots, n}_{m\,\text{times}}, \mu^T_1, \ldots, \mu^T_N)$. We take S_N to be the subgroup of S_{m+N} which fixes the first

m entries, then

$$K_{(\mu')^T,(\lambda')^T}(t) = \sum_{\sigma \in S_{m+N}} \operatorname{sgn}(\sigma)\, P(\sigma((\mu')^T + \delta) - ((\lambda')^T + \delta); t). \tag{5.40}$$

Notice that summand is zero if $\sigma \in S_{m+N} \setminus S_N$. Therefore, we can write (5.40) as

$$K_{(\mu')^T,(\lambda')^T}(t) = \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma)\, P(\sigma((\mu')^T + \delta) - ((\lambda')^T + \delta); t) = \tag{5.41}$$

$$= \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma)\, P(\sigma(\mu^T + \delta) - (\lambda^T + \delta); t) = K_{\mu^T,\lambda^T}(t).$$

□

The graded dimension of C_ν^λ is also given by Brundan [13]:

$$\sum_{r \geq 0} \dim_{\mathbb{C}} C_\nu^\lambda\{2r\}\, t^r = t^{d_\nu^\lambda/2} \sum_{\tau \in \mathcal{P}^+(n,N)} K_{\tau,\nu}\, K_{\tau^T,\lambda^T}(t^{-1}), \tag{5.42}$$

where $K_{\tau,\nu} = K_{\tau,\nu}(1)$ denotes the Kostka number. The Kostka-Foulkes polynomial $K_{\tau,\nu}(t)$ for a composition ν is defined to be $K_{\tau,\widehat{\nu}}(t)$, where $\widehat{\nu}$ is the partition obtained by permuting the entries of ν.

By [5] and [56], the center $Z(R^{\bar{\lambda}})$ of the cyclotomic KLR algebra $R^{\bar{\lambda}}$ is isomorphic to the dual local Weyl module $W^*(\bar{\lambda})$. The next theorem is proved independently by Shan-Vasserot-Varagnolo [56] and Webster [60], section 3.

Theorem 5.4.5. *[60] The current algebra module $\bigoplus_\nu C_\nu^\lambda$ is isomorphic to the dual local Weyl module $W^*(\bar{\lambda})$.*

Proof. We give a proof of existence of an injective map $\varphi\colon \bigoplus_\nu C_\nu^\lambda \to W^*(\bar{\lambda})$. The module $\bigoplus_\nu C_\nu^\lambda$ is cocyclic, cogenerated by the identity element 1_λ of the highest weight space $C_\lambda^\lambda \simeq \mathbb{C}$. This means that for any $v \in \bigoplus_\nu C_\nu^\lambda$, there exist $u \in \mathbf{U}(\mathfrak{sl}_n[t])$ such that $uv = 1_\lambda$. Moreover, $\bigoplus_\nu C_\nu^\lambda$ is a finite-dimensional highest weight module with the highest weight $\bar{\lambda}$. The remaining part of the proof follows from the dualization of the Theorem 3.1.2 – the universality property of local Weyl modules. More explicitly, any cocyclic finite-dimensional highest weight module of highest weight $\bar{\lambda}$ injects into a dual local Weyl module $W^*(\bar{\lambda})$. The injective, degree zero $\mathbf{U}(\mathfrak{sl}_n[t])$-invariant map φ, which is uniquely defined $\varphi(1_{\bar{\lambda}}) = \delta_{v_{\bar{\lambda}}}$.

In order to prove surjectivity, it must shown that the co-kernel $W^*(\bar{\lambda})/\varphi(\bigoplus_\nu C_\nu^\lambda)$ is zero.

Socle of a $\mathbf{U}(\mathfrak{sl}_n[t])$-module M is the largest semisimple submodule of M. Cosocle of M is the largest semisimple quotient of M. The dual of the socle of M is isomorphic to the cosocle of the dual M^*. Kodera-Naoi [37] show that the socle of the local Weyl module $W(\bar{\lambda})$ is the degree d_ν^λ homogeneous piece, and it is isomorphic to the simple $\mathfrak{sl}_n$-module $V(\bar{\lambda}_{\min})$ where $\bar{\lambda}_{\min}$ is the unique minimal dominant weight amongst those $\leq \bar{\lambda}$.

Dually, the cosocle of $W(\bar{\lambda})$ is $V(\bar{\lambda}_{\min})$, consisting of degree zero elements. By the multiplicity formula (5.37), the cosocle of $\bigoplus_\nu C_\nu^\lambda$ is also isomorphic to $V(\bar{\lambda}_{\min})$ in degree zero, generated by the unit of the algebra $C^\lambda_{\bar{\lambda}_{\min}}$ over $\mathfrak{sl}_n$. Hence, $W^*(\bar{\lambda})/\varphi(\bigoplus_\nu C_\nu^\lambda)$ has no degree zero elements.

The dual local Weyl module $W^*(\bar{\lambda})$ has a finite length, therefore every quotient of $W^*(\bar{\lambda})$ contains a simple submodule, which lies in the cosocle, by the definition of the cosocle. Since the cosocle of $W^*(\bar{\lambda})$ is simple, it is contained in the co-kernel $W^*(\bar{\lambda})/\varphi(\bigoplus_\nu C_\nu^\lambda)$. However, $W^*(\bar{\lambda})/\varphi(\bigoplus_\nu C_\nu^\lambda)$ has no degree zero elements, and hence is zero. □

Shan-Vasserot-Varagnolo [56] and Webster [60] also prove that there is a graded algebra isomorphism between the C_ν^λ and the center $Z\left(R^{\bar{\lambda}}(\bar{\lambda}-\bar{\nu})\right)$ of the cyclotomic KLR algebra $R^{\bar{\lambda}}(\bar{\lambda}-\bar{\nu})$ of rank $\bar{\lambda}-\bar{\nu}$. Theorem 7.23 and graded dimension formula in [13] gives the grade dimension of the center $Z\left(R^{\bar{\lambda}}(\bar{\lambda}-\bar{\nu})\right)$:

$$\sum_{r\geq 0}\dim_{\mathbb{C}} Z\left(R^{\bar{\lambda}}(\bar{\lambda}-\bar{\nu})\right)\{2r\}\, t^r = t^{d_\nu^\lambda/2}\sum_{\tau\in\mathcal{P}^+(n,N)} K_{\tau,\nu}\, K_{\tau^T,\lambda^T}(t^{-1}). \tag{5.43}$$

By Proposition 5.38, the right hand side of the equation (5.43) depend on $\bar{\lambda}$ and $\bar{\nu}$ rather than actual λ and ν.

Theorem 5.4.6. *The graded character of the local Weyl module is given by the formula*

$$ch_t\, W(\bar{\lambda}) = \sum_{\tau\in\mathcal{P}^+(n,N)} K_{\tau^T,\lambda^T}(t)\; ch\, V(\bar{\tau}), \tag{5.44}$$

and the graded dimension of the weight space $\bar{\nu}$ is given by

$$\dim_{\mathbb{C}} W_{\bar{\nu}}(\bar{\lambda}) = \sum_{r\geq 0}\dim_{\mathbb{C}} W_{\bar{\nu}}(\bar{\lambda})\{2r\}\, t^r = \sum_{\tau\in\mathcal{P}^+(n,N)} K_{\tau,\nu}\, K_{\tau^T,\lambda^T}(t). \tag{5.45}$$

Proof. Combining the muliplicity formula (5.37) with the Theorem 7.23, we have

$$\sum_{r\geq 0}\left[\textstyle\bigoplus_\nu W^*_{\bar{\nu}}(\bar{\lambda})\{d_\nu^\lambda - 2r\} : V(\bar{\tau})\right] t^r = K_{\tau^T,\lambda^T}(t), \tag{5.46}$$

Since $W^*_{\bar\nu}(\bar\lambda)\{d^\lambda_\nu - 2r\} = (W_{\bar\nu}(\bar\lambda)\{2r\})^*$ and $V(\bar\tau) \simeq V^*(\bar\tau)$, the multiplicity formula for the local Weyl module is

$$\sum_{r\geq 0} \left[\bigoplus\nolimits_\nu W_{\bar\nu}(\bar\lambda)\{2r\} : V(\bar\tau)\right] t^r = K_{\tau^T,\lambda^T}(t), \tag{5.47}$$

which implies (5.44). The graded dimension formula follows similarly from the equation (5.42). □

Remark 5.4.7. *Let $\lambda \in \mathcal{P}^+(n,N)$ and $W(\bar\lambda)$ be the local Weyl module with the highest weight $\bar\lambda$. If we set $t=1$ in the formula* (5.44)*, we get*

$$ch_t\, W(\bar\lambda)\,|_{t=1} = \sum_{\tau\in\mathcal{P}^+(n,N)} K_{\tau^T,\lambda^T}(1)\; ch\, V(\bar\tau) = ch\left(\bigwedge^{\lambda_1^T}\mathbb{C}^n \otimes \bigwedge^{\lambda_2^T}\mathbb{C}^n \otimes \cdots \otimes \bigwedge^{\lambda_p^T}\mathbb{C}^n\right), \tag{5.48}$$

where the $\lambda^T = (\lambda_1^T, \lambda_2^T, \ldots, \lambda_p^T)$ is the transpose of λ. The equation (5.48) *also follows from Chari-Loktev [24], where they prove that $W(\bar\lambda)$ is isomorphic to a certain graded tensor product of fundamental representations, called the fusion product, where recover the usual tensor product*

$$\bigwedge^{\lambda_1^T}\mathbb{C}^n \otimes \bigwedge^{\lambda_2^T}\mathbb{C}^n \otimes \cdots \otimes \bigwedge^{\lambda_p^T}\mathbb{C}^n = \bigoplus_{\tau\in\mathcal{P}^+(n,N)} (V(\bar\tau))^{\oplus K_{\tau^T,\lambda^T}} \tag{5.49}$$

if we set $t=1$.

A computation shows that if $\lambda_0 = (N,0,\ldots,0)$, the graded dimension of $C^{\lambda_0}_\nu$ is given by the quantum multinomial coefficient

$$\binom{N}{\nu}_t = \binom{N}{\nu_1,\ldots,\nu_n}_t = \frac{[N]_t!}{[\nu_1]_t!\cdots[\nu_n]_t!},$$

where $[a]_t!$ denotes the t-factorial

$$\frac{(1-t)(1-t^2)\cdots(1-t^a)}{(1-t)^a} = (1+t)(1+t+t^2)\cdots(1+t+t^2+\cdots+t^{a-1}).$$

Example 5.4.8. *Let $N=9$, $n=4$, $\lambda=(5,2,1,1)$ and $\nu=(3,1,2,3)$. We compute the graded dimension of $Z\left(R^{\bar\lambda}(\bar\lambda-\bar\nu)\right)$. We have $\lambda^T=(4,2,1,1,1)$, $d^\lambda_\nu = 31-23 = 8$.*

$$\sum_{r\geq 0} \dim\, Z\left(R^{\bar\lambda}(\bar\lambda-\bar\nu)\right)\{2r\}\, t^r = t^4 \sum_{\tau\in\mathcal{P}^+(4,9)} K_{\tau,(3,1,2,3)} \cdot K_{\tau^T,(4,2,1,1,1)}(t^{-1}) = \tag{5.50}$$

$$= t^4[K_{(3,3,2,1),(3,1,2,3)} \cdot K_{(4,3,2,0,0),(4,2,1,1,1)}(t^{-1}) + K_{(5,2,1,1),(3,1,2,3)} \cdot K_{(4,2,1,1,1),(4,2,1,1,1)}(t^{-1})+$$

$$+K_{(4,3,1,1),(3,1,2,3)} \cdot K_{(4,2,2,1,0),(4,2,1,1,1)}(t^{-1}) + K_{(4,2,2,1),(3,1,2,3)} \cdot K_{(4,3,1,1,0),(4,2,1,1,1)}(t^{-1})] =$$

$$= t^4(t^{-2} + t^{-3} + t^{-4} + 2 + 2(t^{-2} + t^{-1}) + (t^{-3} + t^{-2} + t^{-1})) = 2t^4 + 3t^3 + 4t^2 + 2t + 1.$$

It is easy to check that we get the same graded dimension if we choose $\lambda' = (4,1,0,0)$ *and* $\nu' = (2,0,1,2)$. *In both cases,* $\bar{\lambda} = \bar{\lambda}' = (3,1,0)$ *and* $\bar{\nu} = \bar{\nu}' = (2,-1,-1)$.

Example 5.4.9. *Let* $N = 5$, $n = 3$, $\lambda = (5,0,0)$ *and* $\nu = (3,1,1)$. *The graded dimension of the center* $Z\left(R^{\bar{\lambda}}(\bar{\lambda} - \bar{\nu})\right)$ *is computed as follows. We have* $\lambda^T = (1,1,1,1,1)$, $d_\nu^\lambda = 14$.

$$\sum_{r \geq 0} \dim_{\mathbb{C}} Z\left(R^{\bar{\lambda}}(\bar{\lambda} - \bar{\nu})\right)\{2r\}\, t^r = t^7 \sum_{\tau \in \mathcal{P}^+(3,5)} K_{\tau,(3,1,1)} \cdot K_{\tau^T,(1,1,1,1,1)}(t^{-1}) = \tag{5.51}$$

$$= t^7(K_{(5,0,0),(3,1,1)} \cdot K_{(1,1,1,1,1),(1,1,1,1,1)}(t^{-1}) + K_{(4,1,0),(3,1,1)} \cdot K_{(2,1,1,1),(1,1,1,1,1)}(t^{-1})+$$

$$+K_{(3,2,0),(3,1,1)} \cdot K_{(2,2,1,0,0),(1,1,1,1,1)}(t^{-1}) + K_{(3,1,1),(3,1,1)} \cdot K_{(3,1,1,0,0),(1,1,1,1,1)}(t^{-1})) =$$

$$= t^7(1+2(t^{-1}+t^{-2}+t^{-3}+t^{-4})+(t^{-2}+t^{-3}+t^{-4}+t^{-5}+t^{-6})+(t^{-3}+t^{-4}+2t^{-5}+t^{-6}+t^{-7})) =$$

$$= t^7 + 2t^6 + 3t^5 + 4t^4 + 4t^3 + 3t^2 + 2t^1 + 1 = \binom{5}{3,1,1}_t.$$

Chapter III

Trace of categorified quantum symmetric pairs

This chapter consists of three sections. We give a brief definition of the coideal algebra and set our notation in the first section. Next we present Bao-Shan-Wang-Webster's [2] diagrammatic categorification of the coideal algebra. The current coideal algebra is defined in the last section. We compute the trace of Bao-Shan-Wang-Webster's 2-category and compare it to the current coideal algebra.

6 Categorification of the coideal algebra

6.1 The coideal algebra $\mathbf{U}^{\jmath}$

Fix an integer $n \geq 1$. We will denote $\diamond := \frac{1}{2}$ and define the set

$$\mathbb{I} = \mathbb{I}_{2n} = \big\{ i \in \mathbb{Z} + \diamond \mid -n < i < n \big\}. \tag{6.1}$$

and a subset

$$\mathbb{I}^{+} = \big\{ i \in \mathbb{Z} + \diamond \mid 0 \leq i < n \big\} \subset \mathbb{I}. \tag{6.2}$$

We consider the Cartan datum of type A_{2n} with Cartan matrix $\{a_{ij}\}_{i,j\in\mathbb{I}}$, simple roots $\{\alpha_i\}_{i\in\mathbb{I}}$, simple coroots $\{\alpha_i^\vee\}_{i\in\mathbb{I}}$. Let X be the $\mathfrak{sl}_{2n+1}$ weight lattice, and let $Y = \bigoplus_{i\in\mathbb{I}} \mathbb{Z}\alpha_i^\vee$ be the coroot lattice. The entries of the Cartan matrix are given by $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ for $i, j \in \mathbb{I}$.

Let $\bigoplus_{a=-n}^{n} \mathbb{Z}\varepsilon_a$ be an integer lattice with the orthonormal basis $\{\varepsilon_i, -n \le i \le n\}$. We can identify X

$$X = \bigoplus_{a=-n}^{n} \mathbb{Z}\varepsilon_a \Big/ \mathbb{Z}(\sum_{a=-n}^{n} \varepsilon_a).$$

Let ϑ be the involution on the weight lattice X defined by $\vartheta(\varepsilon_a) = -\varepsilon_{-a}$ for $-n \leqslant a \leqslant n$. We can write $\mu^\vartheta = \vartheta(\mu)$, for $\mu \in X$. Let X^ϑ be the sublattice of ϑ-fixed points of X. Since $\alpha_i^\vartheta = \alpha_{-i}$ for all $i \in \mathbb{I}$, ϑ induces an automorphism of the root system. We set ${}^\vartheta\alpha_i^\vee = \alpha_i^\vee - \alpha_{-i}^\vee$ for $i \in \mathbb{I}^+$ and define

$$X_\jmath := X/X^\vartheta, \qquad Y^\jmath := \bigoplus_{i \in \mathbb{I}^+} \mathbb{Z}\, {}^\vartheta\alpha_i^\vee. \tag{6.3}$$

The pairing $\langle \cdot, \cdot \rangle : Y \times X \longrightarrow \mathbb{Z}$ induces a nondegenerate pairing

$$\langle \cdot, \cdot \rangle : Y^\jmath \times X_\jmath \longrightarrow \mathbb{Z}.$$

For $\mu \in X_\jmath$, we write

$$\mu_i = \langle {}^\vartheta\alpha_i^\vee, \mu \rangle, \qquad \text{for } i \in \mathbb{I}^+.$$

We sometimes write the image of $\mu \in X$ in $X_\jmath$ again by μ. Let $\dot{\mathbf{U}}^\jmath$ be the $\mathbb{Q}(q)$-linear category with the object set $X_\jmath$ and morphisms generated by $\mathrm{E}_i : \mu \mapsto \mu + \alpha_i = \mu - \alpha_{-i}$, $\mathrm{F}_i : \mu \mapsto \mu - \alpha_i = \mu + \alpha_{-i}$, for all $i \in \mathbb{I}^+$, subject to the following relations for $i \neq j$:

$$[\mathrm{E}_i, \mathrm{F}_j]1_\mu = 0, \tag{6.4}$$

$$[\mathrm{E}_i, \mathrm{F}_i]1_\mu = [\mu_i]1_\mu, \quad \forall\, i \neq \diamond, \tag{6.5}$$

$$\sum_{a+b=1-a_{ij}} (-1)^a \mathrm{E}_i^{(a)} \mathrm{E}_j \mathrm{E}_i^{(b)} = \sum_{a+b=1-a_{ij}} (-1)^a \mathrm{F}_i^{(a)} \mathrm{F}_j \mathrm{F}_i^{(b)} = 0, \tag{6.6}$$

$$(\mathrm{E}_\diamond^{(2)}\mathrm{F}_\diamond - \mathrm{E}_\diamond\mathrm{F}_\diamond\mathrm{E}_\diamond + \mathrm{F}_\diamond\mathrm{E}_\diamond^{(2)})1_\mu = -(q^{\mu_\diamond+2} + q^{-\mu_\diamond-2})\mathrm{E}_\diamond 1_\mu, \tag{6.7}$$

$$(\mathrm{F}_\diamond^{(2)}\mathrm{E}_\diamond - \mathrm{F}_\diamond\mathrm{E}_\diamond\mathrm{F}_\diamond + \mathrm{E}_\diamond\mathrm{F}_\diamond^{(2)})1_\mu = -(q^{\mu_\diamond-1} + q^{-\mu_\diamond+1})\mathrm{F}_\diamond 1_\mu, \tag{6.8}$$

where $\mathrm{E}_\diamond^{(2)}1_\mu$ and $\mathrm{F}_\diamond^{(2)}1_\mu$ are divided powers. The algebra $\dot{\mathbf{U}}^\jmath$ is called *(idempotented) coideal algebra*. The special relations (6.7) and (6.8), called *$\jmath$-Serre relations*, make the coideal algebra different from the usual quantum group.

6.2 Coideal 2-category $\mathcal{U}^\jmath$

Here we introduce the categorification of $\dot{\mathbf{U}}^\jmath$ by Bao-Shan-Wang-Webster [2]. For $i, j \in \mathbb{I}^+$ we set

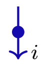
$$t_{ij} = \begin{cases} -1, & \text{if } j = i-1, \\ 1, & \text{otherwise.} \end{cases}$$

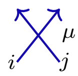
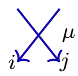
Definition 6.2.1. *The coideal 2-category $\mathfrak{U}^\jmath$ is the graded, additive $\Bbbk$-linear 2-category with*

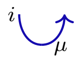
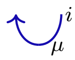
- *objects μ for all $\mu \in X_\jmath$.*
- *The 1-morphisms are the direct sums of compositions of shifts of these 1-morphisms.*

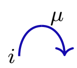
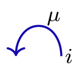
$$\mathcal{E}_i \colon \mu \to \mu + \alpha_i, \quad \mathcal{F}_i \colon \mu \to \mu - \alpha_i, \qquad \textit{for } i \in \mathbb{I}^+.$$

- *2-morphisms are generated by diagrams. We denote the identity 2-morphisms of $\mathcal{E}_i 1_\mu$ and $\mathcal{F}_i 1_\mu$ by ${}^{\mu+\alpha_i}\uparrow_i{}^{\mu}$ and ${}^{\mu-\alpha_i}\downarrow_i{}^{\mu}$ respectively. The other generators of 2-morphims are*

 $\colon \mathcal{E}_i 1_\mu \to \mathcal{E}_i 1_\mu \langle 2 \rangle$,  $\colon \mathcal{F}_i 1_\mu \to \mathcal{F}_i 1_\mu \langle 2 \rangle$,

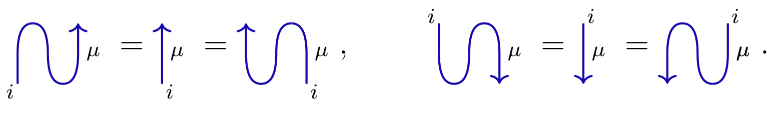
 $\colon \mathcal{E}_i \mathcal{E}_j 1_\mu \to \mathcal{E}_j \mathcal{E}_i 1_\mu \langle -a_{ij} \rangle$,  $\colon \mathcal{F}_i \mathcal{F}_j 1_\mu \to \mathcal{F}_j \mathcal{F}_i 1_\mu \langle -a_{ij} \rangle$,

 $\colon 1_\mu \to \mathcal{F}_i \mathcal{E}_i 1_\mu \langle \mu_i + a_{ii} - 1 \rangle$,  $\colon 1_\mu \to \mathcal{E}_i \mathcal{F}_i 1_\mu \langle 1 - \mu_i \rangle$,

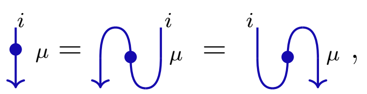
 $\colon \mathcal{E}_i \mathcal{F}_i 1_\mu \to 1_\mu \langle 1 - \mu_i \rangle$,  $\colon \mathcal{F}_i \mathcal{E}_i 1_\mu \to 1_\mu \langle \mu_i + a_{ii} - 1 \rangle$.

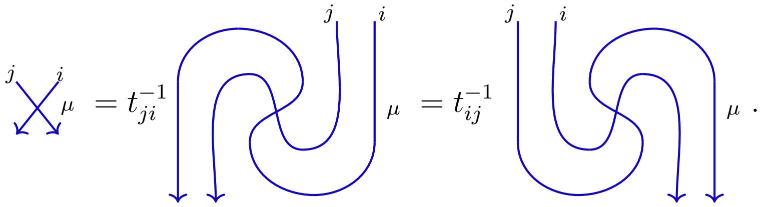
The 2-morphisms are subject to the following relations (1)–(8):

1. *Adjunction:*

 (6.9)

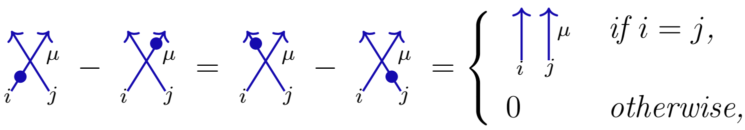
2. *Cyclicity of x and τ:*

 (6.10)

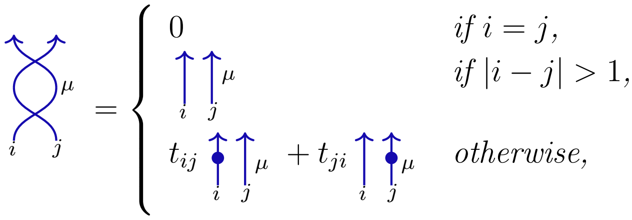


(6.11)

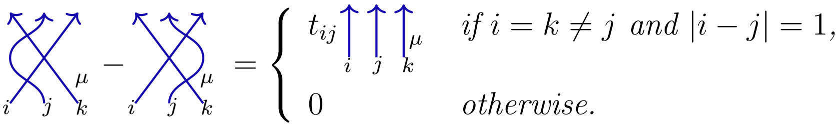
3. KLR relations:



(6.12)

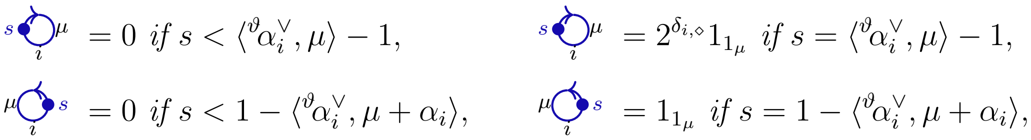


(6.13)



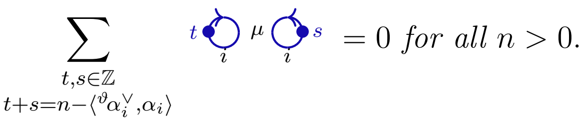
(6.14)

4. Bubble relations:



(6.15)

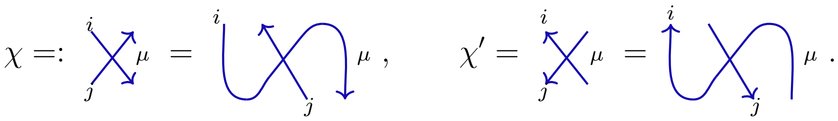
(6.16)



(6.17)

The bubbles with negative 'number' of dots are determined by the relation (6.17) *and called fake bubbles.*

5. *Nodal relations :*
Set



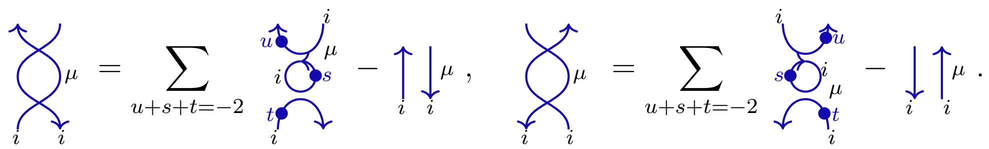
(6.18)

Then for any i we have

(6.19)

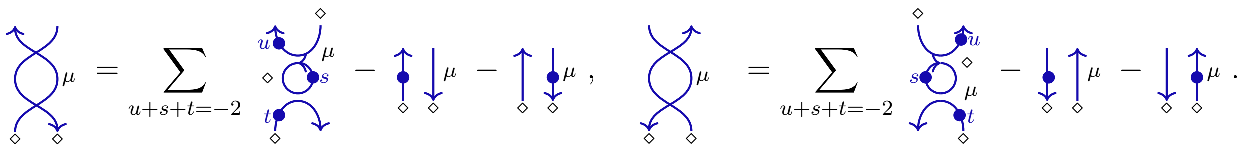
6. *Bicross relations:*

For $i \neq \diamond$, we have



(6.20)

For $i = \diamond$, we have



(6.21)

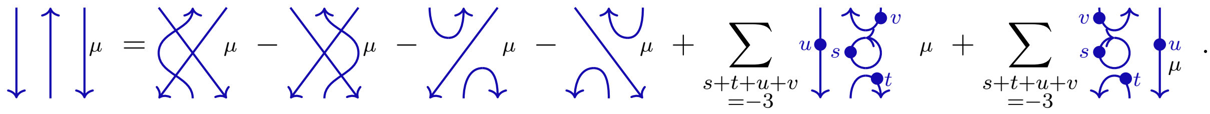
In all the sums above, the indices that are not on a circle run over non-negative integers.

7. *Mixed relations:*

For any $i \neq j$,

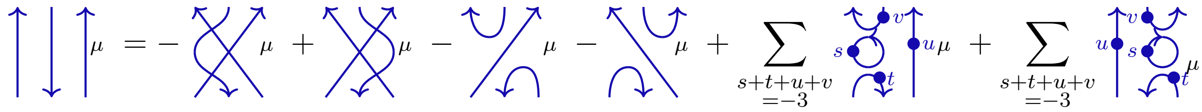
(6.22)

8. *For the sake of simplicity, we will omit the index* $\diamond$ *from the diagrams, that is, all the strands without labels should be viewed as labeled by* $\diamond$. *We have the following relation:*



(6.23)

The relation



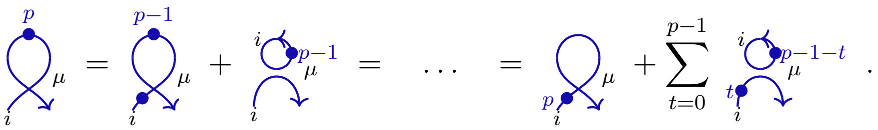
holds in $\mathcal{U}^\jmath$, since it is the image of (6.23) under the isomorphism $\mathrm{End}(\mathcal{F}_\diamond\mathcal{E}_\diamond\mathcal{F}_\diamond) \cong \mathrm{End}(\mathcal{E}_\diamond\mathcal{F}_\diamond\mathcal{E}_\diamond)$ given by adjunction.

We generalize the nodal relations (6.19) and the bicross relations (6.21) for $\diamond$-labeled strands in the following two lemmas.

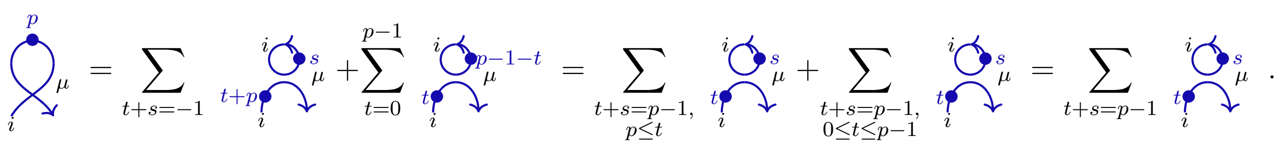
Lemma 6.2.2. *The following relations hold for any* $p \geq 0$*:*

(6.24)

Proof. We prove the equality on the left, the one on the right follows similarly. Using the relation (6.12), we inductively move the dots through crossings:

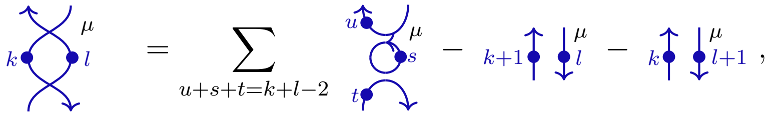


We can simplify further using the relation (6.19):

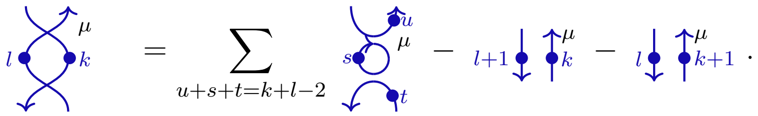


□

Lemma 6.2.3. *The following relations hold for any* $k, l \geq 0$*:*

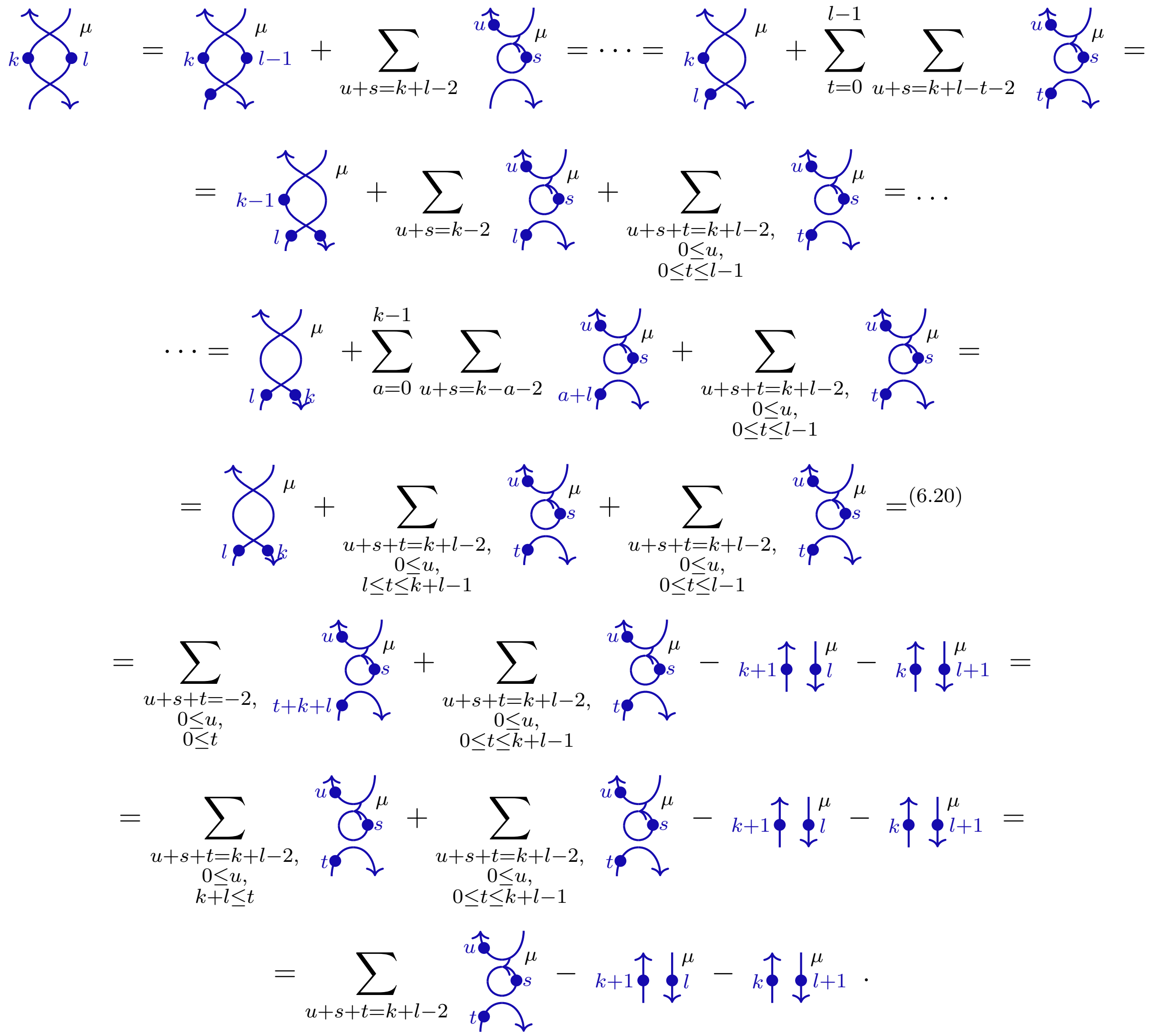


(6.25)



(6.26)

Proof. *We prove the relation* (6.25)*. We slide the dots downwards through the crossing inductively:*



The relation (6.26) *can be proven in a similar way.* □

It can be easily checked that all the relations between 2-morphisms are homogeneous. Thus, the grading on $\mathfrak{U}^\jmath$ is well-defined. Let $\dot{\mathfrak{U}}^\jmath$ be the the Karoubi envelope of $\mathfrak{U}^\jmath$ and ${}_{\mathcal{A}}\dot{\mathbf{U}}^\jmath$ the $\mathcal{A}$-linear subcategory of $\dot{\mathbf{U}}^\jmath$ with the same objects and with morphisms generated by divided powers $\mathrm{E}_i^{(a)}1_\mu$, $\mathrm{F}_i^{(a)}1_\mu$, for all $i \in \mathbb{I}^+$, $\mu \in X_\jmath$, and $a \geq 0$. The next statement is due to Bao-Shan-Wang-Webster [2], Theorem 6.5.

Theorem 6.2.4. *There is a* $\mathbb{Q}(q)$*-module isomorphism between the split Grothendieck group* $K_0(\dot{\mathfrak{U}}^\jmath)$ *and* ${}_{\mathcal{A}}\dot{\mathbf{U}}^\jmath$.

Recall the definition of the $\mathcal{C}^\star$ for given $\Bbbk$-linear 2-category $\mathcal{C}$. We define the $\mathcal{U}^\jmath = (\mathfrak{U}^\jmath)^\star$, that is, $\mathcal{U}^\jmath$ is the 2-category with the same objects and 1-morphisms as $\mathfrak{U}^\jmath$, and with 2-morphisms

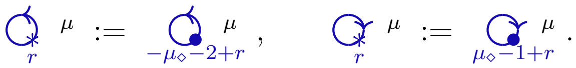
$$\mathcal{U}^\jmath(A,B)(f,g) := \bigoplus_{t\in\mathbb{Z}} \mathfrak{U}^\jmath(A,B)(f,g\langle t\rangle) \tag{6.27}$$

for given 1-morphisms $f, g\colon A \to B$.

7 Trace of $\mathcal{U}^\jmath$

7.1 Current coideal algebra

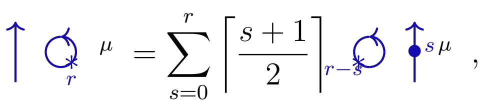
We recall our convention that all the strands without labels should be viewed as labeled by $\diamond$. Let $\mu_\diamond = \langle {}^\vartheta\alpha_\diamond^\vee, \mu\rangle$, and define degree $2r$ bubbles



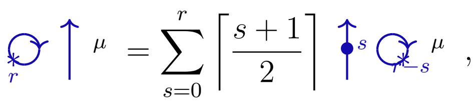
(7.1)

The sliding equations are given by Bao-Shan-Wang-Webster [2], Lemma A.1 and A.2.

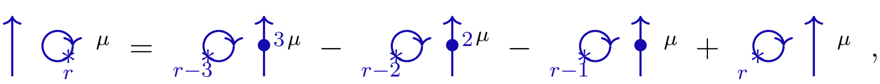
Lemma 7.1.1 (Bubble slides)**.** *The following bubble sliding relations hold for all* $\mu \in X_\jmath$*:*



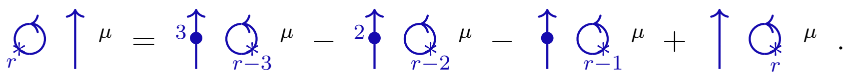
(7.2)



(7.3)

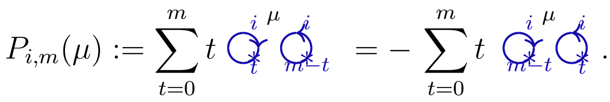


(7.4)



(7.5)

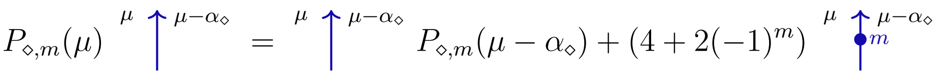
For $m \geq 1$, we define



(7.6)

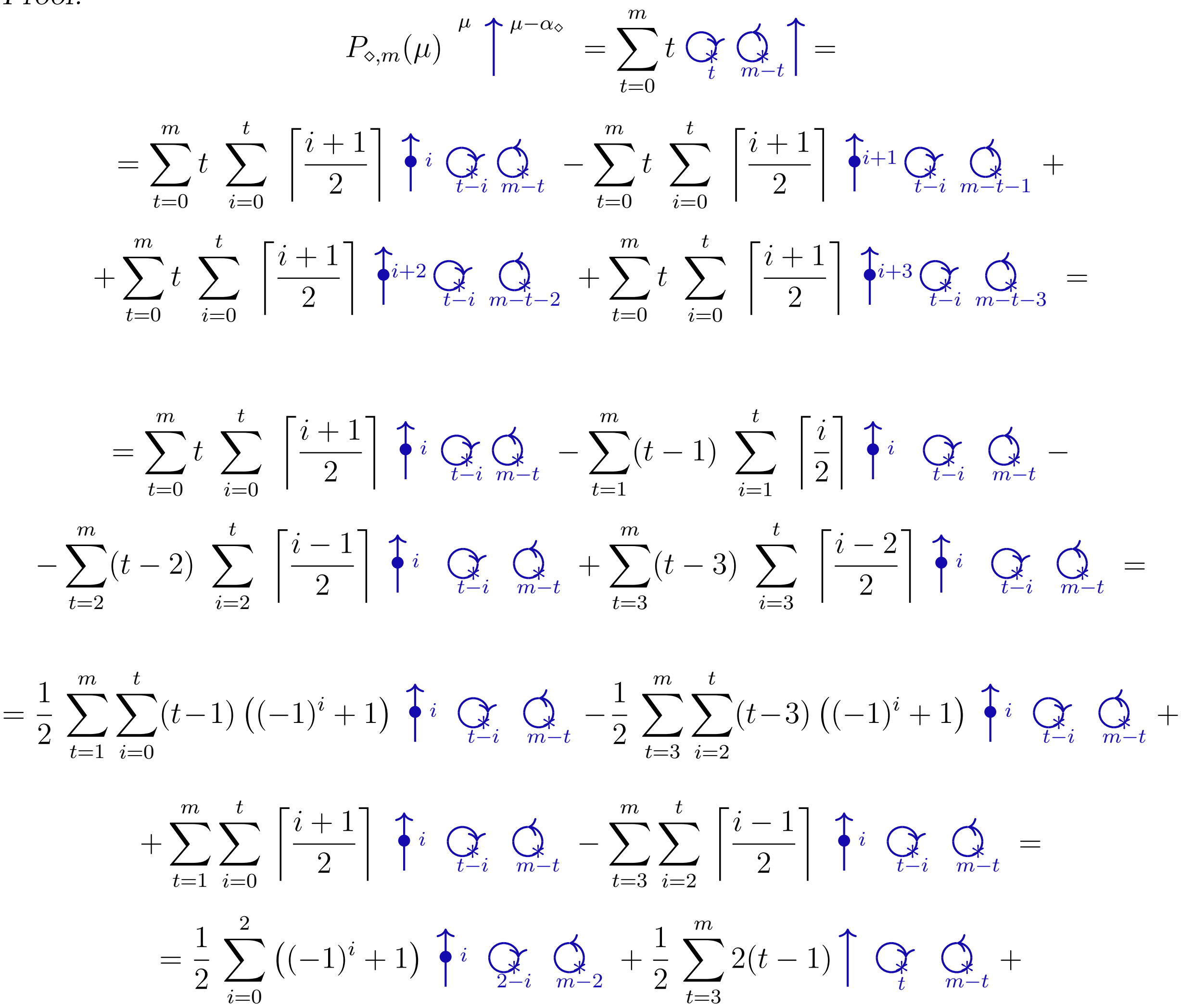
We will need the following crucial lemma for the proof of our main result.

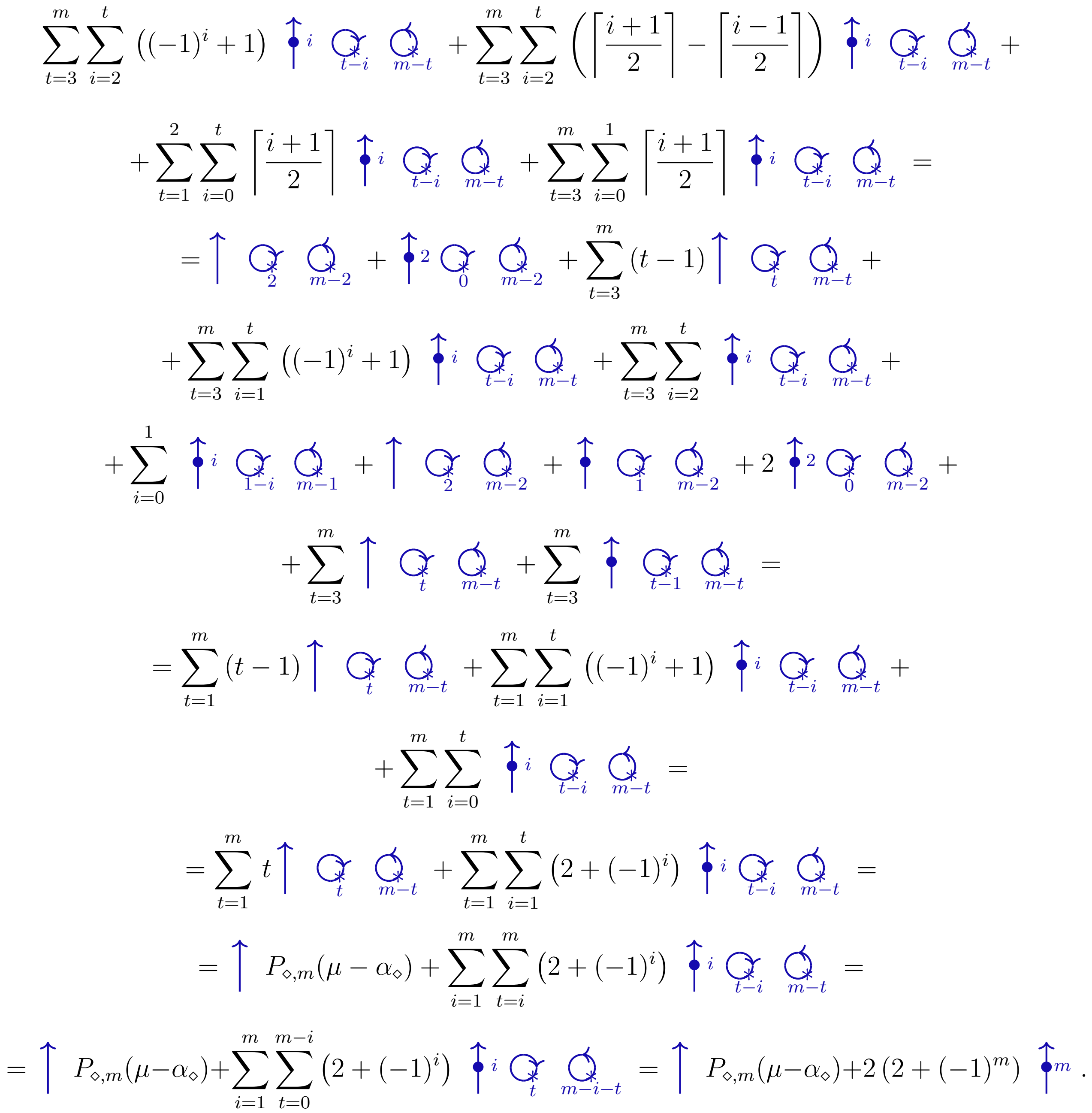
Lemma 7.1.2. *The following relation holds for $m \geq 1$:*



(7.7)

Proof.





□

Definition 7.1.3. *The current coideal algebra* $\mathbf{U}_t^\jmath$ *is the* $\Bbbk$*-algebra generated by* $\mathtt{E}_{i,r}, \mathtt{F}_{i,r}, \mathtt{H}_{i,r}$ *over all* $i \in \mathbb{I}^+$ *and* $r \geq 0$*, subject to the following relations:*

$$[\mathtt{H}_{i,r}, \mathtt{H}_{j,s}] = 0 \quad \textit{for all } i,j \in \mathbb{I}^\jmath,\ r,s \geq 0, \tag{7.8}$$

$$[\mathtt{E}_{i,r}, \mathtt{E}_{j,s}] = 0 \quad , [\mathtt{F}_{i,r}, \mathtt{F}_{j,s}] = 0 \quad \textit{if } |i-j| \neq 1,\ r,s \geq 0, \tag{7.9}$$

$$[\mathtt{H}_{i,r}, \mathtt{F}_{i,s}] = -2\mathtt{F}_{i,r+s}, \quad [\mathtt{H}_{i,r}, \mathtt{E}_{i,s}] = 2\mathtt{E}_{i,r+s} \ \textit{for all } i \neq \diamond,\ r,s \geq 0, \tag{7.10}$$

$$[\mathtt{H}_{j,r}, \mathtt{F}_{i,s}] = \mathtt{F}_{i,r+s}, \quad [\mathtt{H}_{j,r}, \mathtt{E}_{i,s}] = -\mathtt{E}_{i,r+s} \ \textit{for all } |i-j| = 1,\ r,s \geq 0, \tag{7.11}$$

$$[\mathtt{E}_{i,r+1}, \mathtt{E}_{j,s}] = [\mathtt{E}_{i,r}, \mathtt{E}_{j,s+1}], \quad [\mathtt{F}_{i,r+1}, \mathtt{F}_{j,s}] = [\mathtt{F}_{i,r}, \mathtt{F}_{j,s+1}] \ \textit{for all } i,j \in \mathbb{I}^\jmath,\ r,s \geq 0, \tag{7.12}$$

$$[\mathtt{E}_{i,r}, \mathtt{F}_{i,s}] = \mathtt{H}_{i,r+s} \quad \textit{for all } i \neq \diamond,\ r,s \geq 0, \tag{7.13}$$

$$[\mathtt{E}_{i,r}, \mathtt{F}_{j,s}] = 0 \quad \textit{for all } i \neq j,\ r,s \geq 0, \tag{7.14}$$

$$[\mathtt{E}_{i,p}, [\mathtt{E}_{i,r}, \mathtt{E}_{j,s}] = 0, \quad [\mathtt{F}_{i,p}, [\mathtt{F}_{i,r}, \mathtt{F}_{j,s}] = 0 \quad \textit{for all } |i-j| = 1,\ p,r,s \geq 0, \tag{7.15}$$

$$[\mathtt{H}_{\diamond,r}, \mathtt{F}_{\diamond,s}] = \left(-2 - (-1)^r\right) \mathtt{F}_{\diamond,r+s}, \quad [\mathtt{H}_{\diamond,r}, \mathtt{E}_{\diamond,s}] = \left(2 + (-1)^r\right) \mathtt{E}_{\diamond,r+s} \ \textit{for all } \ r,s \geq 0, \tag{7.16}$$

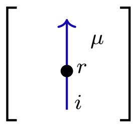
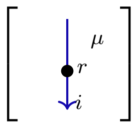
$$[\mathtt{F}_{\diamond,p}, [\mathtt{F}_{\diamond,r}, \mathtt{E}_{\diamond,s}]] = \left(-2 - (-1)^{p+s} - (-1)^{r+s}\right) \mathtt{F}_{\diamond,p+r+s} \quad \textit{for all } \ p,r,s \geq 0. \tag{7.17}$$

$$[\mathtt{E}_{\diamond,p}, [\mathtt{E}_{\diamond,r}, \mathtt{F}_{\diamond,s}]] = \left(-2 - (-1)^{p+s} - (-1)^{r+s}\right) \mathtt{E}_{\diamond,p+r+s}, \quad \textit{for all } \ p,r,s \geq 0, \tag{7.18}$$

Let $\mathbf{U} = \mathbf{U}(\mathfrak{sl}_{2n+1}[t])$ be the current algebra, generated by $E_{i,r}, F_{i,r}, H_{i,r}$ over all $i \in \mathbb{I}$ and $r \geq 0$. There is an embedding of algebras $\jmath \colon \mathbf{U}_t^\jmath \to \mathbf{U}$ such that for all $i \in \mathbb{I}^+$

$$\mathtt{E}_{i,r} \mapsto E_{i,r} + (-1)^r F_{-i,r}, \quad \mathtt{F}_{i,r} \mapsto F_{i,r} + (-1)^r E_{-i,r}, \quad \mathtt{H}_{i,r} \mapsto H_{i,r} - (-1)^r H_{-i,r}. \tag{7.19}$$

The algebra $\mathbf{U}_t^\jmath$ is graded, and it can be viewed as a subalgebra of $\mathbf{U}$ under the embedding $\jmath$. The trace $\mathrm{Tr}\,\mathcal{U}^\jmath$ of the 2-category $\mathcal{U}^\jmath$ is the $\Bbbk$-linear category with the same objects as $\mathcal{U}^\jmath$, and morphisms are obtained by closing the diagrams representing the 2-endomorphisms of $\mathcal{U}^\jmath$ to the right. Let $\mathsf{E}_{i,r}$, $\mathsf{F}_{i,r}$, $\mathsf{H}_{i,r}$ denote the generators of $\mathrm{Tr}\,\mathcal{U}^\jmath$ for all $i \in \mathbb{I}^\jmath$ and $r \geq 0$:

$\mathsf{E}_{i,r}1_\mu =$ , $\quad \mathsf{F}_{i,r}1_\mu :=$ , (7.20)

$$\mathsf{H}_{i,r}1_\mu := \begin{cases} \frac{1}{2}\left[P_{\diamond,r}(\mu)\,\mathrm{Id}_{1_\mu}\right] & \text{if } i = \diamond \text{ and } r > 0, \\ \left[P_{i,r}(\mu)\,\mathrm{Id}_{1_\mu}\right] & \text{if } i \neq \diamond \text{ and } r > 0, \\ \mu_i \text{ if } r = 0, \end{cases} \tag{7.21}$$

where $P_{i,r}(\mu)$ was defined in equation (7.6). We depict the trace classes graphically by closing diagrams around an annulus $\star$. Let $\dot{U}_t^\jmath$ be the idempotent completion of the algebra $U_t^\jmath$. Then the following statement holds:

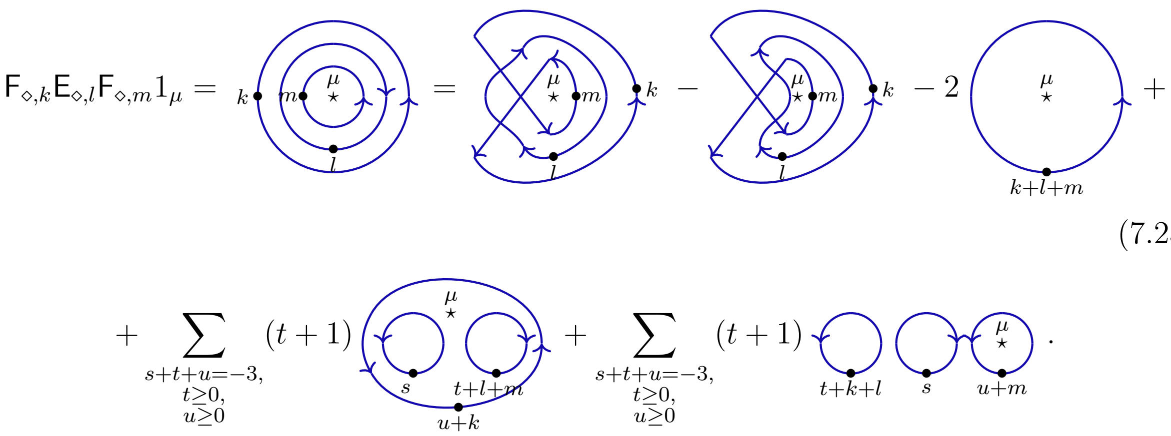
Theorem 7.1.4. *There is a well-defined algebra homomorphism* $h\colon \dot{U}_t^\jmath \to \operatorname{Tr} \mathcal{U}^\jmath$*, given by*

$$\mathtt{E}_{i,r}1_\mu \mapsto \mathsf{E}_{i,r}1_\mu, \qquad \mathtt{F}_{i,r}1_\mu \mapsto \mathsf{F}_{i,r}1_\mu, \qquad \mathtt{H}_{i,r}1_\mu \mapsto \mathsf{H}_{i,r}1_\mu. \tag{7.22}$$

7.2 Proof of the homomorphism theorem

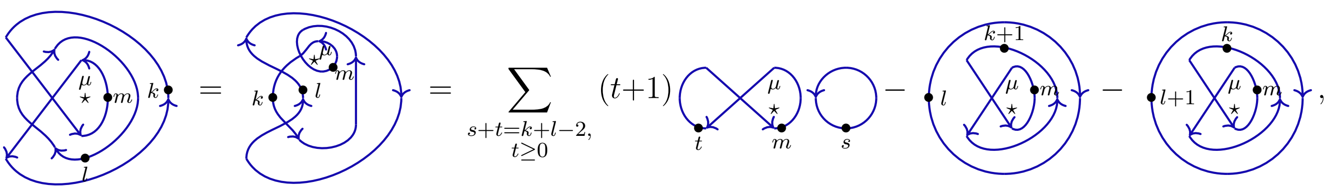
In this final subsection we prove the Theorem 7.1.4.

Proof. In order to show that the homomorphism is well-defined, we will prove the relations (7.8) - (7.17) for the generators of $\operatorname{Tr} \mathcal{U}^\jmath$ using the relation in the 2-category $\mathcal{U}^\jmath$. The relations (7.8) - (7.15) are already proved in Theorem 2.39. We only need to consider the relations between $\diamond$-labeled generators. The relations (7.16) follows directly from the Lemma 7.1.2, by closing the upward oriented strands to the left and right, respectively. We now prove the relation (7.17). By closing the terms of the equation (6.23) around the annulus $\star$, we obtain

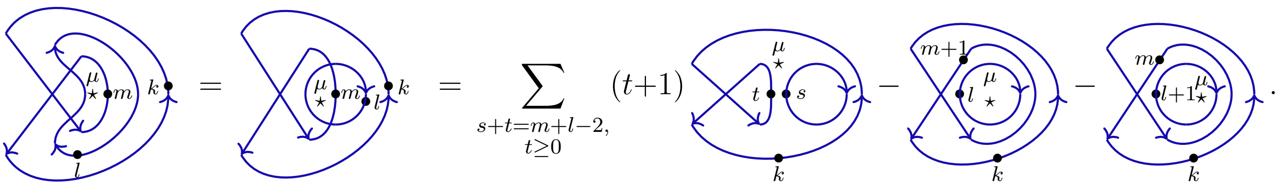


(7.23)

We can simplify the first and the second terms in equation (7.23) using the trace relation and Lemma 6.2.3:

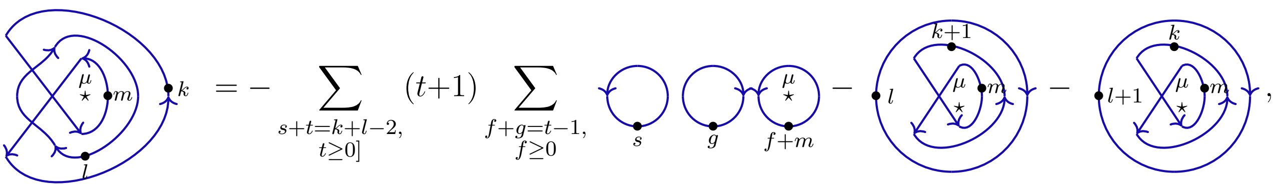


(7.24)

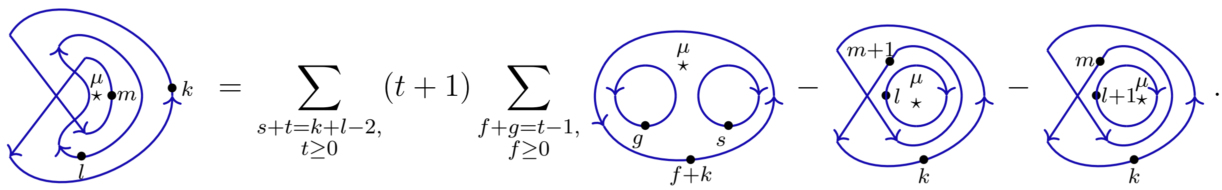


(7.25)

The crossings in the first terms of the equations (7.24) and (7.25) can be further reduced using the relations (6.24), and we get

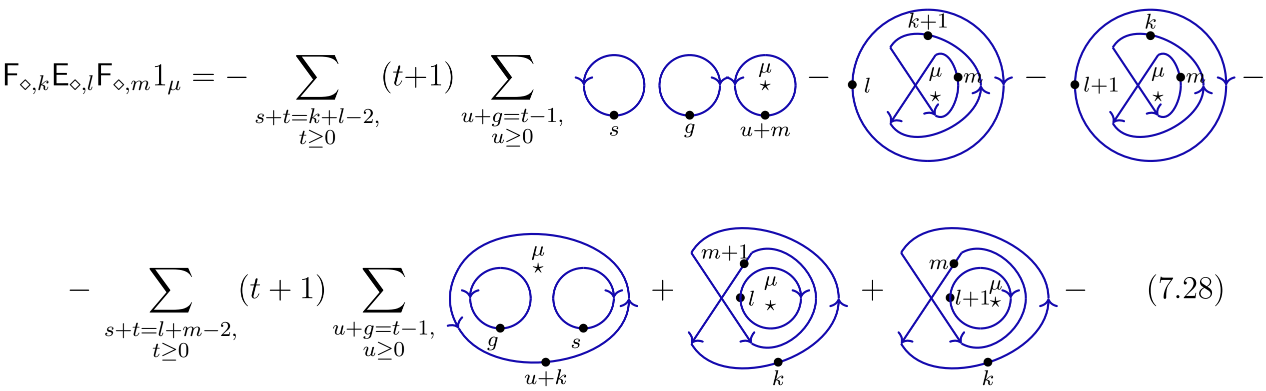


(7.26)

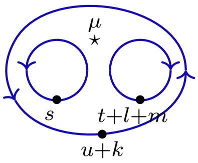


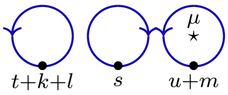
(7.27)

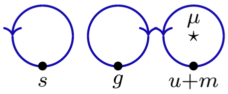
Using the equations (7.26) and (7.27) and changing f to u, we can now rewrite the relation (7.23) as follows:



(7.28)

$$-2\,\mathsf{F}_{\diamond,k+l+m}1_\mu + \sum_{\substack{s+t+u=-3,\\ t\geq 0,\\ u\geq 0}} (t+1)$$


$$+ \sum_{\substack{s+t+u=-3,\\ t\geq 0,\\ u\geq 0}} (t+1)$$


$$= - \sum_{\substack{s+g+u=k+l-3,\\ u+g+1\geq 0,\\ u\geq 0}} (u+g+2)$$


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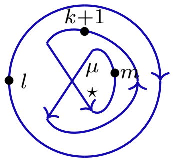

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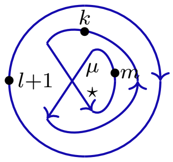
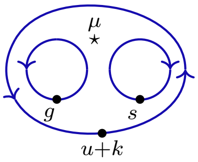
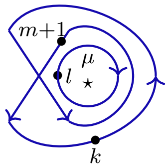
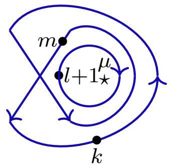


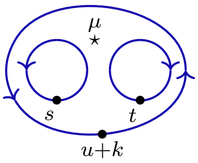
$-$

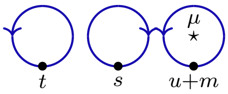
$$- \sum_{\substack{s+g+u=l+m-3,\\ u+g+1\geq 0,\\ u\geq 0}} (u+g+2)$$


$+$


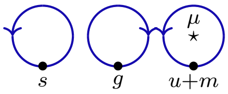
$+$


$-$ (7.29)

$$-2\,\mathsf{F}_{\diamond,k+l+m}1_\mu + \sum_{\substack{s+t+u=l+m-3,\\ t\geq l+m,\\ u\geq 0}} (t-l-m+1)$$


$$+ \sum_{\substack{s+t+u=k+l-3,\\ t\geq k+l,\\ u\geq 0}} (t-k-l+1)$$


$=$

$$= \sum_{\substack{s+g+u=k+l-3,\\ s\leq k+l-2,\\ u\geq 0}} (s-k-l+1)$$


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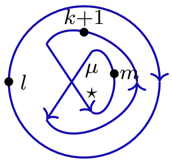

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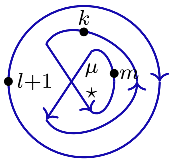
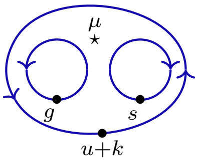
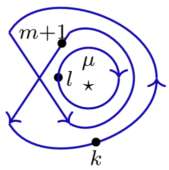


$+$

$$+ \sum_{\substack{s+g+u=l+m-3,\\ s\leq l+m-2,\\ u\geq 0}} (s-l-m+1)$$


$+$


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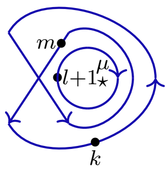
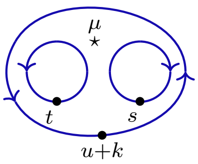
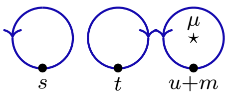
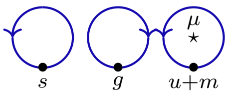


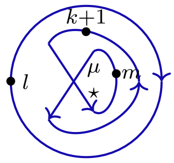
$-$ (7.30)

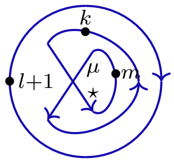
$$-2\,\mathsf{F}_{\diamond,k+l+m}1_\mu + \sum_{\substack{s+t+u=l+m-3,\\ s\geq l+m,\\ u\geq 0}} (s-l-m+1)$$


$$+ \sum_{\substack{s+t+u=k+l-3,\\ s\geq k+l,\\ u\geq 0}} (s-k-l+1)$$


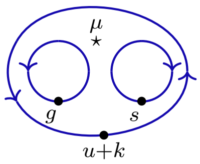
$=$

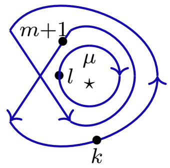
$$= \sum_{\substack{s+g+u=k+l-3,\\ u\geq 0}} (s-k-l+1)$$ 

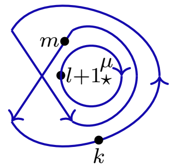
$-$ 

$-$ 

$+$

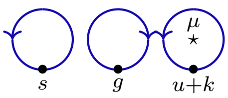
$$+ \sum_{\substack{s+g+u=l+m-3,\\ u\geq 0}} (s-l-m+1)$$ 

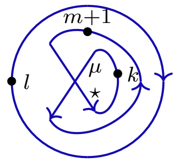
$+$ 

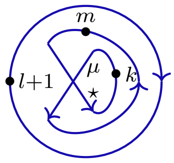
$+$ 

$$-2\,\mathsf{F}_{\diamond,k+l+m}1_\mu. \tag{7.31}$$

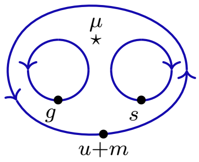
By interchanging k and m in the equation (7.31), we have

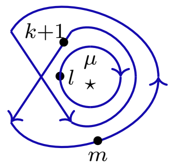
$$\mathsf{F}_{\diamond,m}\mathsf{E}_{\diamond,l}\mathsf{F}_{\diamond,k}1_\mu == \sum_{\substack{s+g+u=m+l-3,\\ u\geq 0}} (s-m-l+1)$$ 

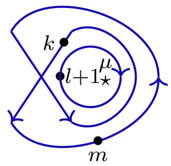
$-$ 

$-$ 

$+$

$$+ \sum_{\substack{s+g+u=l+k-3,\\ u\geq 0}} (s-l-k+1)$$ 

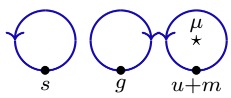
$+$ 

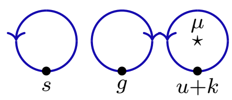
$+$ 

$$-2\,\mathsf{F}_{\diamond,k+l+m}1_\mu. \tag{7.32}$$

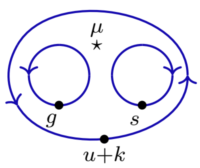
We add the left and right hand sides of the equations (7.31) and (7.32) and using the relations (2.7) and (2.8) in [8], we get the following equality:

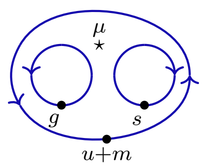
$$\mathsf{F}_{\diamond,k}\mathsf{E}_{\diamond,l}\mathsf{F}_{\diamond,m}1_\mu + \mathsf{F}_{\diamond,m}\mathsf{E}_{\diamond,l}\mathsf{F}_{\diamond,k}1_\mu - \mathsf{F}_{\diamond,k}\mathsf{F}_{\diamond,m}\mathsf{E}_{\diamond,l}1_\mu - \mathsf{E}_{\diamond,l}\mathsf{F}_{\diamond,k}\mathsf{F}_{\diamond,m}1_\mu + 4\mathsf{F}_{\diamond,k+l+m}1_\mu =$$

$$= \sum_{\substack{s+g+u=k+l-3,\\ u\geq 0}} (s-k-l+1)$$ 

$$+ \sum_{\substack{s+g+u=m+l-3,\\ u\geq 0}} (s-m-l+1)$$ 

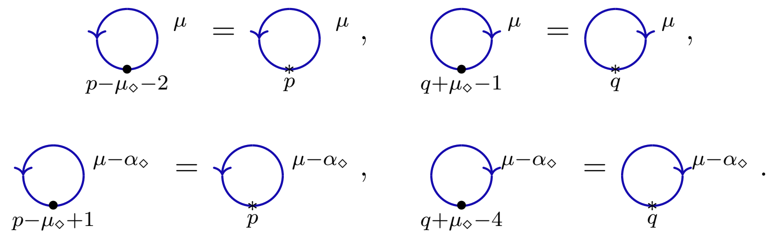
$+$

$$+ \sum_{\substack{s+g+u=l+m-3,\\ u\geq 0}} (s-l-m+1)$$ 

$$+ \sum_{\substack{s+g+u=l+k-3,\\ u\geq 0}} (s-l-k+1)$$ 

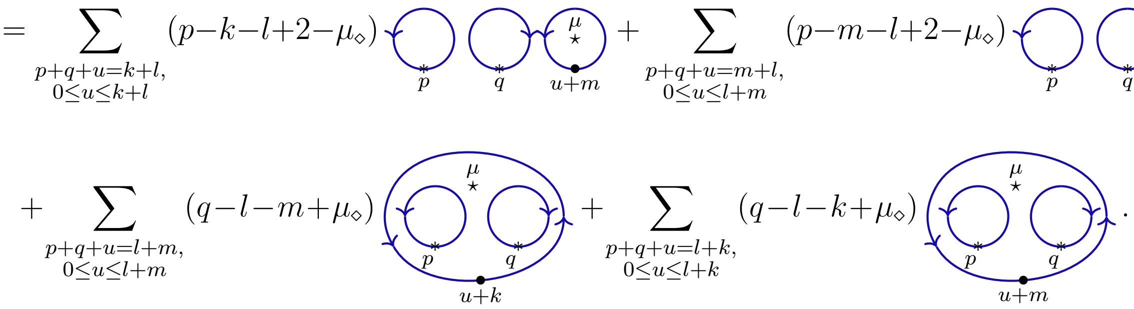
$$. \tag{7.33}$$

We will use the $*$ labels to emphasize the degrees of bubbles:



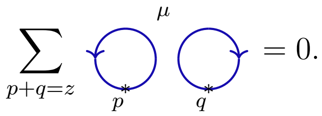
With this notation, the equation (7.33) becomes

$$-[\mathsf{F}_{\diamond,k},[\mathsf{F}_{\diamond,m}\mathsf{E}_{\diamond,l}]]1_\mu + 4\mathsf{F}_{\diamond,k+l+m}1_\mu =$$



(7.34)

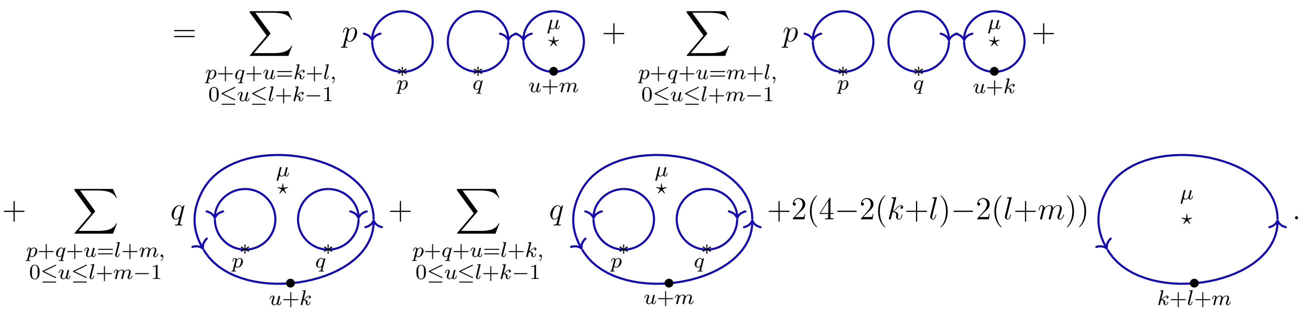
Recall that the following Grassmannian relation (6.17) holds for all $z \geq 1$:



(7.35)

We can split the sums on the right hand side of the equation (7.34) and use the Grassmannian relation (7.35) to write the equality (7.34) as follows:

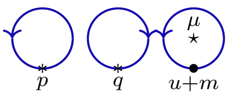
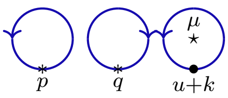
$$-[\mathsf{F}_{\diamond,k},[\mathsf{F}_{\diamond,m}\mathsf{E}_{\diamond,l}]]1_\mu + 4\mathsf{F}_{\diamond,k+l+m}1_\mu =$$

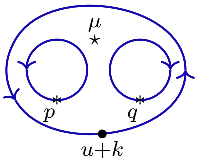
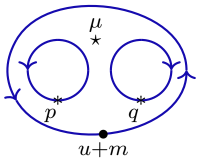


(7.36)

Hence,

$$-[\mathsf{F}_{\diamond,k},[\mathsf{F}_{\diamond,m}\mathsf{E}_{\diamond,l}]]1_\mu - 4\mathsf{F}_{\diamond,k+l+m}1_\mu =$$

$$= -\sum_{\substack{p+q+u=k+l,\\ 0\le u\le l+k-1}} q$$  $$- \sum_{\substack{p+q+u=m+l,\\ 0\le u\le l+m-1}} q$$  $$-$$

$$+\sum_{\substack{p+q+u=l+m,\\ 0\le u\le l+m-1}} q$$  $$+\sum_{\substack{p+q+u=l+k,\\ 0\le u\le l+k-1}} q$$  $$-4(k+2l+m)\,\mathsf{F}_{\diamond,k+l+m}1_\mu\,. \quad (7.37)$$

By using the notation (7.6), we write (7.37) as

$$[\mathsf{F}_{\diamond,k},[\mathsf{F}_{\diamond,m}\mathsf{E}_{\diamond,l}]]1_\mu + 4\mathsf{F}_{\diamond,k+l+m}1_\mu =$$

$$= \sum_{u=0}^{k+l-1} \left(P_{k+l-u}(\mu-\alpha_\diamond)\mathsf{F}_{\diamond,u+m}1_\mu - \mathsf{F}_{\diamond,u+m}1_\mu P_{k+l-u}(\mu)\right) +$$

$$+\sum_{u=0}^{l+m-1} \left(P_{l+m-u}(\mu-\alpha_\diamond)\mathsf{F}_{\diamond,u+k}1_\mu - \mathsf{F}_{\diamond,u+k}1_\mu P_{l+m-u}(\mu)\right) + 4(k+2l+m)\,\mathsf{F}_{\diamond,k+l+m}1_\mu =$$

$$= 2\sum_{u=0}^{k+l-1} [H_{\diamond,k+l-u},\mathsf{F}_{\diamond,u+m}]1_\mu + 2\sum_{u=0}^{l+m-1} [H_{\diamond,l+m-u},\mathsf{F}_{\diamond,u+k}]1_\mu + 4(k+2l+m)\,\mathsf{F}_{\diamond,k+l+m}1_\mu =$$

$$= 2\sum_{u=0}^{k+l-1} (-2-(-1)^{k+l-u})\mathsf{F}_{\diamond,k+l+m}1_\mu + 2\sum_{u=0}^{l+m-1} (-2-(-1)^{l+m-u})\mathsf{F}_{\diamond,k+l+m}1_\mu + 4(k+2l+m)\,\mathsf{F}_{\diamond,k+l+m}1_\mu =$$

$$= -2\sum_{u=0}^{k+l-1} (-1)^{k+l-u}\mathsf{F}_{\diamond,k+l+m}1_\mu - 2\sum_{u=0}^{l+m-1} (-1)^{l+m-u}\mathsf{F}_{\diamond,k+l+m}1_\mu\,.$$

Thus, we finally have

$$[\mathsf{F}_{\diamond,k},[\mathsf{F}_{\diamond,m}\mathsf{E}_{\diamond,l}]]1_\mu = \left(-4 - 2\sum_{u=0}^{k+l-1}(-1)^{k+l-u} - 2\sum_{u=0}^{l+m-1}(-1)^{l+m-u}\right)\mathsf{F}_{\diamond,k+l+m}1_\mu =$$

$$= \left(-4 + 2\sum_{u=0}^{k+l-1}(-1)^{u} + 2\sum_{u=0}^{l+m-1}(-1)^{u}\right)\mathsf{F}_{\diamond,k+l+m}1_\mu = \left(-2-(-1)^{k+l}-(-1)^{l+m}\right)\mathsf{F}_{\diamond,k+l+m}1_\mu.$$

The relation (7.18) can be proved in a similar way. □

We expect the $\Bbbk$-algebra homomorphism (7.1.4) to be an isomorphism. We have the following result about the trace of the 2-category $\mathfrak{U}^\jmath$.

Theorem 7.2.1. *The Chern character map* $h_q\colon K_0(\dot{\mathfrak{U}}^\jmath) \otimes_{\mathbb{Z}} \mathbb{Q}(q) \cong \dot{\mathbf{U}}^\jmath \to \operatorname{Tr} \dot{\mathfrak{U}}^\jmath$ *is an isomorphism of* $\mathbb{Q}(q)$*-modules.*

Proof. The proof is similar to the proof of Theorem 8.1 in [5]. The 2-category $\dot{\mathfrak{U}}^\jmath$ is Krull-Schmidt (see section 4.5 in [2]). Then by Proposition 2.4 in [5], the map h_q is injective.

The trace $\operatorname{Tr} \dot{\mathfrak{U}}^\jmath$ is generated by $\mathsf{E}_{i,0}1_\mu$ and $\mathsf{F}_{i,0}1_\mu$, $\mu \in X^\jmath$ over $\mathbb{Q}(q)$, and they are obviously images of the generators $\mathrm{E}_{i,0}1_\mu$ and $\mathrm{F}_{i,0}1_\mu$ of $\dot{\mathfrak{U}}^\jmath$. Thus, h_q is surjective. □

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