Asymptotic Behaviour of Elliptic Boundary Value Problems in Long Domains

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Abstract

We analyse in this thesis the asymptotic behaviour with respect to the domain for two partial differential equations (pde's) with different boundary conditions. For the third problem we consider the upper bound of the error between the weak solution of a Poisson problem and its approximation. The idea how to choose the approximation comes from a known convergence result.

First we study a linearized, non-stationary Navier-Stokes type problem with the given flux in an infinite periodic pipe of period length L with respect to x_n, $n \in \{2, 3\}$. We show the existence and uniqueness of such a solution in a finite domain. Further we prove that the solution of this problem in 2ℓ periodic cells converges exponentially to the L-periodical extended solution as $\ell \to +\infty$. These results are a joint work with M. Chipot, N. Klovienė and K. Pileckas which have been published in the paper "On a non-stationary fluid flow problem in an infinite periodic pipe", see [19].

In the next chapter we consider a pure Neumann problem in n-dimensional cylinder-like domains. We are interested in the asymptotic behaviour of the solution of this kind or problem when the domain becomes unbounded in p-directions, $1 \leq p < n$. We show, under some restrictions for the initial data and the domain, that this solution converges exponentially to the solution of a Neumann problem in the corresponding infinite domain. The proof of this convergence result varies, depending if $p = 1$ or $1 < p < n$. Further for the case $p = 1$ we analyse in addition the special case where the domain and the initial data are periodic. These results are a joint work with M. Chipot and will be published in the paper "On the asymptotic behaviour of the pure Neumann problem in cylinder-like domains and its applications", see [26].

The idea of the third problem arises from a known result (see [15]) which says that the weak solution of the n-dimensional Poisson equation $-\Delta u(x_1, x') = f(x')$, $x' = (x_2, \ldots, x_n)$ with Dirichlet boundary conditions converges exponentially in the center to the weak solution of the (n-1)-dimensional analogous Poisson problem when the domain converges to infinity in the x_1-direction. Note that the function f needs to be independent of x_1. From this convergence result we receive the idea to approximate the weak solution of the n-dimensional Poisson equation with the weak solution of the (n-1)-dimensional analogous Poisson problem multiplied by a cutoff function depending only on x_1. The goal of this chapter is to find an optimal cutoff function and then to analyse the upper bound of the error between the weak solution of this Poisson problem and its approximation. For the approximation we use the Galerkin finite element method.

Zusammenfassung

In dieser Doktorarbeit analysieren wir das asymptotische Verhalten bezüglich des Gebietes für zwei partielle differential Gleichungen mit unterschiedlichen Randbedingungen. Für das dritte Problem untersuchen wir die obere Schranke des Fehlers zwischen der Lösung eines Poisson Problems und dessen Approximation. Die Idee, wie die Approximation zu wählen ist, kommt von einem bekannten Konvergenzresultat.

Als Erstes analysieren wir ein lineares, nicht stationäres Navier-Stokes Problem mit gegebenem Fluss in einem periodischen unendlich langem Zylinder mit Periode L in Richtung x_n, $n \in \{2,3\}$. Wir zeigen Existenz und Eindeutigkeit der Lösung in einem endlich dimensionalem Raum. Des Weiteren beweisen wir die exponentielle Konvergenz dieser Lösung in 2ℓ Zellen zur L-periodisch erweitern Lösung, wenn ℓ gegen unendlich strebt. Diese Resultate sind eine gemeinsame Arbeit mit M. Chipot, N. Kloviené und K. Pileckas und wurden unter "On a non-stationary fluid flow problem in an infinite periodic pipe" publiziert, siehe [19].

Im nächsten Kapitel betrachten wir ein pures Neumann Problem in n-dimensionalen Zylinder ähnlichen Gebieten. Wir interessieren uns für das asymptotische Verhalten einer Lösung solch eines Problems, wenn das Gebiet unbeschränkt wird in p-Richtungen, $1 \leq p < n$. Wir zeigen, unter einigen Einschränkungen für die anfänglichen Daten und das Gebiet, dass diese Lösung exponentiell zu der Lösung des Neumann Problems im entschrechenden unendlichen Gebiet konvergiert. Der Beweis dieses Konvergenzresultates variert, je nachdem ob $p = 1$ oder $1 < p < n$ ist. Des Weiteren anlaysieren wir für den Fall $p = 1$ zusätzlich den Spezialfall wo das Gebiet und die anfänglichen Daten periodisch sind. Diese Resultate sind eine gemeinsame Arbeit mit M. Chipot und werden in "On the asymptotic behaviour of the pure Neumann problem in cylinder-like domains and its applications" publiziert, siehe [26].

Die Idee für das dritte Problem kommt von einem bekannten Resultat (siehe [15]) welches sagt, dass die schwache Lösung der n-dimensionalen Poisson Gleichung $-\Delta u(x_1, x') = f(x')$, $x' = (x_2, \ldots, x_n)$ mit Dirichlet Randbedingungen im Zentrum gegen die analoge (n-1)-dimensionale Poisson Gleichung konvergiert, wobei das Gebiet gegen unendlich in der x_1-Richtung strebt. Wir nehmen an, dass die Funktion f unabhängig von der Variabel x_1 ist. Dieses Konvergenzresultat impliziert die Idee, die schwache Lösung des n-dimensionalen Poisson Problems mit der (n-1)-dimensionalen analogen Poisson Lösung multipliziert mit einer nur von x_1 abhängigen Funktion zu approximieren. Das Ziel dieses Kapitels ist es, eine optimale Abschneidefunktion zu finden und anschliessend analysieren wir die obere Schranke des Fehlers zwischen der schwachen Lösung des Poisson Problems und dessen Approximation. Für die Approximation benützen wir die Galerkin finite endliche Methode.

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Contents

Introduction

Asymptotic behaviour in domains becoming unbounded, especially in long cylinders for varies partial differential equations (pde's) have been already analysed in many different papers (see [7]-[25] or [39]). Also the case where one assumes at the beginning that the data are periodic, see e.g. [19]-[21] or [25]. Very often the problems have Dirichlet boundary conditions but if one changes to Neumann boundary conditions, some other difficulties may occur, also when we consider linear elliptic equations, see [21].
As well lot of research concerning approximation of a pde with finite element method and their evaluation of the error estimate has been done, see for instance [3]-[4] or [57]-[62].

Focusing on this thesis, we study first the asymptotic behaviour for two different pde problems of second order (the first one is non-stationary) and secondly we analyse the upper bound of the error between the weak solution of a Poisson problem and their approximation. The idea for the approximation arises from a known asymptotic behaviour result.

First we analyse an incompressible, linearized, non-stationary Navier-Stokes problem in a n-dimensional infinite pipe of periodicity $L \in \mathbb{R}^+$ into x_n-direction (n=2 or 3). More precisely, we consider the problem

$$\begin{cases} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{U}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Pi_L, \\ \operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Pi_L, \qquad \mathbf{u}(x,0) = \mathbf{a}(x), \\ \mathbf{u}|_{S_L} = 0 \quad t \in (0,T), \\ \mathbf{u}(x',0,t) = \mathbf{u}(x',L,t) \quad t \in (0,T) \end{cases} \tag{0.1}$$

with the prescribed flux condition

$$\int_{\sigma_{x_n}} u_n(x', x_n, t)\, dx' = F(t) \quad \forall x_n \in (0,L), \tag{0.2}$$

where

$$\Pi_L = \{x = (x', x_n) \in \sigma_{x_n} \times \{x_n\},\, x_n \in (0,L)\}, \quad S_L = \partial\Pi_L \backslash (\sigma_0 \cup \sigma_L),$$

σ_{x_n} being an open bounded connected cross-section, $\mathbf{f} \in \mathbf{L}_2(0,T;\mathbf{L}_2(\Pi_L))$, $\mathbf{a} \in \mathbf{H}^1(\Pi_L)$ and $\mathbf{U} \in \mathbf{L}_\infty(0,T;\mathbf{H}^1(\Pi_L) \cap \mathbf{L}_\infty(\Pi_L))$. In addition we need also to assume that $\mathbf{U}$ fulfils $\operatorname{div}(\mathbf{U}) = 0$, $\mathbf{U}|_{S_L} = 0$ and is L-periodic in x_n. We prove that we can take the pressure function $p(x,t)$ of the form

$$p(x,t) = -q(t)x_n + p_0(t) + \widetilde{p}(x,t), \tag{0.3}$$

where $p_0(t)$ is an arbitrary function, $\widetilde{p}(x,t)$ is a L-periodic function with respect to x_n and $q(t)$ is associated to the flux condition. We substitute (0.3) into the problem (0.1), (0.2) and obtain the weak form

$$\begin{cases} \mathbf{u} \in \mathbf{L}_2(0,T;\mathbf{H}(\Pi_L)), \quad \partial_t\mathbf{u} \in \mathbf{L}_2(0,T;\mathbf{L}_2(\Pi_L)), \quad \mathbf{u}\,|_{t=0} = \mathbf{a}, \quad q \in L_2(0,T), \\ \int_0^t \int_{\Pi_L} \partial_\tau \mathbf{u}\cdot\boldsymbol{\eta}\, dx d\tau + \mu \int_0^t \int_{\Pi_L} \nabla\mathbf{u}\cdot\nabla\boldsymbol{\eta}\, dx d\tau + \int_0^t \int_{\Pi_L} (\mathbf{U}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\eta}\, dx d\tau \\ = \int_0^t \int_{\Pi_L} \mathbf{f}\cdot\boldsymbol{\eta}\, dx d\tau + L\int_0^t q(\tau) \int_{\sigma_{x_n}} \eta_n\, dx' d\tau \quad \forall \boldsymbol{\eta} \in \mathbf{L}_2(0,T;\mathbf{H}(\Pi_L)), \quad \forall t \in (0,T], \\ \int_{\sigma_{x_n}} u_n(x', x_n, t)\, dx' = F(t), \end{cases} \tag{0.4}$$

where

$$\mathbf{H}(\Pi_L) = \{\boldsymbol{\eta} \in \mathbf{H}^1(\Pi_L) = (H^1(\Pi_L))^n \mid \operatorname{div}(\boldsymbol{\eta}) = 0,\, \boldsymbol{\eta}(x',0) = \boldsymbol{\eta}(x',L),\, \boldsymbol{\eta}\,|_{S_L} = 0\}.$$

We show the existence and uniqueness of the solution $(\mathbf{u}, q) \in \mathbf{L}_2(0,T;\mathbf{H}(\Pi_L)) \times L_2(0,T)$ of the problem (0.4) and for that we use the linearity of this problem. In addition we are interested in the asymptotic behaviour in domains, i.e. we extend the domain Π_L and the functions $\mathbf{f}, \mathbf{a}, \mathbf{U}$ L-periodically into x_n-direction. We prove the convergence of the solution in a finite pipe of length $2\ell L$ to the L-periodic solution as $\ell \to +\infty$. Note the case $\ell \in \mathbb{N}^+$ works easier than the case $\ell \in \mathbb{R}^+ \backslash \mathbb{N}$.

In the second chapter we analyse the pure Neumann problem. If Ω is an open connected domain in $\mathbb{R}^n$, then a solution to the pure Neumann problem is a function u such that

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = f & \text{in } \Omega, \\ A(x)\nabla u \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.5}$$

where f is some function or distribution, ν is the outward unit normal to $\partial\Omega$, the boundary of Ω and $A = (a_{ij}(x))$ is a uniformly elliptic matrix with entries a_{ij} belonging to $L_\infty(\mathbb{R}^n)$. We consider domains contained in $\mathbb{R}^p \times \omega_2 \subset \mathbb{R}^n$ and we distinguish between the cases $p = 1$ and $1 < p < n$.
Case $p = 1$: We analyse the problem

$$\begin{cases} -\operatorname{div}(A \cdot \nabla u_\ell) = f & \text{in } \Omega_\ell, \\ A\nabla u_\ell \cdot \nu = 0 & \text{on } \partial\Omega_\ell \end{cases} \tag{0.6}$$

where

$$\Omega_\ell = (-\ell, \ell) \times \omega_2 \cap \Omega, \quad \Omega \subset \mathbb{R} \times \omega_2$$

and ω_2 being a bounded domain of $\mathbb{R}^{n-p}$. Let us denote by σ_{x_1} the section of Ω at the level x_1 and we assume

$$\begin{aligned} &\text{there exists } \ell_\sigma > 0 \text{ such that } \forall |x_1| \geq \ell_\sigma \\ &\Omega_{|x_1|} \text{ and } \sigma_{x_1} \text{ are domains in } \mathbb{R}^n, \mathbb{R}^{n-1} \text{ and } C_{\sigma_{x_1}} \leq \Lambda. \end{aligned}$$

Now if $f \in L_{2,loc}(\overline{\Omega})$ with

$$\int_{\sigma_{x_1}} f(x_1, X_2)\, dX_2 = 0 \quad \text{a.e. } x_1 \in \mathbb{R}, \tag{0.7}$$

then the weak solution $u_\ell \in H^1(\Omega_\ell)$ of (0.6) exists (unique up to a constant). Further we prove that u_ℓ converges exponentially in $H^1(\Omega_{\ell_0})/\mathbb{R}$, $\ell_0 \in \mathbb{R}^+$ towards u_∞, the weak solution of the pure Neumann problem in the unbounded domain Ω, if $|f|_{L_2(\Omega_\ell)} = O(e^{\gamma\ell})$ with γ small enough. In addition we also study the special case where the initial data and the domain are L-periodic.
Case $1 < p < n$: Now we consider the problem

$$\begin{cases} -\operatorname{div}(A \cdot \nabla u_\ell) = f & \text{in } \Omega_\ell, \\ A\nabla u_\ell \cdot \nu = 0 & \text{on } \partial\Omega_\ell \end{cases} \tag{0.8}$$

where

$$\Omega_\ell = \ell\omega_1 \times \omega_2, \quad \ell\omega_1 = \{\ell X_1 \mid X_1 \in \omega_1\}$$

with ω_1 being a bounded convex domain of $\mathbb{R}^p$ containing 0 and ω_2 being a bounded domain of $\mathbb{R}^{n-p}$. We suppose that $f \in L_2(\Omega_\ell)$ with

$$\int_{\omega_2} f(X_1, X_2)\, dX_2 = 0 \quad \text{a.e. } X_1 \in \mathbb{R}^p$$

and the matrix A has the special form

$$A = \begin{pmatrix} a(X_2)A_{11}(X_1) & 0 \\ A_{21}(x) & A_{22}(x) \end{pmatrix}, \quad x = (X_1, X_2) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$$

where A_{11} is a $p \times p$ matrix, a a positive function and A_{22} a $(n-p) \times (n-p)$ matrix. In this case we show that if $|f|_{L_2(\Omega_\ell)} = O(e^{\gamma \ell})$ with γ small enough, then the weak solution u_ℓ of (0.8) converges exponentially in $H^1(\Omega_{\ell_0})/\mathbb{R}$ towards the analogous Neumann problem in the unbounded domain $\Omega_\infty = \mathbb{R}^p \times \omega_2$.

In the third problem we consider the n-dimensional Poisson problem

$$\begin{cases} -\Delta u_\ell(x_1, x') = f(x') & \text{in } \Omega_\ell = (-\ell, \ell) \times \omega, \\ u_\ell = 0 & \text{on } \partial\Omega_\ell, \end{cases} \tag{0.9}$$

where ω is a bounded connected Lipschitz domain in $\mathbb{R}^{n-1}$, $x = (x_1, x')$ with $x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$ and $f \in L_2(\omega)$. We approximate u_ℓ by $\psi \otimes u_\infty$, where $\psi(x_1)$ is a cutoff function and u_∞ is the unique weak solution of

$$\begin{cases} -\Delta' u_\infty(x') = f(x') & \text{in } \omega, \\ u_\infty = 0 & \text{on } \partial\omega. \end{cases} \tag{0.10}$$

We got the idea to approximate u_ℓ by $\psi \otimes u_\infty$ from the result

$$\|\nabla(u_\ell - 1 \otimes u_\infty)\|_{L_2(\Omega_{\ell_0})} \le \sqrt{2}\|u_\infty\|_{H^1(\omega)} e^{\sqrt{\lambda_1}(\ell_0 - \ell)} \tag{0.11}$$

for $n = 2$, $\omega = (-1, 1)$ and $0 < \ell_0 \le \ell$ with λ_1 being the first eigenvalue of the operator $-\Delta' = \partial^2_{x_2x_2} + \dots + \partial^2_{x_nx_n}$ for the Dirichlet problem (see Theorem 6.3 in [15]). Such an asymptotic behaviour result can also be shown for any $n \in \mathbb{N}_{\ge 2}$ with $\omega \subset \mathbb{R}^{n-1}$ being an open bounded connected Lipschitz domain. So we have that u_ℓ converges to $1 \otimes u_\infty$ in the center if the domain increases to infinity in the x_1-direction. The goal of this chapter is to find an optimal cutoff function ψ such that $\psi \otimes u_\infty$ is a good approximation of u_ℓ, the unique weak solution of problem (0.9). For the approximation we use the Galerkin finite element method. We compute an upper bound of the error between the approximation and u_ℓ. For the upper bound of the error we use the following technique

$$\|\nabla(u - v)\|^2_{L_2(\Omega)} \le (1 + \beta)\|\mathbf{Y} - \nabla v\|^2_{L_2(\Omega)} + (1 + \frac{1}{\beta})C_F(\Omega)^2\|\text{div}(\mathbf{Y}) + f\|^2_{L_2(\Omega)} \tag{0.12}$$

for any $\beta > 0$, $v \in H$, $\mathbf{Y} \in \mathbf{H}(\Omega; \text{div})$ and $u \in H$ is the unique weak solution of the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $f \in L_2(\Omega)$, $H \subseteq H^1_0(\Omega)$ being a Hilbert space and Ω being an open bounded connected Lipschitz domain in $\mathbb{R}^n$.

Finally before the list of symbols and the bibliography we have the appendix which includes used spaces of functions, as Sobolev spaces, useful results for pde's, as the Lax-Milgram Theorem, and at the end the finite element method which we use for the third problem.

Chapter 1

On a non-stationary fluid flow problem in an infinite periodic pipe

In this chapter we study the mathematical model of an incompressible, homogeneous flow in a n-dimensional infinite pipe which is periodic in the x_n-direction ($n = 2$ or 3) of period L, an arbitrary fixed positive number. The results of this chapter have been published in the paper "On a non-stationary fluid flow problem in an infinite periodic pipe", see [19], which is a joint work with M. Chipot, N. Kloviénė and K. Pileckas.

1.1 Problem formulation

For every x_n we denote by σ_{x_n} the section of the pipe at the level x_n. σ_{x_n} is a one or a two dimensional open, bounded subset and we denote by D the largest diameter of σ_{x_n}. The period of the pipe is a cell Π_L of width L defined as the open set

$$\Pi_L = \{x \in \sigma_{x_n} \times \{x_n\},\, x_n \in (0, L)\}. \tag{1.1}$$

$\mathbf{e}_n$ denotes the unit vector in the x_n-direction and the pipe itself is the set

$$\Pi_\infty = \cup_{z\in\mathbb{Z}}(\Pi_L \cup \sigma_0 \cup \sigma_L + zL\mathbf{e}_n).$$

We use in this chapter the notation $x' = (x_1)$ for the 2D problem and $x' = (x_1, x_2)$ for the 3D problem where $x = (x', x_n) \in \mathbb{R}^n$. S_L is the lateral boundary of the cell Π_L, i.e.

$$S_L = \partial\Pi_L \backslash (\sigma_0 \cup \sigma_L) = \partial\Pi_L \cap \partial\Pi_\infty. \tag{1.2}$$

Furthermore we assume that we have Lipschitz boundary conditions for the domain Π_L. We would like to analyse the following non-stationary problem

$$\begin{cases} \partial_t\mathbf{u} - \mu\Delta\mathbf{u} + (\mathbf{U}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Pi_L, \\ \mathrm{div}(\mathbf{u}) = 0 \quad \text{in } \Pi_L, \qquad \mathbf{u}(x,0) = \mathbf{a}(x), \\ \mathbf{u}|_{S_L} = 0 \quad t \in (0,T), \\ \mathbf{u}(x', 0, t) = \mathbf{u}(x', L, t) \quad t \in (0,T) \end{cases} \tag{1.3}$$

with the prescribed flux condition

$$\int_{\sigma_{x_n}} u_n(x', x_n, t)\, dx' = F(t) \quad \forall x_n \in (0, L). \tag{1.4}$$

In (1.3), (1.4) we have $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ when $n = 2$ and $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ when $n = 3$. $\mathbf{u}$ is the velocity of the fluid, p describes the pressure field and μ is the viscosity. The given function $\mathbf{U}$ satisfies the following conditions:

$$\begin{cases} \mathbf{U} \in \mathbf{L}_\infty(0,T;\mathbf{H}^1(\Pi_L) \cap \mathbf{L}_\infty(\Pi_L)), \quad \operatorname{div}(\mathbf{U}) = 0, \quad \mathbf{U}|_{\mathrm{S_L}} = 0, \\ \mathbf{U} \text{ is } L\text{-periodic in } x_n, \text{ i.e. } \mathbf{U}(x', x_n, t) = \mathbf{U}(x', x_n + L, t). \end{cases} \tag{1.5}$$

The exterior force $\mathbf{f} \in \mathbf{L}_2(0,T;\mathbf{L}_2(\Pi_L))$ and the initial velocity $\mathbf{a} \in \mathbf{H}^1(\Pi_L) = (H^1(\Pi_L))^n$ is L-periodic with respect to x_n, solenoidal and equal to zero on the lateral boundary of the pipe.

Remark 1.1.1. *Let $0 < \alpha \leq L$. Assume that $\boldsymbol{\eta}$ satisfies the Dirichlet homogeneous boundary conditions on S_α and is a solenoidal vector field. Applying the divergence formula we get*

$$0 = \int_{\Pi_\alpha} \operatorname{div}(\boldsymbol{\eta})\, dx = \int_{\sigma_0} \eta_n\, dx' + \int_{S_\alpha} \boldsymbol{\eta} \cdot \boldsymbol{n}\, dS_\alpha - \int_{\sigma_\alpha} \eta_n\, dx',$$

therefore $\int_{\sigma_0} \eta_n(x', 0, t)\, dx' = \int_{\sigma_\alpha} \eta_n(x', \alpha, t)\, dx'$ for every $0 < \alpha \leq L$ where σ_α is a short cut for the set $\sigma_\alpha \times \{\alpha\}$. We denote in what follows by $\int_\sigma \eta_n\, dx'$ the common value of this integral.

Let us notice that the case $\mathbf{U}(x,t) = 0$ coincides with the non-stationary Stokes problem.
We will prove that we can take the pressure function $p(x,t)$ of the form

$$p(x,t) = -q(t)x_n + p_0(t) + \widetilde{p}(x,t), \tag{1.6}$$

where $p_0(t)$ is an arbitrary function and $\widetilde{p}(x,t)$ is a L-periodic function with respect to x_n. To separate the pressure function term with $q(t)$ (which will be shown is associated to the flux condition) is important for the problems in domains which have several outlets to infinity (see [53]). The Poiseuille type solution (the velocity has a prescribed flux, and the pressure function is expressed by (1.6)) for the stationary Stokes and Navier-Stokes problems in domains with periodically varying section was found in [41], [48]. The non-stationary solutions with prescribed fluxes to Stokes and Navier-Stokes problems in infinite cylinders and in unbounded domains with cylindrical outlets to infinity were studied in [31], [50], [51], [56], [66]. The existence of time periodic Poiseuille type solutions in the infinite cylinders was proved in [9], [35].
Substituting expression (1.6) into the problem (1.3), (1.4) we get:

$$\begin{cases} \partial_t \mathbf{u} - \mu \Delta \mathbf{u} + (\mathbf{U} \cdot \nabla)\mathbf{u} + \nabla \widetilde{p} = \mathbf{q} + \mathbf{f} \quad \text{in } \Pi_L, \\ \operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Pi_L, \qquad \mathbf{u}(x,0) = \mathbf{a}(x), \\ \mathbf{u}(t)|_{S_L} = 0 \quad t \in (0,T), \\ \mathbf{u}(x', 0, t) = \mathbf{u}(x', L, t) \quad t \in (0,T), \\ \int_\sigma u_n(x', x_n, t)\, dx' = F(t), \end{cases} \tag{1.7}$$

where $\mathbf{q}(t) = \begin{pmatrix} 0 \\ q(t) \end{pmatrix}$ for the 2D problem and $\mathbf{q}(t) = \begin{pmatrix} 0 \\ 0 \\ q(t) \end{pmatrix}$ for the 3D problem. Here we are interested in $(\mathbf{u}, \mathbf{q}, \tilde{p})$ the solution of problem (1.7). Remark the function $\mathbf{q}(t)$ is picked up such that to have the given flux condition satisfied (this will be detailed later).
A weak solution of the previous problem can be written as $(\mathbf{u}, q) : \mathbb{R}^3(\text{resp. } \mathbb{R}^2) \times \mathbb{R} \to \mathbb{R}^3(\text{resp. } \mathbb{R}^2)$ solution to

$$\begin{cases} \mathbf{u} \in \mathbf{L}_2(0,T;\mathbf{H}(\Pi_L)), \quad \partial_t \mathbf{u} \in \mathbf{L}_2(0,T;\mathbf{L}_2(\Pi_L)), \quad \mathbf{u}\,|_{t=0} = \mathbf{a}, \quad q \in L_2(0,T), \\ \int_0^t \int_{\Pi_L} \partial_\tau \mathbf{u} \cdot \boldsymbol{\eta}\, dx d\tau + \mu \int_0^t \int_{\Pi_L} \nabla \mathbf{u} \cdot \nabla \boldsymbol{\eta}\, dx d\tau + \int_0^t \int_{\Pi_L} (\mathbf{U} \cdot \nabla)\mathbf{u} \cdot \boldsymbol{\eta}\, dx d\tau \\ = \int_0^t \int_{\Pi_L} \mathbf{f} \cdot \boldsymbol{\eta}\, dx d\tau + L \int_0^t q(\tau) \int_\sigma \eta_n\, dx' d\tau \quad \forall \boldsymbol{\eta} \in \mathbf{L}_2(0,T;\mathbf{H}(\Pi_L)), \quad \forall t \in (0,T], \\ \int_\sigma u_n(x', x_n, t)\, dx' = F(t), \end{cases} \tag{1.8}$$

where

$$\mathbf{H}(\Pi_L) = \{\boldsymbol{\eta} \in \mathbf{H}^1(\Pi_L) = (H^1(\Pi_L))^n \mid \operatorname{div}(\boldsymbol{\eta}) = 0,\, \boldsymbol{\eta}(x', 0) = \boldsymbol{\eta}(x', L),\, \boldsymbol{\eta}\,|_{S_L} = 0\} \tag{1.9}$$

and

$$\nabla \mathbf{h} \cdot \nabla \mathbf{w} = \sum_{i=1}^n \nabla h_i \cdot \nabla w_i$$

for $\mathbf{h}$, $\mathbf{w} \in \mathbf{H}^1(\Pi_L)$ with having the usual scalar product in $\mathbb{R}^n$ between ∇h_i and ∇w_i.

Remark 1.1.2. *The right hand side of the first integral equation in (1.8) is obtained by using the fact that*

$$\int_{\Pi_L} \boldsymbol{q}\cdot\boldsymbol{\eta}\,dx = \int_{\Pi_L} q\eta_n\,dx = q(t)\int_0^L\int_{\sigma_{x_n}} \eta_n\,dx'dx_n = q(t)\int_0^L\int_{\sigma}\eta_n\,dx'dx_n = Lq(t)\int_\sigma \eta_n\,dx'.$$

Since the problem (1.7) is linear, we can look for the solution $(\mathbf{u}(x,t),\mathbf{q}(t),\tilde{p}(x,t))$ in the form

$$(\mathbf{u}(x,t),\mathbf{q}(t),\tilde{p}(x,t)) = (\mathbf{V}(x,t),0,\tilde{p}_1(x,t)) + (\mathbf{v}(x,t),\mathbf{q}(t),\tilde{p}_2(x,t)).$$

$(\mathbf{V}(x,t),\tilde{p}_1(x,t))$ is the solution of the problem

$$\begin{cases} \partial_t\mathbf{V} - \mu\Delta\mathbf{V} + (\mathbf{U}\cdot\nabla)\mathbf{V} + \nabla\widetilde{p}_1 = \mathbf{f} \quad \text{in } \Pi_L, \\ \operatorname{div}(\mathbf{V}) = 0 \quad \text{in } \Pi_L, \qquad \mathbf{V}(x,0) = \mathbf{a}(x), \\ \mathbf{V}|_{S_L} = 0 \quad t\in(0,T), \\ \mathbf{V}(x',0,t) = \mathbf{V}(x',L,t) \quad t\in(0,T) \end{cases} \tag{1.10}$$

and $(\mathbf{v}(x,t),\mathbf{q}(t),\tilde{p}_2(x,t))$ is the solution of the problem

$$\begin{cases} \partial_t\mathbf{v} - \mu\Delta\mathbf{v} + (\mathbf{U}\cdot\nabla)\mathbf{v} + \nabla\widetilde{p}_2 = \mathbf{q} \quad \text{in } \Pi_L, \\ \operatorname{div}(\mathbf{v}) = 0 \quad \text{in } \Pi_L, \qquad \mathbf{v}(x,0) = 0, \\ \mathbf{v}|_{S_L} = 0 \quad t\in(0,T), \\ \mathbf{v}(x',0,t) = \mathbf{v}(x',L,t) \quad t\in(0,T), \\ \int_\sigma v_n\,dx' = \widetilde{F}(t), \end{cases} \tag{1.11}$$

where $\widetilde{F}(t) = F(t) - \int_\sigma V_n(x,t)\,dx'$. We assume that the necessary compatibility condition $\widetilde{F}(0) = 0$ holds, i.e. $F(0) = \int_\sigma V_n(x',x_n,0)\,dx'$. The pressures $\widetilde{p}_1$ and $\widetilde{p}_2$ are L-periodic in x_n. In the second section we prove existence and uniqueness of a weak solution of problem (1.7) by getting the solution of the weak formulation of problems (1.10) and (1.11). In the third section we analyse the spatial asymptotic behaviour of the prescribed problem solution to the problem (1.51), the weak solution $(\mathbf{u}_\ell, q_\ell)$ of an analogous problem in a union of 2ℓ periodic cells Π_L.

1.2 Solvability of the problem

1.2.1 Analysis of problem (1.10)

In this paragraph we analyse the weak formulation of the problem (1.10):

$$\begin{cases} \mathbf{V}\in\mathbf{L}_2(0,T;\mathbf{H}(\Pi_L)), \quad \partial_t\mathbf{V}\in\mathbf{L}_2(0,T;\mathbf{L}_2(\Pi_L)), \quad \mathbf{V}\,|_{t=0} = \mathbf{a}, \\ \int_0^t\int_{\Pi_L}\partial_\tau\mathbf{V}\cdot\boldsymbol{\eta}\,dxd\tau + \mu\int_0^t\int_{\Pi_L}\nabla\mathbf{V}\cdot\nabla\boldsymbol{\eta}\,dxd\tau + \int_0^t\int_{\Pi_L}(\mathbf{U}\cdot\nabla)\mathbf{V}\cdot\boldsymbol{\eta}\,dxd\tau \\ = \int_0^t\int_{\Pi_L}\mathbf{f}\cdot\boldsymbol{\eta}\,dxd\tau \quad \forall\boldsymbol{\eta}\in\mathbf{L}_2(0,T;\mathbf{H}(\Pi_L)), \quad \forall t\in(0,T], \end{cases} \tag{1.12}$$

where $\mathbf{H}(\Pi_L)$ is defined in (1.9). Note for this problem we have no flux condition.
Let λ_k be the eigenvalues and $\mathbf{v}_k(x)$ be the eigenfunctions of the following boundary value problem:

$$\begin{cases} \mathbf{v}_k\in\mathbf{H}(\Pi_L), \\ \mu\int_{\Pi_L}\nabla\mathbf{v}_k(x)\cdot\nabla\mathbf{v}\,dx = \lambda_k\int_{\Pi_L}\mathbf{v}_k(x)\mathbf{v}\,dx \quad \forall\mathbf{v}\in\mathbf{H}(\Pi_L). \end{cases} \tag{1.13}$$

For our set Π_L holds the following:

- (1.13) defines a countable set of eigenvalues $\lambda_k > 0, \quad \lambda_k\to\infty, \quad k = 1,2,...$; the corresponding eigenfunctions $\mathbf{v}_k$ constitute a basis $\{\mathbf{v}_k\}_{k\geq 1}$ in $\mathbf{H}(\Pi_L)$.

- The eigenfunctions $\mathbf{v}_k$ can be orthonormalized:

$$\int_{\Pi_L} \mathbf{v}_k \cdot \mathbf{v}_\ell \, dx = \begin{cases} 1 & \text{for } k = \ell, \\ 0 & \text{for } k \neq \ell \end{cases} \tag{1.14}$$

- Moreover

$$\int_{\Pi_L} \nabla \mathbf{v}_k \cdot \nabla \mathbf{v}_\ell \, dx = \begin{cases} \frac{\lambda_k}{\mu} & \text{for } k = \ell, \\ 0 & \text{for } k \neq \ell \end{cases} \tag{1.15}$$

(For this result compare Lemma 8.1 in [29] or see [43].)
We look for an approximate solution $\mathbf{V}^{(N)}(x,t)$ of the problem (1.12) in the form

$$\mathbf{V}^{(N)}(x,t) = \sum_{k=1}^{N} y_k^{(N)}(t)\mathbf{v}_k(x),$$

where the coefficients $y_k^{(N)}(t)$ are found from the differential equations

$$\int_{\Pi_L} \partial_t \mathbf{V}^{(N)} \cdot \mathbf{v}_k \, dx + \mu \int_{\Pi_L} \nabla \mathbf{V}^{(N)} \cdot \nabla \mathbf{v}_k \, dx + \int_{\Pi_L} (\mathbf{U} \cdot \nabla) \mathbf{V}^{(N)} \cdot \mathbf{v}_k \, dx = \int_{\Pi_L} \mathbf{f} \cdot \mathbf{v}_k \, dx \tag{1.16}$$

$k = 1, 2, \ldots, N$ and the initial conditions $\mathbf{V}^{(N)}(x,0) = \sum_{k=1}^{N} (\int_{\Pi_L} \mathbf{a} \cdot \mathbf{v}_k \, dx) \mathbf{v}_k$. Using the properties (1.14) and (1.15) of the basis we then derive from the above equality the following Cauchy problem for the system of linear ordinary differential equations

$$\begin{cases} \mathbf{Y}^{(N)'}(t) + (\mathbb{J}^{(N)} + \mathbb{A}^{(N)}(t))\mathbf{Y}^{(N)}(t) = \mathcal{B}^{(N)}(t), \\ \mathbf{Y}^{(N)}(0) = \mathbf{C}^{(N)}, \end{cases} \tag{1.17}$$

where

$$\mathbf{Y}^{(N)}(t) = \begin{pmatrix} y_1^{(N)}(t) \\ \vdots \\ y_N^{(N)}(t) \end{pmatrix}, \quad \mathcal{B}^{(N)}(t) = \begin{pmatrix} B_1(t) \\ \vdots \\ B_N(t) \end{pmatrix}, \quad \mathbf{C}^{(N)} = \begin{pmatrix} C_1 \\ \vdots \\ C_N \end{pmatrix},$$

$$\mathbb{J}^{(N)} = diag(\lambda_1, \ldots, \lambda_N), \quad B_i(t) = \int_{\Pi_L} \mathbf{f} \cdot \mathbf{v}_i \, dx, \quad C_i = \int_{\Pi_L} \mathbf{a} \cdot \mathbf{v}_i \, dx,$$

$$\mathbb{A}^{(N)}(t) = (\alpha_{ij}(t))_{i,j=1,\ldots,N} \text{ with } \alpha_{ij}(t) = \int_{\Pi_L} (\mathbf{U} \cdot \nabla)\mathbf{v}_i \cdot \mathbf{v}_j \, dx.$$

Lemma 1.2.1. *Let* $\mathbf{f} \in \boldsymbol{L}_2(0,T;\boldsymbol{L}_2(\Pi_L))$, $T \in (0,\infty)$ *and suppose that* $\boldsymbol{U}$ *satisfies* (1.5), *then there exists a unique solution* $\mathbf{Y}^{(N)} \in \boldsymbol{H}^1(0,T)$ *of the problem* (1.17).

Proof. Let us prove that the entries $\alpha_{ij}(t)$ of the matrix $\mathbb{A}^{(N)}(t)$ are bounded. We have

$$|\alpha_{ij}(t)| = |\int_{\Pi_L} (\mathbf{U} \cdot \nabla)\mathbf{v}_i \cdot \mathbf{v}_j \, dx| \leq |\mathbf{U}|_{\mathbf{L}_\infty(\Pi_L)} \big|\big|\nabla \mathbf{v}_i\big|\big|_{\mathbf{L}_2(\Pi_L)} |\mathbf{v}_j|_{\mathbf{L}_2(\Pi_L)}. \tag{1.18}$$

Thus, all the entries of the matrix $\mathbb{A}^{(N)}(t)$ are bounded functions and, therefore, the existence of the unique solution of the problem (1.17) follows from standard results for linear ordinary differential equations (see, for example, [69]). □

We prove now an a priori estimate for the solution of problem (1.12).

Theorem 1.2.2. *Suppose that* $\mathbf{f} \in \boldsymbol{L}_2(0,T;\boldsymbol{L}_2(\Pi_L))$, $T \in (0,\infty)$, $\boldsymbol{a} \in \boldsymbol{H}^1(\Pi_L)$ *and* $\boldsymbol{U}$ *satisfies* (1.5)*, then for the approximate solution* $\boldsymbol{V}^{(N)}(x,t)$ *of the problem* (1.12) *holds the following estimate*

$$\begin{aligned}&\sup_{t\in[0,T]} ||\,\boldsymbol{V}^{(N)}(\cdot,t)||^2_{\boldsymbol{H}^1(\Pi_L)} + \int_0^t |\partial_\tau \boldsymbol{V}^{(N)}|^2_{\boldsymbol{L}_2(\Pi_L)}\, d\tau + \int_0^t \big|\big|\nabla \boldsymbol{V}^{(N)}\big|\big|^2_{\boldsymbol{L}_2(\Pi_L)}\, d\tau \\ &\leq c\left(||\boldsymbol{a}||^2_{\boldsymbol{H}^1(\Pi_L)} + \int_0^T |\mathbf{f}|^2_{\boldsymbol{L}_2(\Pi_L)}\, d\tau\right)\end{aligned} \tag{1.19}$$

where c depends on L, D, T, μ *and the function* $\boldsymbol{U}$*, and is independent of* N *and* t.

Proof. Multiplying (1.16) by $y_k^{(N)}(t)$ and summing up for k from 1 to N, we obtain by using Young's and Poincaré's inequalities

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Pi_L} |\mathbf{V}^{(N)}|^2\, dx + \mu\int_{\Pi_L} |\nabla\mathbf{V}^{(N)}|^2\, dx + \int_{\Pi_L} (\mathbf{U}\cdot\nabla)\mathbf{V}^{(N)}\cdot\mathbf{V}^{(N)}\, dx \\ &= \int_{\Pi_L} \mathbf{f}\cdot\mathbf{V}^{(N)}\, dx \leq \frac{1}{2\epsilon}\int_{\Pi_L} |\mathbf{f}|^2\, dx + \frac{\epsilon}{2}\int_{\Pi_L} |\mathbf{V}^{(N)}|^2\, dx \\ &\leq \frac{1}{2\epsilon}\int_{\Pi_L} |\mathbf{f}|^2\, dx + \frac{\epsilon C_F}{2}\int_{\Pi_L} |\nabla\mathbf{V}^{(N)}|^2\, dx,\end{aligned}$$

here C_F, depending on D, is the constant of the Friedrich's inequality. (Remark: Since $\mathbf{U}$ is L-periodic in x_n and $\mathbf{V}^{(N)}|_{S_L} = 0$, we get by applying the Divergence formula that $\int_{\Pi_L} (\mathbf{U}\cdot\nabla)\mathbf{V}^{(N)}\cdot\mathbf{V}^{(N)}\, dx = \sum_{i,j}\int_{\Pi_L} \frac{1}{2}\mathbf{U}^i\partial_{x_i}((\mathbf{V}^{(N)})^j)^2\, dx = 0$.) Integrating in t we derive

$$\frac{1}{2}\int_{\Pi_L} |\mathbf{V}^{(N)}(t)|^2\, dx + (\mu - \frac{\epsilon C_F}{2})\int_0^t \big|\big|\nabla\mathbf{V}^{(N)}\big|\big|^2_{\mathbf{L}_2(\Pi_L)}\, d\tau \leq \frac{1}{2\epsilon}\int_0^t |\mathbf{f}|^2_{\mathbf{L}_2(\Pi_L)}\, d\tau + \frac{1}{2}|\mathbf{a}|^2_{\mathbf{L}_2(\Pi_L)} \tag{1.20}$$

and we choose ϵ small enough such that $\epsilon < \frac{2\mu}{C_F}$.
Next we multiply the equality (1.16) by $\frac{d}{dt}y_k^{(N)}(t)$, then we sum up and so we obtain

$$\begin{aligned}&\int_{\Pi_L} |\partial_t\mathbf{V}^{(N)}|^2\, dx + \frac{\mu}{2}\frac{d}{dt}\int_{\Pi_L} |\nabla\mathbf{V}^{(N)}|^2\, dx + \int_{\Pi_L} (\mathbf{U}\cdot\nabla)\mathbf{V}^{(N)}\cdot\partial_t\mathbf{V}^{(N)}\, dx \\ &= \int_{\Pi_L} \mathbf{f}\cdot\partial_t\mathbf{V}^{(N)}\, dx \leq \frac{1}{2\delta}|\mathbf{f}|^2_{\mathbf{L}_2(\Pi_L)} + \frac{\delta}{2}|\partial_t\mathbf{V}^{(N)}|^2_{\mathbf{L}_2(\Pi_L)}\end{aligned}$$

which leads to

$$\begin{aligned}&(1-\frac{\delta}{2})\int_{\Pi_L} |\partial_t\mathbf{V}^{(N)}|^2\, dx + \frac{\mu}{2}\frac{d}{dt}\int_{\Pi_L} |\nabla\mathbf{V}^{(N)}|^2\, dx \\ &\leq \int_{\Pi_L} |(\mathbf{U}\cdot\nabla)\mathbf{V}^{(N)}\cdot\partial_t\mathbf{V}^{(N)}|\, dx + \frac{1}{2\delta}|\mathbf{f}|^2_{\mathbf{L}_2(\Pi_L)} \\ &\leq \frac{1}{2}\sup_{x\in\Pi_L} |\mathbf{U}|^2\int_{\Pi_L} |\nabla\mathbf{V}^{(N)}|^2\, dx + \frac{1}{2}\int_{\Pi_L} |\partial_t\mathbf{V}^{(N)}|^2\, dx + \frac{1}{2\delta}|\mathbf{f}|^2_{\mathbf{L}_2(\Pi_L)}.\end{aligned} \tag{1.21}$$

Let us choose δ small enough such that $\delta < 1$, hence it holds

$$\frac{d}{dt}\int_{\Pi_L} |\nabla\mathbf{V}^{(N)}|^2\, dx \leq \frac{1}{\mu}m(t)\int_{\Pi_L} |\nabla\mathbf{V}^{(N)}|^2\, dx + \frac{1}{\mu\delta}|\mathbf{f}|^2_{\mathbf{L}_2(\Pi_L)}$$

with

$$m(t) = \sup_{x\in\Pi_L} |\mathbf{U}|^2. \tag{1.22}$$

Applying Gronwall's inequality, we get

$$\begin{aligned}
\big|\big|\nabla \mathbf{V}^{(N)}\big|\big|^2_{\mathbf{L}_2(\Pi_L)} &\leq e^{\frac{1}{\mu}\int_0^t m(\tau)\,d\tau}\Big(\frac{1}{\mu\delta}\int_0^t |\mathbf{f}|^2_{\mathbf{L}_2(\Pi_L)}\,d\tau + \big|\big|\nabla \mathbf{a}\big|\big|^2_{\mathbf{L}_2(\Pi_L)}\Big)\\
&\leq c_1 \int_0^t |\mathbf{f}|^2_{\mathbf{L}_2(\Pi_L)}\,d\tau + c_2 \big|\big|\nabla \mathbf{a}\big|\big|^2_{\mathbf{L}_2(\Pi_L)}.
\end{aligned}$$

Integrating (1.21) in t and using the above estimate, we derive

$$\begin{aligned}
&(1-\delta)\int_0^t |\partial_\tau \mathbf{V}^{(N)}|^2_{\mathbf{L}_2(\Pi_L)}\,d\tau + \mu \int_{\Pi_L} |\nabla \mathbf{V}^{(N)}|^2\,dx \leq \int_0^t m(\tau)\Big(c_1 \int_0^\tau |\mathbf{f}|^2_{\mathbf{L}_2(\Pi_L)}\,dt\\
&+ c_2 \int_{\Pi_L} |\nabla \mathbf{a}|^2\,dx\Big)\,d\tau + \frac{1}{\delta}\int_0^t |\mathbf{f}|^2_{\mathbf{L}_2(\Pi_L)}\,d\tau + \mu \int_{\Pi_L} |\nabla \mathbf{a}|^2\,dx\\
&\leq c\Big(\int_0^T |\mathbf{f}|^2_{\mathbf{L}_2(\Pi_L)}\,d\tau + \big|\big|\nabla \mathbf{a}\big|\big|^2_{\mathbf{L}_2(\Pi_L)}\Big)
\end{aligned}$$

which completes the proof. Remark that constants depend on D, μ, L, T and $\mathbf{U}$. The inequality of the theorem is a combination of the previous inequality and (1.20). □

Remark 1.2.3. *In the inequality (1.19) and the inequalities below we keep both terms* $\sup_{t\in[0,T]} ||\boldsymbol{V}^{(N)}(\cdot,t)||^2_{\boldsymbol{H}^1(\Pi_L)}$ *and* $\int_0^t \big|\big|\nabla \boldsymbol{V}^{(N)}\big|\big|^2_{\boldsymbol{L}_2(\Pi_L)}\,d\tau$ *although the second term can be estimated by the first one:*

$$\int_0^t \big|\big|\nabla \boldsymbol{V}^{(N)}\big|\big|^2_{\boldsymbol{L}_2(\Pi_L)}\,d\tau \leq T \sup_{t\in[0,T]} ||\boldsymbol{V}^{(N)}(\cdot,t)||^2_{\boldsymbol{H}^1(\Pi_L)}.$$

The reason is that the constant in (1.19) depends on T only in the term of the norm of the function $\boldsymbol{U}$ and if the corresponding norm is finite for $T \in [0,+\infty]$, this constant becomes independent of T. Therefore, having in mind further applications, we keep in our estimates both mentioned above terms.

Theorem 1.2.4. *Suppose that* $\mathbf{f} \in \boldsymbol{L}_2(0,T;\boldsymbol{L}_2(\Pi_L))$, $T \in (0,\infty)$, $\boldsymbol{a} \in \boldsymbol{H}^1(\Pi_L)$ *and* $\boldsymbol{U}$ *satisfies* (1.5). *Then the problem* (1.10) *admits a unique weak solution* $\boldsymbol{V}(x,t) \in \boldsymbol{L}_2(0,T;\boldsymbol{H}(\Pi_L))$ *and the following estimate*

$$\begin{aligned}
&\sup_{t\in[0,T]} ||\boldsymbol{V}||^2_{\boldsymbol{H}^1(\Pi_L)} + \int_0^t |\partial_\tau \boldsymbol{V}|^2_{\boldsymbol{L}_2(\Pi_L)}\,d\tau + \int_0^t \big|\big|\nabla \boldsymbol{V}\big|\big|^2_{\boldsymbol{L}_2(\Pi_L)}\,d\tau\\
&\leq c(||\boldsymbol{a}||^2_{\boldsymbol{H}^1(\Pi_L)} + \int_0^T |\mathbf{f}|^2_{\boldsymbol{L}^2(\Pi_L)}\,d\tau)
\end{aligned}\tag{1.23}$$

is valid, where the constant c is depending on μ, T, L, D and the function $\boldsymbol{U}$.

Proof. From the previous theorem we get that $\{\mathbf{V}^{(N)}\}$ resp. $\{\partial_t \mathbf{V}^{(N)}\}$ are bounded in the space $\mathbf{L}_2(0,T;\mathbf{H}^1(\Pi_L))$ resp. $\mathbf{L}_2(0,T;\mathbf{L}_2(\Pi_L))$ hence there exist subsequences $\{\mathbf{V}^{(N_k)}\}$ resp. $\{\partial_t \mathbf{V}^{(N_k)}\}$ which converge weakly to $\mathbf{V}$ resp. $\partial_t \mathbf{V}$ in the space $\mathbf{L}_2(0,T;\mathbf{H}^1(\Pi_L))$ resp. $\mathbf{L}_2(0,T;\mathbf{L}_2(\Pi_L))$.
For the approximate solution we have for k large enough, i.e. $N_k \geq M$

$$\begin{aligned}
&\int_0^t \int_{\Pi_L} \partial_\tau \mathbf{V}^{(N_k)} \cdot \boldsymbol{\eta}\,dxd\tau + \mu \int_0^t \int_{\Pi_L} \nabla \mathbf{V}^{(N_k)} \cdot \nabla \boldsymbol{\eta}\,dxd\tau\\
&+ \int_0^t \int_{\Pi_L} (\mathbf{U}\cdot\nabla)\mathbf{V}^{(N_k)} \cdot \boldsymbol{\eta}\,dxd\tau = \int_0^t \int_{\Pi_L} \mathbf{f}\cdot\boldsymbol{\eta}\,dxd\tau
\end{aligned}\tag{1.24}$$

for $\boldsymbol{\eta} \in \mathbf{L}_2(0,T;[\mathbf{v}_1,\cdots,\mathbf{v}_M])$, where $[\mathbf{v}_1,\cdots,\mathbf{v}_M]$ denotes the span of $\mathbf{v}_1,\cdots,\mathbf{v}_M$. Passing to the limit as $N_k \to \infty$ we get

$$\begin{aligned}
&\int_0^t \int_{\Pi_L} \partial_\tau \mathbf{V}\cdot\boldsymbol{\eta}\,dxd\tau + \mu \int_0^t \int_{\Pi_L} \nabla \mathbf{V}\cdot\nabla\boldsymbol{\eta}\,dxd\tau + \int_0^t \int_{\Pi_L} (\mathbf{U}\cdot\nabla)\mathbf{V}\cdot\boldsymbol{\eta}\,dxd\tau\\
&= \int_0^t \int_{\Pi_L} \mathbf{f}\cdot\boldsymbol{\eta}\,dxd\tau \qquad \forall \boldsymbol{\eta} \in \mathbf{L}_2(0,T;[\mathbf{v}_1,\cdots,\mathbf{v}_M]).
\end{aligned}\tag{1.25}$$

So the existence of the weak solution of the problem (1.10) is shown, since $\mathbf{L}_2(0,T;[\mathbf{v}_1,\cdots,\mathbf{v}_M])$ is dense in $\mathbf{L}_2(0,T;\mathbf{H}(\Pi_L))$ when $M\to\infty$.
Now let us show the uniqueness. For this we assume that $\mathbf{V}_1$ and $\mathbf{V}_2$ are two solutions of the problem (1.12) and then the following equality holds

$$\int_0^t\int_{\Pi_L}\partial_\tau(\mathbf{V}_1-\mathbf{V}_2)\cdot\boldsymbol{\eta}\,dxd\tau+\mu\int_0^t\int_{\Pi_L}\nabla(\mathbf{V}_1-\mathbf{V}_2)\cdot\nabla\boldsymbol{\eta}\,dxd\tau$$
$$+\int_0^t\int_{\Pi_L}(\mathbf{U}\cdot\nabla)(\mathbf{V}_1-\mathbf{V}_2)\cdot\boldsymbol{\eta}\,dxd\tau=0.$$

Let us take $\boldsymbol{\eta}=\mathbf{V}_1-\mathbf{V}_2$ and so we get

$$\frac{1}{2}\int_{\Pi_L}|\mathbf{V}_1-\mathbf{V}_2|^2\,dx+\mu\int_0^t\int_{\Pi_L}\nabla(\mathbf{V}_1-\mathbf{V}_2)\cdot\nabla(\mathbf{V}_1-\mathbf{V}_2)\,dxd\tau$$
$$+\int_0^t\int_{\Pi_L}(\mathbf{U}\cdot\nabla)(\mathbf{V}_1-\mathbf{V}_2)\cdot(\mathbf{V}_1-\mathbf{V}_2)\,dxd\tau=0,$$

i.e.

$$\frac{1}{2}\int_{\Pi_L}|\mathbf{V}_1-\mathbf{V}_2|^2\,dx+\mu\int_0^t\int_{\Pi_L}|\nabla(\mathbf{V}_1-\mathbf{V}_2)|^2\,dxd\tau=0$$

and therefore the uniqueness follows. For the proof of the estimate see Theorem 1.2.2. □

1.2.2 Analysis of problem (1.11)

This paragraph is devoted to the analysis of the weak formulation of the problem (1.11):

$$\begin{cases}\mathbf{v}\in\mathbf{L}_2(0,T;\mathbf{H}(\Pi_L)),\quad \partial_t\mathbf{v}\in\mathbf{L}_2(0,T;\mathbf{L}_2(\Pi_L)),\quad \mathbf{v}\,|_{t=0}=0,\quad q\in L_2(0,T),\\ \int_0^t\int_{\Pi_L}\partial_\tau\mathbf{v}\cdot\boldsymbol{\eta}\,dxd\tau+\mu\int_0^t\int_{\Pi_L}\nabla\mathbf{v}\cdot\nabla\boldsymbol{\eta}\,dxd\tau+\int_0^t\int_{\Pi_L}(\mathbf{U}\cdot\nabla)\mathbf{v}\cdot\boldsymbol{\eta}\,dxd\tau\\ =L\int_0^t q(\tau)\int_\sigma\eta_n\,dx'd\tau\quad\forall\boldsymbol{\eta}\in\mathbf{L}_2(0,T;\mathbf{H}(\Pi_L)),\quad\forall t\in(0,T],\\ \int_\sigma v_n\,dx'=\widetilde{F}(t),\end{cases}\tag{1.26}$$

where $\widetilde{F}(t)=F(t)-\int_\sigma V_n\,dx'$ and $\mathbf{V}$ is the solution of the problem (1.12) and the compatibility condition $\widetilde{F}(0)=0$ holds.
We are looking for an approximate solution $(\mathbf{v}^{(N)}(x,t),q^{(N)}(t))$ of the problem (1.26) in the form

$$\mathbf{v}^{(N)}(x,t)=\sum_{k=1}^N y_k^{(N)}(t)\mathbf{v}_k(x),\tag{1.27}$$

where $\mathbf{v}_k(x)$ are the eigenfunctions of the problem (1.13) and the coefficients $y_k^{(N)}(t)$ are found from the differential equation

$$\int_{\Pi_L}\partial_t\mathbf{v}^{(N)}\cdot\mathbf{v}_k\,dx+\mu\int_{\Pi_L}\nabla\mathbf{v}^{(N)}\cdot\nabla\mathbf{v}_k\,dx+\int_{\Pi_L}(\mathbf{U}\cdot\nabla)\mathbf{v}^{(N)}\cdot\mathbf{v}_k\,dx=Lq^{(N)}(t)\int_\sigma v_{kn}\,dx',\tag{1.28}$$

$k=1,...,N$ and the initial condition $\mathbf{v}^{(N)}(x,0)=0$. In (1.28) v_{kn} means the last component of the vector $\mathbf{v}_k$. Function $q^{(N)}(t)$ is picked up in order to satisfy the flux condition

$$\int_\sigma v_n^{(N)}(x,t)\,dx'=\widetilde{F}(t).\tag{1.29}$$

Using the properties (1.14) and (1.15) of the basis combining with (1.28) we derive the following Cauchy problem for the system of linear ordinary differential equations

$$\begin{cases} \mathbf{Y}^{(N)'}(t) + (\mathbb{J}^{(N)} + \mathbb{A}^{(N)}(t))\mathbf{Y}^{(N)}(t) = L\boldsymbol{\beta}^{(N)}q^{(N)}(t), \\ \mathbf{Y}^{(N)}(0) = 0, \end{cases} \tag{1.30}$$

where

$$\mathbf{Y}^{(N)}(t) = \begin{pmatrix} y_1^{(N)}(t) \\ \vdots \\ y_N^{(N)}(t) \end{pmatrix}, \quad \boldsymbol{\beta}^{(N)} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix},$$

$\mathbb{J}^{(N)} = \mathrm{diag}(\lambda_1, ..., \lambda_N)$ - diagonal matrix, $\mathbb{A}^{(N)}(t)$ is a $N \times N$ matrix with the elements $\alpha_{ij}(t) = \int_{\Pi_L} (\mathbf{U} \cdot \nabla)\mathbf{v}_i \cdot \mathbf{v}_j \, dx$ and $\beta_i = \int_\sigma v_{in} \, dx'$ with $i, j = 1, ..., N$.

Lemma 1.2.5. *Let $q^{(N)} \in L_2(0,T)$, $T \in (0, \infty)$ and suppose that $\boldsymbol{U}$ satisfies* (1.5)*, then there exists a unique solution $\mathbf{Y}^{(N)} \in \boldsymbol{H}^1(0,T)$ of the problem* (1.30).

Proof. The proof of this lemma is similar to the one of Lemma 1.2.1. □

The elements $\alpha_{ij}(t)$ of the matrix $\mathbb{A}^{(N)}(t)$ are bounded (see the proof of Lemma 1.2.1). The fundamental matrix $\mathbb{Z}^{(N)}(t)$ of the problem (1.30) is the solution of the matrix Cauchy problem

$$\begin{cases} \mathbb{Z}^{(N)'}(t) + (\mathbb{J}^{(N)} + \mathbb{A}^{(N)}(t))\mathbb{Z}^{(N)}(t) = \mathbb{O}, \\ \mathbb{Z}^{(N)}(0) = \mathbb{E}^{(N)}, \end{cases} \tag{1.31}$$

where $\mathbb{E}^{(N)}$ is the unit matrix and $\mathbb{O}$ is the zero matrix. This problem is equivalent to the integral equation of the type:

$$\exp(\mathbb{J}^{(N)}t)\mathbb{Z}^{(N)}(t) = \mathbb{E}^{(N)} - \int_0^t \mathbb{A}^{(N)}(\tau) \exp(\mathbb{J}^{(N)}\tau)\mathbb{Z}^{(N)}(\tau) \, d\tau.$$

$\| \cdot \|$ denotes an operator norm for the matrices and we deduce

$$\| \exp(\mathbb{J}^{(N)}t)\mathbb{Z}^{(N)}(t)\| \le \|\mathbb{E}^{(N)}\| + \int_0^t \|\mathbb{A}^{(N)}(\tau)\| \| \exp(\mathbb{J}^{(N)}\tau)\mathbb{Z}^{(N)}(\tau)\| \, d\tau,$$

i.e.

$$\| \exp(\mathbb{J}^{(N)}t)\mathbb{Z}^{(N)}(t)\| \le 1 + \int_0^t \|\mathbb{A}^{(N)}(\tau)\| \| \exp(\mathbb{J}^{(N)}\tau)\mathbb{Z}^{(N)}(\tau)\| \, d\tau$$

and by the Gronwall inequality

$$\| \exp(\mathbb{J}^{(N)}t)\mathbb{Z}^{(N)}(t)\| \le \exp(\int_0^t \|\mathbb{A}^{(N)}(\tau)\| \, d\tau).$$

Thus for $t \in (0,T)$

$$\|\mathbb{Z}^{(N)}(t)\| \le \exp(-\lambda_1 t + \int_0^t \|\mathbb{A}^{(N)}(\tau)\| \, d\tau) \le C(T). \tag{1.32}$$

From the theory of ordinary differential equations we get that

$$\det(\mathbb{Z}^{(N)}(t)) = \det \mathbb{Z}^{(N)}(0) \exp\big(- \int_0^t \mathrm{tr}(\mathbb{J}^{(N)} + \mathbb{A}^{(N)}(\tau)) \, d\tau\big) = \exp\big(- \int_0^t \mathrm{tr}\ \mathbb{J}^{(N)} \, d\tau\big).$$

Since $\mathbb{J}^{(N)}$ is formed from the eigenvalues, it follows that $\det(\mathbb{Z}^{(N)}(t)) \ge C_N(T)$. Since for a fixed N the matrix $\mathbb{Z}^{(N)}$ is bounded, we see easily using the form of $(\mathbb{Z}^{(N)})^{-1}$ that $\|(\mathbb{Z}^{(N)})^{-1}(t)\| \le C_N(T)$.
The solution $\mathbf{Y}^{(N)}(t)$ of the problem (1.30) is given by

$$\mathbf{Y}^{(N)}(t) = L \textstyle\int_0^t \mathbb{Z}^{(N)}(t)(\mathbb{Z}^{(N)}(\tau))^{-1}\boldsymbol{\beta}^{(N)}q^{(N)}(\tau) \, d\tau. \tag{1.33}$$

We find the functions $q^{(N)}(t)$ by using the flux condition (1.29). Substituting $\mathbf{v}^{(N)}(x,t)$ into (1.29) gives

$$\widetilde{F}(t) = \int_\sigma v_n^{(N)}(x',t)\,dx' = \sum_{k=1}^N y_k^{(N)}(t)\int_\sigma v_{kn}\,dx' = \mathbf{Y}^{(N)}(t)\cdot\boldsymbol{\beta}^{(N)}$$
$$= L\boldsymbol{\beta}^{(N)}\cdot\int_0^t \mathbb{Z}^{(N)}(t)(\mathbb{Z}^{(N)}(\tau))^{-1}\boldsymbol{\beta}^{(N)}q^{(N)}(\tau)\,d\tau.$$

Thus, $q^{(N)}(t)$ has to be found as the solution of the Volterra integral equation of the first kind

$$L\int_0^t \boldsymbol{\beta}^{(N)}\cdot\mathbb{Z}^{(N)}(t)(\mathbb{Z}^{(N)}(\tau))^{-1}\boldsymbol{\beta}^{(N)}q^{(N)}(\tau)\,d\tau = \widetilde{F}(t).$$

Since both functions vanish at 0, differentiating and using (1.31), we reduce the equation above to a Volterra integral equation of the second kind

$$q^{(N)}(t) - \int_0^t \mathcal{K}^{(N)}(t,\tau)q^{(N)}(\tau)\,d\tau = \vartheta^{(N)}(t) \tag{1.34}$$

with the kernel

$$\mathcal{K}^{(N)}(t,\tau) = \frac{\boldsymbol{\beta}^{(N)}}{\kappa_N}\cdot(\mathbb{J}^{(N)} + \mathbb{A}^{(N)}(t))\mathbb{Z}^{(N)}(t)(\mathbb{Z}^{(N)}(\tau))^{-1}\boldsymbol{\beta}^{(N)},$$

where $\kappa_N = |\boldsymbol{\beta}^{(N)}|^2$ and $\vartheta^{(N)}(t) = \dfrac{1}{L\kappa_N}\dfrac{d}{dt}\widetilde{F}(t)$.
In virtue of (1.32) we have that for any fixed N the kernel $\mathcal{K}^{(N)}(t,\tau)$ is bounded for all $0 \le \tau \le t$ and hence $\mathcal{K}^{(N)} \in L_2(\mathbb{Q}^T)$ with $\mathbb{Q}^T = (0,T)\times(0,T)$. Therefore, for any $\dfrac{d}{dt}\widetilde{F} \in L_2(0,T)$ there exists a unique solution $q^{(N)} \in L_2(0,T)$ of the integral equation (1.34) and the following estimate

$$|q^{(N)}|_{L_2(0,T)} \le C_N\Big|\frac{d}{dt}\widetilde{F}\Big|_{L_2(0,T)} \le C_N\Big(\Big|\frac{d}{dt}F\Big|_{L_2(0,T)} + |\partial_t\mathbf{V}|_{L_2(\Pi_L\times(0,T))}\Big) \tag{1.35}$$

holds (see, for example, [70]). The constant C_N in (1.35) depends on the kernel $\mathcal{K}^{(N)}(t,\tau)$ and we can not say in advance that C_N stays bounded as $N\to\infty$.

Remark 1.2.6. *In the estimate (1.35) we use the function* $\dfrac{d}{dt}\widetilde{F} = \dfrac{d}{dt}F - \int_\sigma \partial_t V_n\,dx'$. *But* $\partial_t\mathbf{V}$ *is only a* L_2*-function, and we can not speak about the usual trace estimate of it. However,* $\partial_t\mathbf{V}$ *is divergence free, therefore the trace of the normal component* $\partial_t V_n$ *is defined on* σ *as an element of the dual space* $W_2^{-1/2}(\sigma)$ *and the corresponding estimate holds true (see [68], Chapter 1).*

So we have now shown the existence of a unique approximate weak solution $(\mathbf{v}^{(N)}(x,t), q^{(N)}(t))$. Next we will obtain an a priori estimate for the approximate solution.

Theorem 1.2.7. *Suppose that* $\widetilde{F}(t) \in H^1(0,T)$, $\widetilde{F}(0) = 0$, $T\in(0,\infty)$ *and* $\boldsymbol{U}$ *satisfies* (1.5). *Then for the approximate weak solution* $(\mathbf{v}^{(N)}(x,t), q^{(N)}(t))$ *of the problem* (1.11) *the following estimate*

$$\begin{aligned}&\sup_{t\in[0,T]}\|\mathbf{v}^{(N)}(\cdot,t)\|^2_{\boldsymbol{H}^1(\Pi_L)} + \int_0^t |\partial_\tau\mathbf{v}^{(N)}|^2_{\boldsymbol{L}_2(\Pi_L)}\,d\tau\\ &+\int_0^t \big|\nabla\mathbf{v}^{(N)}\big|^2_{\boldsymbol{L}_2(\Pi_L)}\,d\tau + |q^{(N)}|^2_{L_2(0,T)} \le c\|\widetilde{F}\|^2_{H^1(0,T)}\end{aligned} \tag{1.36}$$

holds. The constant c *is depending on* L, D, T, μ *and the function* $\boldsymbol{U}$ *and is independent of* N *and* t.

Proof. Multiplying the equality (1.28) by $y_k^{(N)}(t)$ and summing up by k from 1 to N we get:

$$\frac{1}{2}\frac{d}{dt}\int_{\Pi_L}|\mathbf{v}^{(N)}|^2\,dx + \mu\int_{\Pi_L}|\nabla\mathbf{v}^{(N)}|^2\,dx + \int_{\Pi_L}(\mathbf{U}\cdot\nabla)\mathbf{v}^{(N)}\cdot\mathbf{v}^{(N)}\,dx = Lq^{(N)}\int_\sigma v_n^{(N)}\,dx'. \tag{1.37}$$

Having in mind the flux condition (1.29), we derive the following estimate

$$\frac{1}{2}\frac{d}{dt}\int_{\Pi_L}|\mathbf{v}^{(N)}|^2\,dx+\mu\int_{\Pi_L}|\nabla\mathbf{v}^{(N)}|^2\,dx\le\varepsilon|q^{(N)}|^2+\frac{L^2}{\varepsilon}|\widetilde{F}|^2. \tag{1.38}$$

Integrating the inequality above in t we get

$$\frac{1}{2}\int_{\Pi_L}|\mathbf{v}^{(N)}|^2\,dx+\mu\int_0^t\int_{\Pi_L}|\nabla\mathbf{v}^{(N)}|^2\,dxd\tau\le\varepsilon\int_0^t|q^{(N)}|^2\,d\tau+\frac{L^2}{\varepsilon}\int_0^t|\widetilde{F}|^2\,d\tau. \tag{1.39}$$

Let us now multiply (1.28) by $\frac{d}{dt}y_k^{(N)}(t)$ and sum up in k from 1 to N then we obtain

$$\int_{\Pi_L}|\partial_t\mathbf{v}^{(N)}|^2\,dx+\frac{\mu}{2}\frac{d}{dt}\int_{\Pi_L}|\nabla\mathbf{v}^{(N)}|^2\,dx+\int_{\Pi_L}(\mathbf{U}\cdot\nabla)\mathbf{v}^{(N)}\cdot\partial_t\mathbf{v}^{(N)}\,dx=Lq^{(N)}\int_\sigma\partial_t v_n^{(N)}\,dx'.$$

Using Cauchy-Schwarz and Young inequalities we can get the estimate

$$\begin{aligned}&\textstyle\int_{\Pi_L}|\partial_t\mathbf{v}^{(N)}|^2\,dx+\frac{\mu}{2}\frac{d}{dt}\int_{\Pi_L}|\nabla\mathbf{v}^{(N)}|^2\,dx\le\int_{\Pi_L}|(\mathbf{U}\cdot\nabla)\mathbf{v}^{(N)}\cdot\partial_t\mathbf{v}^{(N)}|\,dx+L\left|q^{(N)}\frac{d}{dt}\widetilde{F}\right|\\&\textstyle\le\frac{1}{2}\sup_{x\in\Pi_L}|\mathbf{U}|^2\int_{\Pi_L}|\nabla\mathbf{v}^{(N)}|^2\,dx+\frac{1}{2}\int_{\Pi_L}|\partial_t\mathbf{v}^{(N)}|^2\,dx+\varepsilon|q^{(N)}|^2+\frac{L^2}{\varepsilon}|\frac{d}{dt}\widetilde{F}|^2.\end{aligned} \tag{1.40}$$

Hence

$$\frac{d}{dt}\int_{\Pi_L}|\nabla\mathbf{v}^{(N)}|^2\,dx\le\frac{m(t)}{\mu}\int_{\Pi_L}|\nabla\mathbf{v}^{(N)}|^2\,dx+\frac{2\varepsilon}{\mu}|q^{(N)}|^2+\frac{2L^2}{\varepsilon\mu}|\frac{d}{dt}\widetilde{F}|^2$$

(recall that $m(t)=\sup\limits_{x\in\Pi_L}|\mathbf{U}|^2$) and so integrating in t

$$\begin{aligned}&\int_{\Pi_L}|\nabla\mathbf{v}^{(N)}|^2\,dx\le e^{\frac{1}{\mu}\int_0^t m(\tau)\,d\tau}\int_0^t(\frac{2\varepsilon}{\mu}|q^{(N)}|^2+\frac{2L^2}{\varepsilon\mu}|\frac{d}{d\tau}\widetilde{F}|^2)e^{-\frac{1}{\mu}\int_0^\tau m(s)\,ds}\,d\tau\\&\le c_1\varepsilon\int_0^t|q^{(N)}|^2\,d\tau+\frac{c_2}{\varepsilon}\int_0^t|\frac{d}{d\tau}\widetilde{F}|^2\,d\tau.\end{aligned}$$

With this estimate and going back to (1.40) we derive by integrating in t

$$\begin{aligned}&\frac{1}{2}\int_0^t\int_{\Pi_L}|\partial_\tau\mathbf{v}^{(N)}|^2\,dxd\tau+\frac{\mu}{2}\int_{\Pi_L}|\nabla\mathbf{v}^{(N)}|^2\,dx\\&\le\frac{1}{2}\int_0^t m(\tau)\,d\tau(c_1\varepsilon\int_0^\tau|q^{(N)}|^2\,d\tau+\frac{c_2}{\varepsilon}\int_0^\tau|\frac{d}{d\tau}\widetilde{F}|^2\,d\tau)+\varepsilon\int_0^t|q^{(N)}|^2\,d\tau+\frac{L^2}{\varepsilon}\int_0^t|\frac{d}{d\tau}\widetilde{F}|^2\,d\tau\\&\le c_1(\varepsilon\int_0^t|q^{(N)}|^2\,d\tau+\frac{1}{\varepsilon}\int_0^t|\frac{d}{d\tau}\widetilde{F}|^2\,d\tau).\end{aligned} \tag{1.41}$$

Our next aim is to get an estimate for $q^{(N)}(t)$. Let $\boldsymbol{\omega}$ be the solution of the problem:

$$\begin{cases}\boldsymbol{\omega}\in\mathbf{H}(\Pi_L),\\ \mu\int_{\Pi_L}\nabla\boldsymbol{\omega}\cdot\nabla\boldsymbol{\eta}\,dx=L\int_\sigma\eta_n\,dx'\quad\forall\boldsymbol{\eta}\in\mathbf{H}(\Pi_L).\end{cases} \tag{1.42}$$

The existence and uniqueness of the solution of the problem (1.42) follows from the Lax-Milgram Theorem (cf. Theorem 4.2.2). We claim that

$$L\int_\sigma\omega_n\,dx'=\xi_0>0.$$

Indeed if we take $\boldsymbol{\eta} = \boldsymbol{\omega}$ in (1.42) we obtain

$$\mu \int_{\Pi_L} |\nabla \boldsymbol{\omega}|^2 \, dx = L \int_\sigma \omega_n \, dx' = \xi_0 \geq 0.$$

From this estimate follows that the integral in the right hand side does not depend on x_n and therefore

$$\int_{\Pi_L} \boldsymbol{\omega}_n \, dx = L \int_\sigma \omega_n \, dx'.$$

If $\xi_0 = 0$ then we derive $\int_{\Pi_L} |\nabla \boldsymbol{\omega}|^2 \, dx = 0$, so $\nabla \boldsymbol{\omega} = 0$ and then from (1.42) we get $\int_\sigma \eta_n \, dx' = 0 \, \forall \boldsymbol{\eta} \in \mathbf{H}(\Pi_L)$ which is impossible.

Since $\boldsymbol{\omega} \in \mathbf{H}(\Pi_L)$ we have that $\boldsymbol{\omega}(x) = \sum\limits_{k=1}^{\infty} \gamma_k \mathbf{v}_k$, where $\gamma_k = \int_{\Pi_L} \boldsymbol{\omega} \cdot \mathbf{v}_k \, dx$. Multiplying (1.28) by γ_k and summing up from 1 to N, we obtain

$$\begin{aligned} &\textstyle\int_{\Pi_L} \partial_t \mathbf{v}^{(N)} \cdot \boldsymbol{\omega}^{(N)} \, dx + \mu \int_{\Pi_L} \nabla \mathbf{v}^{(N)} \cdot \nabla \boldsymbol{\omega}^{(N)} \, dx + \int_{\Pi_L} (\mathbf{U} \cdot \nabla) \mathbf{v}^{(N)} \cdot \boldsymbol{\omega}^{(N)} \, dx \\ &= L q^{(N)}(t) \textstyle\int_\sigma \omega_n^{(N)} \, dx' \end{aligned} \tag{1.43}$$

with $\boldsymbol{\omega}^{(N)}(x) = \sum\limits_{k=1}^{N} \gamma_k \mathbf{v}_k$. In virtue of (1.42) and the flux condition (1.29) we have

$$\begin{aligned} &\mu \textstyle\int_{\Pi_L} \nabla \mathbf{v}^{(N)} \cdot \nabla \boldsymbol{\omega}^{(N)} \, dx = \mu \int_{\Pi_L} \nabla \mathbf{v}^{(N)} \cdot \nabla \boldsymbol{\omega} \, dx + \mu \int_{\Pi_L} \nabla \mathbf{v}^{(N)} \cdot \nabla (\boldsymbol{\omega}^{(N)} - \boldsymbol{\omega}) \, dx \\ &= L \textstyle\int_\sigma v_n^{(N)} \, dx + \mu \int_{\Pi_L} \nabla \mathbf{v}^{(N)} \cdot \nabla (\boldsymbol{\omega}^{(N)} - \boldsymbol{\omega}) \, dx \\ &= L \widetilde{F}(t) + \mu \textstyle\int_{\Pi_L} \nabla \mathbf{v}^{(N)} \cdot \nabla (\boldsymbol{\omega}^{(N)} - \boldsymbol{\omega}) \, dx. \end{aligned}$$

We have also

$$L \int_\sigma \omega_n^{(N)} \, dx' = L \int_\sigma \omega_n \, dx' + L \int_\sigma (\omega_n^{(N)} - \omega_n) \, dx' = \xi_0 + L \int_\sigma (\omega_n^{(N)} - \omega_n) \, dx'.$$

Therefore the equation (1.43) can be rewritten as

$$\begin{aligned} &\textstyle\int_{\Pi_L} \partial_t \mathbf{v}^{(N)} \cdot \boldsymbol{\omega}^{(N)} \, dx + L \widetilde{F}(t) + \mu \int_{\Pi_L} \nabla \mathbf{v}^{(N)} \cdot \nabla (\boldsymbol{\omega}^{(N)} - \boldsymbol{\omega}) \, dx \\ &+ \textstyle\int_{\Pi_L} (\mathbf{U} \cdot \nabla) \mathbf{v}^{(N)} \cdot \boldsymbol{\omega}^{(N)} \, dx = q^{(N)}(t) \xi_0 + L q^{(N)}(t) \int_\sigma (\omega_n^{(N)} - \omega_n) \, dx'. \end{aligned}$$

Next we look for an estimate of the function $q^{(N)}(t)$. From the last equality it follows that

$$\begin{aligned} &\xi_0 q^{(N)}(t) = L q^{(N)}(t) \textstyle\int_\sigma (\omega_n - \omega_n^{(N)}) \, dx' + \mu \int_{\Pi_L} \nabla \mathbf{v}^{(N)} \cdot \nabla (\boldsymbol{\omega}^{(N)} - \boldsymbol{\omega}) \, dx \\ &+ \textstyle\int_{\Pi_L} \partial_t \mathbf{v}^{(N)} \cdot \boldsymbol{\omega}^{(N)} \, dx + \int_{\Pi_L} (\mathbf{U} \cdot \nabla) \mathbf{v}^{(N)} \cdot \boldsymbol{\omega}^{(N)} \, dx + L \widetilde{F}(t). \end{aligned}$$

Because of the orthogonality of the eigenfunctions $\mathbf{v}_k$ and since $\boldsymbol{\omega}^{(N)} - \boldsymbol{\omega} = -\sum\limits_{k=N+1}^{\infty} \gamma_k \mathbf{v}_k$ we derive

$$\int_{\Pi_L} \nabla \mathbf{v}^{(N)} \cdot \nabla (\boldsymbol{\omega}^{(N)} - \boldsymbol{\omega}) \, dx = 0.$$

So we have, by using Young's and Hölder's inequalities

$$\begin{aligned} &\int_0^t \xi_0^2 |q^{(N)}|^2 \, d\tau \leq \int_0^t \Big\{ L |q^{(N)} \int_\sigma (\omega_n - \omega_n^{(N)}) \, dx'| + |\partial_\tau \mathbf{v}^{(N)}|_{\mathbf{L}_2(\Pi_L)} |\boldsymbol{\omega}^{(N)}|_{\mathbf{L}_2(\Pi_L)} \\ &+ |\mathbf{U}|_{\mathbf{L}_4(\Pi_L)} \big| |\nabla \mathbf{v}^{(N)}| \big|_{\mathbf{L}_2(\Pi_L)} |\boldsymbol{\omega}^{(N)}|_{\mathbf{L}_4(\Pi_L)} + L |\widetilde{F}| \Big\}^2 \, d\tau \\ &\leq c \Big\{ \int_0^t |q^{(N)}(\tau)|^2 \, d\tau \int_{\Pi_L} |\boldsymbol{\omega} - \boldsymbol{\omega}^{(N)}|^2 \, dx + \int_0^t \int_{\Pi_L} |\partial_\tau \mathbf{v}^{(N)}|^2 \, dx d\tau \int_{\Pi_L} |\boldsymbol{\omega}^{(N)}|^2 \, dx \\ &+ \sup_{\tau \in [0,t]} (\int_{\Pi_L} |\nabla \mathbf{U}|^2 \, dx) \int_0^t \int_{\Pi_L} |\nabla \mathbf{v}^{(N)}|^2 \, dx \int_{\Pi_L} |\nabla \boldsymbol{\omega}^{(N)}|^2 \, dx d\tau + \int_0^t |\widetilde{F}|^2 \, d\tau \Big\}. \end{aligned}$$

Since

$$\int_{\Pi_L} |\boldsymbol{\omega}^{(N)}|^2\,dx \le \int_{\Pi_L} |\boldsymbol{\omega}|^2\,dx, \quad \int_{\Pi_L} |\nabla\boldsymbol{\omega}^{(N)}|^2\,dx \le \int_{\Pi_L} |\nabla\boldsymbol{\omega}|^2\,dx$$

and $|\boldsymbol{\omega} - \boldsymbol{\omega}^{(N)}|_{\mathbf{L}_2(\Pi_L)} \to 0$ when $N \to \infty$ we get

$$\int_0^t |q^{(N)}(\tau)|^2\,d\tau \le c\int_0^t \Big(|\partial_\tau \mathbf{v}^{(N)}|^2_{\mathbf{L}_2(\Pi_L)} + \big|\nabla\mathbf{v}^{(N)}\big|^2_{\mathbf{L}_2(\Pi_L)} + |\widetilde{F}|^2\Big)\,d\tau. \tag{1.44}$$

On the other hand from (1.39) and (1.41) we establish

$$\begin{aligned}&\textstyle\int_{\Pi_L} |\mathbf{v}^{(N)}|^2\,dx + \int_0^t\int_{\Pi_L} |\nabla\mathbf{v}^{(N)}|^2\,dxd\tau + \int_0^t\int_{\Pi_L} |\partial_\tau\mathbf{v}^{(N)}|^2\,dxd\tau + \int_{\Pi_L} |\nabla\mathbf{v}^{(N)}|^2\,dx\\ &\le c(\tfrac{1}{\varepsilon}\|\widetilde{F}\|^2_{H^1(0,t)} + \varepsilon\textstyle\int_0^t |q^{(N)}|^2\,d\tau).\end{aligned} \tag{1.45}$$

For sufficiently small ε we get from (1.44) and (1.45) that

$$\textstyle\int_0^t |q^{(N)}(\tau)|^2\,d\tau \le c\int_0^t \big(|\widetilde{F}(\tau)|^2 + |\frac{d}{d\tau}\widetilde{F}(\tau)|^2\big)\,d\tau. \tag{1.46}$$

The estimate (1.36) is a consequence of (1.45) and (1.46). □

Theorem 1.2.8. *Suppose that $\widetilde{F} \in H^1(0,T)$, $\widetilde{F}(0) = 0$, $T \in (0,\infty)$ and $\boldsymbol{U}$ satisfies (1.5). Then the problem (1.26) admits a unique solution $(\mathbf{v}, q) \in \boldsymbol{L}_2(0,T;\boldsymbol{H}(\Pi_L)) \times L_2(0,T)$ and the following estimate*

$$\sup_{t\in[0,T]} \|\mathbf{v}\|^2_{\boldsymbol{H}^1(\Pi_L)} + \int_0^t |\partial_\tau\mathbf{v}|^2_{\boldsymbol{L}_2(\Pi_L)}\,d\tau + \int_0^t \big|\nabla\mathbf{v}\big|^2_{\boldsymbol{L}_2(\Pi_L)}\,d\tau + |q|^2_{L_2(0,T)} \le c\|\widetilde{F}\|^2_{H^1(0,T)} \tag{1.47}$$

is valid. Here the constant c depends on L, D, T, μ and the function $\boldsymbol{U}$.

Proof. According to the estimate see Theorem 1.2.7 and the proof of the existence is equivalent to the proof of Theorem 1.2.4. Next we would like to prove the uniqueness of the solution. For this we consider $\mathbf{v}(x,t) = \mathbf{v}_1(x,t) - \mathbf{v}_2(x,t)$ and $q(t) = q_1(t) - q_2(t)$, where $(\mathbf{v}_i, q_i)$ are solutions of the problem (1.26). Since both solutions have to fulfil the flux condition, we obtain that

$$\int_\sigma v_n\,dx' = \int_\sigma (v_{1n} - v_{2n})\,dx' = \widetilde{F}(t) - \widetilde{F}(t) = 0$$

and so by subtracting the equations satisfied by $(\mathbf{v}_1, q_1)$ and $(\mathbf{v}_2, q_2)$, and taking $\mathbf{v}$ as a test function we get

$$\frac{1}{2}\int_{\Pi_L} |\mathbf{v}|^2\,dx + \mu\int_0^t\int_{\Pi_L} |\nabla\mathbf{v}|^2\,dxd\tau = L\int_0^t q\int_\sigma v_n\,dx'd\tau = 0.$$

This implies that $\mathbf{v}(x,t) = 0$ and therefore $\mathbf{v}_1 = \mathbf{v}_2$. Going back to (1.26) we obtain that

$$0 = L\int_0^t q(\tau)\int_\sigma \eta_n\,dx'd\tau$$

for all possible $\boldsymbol{\eta}$, which implies $q = 0$ and so $q_1 = q_2$. This completes the proof of the uniqueness. □

1.2.3 Analysis of problem (1.7)

Combining the obtained results of the problems (1.10) and (1.11) (see Theorem 1.2.2 and Theorem 1.2.4) we can state the following:

Theorem 1.2.9. *Suppose that* $\mathbf{f} \in \boldsymbol{L}_2(0,T;\boldsymbol{L}_2(\Pi_L))$, $\boldsymbol{a} \in \boldsymbol{H}^1(\Pi_L)$, $F \in H^1(0,T)$, $T \in (0,\infty)$, $F(0) = \int_\sigma a_n\, dx'$ *and* $\boldsymbol{U}$ *satisfies* (1.5). *Then the problem* (1.7) *admits a unique weak solution* $(\boldsymbol{u}, q) \in \boldsymbol{L}_2(0,T;\boldsymbol{H}(\Pi_L)) \times L_2(0,T)$ *and the following estimate*

$$\begin{aligned}&\sup_{t\in[0,T]} \|\boldsymbol{u}(\cdot,t)\|^2_{\boldsymbol{H}^1(\Pi_L)} + \int_0^t |\partial_\tau \boldsymbol{u}|^2_{\boldsymbol{L}_2(\Pi_L)}\, d\tau + \int_0^t \left||\nabla \boldsymbol{u}|\right|^2_{\boldsymbol{L}_2(\Pi_L)}\, d\tau + |q|^2_{L_2(0,T)} \\ &\le c(\|F\|^2_{H^1(0,T)} + \int_0^T |\mathbf{f}|^2_{\mathbf{L}_2(\Pi_L)} d\tau + \|\boldsymbol{a}\|^2_{\boldsymbol{H}^1(\Pi_L)})\end{aligned} \tag{1.48}$$

is valid. Here the constant c *depends on* L, D, T, μ *and the function* $\boldsymbol{U}$.

Proof. For the proof of this theorem see Theorem 1.2.4 and Theorem 1.2.8. □

1.3 Asymptotic behaviour

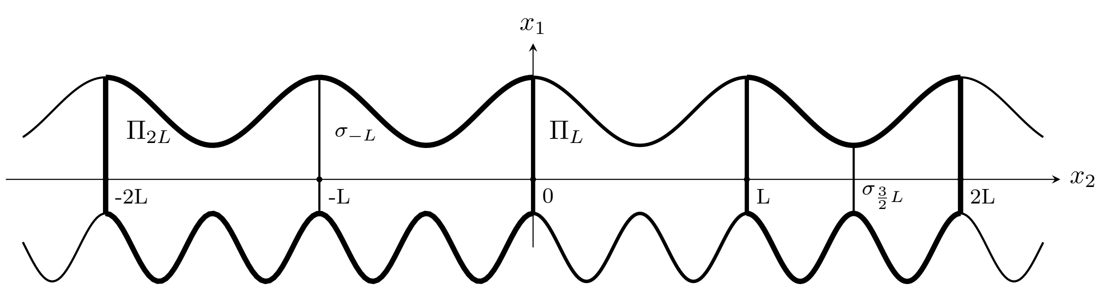
In this section we will analyse the asymptotic behaviour in space between the problem (1.7), defined in one periodicity cell Π_L and the analogous problem, defined in the bigger domain

$$\Pi_{\ell L} = \{x \in \sigma_{x_n} \times \{x_n\},\, x_n \in (-\ell L, \ell L)\} = \bigcup_{z=-\ell}^{\ell-1} (\Pi_L \cup \sigma_0 \cup \sigma_L + zL\mathbf{e}_n). \tag{1.49}$$

Let us define the lateral boundary of this domain by

$$S_{\ell L} = \partial \Pi_{\ell L} \backslash (\sigma_{-\ell L} \cup \sigma_{\ell L}). \tag{1.50}$$

A possible domain might be for $n = 2$:



We will compare the weak solution $(\mathbf{u}, q)$ of the problem (1.7) to the solution $(\mathbf{u}_\ell, q_\ell)$ of the problem

$$\begin{cases} \mathbf{u}_\ell \in \mathbf{L}_2(0,T;\mathbf{H}(\Pi_{\ell L})), \quad \partial_t \mathbf{u}_\ell \in \mathbf{L}_2(0,T;\mathbf{H}(\Pi_{\ell L})), \quad \mathbf{u}_\ell\mid_{t=0} = \mathbf{a}, \quad q_\ell \in L_2(0,T), \\ \int_0^t \int_{\Pi_{\ell L}} \partial_\tau \mathbf{u}_\ell \cdot \boldsymbol{\eta}\, dx d\tau + \mu \int_0^t \int_{\Pi_{\ell L}} \nabla \mathbf{u}_\ell \cdot \nabla \boldsymbol{\eta}\, dx d\tau + \int_0^t \int_{\Pi_{\ell L}} (\mathbf{U}\cdot\nabla)\mathbf{u}_\ell \cdot \boldsymbol{\eta}\, dx d\tau \\ = \int_0^t \int_{\Pi_{\ell L}} \mathbf{f}\cdot\boldsymbol{\eta}\, dx d\tau + 2L\ell \int_0^t q_\ell(\tau) \int_\sigma \eta_n\, dx' d\tau \quad \forall \boldsymbol{\eta} \in \mathbf{L}_2(0,T;\mathbf{H}(\Pi_{\ell L})), \quad \forall t \in (0,T], \\ \int_\sigma u_{\ell,n}(x', x_n, t)\, dx' = F(t) \quad \forall x_n \in (-\ell L, \ell L), \end{cases} \tag{1.51}$$

where

$$\mathbf{H}(\Pi_{\ell L}) = \{\boldsymbol{\eta} \in \mathbf{H}^1(\Pi_{\ell L}) \mid \operatorname{div}(\boldsymbol{\eta}) = 0,\, \boldsymbol{\eta}(x', -\ell L) = \boldsymbol{\eta}(x', \ell L),\, \boldsymbol{\eta}\mid_{S_{\ell L}} = 0\}, \tag{1.52}$$

$\mathbf{f}$, $\mathbf{a}$ and $\mathbf{U}$ are L-periodically extended functions in x_n and the given function $\mathbf{U}$ satisfies (1.5). For the existence of the solution of the problem (1.51) compare previous section. If $\ell \in \mathbb{N}^+$ then it is obvious that we get the existence and uniqueness from the previous section. If $\ell \in \mathbb{R}^+\backslash\mathbb{N}^+$ we need to assume as well that Π_L is a symmetric set with respect to the hyperplane $x_n = \frac{L}{2}$ and that $\mathbf{U}$ and $\mathbf{a}$ are L-periodic symmetric functions with respect to $x_n = \frac{L}{2}$ in the set $\Pi_{\ell L}$. Note that we then have $\mathbf{U}(x', -\ell L) = \mathbf{U}(x', \ell L)$ and $\mathbf{a}(x', -\ell L) = \mathbf{a}(x', \ell L)$.

We will analyse two different cases of the asymptotic behaviour. First we will assume that $\ell \in \mathbb{N}^+$, i.e. the domain $\Pi_{\ell L}$ consists exactly of 2ℓ periodicity cells Π_L. Under this assumption we can prove the equality between the weak solution $(\mathbf{u}, q)$ of the problem (1.7) to the solution $(\mathbf{u}_\ell, q_\ell)$ of the problem (1.51). Secondly we will take $\ell \in \mathbb{R}^+$ and we will prove that $\mathbf{u}_\ell$ of the problem (1.51) converges exponentially to $\mathbf{u}$ of the problem (1.7).

1.3.1 Case $\ell \in \mathbb{N}^+$

First let us prove an auxiliary result for the weak solution $(\mathbf{u}, q)$ of the problem (1.7).

Lemma 1.3.1. *Let $\ell \in \mathbb{N}^+$, $T \in (0,\infty)$, $\mathbf{f} \in \boldsymbol{L}_2(0,T;\boldsymbol{L}_2(\Pi_L))$, $\boldsymbol{a} \in \boldsymbol{H}^1(\Pi_L)$ and $\boldsymbol{U}$ satisfies (1.5). Let $(\boldsymbol{u}, q)$ be the weak solution of the problem (1.7) and $\boldsymbol{u}$, $\boldsymbol{a}$, $\boldsymbol{U}$ and $\mathbf{f}$ in the domain $\Pi_{\ell L}$ are L-periodically extended with respect to x_n. Then for every $\mathbf{v} \in \boldsymbol{L}_2(0,T;\boldsymbol{H}(\Pi_{\ell L}))$ the following identity*

$$
\begin{aligned}
&\int_0^t \int_{\Pi_{\ell L}} \partial_\tau \boldsymbol{u} \cdot \mathbf{v}\, dx d\tau + \mu \int_0^t \int_{\Pi_{\ell L}} \nabla \boldsymbol{u} \cdot \nabla \mathbf{v}\, dx d\tau + \int_0^t \int_{\Pi_{\ell L}} (\boldsymbol{U} \cdot \nabla)\boldsymbol{u} \cdot \mathbf{v}\, dx d\tau \\
&= \int_0^t \int_{\Pi_{\ell L}} \boldsymbol{f} \cdot \mathbf{v}\, dx d\tau + 2L\ell \int_0^t q \int_\sigma v_n\, dx' d\tau
\end{aligned}
\tag{1.53}
$$

holds.

Proof. Let us extend the functions $\mathbf{u}$ and $\mathbf{f}$ L-periodically with respect to x_n in the domain $\Pi_{\ell L}$. Having in mind that the weak solution of the problem (1.7) is unique (see Theorem 1.2.9) and the function q does not depend on x_n, it is obvious that the L-periodically extended function q, with respect to x_n, is the same in $\Pi_{\ell L}$ as in the domain Π_L.
Let us take in the integral identity (1.8) a test function $\mathbf{v} \in \mathbf{L}_2(0,T;\mathbf{H}(\Pi_{\ell L}))$. Using the periodicity condition of the functions $\mathbf{u}$ and $\mathbf{U}$, then we obtain on the left side of the integral identity:

$$
\begin{aligned}
&\int_0^t \int_{\Pi_{\ell L}} \partial_\tau \mathbf{u} \cdot \mathbf{v}\, dx d\tau + \mu \int_0^t \int_{\Pi_{\ell L}} \nabla \mathbf{u} \cdot \nabla \mathbf{v}\, dx d\tau + \int_0^t \int_{\Pi_{\ell L}} (\mathbf{U} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}\, dx d\tau \\
&= \sum_{z=-\ell}^{\ell-1} \int_0^t \int_{\Pi_L + zL\mathbf{e}_n} \partial_\tau \mathbf{u} \cdot \mathbf{v}\, dx d\tau + \mu \int_0^t \int_{\Pi_L + zL\mathbf{e}_n} \nabla \mathbf{u} \cdot \nabla \mathbf{v}\, dx d\tau + \int_0^t \int_{\Pi_L + zL\mathbf{e}_n} (\mathbf{U} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}\, dx d\tau \\
&= \sum_{z=-\ell}^{\ell-1} \Big(\int_0^t \int_{\Pi_L} \partial_\tau \mathbf{u}(x,\tau) \cdot \mathbf{v}(x + Lz\mathbf{e}_n, \tau)\, dx d\tau + \mu \int_0^t \int_{\Pi_L} \nabla \mathbf{u}(x,\tau) \cdot \nabla \mathbf{v}(x + Lz\mathbf{e}_n, \tau)\, dx d\tau \\
&+ \int_0^t \int_{\Pi_L} (\mathbf{U}(x,\tau) \cdot \nabla)\mathbf{u}(x,\tau) \cdot \mathbf{v}(x + Lz\mathbf{e}_n, \tau)\, dx d\tau \Big).
\end{aligned}
$$

Note that in the above calculations it is important that $\ell \in \mathbb{N}^+$, otherwise the domain $\Pi_{\ell L}$ could not be splitted into an integer number of periodicity cells Π_L.
Let us set $\tilde{\mathbf{v}} = \sum_{z=-\ell}^{\ell-1} \mathbf{v}(x + Lz\mathbf{e}_n, t)$. Since $\mathbf{v}|_{S_{\ell L}} = 0$ it is clear that $\tilde{\mathbf{v}}|_{S_L} = 0$. Moreover, we have

$$
\operatorname{div}(\tilde{\mathbf{v}}) = \sum_{z=-\ell}^{\ell-1} \operatorname{div}(\mathbf{v}(x + zL\mathbf{e}_n, t)) = 0
$$

and

$$
\tilde{\mathbf{v}}(x', 0, t) = \sum_{z=-\ell}^{\ell-1} \mathbf{v}(x', 0 + zL, t) = \sum_{z=-\ell+1}^{\ell} \mathbf{v}(x', zL, t) = \sum_{z=-\ell}^{\ell-1} \mathbf{v}(x', L + zL, t) = \tilde{\mathbf{v}}(x', L, t).
$$

Above, we used the periodicity condition $\mathbf{v}(x', -\ell L, t) = \mathbf{v}(x', \ell L, t)$. So we get $\tilde{\mathbf{v}} \in \mathbf{L}_2(0,T;\mathbf{H}(\Pi_L))$ and therefore

$$\begin{aligned}
&\int_0^t \int_{\Pi_{\ell L}} \partial_\tau \mathbf{u}\cdot\mathbf{v}\,dxd\tau + \mu\int_0^t \int_{\Pi_{\ell L}} \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dxd\tau + \int_0^t \int_{\Pi_{\ell L}} (\mathbf{U}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,dxd\tau \\
&= \int_0^t \int_{\Pi_L} \partial_\tau \mathbf{u}\cdot\tilde{\mathbf{v}}\,dxd\tau + \mu\int_0^t \int_{\Pi_L} \nabla\mathbf{u}\cdot\nabla\tilde{\mathbf{v}}\,dxd\tau + \int_0^t \int_{\Pi_L} (\mathbf{U}\cdot\nabla)\mathbf{u}\cdot\tilde{\mathbf{v}}\,dxd\tau \\
&= \int_0^t \int_{\Pi_L} (\mathbf{f}+\mathbf{q})\cdot\tilde{\mathbf{v}}\,dxd\tau = \sum_{z=-\ell}^{\ell-1}\int_0^t \int_{\Pi_L} (\mathbf{f}(x+Lz\mathbf{e}_n,\tau)+\mathbf{q})\cdot\mathbf{v}(x+Lz\mathbf{e}_n,\tau)\,dxd\tau \\
&= \sum_{z=-\ell}^{\ell-1}\int_0^t \int_{\Pi_L+zL\mathbf{e}_n} (\mathbf{f}+\mathbf{q})\cdot\mathbf{v}\,dxd\tau = \int_0^t \int_{\Pi_{\ell L}} (\mathbf{f}+\mathbf{q})\cdot\mathbf{v}\,dxd\tau.
\end{aligned}$$

This identity completes the proof of the lemma. □

From the Theorem 1.2.9 we have that the solutions of the problems (1.8) and (1.51) are unique, therefore combining with the result of Lemma 1.3.1 we derive that $(\mathbf{u}, q) = (\mathbf{u}_\ell, q_\ell)$.

1.3.2 Case $\ell \in \mathbb{R}^+\backslash\mathbb{N}^+$

The assumptions for the case $\ell \in \mathbb{R}^+\backslash\mathbb{N}^+$ requires more restrictions. For the existence of $(\mathbf{u}_\ell, q_\ell)$ (see previous section) we need that the domain Π_L and the functions $\mathbf{U}$ and $\mathbf{a}$ are symmetric with respect to $x_n = \frac{L}{2}$. We are able to prove the convergence of $\mathbf{u}_\ell$ to $\mathbf{u}$, when we take the test functions in (1.8) in addition equal to zero on the whole boundary of $\Pi_{\ell L}$ (see next lemma).
First let us define the new function space

$$\widehat{\mathbf{H}}(\Pi_{\ell L}) = \{\boldsymbol{\eta} \in \mathbf{H}_0^1(\Pi_{\ell L}) \mid \operatorname{div}(\boldsymbol{\eta}) = 0\}. \tag{1.54}$$

Lemma 1.3.2. *Let* $\ell \in \mathbb{R}^+\backslash\mathbb{N}^+$, $T \in (0,\infty)$, $\mathbf{f} \in \boldsymbol{L}_2(0,T;\boldsymbol{L}_2(\Pi_L))$, $\boldsymbol{a} \in \boldsymbol{H}^1(\Pi_L)$ *and* $\boldsymbol{U}$ *satisfies* (1.5). *Let* $(\boldsymbol{u}, q)$ *resp.* $(\boldsymbol{u}_\ell, q_\ell)$ *be the weak solution of the problem* (1.7) *resp.* (1.51) *and* $\boldsymbol{u}$, $\boldsymbol{a}$, $\boldsymbol{U}$ *and* $\mathbf{f}$ *are extended L-periodically with respect to* x_n. *Then for every* $\mathbf{v} \in \boldsymbol{L}_2(0,T;\widehat{\boldsymbol{H}}(\Pi_{\ell L}))$ *the following integral identity*

$$\int_0^t \int_{\Pi_{\ell L}} \partial_\tau \boldsymbol{u}\cdot\mathbf{v}\,dxd\tau + \mu\int_0^t \int_{\Pi_{\ell L}} \nabla\boldsymbol{u}\cdot\nabla\mathbf{v}\,dxd\tau + \int_0^t \int_{\Pi_{\ell L}} (\boldsymbol{U}\cdot\nabla)\boldsymbol{u}\cdot\mathbf{v}\,dxd\tau = \int_0^t \int_{\Pi_{\ell L}} \mathbf{f}\cdot\mathbf{v}\,dxd\tau \tag{1.55}$$

is valid.

Proof. Let us extend the function $\mathbf{v}$ by zero outside the domain $\Pi_{\ell L}$. By (1.53) and since $\int_\sigma v_n\,dx' = 0$ one has

$$\begin{aligned}
&\int_0^t \int_{\Pi_{\ell L}} \partial_\tau \mathbf{u}\cdot\mathbf{v}\,dxd\tau + \mu\int_0^t \int_{\Pi_{\ell L}} \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dxd\tau + \int_0^t \int_{\Pi_{\ell L}} (\mathbf{U}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,dxd\tau \\
&= \int_0^t \int_{\Pi_{([\ell]+1)L}} \partial_\tau \mathbf{u}\cdot\mathbf{v}\,dxd\tau + \mu\int_0^t \int_{\Pi_{([\ell]+1)L}} \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dxd\tau + \int_0^t \int_{\Pi_{([\ell]+1)L}} (\mathbf{U}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,dxd\tau \\
&= \int_0^t \int_{\Pi_{([\ell]+1)L}} \mathbf{f}\cdot\mathbf{v}\,dxd\tau = \int_0^t \int_{\Pi_{\ell L}} \mathbf{f}\cdot\mathbf{v}\,dxd\tau,
\end{aligned}$$

where the symbol [ ] means the integer part of the number. This completes the proof of the lemma. □

Note since now the test functions are zero on the whole boundary we loose the term including q, i.e. $\int_0^t q\int_\sigma v_n\,dx'd\tau = 0$.

Theorem 1.3.3. *Let* $\mathbf{f} \in \boldsymbol{L}_2(0,T;\boldsymbol{L}_2(\Pi_L))$*,* $\boldsymbol{a} \in \boldsymbol{H}^1(\Pi_L)$*,* $T \in (0,\infty)$*,* $\boldsymbol{U}$ *satisfies* (1.5) *and in addition we assume* $\sup\limits_{t\in[0,T]} \sup\limits_{x\in\Pi_L} |\boldsymbol{U}|^2 < 4\mu$*. Let* $(\boldsymbol{u}, q)$ *resp.* $(\boldsymbol{u}_\ell, q_\ell)$ *be the weak solution of the problem* (1.7) *resp.* (1.51) *and* $\boldsymbol{u}$*,* $\boldsymbol{a}$*,* $\boldsymbol{U}$ *and* $\mathbf{f}$ *are extended L-periodically with respect to* x_n*. Then the following estimate*

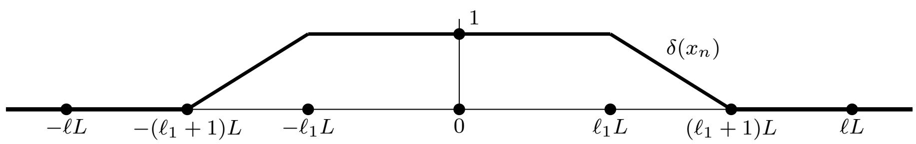
$$\begin{aligned} &\sup_{t\in[0,T]} \|(\boldsymbol{u}_\ell - \boldsymbol{u})(\cdot,t)\|^2_{\boldsymbol{H}^1(\Pi_{\frac{\ell}{2}L})} + \int_0^T |\partial_t(\boldsymbol{u}_\ell - \boldsymbol{u})|^2_{\boldsymbol{L}_2(\Pi_{\frac{\ell}{2}L})}\,dt \\ &+ \int_0^T \left|\left|\nabla(\boldsymbol{u}_\ell - \boldsymbol{u})\right|\right|^2_{\boldsymbol{L}_2(\Pi_{\frac{\ell}{2}L})}\,dt \le Ce^{-\ell\alpha} \end{aligned} \tag{1.56}$$

holds for some constants C*,* α *independent of* ℓ*.*

Proof. Subtracting the integral identity (1.55) from (1.51) we get

$$\begin{aligned} &\int_0^t \int_{\Pi_{\ell L}} \partial_\tau(\mathbf{u}_\ell - \mathbf{u}) \cdot \mathbf{v}\,dxd\tau + \mu \int_0^t \int_{\Pi_{\ell L}} \nabla(\mathbf{u}_\ell - \mathbf{u}) \cdot \nabla \mathbf{v}\,dxd\tau \\ &+ \int_0^t \int_{\Pi_{\ell L}} (\mathbf{U}\cdot\nabla)(\mathbf{u}_\ell - \mathbf{u}) \cdot \mathbf{v}\,dxd\tau = 0 \quad \forall \mathbf{v} \in \mathbf{L}_2(0,T;\widehat{\mathbf{H}}(\Pi_{\ell L})). \end{aligned} \tag{1.57}$$

Note that in the above identity it is not possible to take $\mathbf{w}_\ell = \mathbf{u}_\ell - \mathbf{u}$ as a test function since $\mathbf{u}_\ell - \mathbf{u} \notin \widehat{\mathbf{H}}(\Pi_{\ell L})$. Next, let us introduce the function δ such that



$$\delta(x_n) = \begin{cases} 1 & \text{on } [-\ell_1 L, \ell_1 L], \\ 0 & \text{on } \mathbb{R}\backslash((-\ell_1 - 1)L, (\ell_1 + 1)L), \\ ((\ell_1+1)L - |x_n|)/L & \text{on } ((-\ell_1 - 1)L, -\ell_1 L) \cup (\ell_1 L, (\ell_1+1)L), \end{cases} \tag{1.58}$$

where $0 < \ell_1 \le \ell - 1$. Further we have for almost every t

$$\operatorname{div}(\delta \mathbf{w}_\ell) = \begin{cases} 0 & \text{on } \Pi_{\ell L}\backslash D_{\ell_1}, \\ \partial_{x_n}\delta w_{\ell,n} & \text{on } D_{\ell_1} \end{cases}$$

with $D_{\ell_1} = \Pi_{(\ell_1+1)L}\backslash\Pi_{\ell_1 L}$. Using the flux condition i.e. the fact that $0 = \int_\sigma (u_{\ell,n} - u_n)\,dx' = \int_\sigma w_{\ell,n}\,dx'$ and Remark 1.1.1, we get

$$\begin{aligned} \int_{D^-_{\ell_1}} \partial_{x_n}\delta w_{\ell,n}\,dx &= \int_{-(\ell_1+1)L}^{-\ell_1 L} \int_\sigma \frac{1}{L} w_{\ell,n}\,dx'dx_n = 0, \\ \int_{D^+_{\ell_1}} \partial_{x_n}\delta w_{\ell,n}\,dx &= \int_{\ell_1 L}^{(\ell_1+1)L} \int_\sigma -\frac{1}{L} w_{\ell,n}\,dx'dx_n = 0, \end{aligned}$$

where

$$D^+_{\ell_1} = D_{\ell_1} \cap \{x \in \mathbb{R}^n | x_n > 0\}, \quad D^-_{\ell_1} = D_{\ell_1} \cap \{x \in \mathbb{R}^n | x_n < 0\}.$$

Therefore there exists a function $\boldsymbol{\beta}$ (see [6] and [33]) such that

$$\begin{cases} \operatorname{div}(\boldsymbol{\beta}) = \partial_{x_n}\delta w_{\ell,n} & \text{in } D_{\ell_1}, \\ \left|\left|\nabla\boldsymbol{\beta}\right|\right|_{\mathbf{L}_2(D_{\ell_1})} \le \hat{c}|w_{\ell,n}|_{L_2(D_{\ell_1})} \end{cases} \tag{1.59}$$

with $\boldsymbol{\beta} \in \mathbf{L}_2(0,T;\mathbf{H}_0^1(D_{\ell_1}))$ and the constant $\hat{c}$ depends on L and D. Note $\boldsymbol{\beta}$ is measurable with respect to t. Indeed, let us define $G := \partial_{x_n}\delta w_{\ell,n} = \mp\frac{1}{L}w_{\ell,n}$ in $D_{\ell_1}^{\pm}$. It holds $\int_{D_{\ell_1}^{\pm}} G\,dx = 0$ and $w_{\ell,n} \in L_2(0,T;H^1(\Pi_{\ell L}))$. Let $\{G_m\} \subset \mathcal{D}(D_{\ell_1}^{\pm}\times(0,T))$ be a sequence approximating G in $L_2(0,T;L_2(\Pi_{\ell L}))$ and let us choose

$$G_m^* = G_m - \phi\int_{D_{\ell_1}^{\pm}} G_m\,dx, \quad m \in \mathbb{N}$$

with

$$\phi \in \mathcal{D}(D_{\ell_1}^{\pm}), \quad \int_{D_{\ell_1}^{\pm}} \phi\,dx = 1.$$

Then $\{G_m^*\} \subset \mathcal{D}(D_{\ell_1}^{\pm}\times(0,T))$ still approximates G in $L_2(0,T;L_2(\Pi_{\ell L}))$ and at the same time the condition $\int_{D_{\ell_1}^{\pm}} G_m^*\,dx = 0$ is fulfilled. Therefore we can find a measurable $\boldsymbol{\beta}_m$ (see equation III.3.8 and Lemma III.3.1 in [33]) with

$$\begin{cases} \operatorname{div}(\boldsymbol{\beta}_m) = G_m^* \text{ in } D_{\ell_1}^{\pm} \\ \big||\nabla\boldsymbol{\beta}_m|\big|_{\mathbf{L}_2(D_{\ell_1}^{\pm})} \le \hat{c}|G_m^*|_{L_2(D_{\ell_1}^{\pm})} \end{cases} \tag{1.60}$$

and $\boldsymbol{\beta}_m \in L_2(0,T;H_0^1(D_{\ell_1}^{\pm}))$. Passing to the limit we obtain that $\boldsymbol{\beta}$ is measurable with respect to t. Next, we extend $\boldsymbol{\beta}$ by zero outside D_{ℓ_1}. We take now $\mathbf{v} = \delta\mathbf{w}_\ell - \boldsymbol{\beta} \in \mathbf{L}_2(0,T;\widehat{\mathbf{H}}(\Pi_{\ell L}))$ in (1.57) and obtain

$$\int_0^t\int_{\Pi_{\ell L}} \partial_\tau\mathbf{w}_\ell\cdot(\delta\mathbf{w}_\ell-\boldsymbol{\beta})\,dxd\tau + \mu\int_0^t\int_{\Pi_{\ell L}} \nabla\mathbf{w}_\ell\cdot\nabla(\delta\mathbf{w}_\ell-\boldsymbol{\beta})\,dxd\tau + \int_0^t\int_{\Pi_{\ell L}} (\mathbf{U}\cdot\nabla)\mathbf{w}_\ell\cdot(\delta\mathbf{w}_\ell-\boldsymbol{\beta})\,dxd\tau = 0.$$

Since $\mathbf{w}_\ell = \delta\mathbf{w}_\ell + (1-\delta)\mathbf{w}_\ell$ and $\int_0^t\int_{\Pi_{\ell L}} (\mathbf{U}\cdot\nabla)\delta\mathbf{w}_\ell\cdot\delta\mathbf{w}_\ell\,dxd\tau = 0$ we get

$$\begin{aligned}
&\int_0^t\int_{\Pi_{\ell_1 L}} \partial_\tau\mathbf{w}_\ell\cdot\mathbf{w}_\ell\,dxd\tau + \mu\int_0^t\int_{\Pi_{\ell_1 L}} |\nabla\mathbf{w}_\ell|^2\,dxd\tau = -\int_0^t\int_{D_{\ell_1}} \partial_\tau\mathbf{w}_\ell\cdot(\delta\mathbf{w}_\ell-\boldsymbol{\beta})\,dxd\tau \\
&-\mu\int_0^t\int_{D_{\ell_1}} \nabla\mathbf{w}_\ell\cdot\nabla(\delta\mathbf{w}_\ell-\boldsymbol{\beta})\,dxd\tau - \int_0^t\int_{D_{\ell_1}} (\mathbf{U}\cdot\nabla)(1-\delta)\mathbf{w}_\ell\cdot\delta\mathbf{w}_\ell\,dxd\tau \\
&+\int_0^t\int_{D_{\ell_1}} (\mathbf{U}\cdot\nabla)\mathbf{w}_\ell\cdot\boldsymbol{\beta}\,dxd\tau.
\end{aligned}$$

Using the Young inequality $\mathbf{a}\cdot\mathbf{b} \le \frac{1}{2}|\mathbf{a}|^2 + \frac{1}{2}|\mathbf{b}|^2$ we obtain

$$\begin{aligned}
&\int_0^t\int_{\Pi_{\ell_1 L}} \partial_\tau\mathbf{w}_\ell\cdot\mathbf{w}_\ell\,dxd\tau + \mu\int_0^t\int_{\Pi_{\ell_1 L}} |\nabla\mathbf{w}_\ell|^2\,dxd\tau \le \frac{1}{2}\int_0^t\int_{D_{\ell_1}} |\partial_\tau\mathbf{w}_\ell|^2\,dxd\tau \\
&+\frac{1}{2}\int_0^t\int_{D_{\ell_1}} |\delta\mathbf{w}_\ell-\boldsymbol{\beta}|^2\,dxd\tau + \frac{\mu}{2}\int_0^t\int_{D_{\ell_1}} |\nabla\mathbf{w}_\ell|^2\,dxd\tau + \frac{\mu}{2}\int_0^t\int_{D_{\ell_1}} |\nabla(\delta\mathbf{w}_\ell-\boldsymbol{\beta})|^2\,dxd\tau \\
&+\frac{1}{2}|\mathbf{U}|_\infty^2\int_0^t\int_{D_{\ell_1}} |\nabla(1-\delta)\mathbf{w}_\ell|^2\,dxd\tau + \frac{1}{2}\int_0^t\int_{D_{\ell_1}} |\delta\mathbf{w}_\ell|^2\,dxd\tau + \frac{1}{2}|\mathbf{U}|_\infty^2\int_0^t\int_{D_{\ell_1}} |\nabla\mathbf{w}_\ell|^2\,dxd\tau \\
&+\frac{1}{2}\int_0^t\int_{D_{\ell_1}} |\boldsymbol{\beta}|^2\,dxd\tau
\end{aligned}$$

where we have set $|\mathbf{U}|_\infty = \sup_{t\in[0,T]}\sup_{x\in\Pi_L}|\mathbf{U}|$. Using then the Poincaré inequality on the section of the domain, the definition of δ and $\boldsymbol{\beta}$, we arrive easily to

$$\frac{1}{2}\int_{\Pi_{\ell_1 L}} |\mathbf{w}_\ell|^2\,dx + \mu\int_0^t\int_{\Pi_{\ell_1 L}} |\nabla\mathbf{w}_\ell|^2\,dxd\tau \le \frac{1}{2}\int_0^t\int_{D_{\ell_1}} |\partial_\tau\mathbf{w}_\ell|^2\,dxd\tau + C\int_0^t\int_{D_{\ell_1}} |\nabla\mathbf{w}_\ell|^2\,dxd\tau \tag{1.61}$$

where C is a constant. Next we will find an estimate for $|\partial_\tau \mathbf{w}_\ell|$. Let δ be the function defined in (1.58), then we have that

$$\operatorname{div}(\delta\partial_t \mathbf{w}_\ell) = \begin{cases} 0 & \text{on } \Pi_{\ell L}\backslash D_{\ell_1}, \\ \partial_{x_n}\delta\partial_t w_{\ell,n} & \text{on } D_{\ell_1}. \end{cases}$$

Using the flux condition, it holds

$$\int_{D_{\ell_1}^{\pm}} \partial_{x_n}\delta\partial_t w_{\ell,n}\, dx = \partial_t \int_{D_{\ell_1}^{\pm}} \mp\frac{1}{L} w_{\ell,n}\, dx = 0.$$

Hence there exists a function $\boldsymbol{\gamma}$ (see [6] and [33]) such that

$$\begin{cases} \operatorname{div}(\boldsymbol{\gamma}) = \partial_{x_n}\delta\partial_t w_{\ell,n} & \text{in } D_{\ell_1}, \\ \left|\left|\nabla\boldsymbol{\gamma}\right|\right|_{\mathbf{L}_2(D_{\ell_1})} \leq \hat{c}|\partial_t w_{\ell,n}|_{L_2(D_{\ell_1})} \end{cases} \tag{1.62}$$

with $\boldsymbol{\gamma} \in L_2(0,T; H_0^1(D_{\ell_1}))$ and the constant $\hat{c}$ depends on L and D. We extend $\boldsymbol{\gamma}$ by zero outside D_{ℓ_1}. We take now $\mathbf{v} = \delta\partial_t \mathbf{w}_\ell - \boldsymbol{\gamma} \in \mathbf{L}_2(0,T; \widehat{\mathbf{H}}(\Pi_{\ell L}))$ in (1.57) and obtain

$$\begin{aligned} &\int_0^t \int_{\Pi_{\ell L}} \partial_\tau \mathbf{w}_\ell \cdot (\delta\partial_\tau \mathbf{w}_\ell - \boldsymbol{\gamma})\, dx d\tau + \mu \int_0^t \int_{\Pi_{\ell L}} \nabla\mathbf{w}_\ell \cdot \nabla(\delta\partial_\tau \mathbf{w}_\ell - \boldsymbol{\gamma})\, dx d\tau \\ &+ \int_0^t \int_{\Pi_{\ell L}} (\mathbf{U}\cdot\nabla)\mathbf{w}_\ell \cdot (\delta\partial_\tau \mathbf{w}_\ell - \boldsymbol{\gamma})\, dx d\tau = 0. \end{aligned}$$

This leads to

$$\begin{aligned} &\int_0^t \int_{\Pi_{\ell_1 L}} |\partial_\tau \mathbf{w}_\ell|^2\, dx d\tau + \mu \int_0^t \int_{\Pi_{\ell L}} \nabla\mathbf{w}_\ell \cdot \nabla\partial_\tau(\delta\mathbf{w}_\ell)\, dx d\tau = -\int_0^t \int_{D_{\ell_1}} \partial_\tau \mathbf{w}_\ell \cdot (\delta\partial_\tau \mathbf{w}_\ell - \boldsymbol{\gamma})\, dx d\tau \\ &- \mu \int_0^t \int_{D_{\ell_1}} \nabla\mathbf{w}_\ell \cdot \nabla\boldsymbol{\gamma}\, dx d\tau - \int_0^t \int_{\Pi_{\ell_1 L}} (\mathbf{U}\cdot\nabla)\mathbf{w}_\ell \cdot \partial_\tau \mathbf{w}_\ell\, dx d\tau \\ &- \int_0^t \int_{D_{\ell_1}} (\mathbf{U}\cdot\nabla)\mathbf{w}_\ell \cdot (\delta\partial_\tau \mathbf{w}_\ell - \boldsymbol{\gamma})\, dx d\tau. \end{aligned}$$

Using the equality

$$\int_0^t \int_{\Pi_{\ell L}} \nabla\mathbf{w}_\ell \cdot \nabla\partial_\tau(\delta\mathbf{w}_\ell)\, dx d\tau = \int_0^t \int_{\Pi_{\ell L}} \delta\nabla\mathbf{w}_\ell \cdot \nabla\partial_\tau \mathbf{w}_\ell\, dx d\tau + \int_0^t \int_{D_{\ell_1}} \partial_{x_n}\delta\partial_{x_n}\mathbf{w}_\ell \cdot \partial_\tau \mathbf{w}_\ell\, dx d\tau$$

and the Young inequality we derive

$$\begin{aligned} &\int_0^t \int_{\Pi_{\ell_1 L}} |\partial_\tau \mathbf{w}_\ell|^2\, dx d\tau + \frac{\mu}{2}\int_0^t \partial_\tau \int_{\Pi_{\ell L}} \delta|\nabla\mathbf{w}_\ell|^2\, dx d\tau \leq \frac{1}{2}\int_0^t \int_{D_{\ell_1}} |\partial_{x_n}\delta\partial_{x_n}\mathbf{w}_\ell|^2\, dx d\tau \\ &+ \frac{1}{2}\int_0^t \int_{D_{\ell_1}} |\partial_\tau \mathbf{w}_\ell|^2\, dx d\tau + \frac{1}{2}\int_0^t \int_{D_{\ell_1}} |\partial_\tau \mathbf{w}_\ell|^2\, dx d\tau + \frac{1}{2}\int_0^t \int_{D_{\ell_1}} |\delta\partial_\tau \mathbf{w}_\ell - \boldsymbol{\gamma}|^2\, dx d\tau \\ &+ \frac{\mu}{2}\int_0^t \int_{D_{\ell_1}} |\nabla\mathbf{w}_\ell|^2\, dx d\tau + \frac{\mu}{2}\int_0^t \int_{D_{\ell_1}} |\nabla\boldsymbol{\gamma}|^2\, dx d\tau + \frac{|\mathbf{U}|_\infty^2}{2\epsilon}\int_0^t \int_{\Pi_{\ell_1 L}} |\nabla\mathbf{w}_\ell|^2\, dx d\tau \\ &+ \frac{\epsilon}{2}\int_0^t \int_{\Pi_{\ell_1 L}} |\partial_\tau \mathbf{w}_\ell|^2\, dx d\tau + \frac{|\mathbf{U}|_\infty^2}{2}\int_0^t \int_{D_{\ell_1}} |\nabla\mathbf{w}_\ell|^2\, dx d\tau + \frac{1}{2}\int_0^t \int_{D_{\ell_1}} |\delta\partial_\tau \mathbf{w}_\ell - \boldsymbol{\gamma}|^2\, dx d\tau. \end{aligned}$$

Using the properties of $\boldsymbol{\gamma}$ we get

$$\int_0^t \int_{\Pi_{\ell_1 L}} |\partial_\tau \mathbf{w}_\ell|^2 \, dx d\tau + \frac{\mu}{2} \int_{\Pi_{\ell_1 L}} |\nabla \mathbf{w}_\ell|^2 \, dx \le C \int_0^t \int_{D_{\ell_1}} |\partial_\tau \mathbf{w}_\ell|^2 \, dx d\tau + C \int_0^t \int_{D_{\ell_1}} |\nabla \mathbf{w}_\ell|^2 \, dx d\tau$$
$$+ \frac{|\mathbf{U}|_\infty^2}{2\epsilon} \int_0^t \int_{\Pi_{\ell_1 L}} |\nabla \mathbf{w}_\ell|^2 \, dx d\tau + \frac{\epsilon}{2} \int_0^t \int_{\Pi_{\ell_1 L}} |\partial_\tau \mathbf{w}_\ell|^2 \, dx d\tau.$$

Combining this with (1.61) we obtain for some constant C

$$\frac{1}{2} \int_{\Pi_{\ell_1 L}} |\mathbf{w}_\ell|^2 \, dx + \mu \int_0^t \int_{\Pi_{\ell_1 L}} |\nabla \mathbf{w}_\ell|^2 \, dx d\tau + \int_0^t \int_{\Pi_{\ell_1 L}} |\partial_\tau \mathbf{w}_\ell|^2 \, dx d\tau + \frac{\mu}{2} \int_{\Pi_{\ell_1 L}} |\nabla \mathbf{w}_\ell|^2 \, dx d\tau$$
$$\le C \int_0^t \int_{D_{\ell_1}} |\partial_\tau \mathbf{w}_\ell|^2 \, dx d\tau + C \int_0^t \int_{D_{\ell_1}} |\nabla \mathbf{w}_\ell|^2 \, dx d\tau + \frac{|\mathbf{U}|_\infty^2}{2\epsilon} \int_0^t \int_{\Pi_{\ell_1 L}} |\nabla \mathbf{w}_\ell|^2 \, dx d\tau$$
$$+ \frac{\epsilon}{2} \int_0^t \int_{\Pi_{\ell_1 L}} |\partial_\tau \mathbf{w}_\ell|^2 \, dx d\tau.$$

Since we supposed $|\mathbf{U}|_\infty^2 < 4\mu$ we can choose ϵ such that $\frac{|\mathbf{U}|_\infty^2}{2\mu} < \epsilon < 2$ to get for some constant $c > 0$

$$\frac{1}{2} \int_{\Pi_{\ell_1 L}} |\mathbf{w}_\ell|^2 \, dx + \frac{\mu}{2} \int_{\Pi_{\ell_1 L}} |\nabla \mathbf{w}_\ell|^2 \, dx + c \int_0^t \int_{\Pi_{\ell_1 L}} |\nabla \mathbf{w}_\ell|^2 \, dx d\tau + c \int_0^t \int_{\Pi_{\ell_1 L}} |\partial_\tau \mathbf{w}_\ell|^2 \, dx d\tau$$
$$\le C \int_0^t \int_{D_{\ell_1}} |\partial_\tau \mathbf{w}_\ell|^2 \, dx d\tau + C \int_0^t \int_{D_{\ell_1}} |\nabla \mathbf{w}_\ell|^2 \, dx d\tau.$$

Let us define

$$\begin{aligned} B(\ell_1) = \frac{1}{2} \int_{\Pi_{\ell_1 L}} |\mathbf{w}_\ell|^2 \, dx + \frac{\mu}{2} \int_{\Pi_{\ell_1 L}} |\nabla \mathbf{w}_\ell|^2 \, dx + (c + C) \int_0^t \int_{\Pi_{\ell_1 L}} |\nabla \mathbf{w}_\ell|^2 \, dx d\tau \\ + (c + C) \int_0^t \int_{\Pi_{\ell_1 L}} |\partial_\tau \mathbf{w}_\ell|^2 \, dx d\tau, \end{aligned} \tag{1.63}$$

$$0 < b = \frac{C}{c + C} < 1. \tag{1.64}$$

It is clear that

$$B(\ell_1) \le bB(\ell_1 + 1) \quad \forall \ell_1 \le \ell - 1. \tag{1.65}$$

We denote by [ ] the integer part of numbers. Starting from $\ell_1 = \frac{\ell}{2}$ and iterating this inequality $[\frac{\ell}{2}]$-times we get

$$B(\frac{\ell}{2}) \le b^{[\frac{\ell}{2}]} B(\frac{\ell}{2} + [\frac{\ell}{2}]).$$

Since $\frac{\ell}{2} - 1 < [\frac{\ell}{2}] \le \frac{\ell}{2}$ we have

$$B(\frac{\ell}{2}) \le \frac{1}{b} b^{\frac{\ell}{2}} B(\frac{\ell}{2} + \frac{\ell}{2}) = \frac{1}{b} \exp(-\frac{\ell}{2} \ln(\frac{1}{b})) B(\ell). \tag{1.66}$$

(Since $0 < b < 1$ it holds $\ln(\frac{1}{b}) > 0$.) Next we would like to find an estimate for $B(\ell)$. We know from Theorem 1.2.9 that

$$\sup_{t \in [0,T]} \|\mathbf{u}_\ell\|^2_{\mathbf{H}^1(\Pi_{\ell L})} + \int_0^T |\partial_t \mathbf{u}_\ell|^2_{\mathbf{L}_2(\Pi_{\ell L})} \, dt + \int_0^T \left| |\nabla \mathbf{u}_\ell| \right|^2_{\mathbf{L}_2(\Pi_{\ell L})} \, dt$$
$$\le c(\|F\|^2_{H^1(0,T)} + \int_0^T |\mathbf{f}|^2_{\mathbf{L}_2(\Pi_{\ell L})} \, dt + \|\mathbf{a}\|^2_{\mathbf{H}^1(\Pi_{\ell L})})$$

and we have a similar estimate for $\mathbf{u}$. Thus we obtain easily

$$B(\ell) \leq c(\|F\|^2_{H^1(0,T)} + \int_0^T |\mathbf{f}|^2_{\mathbf{L}_2(\Pi_{\ell L})}\, dt + \|\mathbf{a}\|^2_{\mathbf{H}^1(\Pi_{\ell L})})$$

and therefore due to the periodicity of our data

$$B(\ell) \leq C\ell. \tag{1.67}$$

Going back to (1.66) we derive

$$B(\frac{\ell}{2}) \leq C\ell e^{-\ell\alpha'} \leq Ce^{-\ell\alpha} \tag{1.68}$$

for any $0 < \alpha < \alpha' = \frac{1}{2}\ln(\frac{1}{b})$. Hence

$$\sup_{t\in[0,T]} ||\mathbf{u}_\ell - \mathbf{u}||^2_{\mathbf{H}^1(\Pi_{\frac{\ell}{2}L})} + \int_0^T |\partial_t(\mathbf{u}_\ell - \mathbf{u})|^2_{\mathbf{L}_2(\Pi_{\frac{\ell}{2}L})}\, dt + \int_0^T \big||\nabla(\mathbf{u}_\ell - \mathbf{u})|\big|^2_{\mathbf{L}_2(\Pi_{\frac{\ell}{2}L})}\, dt \leq Ce^{-\ell\alpha} \tag{1.69}$$

which completes the proof. □

Chapter 2

On the asymptotic behaviour of the pure Neumann problem in cylinder-like domains and its applications

We consider in this chapter the pure Neumann problem in n-dimensional cylinder-like domains. We are interested in the asymptotic behaviour of the solution of this kind of problem when the domain becomes infinite in p-directions, $1 \leq p < n$. We show that this solution converges exponentially to the solution of a Neumann problem in the corresponding unbounded domain. We distinguish between the case $p = 1$ and $1 < p < n$ and the latter requiring a more involved analysis. For $p = 1$ we consider also the special situation when the domain and the initial data are periodic. The results of this chapter are a joint work with M. Chipot and can be found in [26].

2.1 Introduction

The asymptotic behaviour of problems set in domains becoming unbounded, especially in long cylinders, has been analysed lately for various partial differential equations in many different papers. To quote a few we refer the reader to [7]-[25] or [39] and the references there. Similar analysis can be carried out when the data are periodic, see e.g. [19]-[20] or [25]. When dealing with Dirichlet boundary conditions the Poincaré inequality plays a crucial role. We can not rely on it in the case of Neumann boundary conditions and the problems become more involved even when we consider linear elliptic equations, see [21].

The goal of this note is to study the pure Neumann problem in domains Ω contained in

$$\mathbb{R}^p \times \omega_2 \subset \mathbb{R}^n$$

when ω_2 is a bounded domain of $\mathbb{R}^{n-p}$. The case where $p = 1$ was investigated in [21], but was restricted to domains of the type

$$\Omega_\ell = (-\ell, \ell) \times \omega_2$$

with $\ell > 0$ and possibly $\ell = +\infty$. In what follows we will consider more general domains when $p = 1$, investigate the case of periodic coefficients and finally consider some issues for $p > 1$.
Let us first set some notation and explain what kind of problems we have in mind. Let

$$A = A(x) = (a_{ij}(x))$$

be a $n \times n$ matrix with entries $a_{ij} \in L_\infty(\mathbb{R}^n)$ for $i, j = 1, \dots, n$ and satisfying the usual ellipticity condition

$$A(x)\xi \cdot \xi \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x, \tag{2.1}$$

and

$$|A(x)\xi| \leq \Lambda |\xi| \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x. \tag{2.2}$$

(Here "$\cdot$" denotes the usual euclidean product in $\mathbb{R}^n$, $|\cdot|$ the euclidean norm, λ and Λ some positive constants.) If Ω is a domain in $\mathbb{R}^n$ (i.e. an open and connected subset), then a solution to the pure Neumann problem associated to A is a function u such that

$$\begin{cases} -\text{div}(A(x)\nabla u) = f & \text{in } \Omega, \\ A(x)\nabla u \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Here f denotes some function or distribution, ν is the outward unit normal to $\partial\Omega$, the boundary of Ω. Note that a solution to (2.3) is defined up to a constant moreover -as we will see later- some condition on f is needed for existence for instance of a weak solution. At that stage we do not precise what we mean by weak solution.
Since we will consider domains contained in $\mathbb{R}^p \times \omega_2$ it will be convenient to write the points in these domains as

$$x = (X_1, X_2)$$

where $X_1 \in \mathbb{R}^p$, $X_2 \in \omega_2 \subset \mathbb{R}^{n-p}$. In the case $p = 1$ we will simply write x_1 for X_1. Similarly a matrix A defined on this type of domains can be decomposed into

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{2.4}$$

where A_{11}, A_{22} are respectively $p \times p$ and $(n-p) \times (n-p)$ matrices, A_{12}, A_{21} respectively $p \times (n-p)$ and $(n-p) \times p$ matrices.

The paper is divided as follows. In Section 2 we consider the case $p = 1$. We prove in particular existence of solution in some unbouned domains contained in $\mathbb{R} \times \omega_2$. Note that our arguments there are slightly different of the ones in [21]. We study also in periodic domains the possibility for the solution to be periodic. In the third section we give some results for the case $p > 1$ imposing some restrictions on the matrix A.

2.2 The case $p = 1$

2.2.1 An existence result

If ω_2 is a bounded domain in $\mathbb{R}^{n-1}$ we denote by Ω a domain in $\mathbb{R}^n$ such that

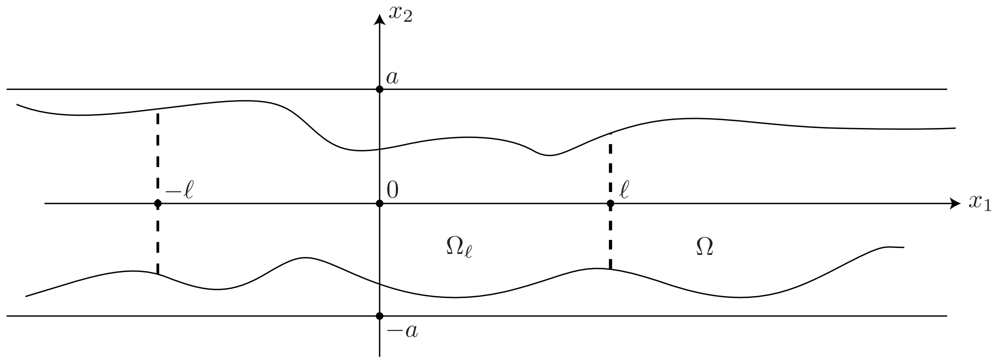
$$\Omega \subset \mathbb{R} \times \omega_2$$

and by Ω_ℓ, $\ell > 0$ the bounded open subset of $\mathbb{R}^n$ defined as

$$\Omega_\ell = ((-\ell, \ell) \times \omega_2) \cap \Omega, \tag{2.5}$$

see the figure below where $n = 2$, $\omega_2 = (-a, a)$, $a \in \mathbb{R}^+$.
We will denote by (x_1, X_2) the points in $\mathbb{R}^n$ and by σ_{x_1} the section of Ω at the level x_1, i.e.

$$\sigma_{x_1} = \{X_2 \mid (x_1, X_2) \in \Omega\}.$$



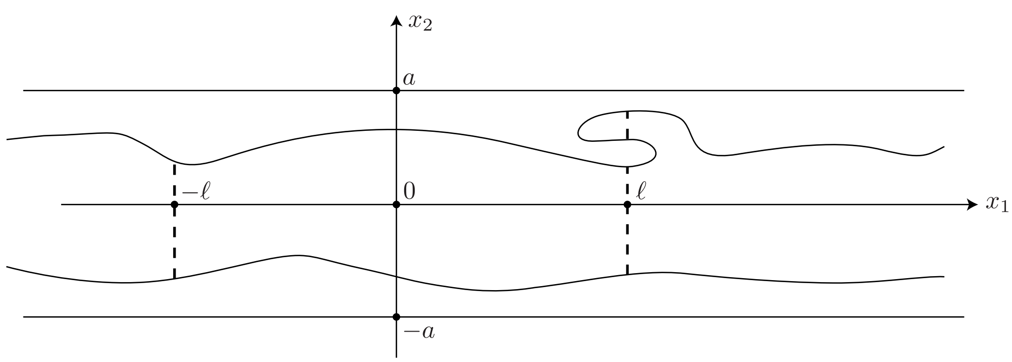
Let us recall the Poincaré-Wirtinger inequality. If D is a bounded domain in $\mathbb{R}^d$ there exists a constant C_D such that (cf. [28], [67])

$$\int_D |u - \overline{u}|^2 \, dx \le C_D^2 \int_D |\nabla u|^2 \, dx \quad \forall u \in H^1(D), \tag{2.6}$$

$\overline{u}$ being the average of u in D that is to say

$$\overline{u} = \frac{1}{|D|} \int_D u \, dx, \tag{2.7}$$

where $|D|$ denotes the measure of D. Note that even if Ω is a domain in $\mathbb{R}^n$, Ω_ℓ is not necessary one. This is clear on the figure below.



Thus we will make the following assumption

$$\begin{aligned} &\text{there exists } \ell_\sigma > 0 \text{ such that } \forall |x_1| \ge \ell_\sigma \\ &\Omega_{|x_1|} \text{ and } \sigma_{x_1} \text{ are domains in } \mathbb{R}^n, \mathbb{R}^{n-1} \text{ and } C_{\sigma_{x_1}} \le \Lambda. \end{aligned} \tag{2.8}$$

($C_{\sigma_{x_1}}$ is the constant in (2.6) corresponding to $D = \sigma_{x_1}$, w.l.o.g. we can take as Λ the same constant as in (2.2).)
Let $f \in L_{2,loc}(\overline{\Omega})$ such that

$$\int_{\sigma_{x_1}} f(x_1, X_2) \, dX_2 = 0 \quad \text{a.e. } x_1 \in \mathbb{R}. \tag{2.9}$$

($L_{2,loc}(\overline{\Omega})$ denotes the space of measurable functions in $L_2(K)$ for any open set $K \subset \Omega$, K bounded in $\mathbb{R}^n$.)
Then one has:

Theorem 2.2.1. *Suppose that A satisfies (2.1), (2.2) for a.e. $x \in \Omega$ and f satisfies (2.9). Under the assumption (2.8) for any $\ell > \ell_\sigma$ there exists u_ℓ (unique up to a constant) solution to*

$$\begin{cases} u_\ell \in H^1(\Omega_\ell), \\ \int_{\Omega_\ell} A\nabla u_\ell \cdot \nabla v \, dx = \int_{\Omega_\ell} fv \, dx \quad \forall v \in H^1(\Omega_\ell). \end{cases} \tag{2.10}$$

Proof. Consider V_ℓ the subspace of $H^1(\Omega_\ell)$ (we refer to [28], [15] for details on the Sobolev spaces used here) defined by

$$V_\ell = \{v \in H^1(\Omega_\ell) \mid \int_{\Omega_\ell} v \, dx = 0\}.$$

Thanks to (2.8), Ω_ℓ is a bounded domain in $\mathbb{R}^n$ (i.e. a bounded connected open subset) and thus by (2.6) one has for some constant C_ℓ

$$\int_{\Omega_\ell} v^2 \, dx \le C_\ell^2 \int_{\Omega_\ell} |\nabla v|^2 \, dx$$

which shows that

$$v \mapsto \Big(\int_{\Omega_\ell} |\nabla v|^2 \, dx\Big)^{1/2}$$

is a norm equivalent to the $H^1(\Omega_\ell)-$norm on V_ℓ. Due to (2.1), (2.2) it follows that

$$(u, v) \to \int_{\Omega_\ell} A\nabla u \cdot \nabla v \, dx$$

is a continuous bilinear coercive form on V_ℓ. It follows from the Lax-Milgram Theorem (cf. Theorem 4.2.2) that there exists a unique u_ℓ solution to

$$\begin{cases} u_\ell \in V_\ell, \\ \int_{\Omega_\ell} A\nabla u_\ell \cdot \nabla v \, dx = \int_{\Omega_\ell} fv \, dx \quad \forall v \in V_\ell. \end{cases}$$

If $v \in H^1(\Omega_\ell)$ one has

$$v = v - \overline{v} + \overline{v} = \tilde{v} + \overline{v} \quad (\tilde{v} = v - \overline{v}, \, \overline{v} = \frac{1}{|\Omega_\ell|} \int_{\Omega_\ell} v \, dx)$$

and thus

$$\int_{\Omega_\ell} A\nabla u_\ell \cdot \nabla v \, dx = \int_{\Omega_\ell} A\nabla u_\ell \cdot \nabla \tilde{v} \, dx = \int_{\Omega_\ell} f\tilde{v} \, dx = \int_{\Omega_\ell} fv \, dx$$

(by (2.9)). This completes the proof of the theorem. □

Our main result in this section is the following.

Theorem 2.2.2. *Suppose that for some $\gamma > 0$*

$$|f|_{L_2(\Omega_\ell)} = O(e^{\gamma\ell}) \tag{2.11}$$

where $|\cdot|_{L_2(A)}$ denotes the usual $L_2(A)-$norm. Under the assumptions of Theorem 2.2.1 for γ small enough, $\ell \ge \ell_\sigma$, u_ℓ is a Cauchy sequence which converges for any ℓ_0 in $H^1(\Omega_{\ell_0})/\mathbb{R}$ towards the unique solution (up to a constant) of the Neumann problem

$$\begin{cases} u_\infty \in H^1_{loc}(\overline{\Omega}), \qquad -\mathrm{div}(A\nabla u_\infty) = f \quad \text{in } \Omega, \\ A\nabla u_\infty \cdot \nu = 0 \quad \text{on } \partial\Omega, \\ \int_{\sigma_{x_1}} A_1 \cdot \nabla u_\infty \, dX_2 = 0 \quad \text{a.e. } x_1 \in \mathbb{R}, \\ \big||\nabla u_\infty|\big|_{L_2(\Omega_\ell)} = O(e^{2\gamma\ell}). \end{cases} \tag{2.12}$$

Moreover for any $\ell \geq 2\ell_\sigma$ one has

$$\big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{L_2(\Omega_{\ell/2})} \leq Ce^{-\alpha\ell} \tag{2.13}$$

for some positive constants C, α.
($H^1_{loc}(\overline{\Omega})$ denotes the space of functions which are in $H^1(K)$ for any open subset $K \subset \Omega$, K bounded, ν denotes the unit outward normal to $\partial\Omega$ and A_1 is the first row of the matrix A.)

Proof. We split the proof in several steps.

(i) A property of u_ℓ.

Taking $v = v(x_1)$ -i.e. v depending on x_1 only- in (2.10) one gets by (2.9)

$$\int_{\Omega_\ell} A_1 \cdot \nabla u_\ell v'(x_1)\, dx_1 dX_2 = \int_{-\ell}^{\ell}\int_{\sigma_{x_1}} f(x_1, X_2) v(x_1)\, dX_2 dx_1 = 0$$

i.e.

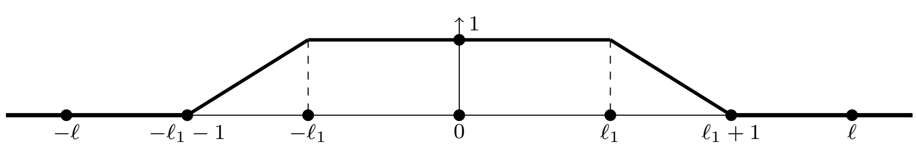
$$\int_{-\ell}^{\ell} v'(x_1) \int_{\sigma_{x_1}} A_1 \cdot \nabla u_\ell\, dX_2 dx_1 = 0 \quad \forall v = v(x_1).$$

It follows that $\int_{\sigma_{x_1}} A_1 \cdot \nabla u_\ell\, dX_2 = \text{cst}$ and choosing $v(x_1) = x_1$ one sees that this constant has to vanish, i.e. one has

$$\int_{\sigma_{x_1}} A_1 \cdot \nabla u_\ell\, dX_2 = 0 \quad \text{a.e. } x_1 \in (-\ell, \ell). \tag{2.14}$$

(ii) Estimate of $u_\ell - u_{\ell+r}$ for $r \in [0, 1]$.

For $\ell_\sigma \leq \ell_1 \leq \ell - 1$ we consider the function $\rho = \rho_{\ell_1}(x_1)$ depicted in the figure below.



Since $(u_\ell - u_{\ell+r})\rho \in H^1(\Omega_\ell)$ and $H^1(\Omega_{\ell+r})$, from the equations satisfied by u_ℓ and $u_{\ell+r}$, one derives easily

$$\int_{\Omega_{\ell_1+1}} A\nabla(u_\ell - u_{\ell+r}) \cdot \nabla\{(u_\ell - u_{\ell+r})\rho\}\, dx = 0. \tag{2.15}$$

This can also be written as

$$\int_{\Omega_{\ell_1+1}} A\nabla(u_\ell - u_{\ell+r}) \cdot \nabla(u_\ell - u_{\ell+r})\rho\, dx = -\int_{D_{\ell_1}} A_1 \cdot \nabla(u_\ell - u_{\ell+r})\rho'(u_\ell - u_{\ell+r})\, dx$$

where $D_{\ell_1} = \Omega_{\ell_1+1} \backslash \Omega_{\ell_1}$ -recall that $\rho = 1$ on Ω_{ℓ_1}. We argue then, for instance, on $D^+_{\ell_1} = D_{\ell_1} \cap \{x_1 > 0\}$. One has (see the graph of ρ)

$$\begin{aligned}
I := &-\int_{D^+_{\ell_1}} A_1 \cdot \nabla(u_\ell - u_{\ell+r})\rho'(u_\ell - u_{\ell+r})\, dx \\
&= \int_{D^+_{\ell_1}} A_1 \cdot \nabla(u_\ell - u_{\ell+r})(u_\ell - u_{\ell+r})\, dx \\
&= \int_{\ell_1}^{\ell_1+1}\int_{\sigma_{x_1}} A_1 \cdot \nabla(u_\ell - u_{\ell+r})(u_\ell - u_{\ell+r} - \fint_{\sigma_{x_1}} u_\ell - u_{\ell+r})\, dX_2 dx_1
\end{aligned} \tag{2.16}$$

(by (2.14)) with

$$\fint_{\sigma_{x_1}} u_\ell - u_{\ell+r} = \frac{1}{|\sigma_{x_1}|}\int_{\sigma_{x_1}} (u_\ell - u_{\ell+r})\, dX_2.$$

Since A is assumed to be bounded one has for some constant a_1

$$|A_1| \le a_1.$$

It follows then from (2.16) and the Cauchy-Young inequality that one has

$$\begin{aligned} I &\le a_1 \int_{\ell_1}^{\ell_1+1}\int_{\sigma_{x_1}} |\nabla(u_\ell - u_{\ell+r})||u_\ell - u_{\ell+r} - \fint_{\sigma_{x_1}} u_\ell - u_{\ell+r}|\, dX_2 dx_1 \\ &\le \frac{a_1}{2}\int_{\ell_1}^{\ell_1+1}\int_{\sigma_{x_1}} |\nabla(u_\ell - u_{\ell+r})|^2 + |u_\ell - u_{\ell+r} - \fint_{\sigma_{x_1}} u_\ell - u_{\ell+r}|^2\, dX_2 dx_1. \end{aligned}$$

Using (2.8) since $\ell_1 \ge \ell_\sigma$, one derives

$$\begin{aligned} I &\le \frac{a_1}{2}\int_{\ell_1}^{\ell_1+1}\{\int_{\sigma_{x_1}} |\nabla(u_\ell - u_{\ell+r})|^2\, dX_2 + \Lambda^2 \int_{\sigma_{x_1}} |\nabla(u_\ell - u_{\ell+r})|^2\, dX_2\}\, dx_1 \\ &\le a_1(1\vee\Lambda^2)\int_{D_{\ell_1}^+} |\nabla(u_\ell - u_{\ell+r})|^2\, dx \end{aligned}$$

where $1\vee\Lambda^2$ denotes the maximum of 1 and Λ^2. Arguing the same way on $D_{\ell_1}^- = D_{\ell_1}\cap\{x_1 < 0\}$ one obtains

$$\int_{\Omega_{\ell_1+1}} A\nabla(u_\ell - u_{\ell+r})\cdot\nabla(u_\ell - u_{\ell+r})\rho\, dx \le \delta\int_{D_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2\, dx$$

with $\delta = a_1(1\vee\Lambda^2)$. Using (2.1) this leads to

$$\lambda\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2\, dx \le \delta\int_{\Omega_{\ell_1+1}\setminus\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2\, dx$$

and to

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2\, dx \le \frac{\delta}{\delta+\lambda}\int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_{\ell+r})|^2\, dx.$$

Starting from $\ell_1 = \frac{\ell}{2} \ge \ell_\sigma$ and iterating this inequality $[\frac{\ell}{2}]-$times, where $[\frac{\ell}{2}]$ denotes the integer part of $\frac{\ell}{2}$ leads to (recall that $\frac{\ell}{2} - 1 < [\frac{\ell}{2}] \le \frac{\ell}{2}$)

$$\begin{aligned} \int_{\Omega_{\ell/2}} |\nabla(u_\ell - u_{\ell+r})|^2\, dx &\le (\frac{\delta}{\delta+\lambda})^{[\frac{\ell}{2}]}\int_{\frac{\ell}{2}+[\frac{\ell}{2}]} |\nabla(u_\ell - u_{\ell+r})|^2\, dx \\ &\le (\frac{\delta+\lambda}{\delta})(\frac{\delta}{\delta+\lambda})^{\frac{\ell}{2}}\int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2\, dx \\ &\le 2(\frac{\delta+\lambda}{\delta})e^{-\frac{\ell}{2}\ln(\frac{\delta+\lambda}{\delta})}\{\int_{\Omega_\ell} |\nabla u_\ell|^2\, dx + \int_{\Omega_{\ell+r}} |\nabla u_{\ell+r}|^2\, dx\} \\ &= Ce^{-\alpha'\ell}\{\int_{\Omega_\ell} |\nabla u_\ell|^2\, dx + \int_{\Omega_{\ell+r}} |\nabla u_{\ell+r}|^2\, dx\} \end{aligned} \tag{2.17}$$

with $C = 2(\frac{\delta+\lambda}{\delta})$, $\alpha' = \frac{1}{2}\ln(\frac{\delta+\lambda}{\delta})$. It remains to estimate these two integrals. Taking $v = u_\ell$ in (2.10) we get for $\ell \geq \ell_\sigma$

$$\begin{aligned}\lambda\int_{\Omega_\ell} |\nabla u_\ell|^2\,dx \leq \int_{\Omega_\ell} A\nabla u_\ell\cdot\nabla u_\ell\,dx = \int_{\Omega_\ell} fu_\ell\,dx\\ = \int_{\Omega_{\ell_\sigma}} fu_\ell\,dx + \int_{\Omega_\ell\setminus\Omega_{\ell_\sigma}} fu_\ell\,dx\\ = \int_{\Omega_{\ell_\sigma}} f(u_\ell-\overline{u}_\ell)\,dx + \int_{\Omega_\ell\setminus\Omega_{\ell_\sigma}} f(u_\ell-\overline{\overline{u}}_\ell)\,dx\end{aligned}$$

(by (2.9)). We have set

$$\overline{u}_\ell = ⨍_{\Omega_{\ell_\sigma}} u_\ell\,dx, \qquad \overline{\overline{u}}_\ell = ⨍_{\sigma_{x_1}} u_\ell\,dX_2 \quad \text{a.e. } x_1.$$

(Recall that $⨍_D = \frac{1}{|D|}\int$.) Using the Cauchy-Schwarz inequality it comes

$$\begin{aligned}\lambda\int_{\Omega_\ell}|\nabla u_\ell|^2\,dx &\leq |f|_{L_2(\Omega_{\ell_\sigma})}|u_\ell-\overline{u}_\ell|_{L_2(\Omega_{\ell_\sigma})} + \int_{|x_1|>\ell_\sigma}\int_{\sigma_{x_1}}|f||u_\ell-\overline{\overline{u}}_\ell|\,dX_2dx_1\\ &\leq C_{\Omega_{\ell_\sigma}}|f|_{L_2(\Omega_{\ell_\sigma})}\big||\nabla u_\ell|\big|_{L_2(\Omega_{\ell_\sigma})} + \int_{|x_1|>\ell_\sigma}|f|_{L_2(\sigma_{x_1})}|u_\ell-\overline{\overline{u}}_\ell|_{L_2(\sigma_{x_1})}\,dx\\ &\leq C_{\Omega_{\ell_\sigma}}|f|_{L_2(\Omega_{\ell_\sigma})}\big||\nabla u_\ell|\big|_{L_2(\Omega_{\ell_\sigma})} + \Lambda|f|_{L_2(\Omega_\ell\setminus\Omega_{\ell_\sigma})}\big||\nabla u_\ell|\big|_{L_2(\Omega_\ell\setminus\Omega_{\ell_\sigma})}\\ &\leq C|f|_{L_2(\Omega_\ell)}\big||\nabla u_\ell|\big|_{L_2(\Omega_\ell)}\end{aligned}$$

with for instance $C = C_{\Omega_{\ell_\sigma}} + \Lambda$. Thus we have -see (2.11)

$$\big||\nabla u_\ell|\big|_{L_2(\Omega_\ell)} \leq \frac{C}{\lambda}|f|_{L_2(\Omega_\ell)} \leq C'e^{\gamma\ell}. \tag{2.18}$$

Going back to (2.17) we have for some constant C''

$$\begin{aligned}\int_{\Omega_{\ell/2}}|\nabla(u_\ell-u_{\ell+r})|^2\,dx &\leq C''e^{-\alpha'\ell}\{e^{2\gamma\ell}+e^{2\gamma(\ell+r)}\}\\ &\leq C''e^{-(\alpha'-2\gamma)\ell}\{1+e^{2\gamma r}\}.\end{aligned}$$

Choosing $2\gamma < \alpha'$ and setting $2\alpha = \alpha' - 2\gamma$ one obtains for some constant C -recall that $r \in [0,1]$- and $\ell \geq 2\ell_\sigma$

$$\int_{\Omega_{\ell/2}}|\nabla(u_\ell-u_{\ell+r})|^2\,dx \leq C^2e^{-2\alpha\ell}. \tag{2.19}$$

(iii) Estimate of $u_\ell - u_{\ell+t}$, $t > 0$.

Let us simplify our notation by setting

$$|u|_\ell = (\int_{\Omega_\ell}|\nabla u|^2\,dx)^{1/2}$$

and remark that for any function u in $H^1(\Omega_{\ell+s})$, where $s \geq 0$,

$$|u|_\ell \leq |u|_{\ell+s}.$$

The inequality (2.19) can be written as

$$|u_\ell - u_{\ell+r}|_{\ell/2} \leq Ce^{-\alpha\ell} \quad \forall r\in[0,1],\ \ell\geq 2\ell_\sigma.$$

Choose then $\ell \geq 2\ell_\sigma$ and t arbitrary one has by the triangular inequality

$$\begin{aligned}
|u_\ell - u_{\ell+t}|_{\ell/2} &\leq |u_\ell - u_{\ell+1}|_{\ell/2} + |u_{\ell+1} - u_{\ell+2}|_{\ell/2} + \cdots + |u_{\ell+[t]} - u_{\ell+t}|_{\ell/2} \\
&\leq |u_\ell - u_{\ell+1}|_{\ell/2} + |u_{\ell+1} - u_{\ell+2}|_{\frac{\ell+1}{2}} + \cdots + |u_{\ell+[t]} - u_{\ell+t}|_{\frac{\ell+[t]}{2}} \\
&\leq Ce^{-\alpha\ell} + Ce^{-\alpha(\ell+1)} + \cdots + Ce^{-\alpha(\ell+[t])} \\
&\leq Ce^{-\alpha\ell}\frac{1}{1-e^{-\alpha}}.
\end{aligned}$$

We have thus arrived, for some constant C independent of ℓ, to

$$\int_{\Omega_{\ell/2}} |\nabla(u_\ell - u_{\ell+t})|^2 \, dx \leq C^2 e^{-2\alpha\ell} \quad \forall t \geq 0, \, \ell \geq 2\ell_\sigma. \tag{2.20}$$

(iv) Construction of u_∞.

We construct u_∞ inductively in $\Omega_{\ell_\sigma+k}$. First we notice that for any $\ell \geq \ell_\sigma$ the norms

$$(\int_{\Omega_\ell} |\nabla u|^2 \, dx)^{1/2} \quad \text{and} \quad \inf_C |u - C|_{L_2(\Omega_\ell)} + (\int_{\Omega_\ell} |\nabla u|^2 \, dx)^{1/2}$$

are equivalent on $H^1(\Omega_\ell)/\mathbb{R}$ (cf. (2.6)).
For $k = 0$ we denote by u_∞ the element of $H^1(\Omega_{\ell_\sigma})$ such that

$$u_\ell \to u_\infty \quad \text{in } H^1(\Omega_{\ell_\sigma})/\mathbb{R}, \int_{\Omega_{\ell_\sigma}} u_\infty \, dx = 0. \tag{2.21}$$

(We identify elements of $H^1(\Omega_{\ell_\sigma})/\mathbb{R}$ with their representatives. Since u_ℓ is a Cauchy sequence in $H^1(\Omega_{\ell_\sigma})/\mathbb{R}$ it is always possible to find such a u_∞ which is uniquely defined.)
Suppose u_∞ defined in $\Omega_{\ell_\sigma+k}$. Since u_ℓ is a Cauchy sequence in $H^1(\Omega_{\ell_\sigma+k+1})/\mathbb{R}$ there exists $\tilde{u}_\infty$ such that

$$u_\ell \to \tilde{u}_\infty \quad \text{in } H^1(\Omega_{\ell_\sigma+k+1})/\mathbb{R}.$$

We can always choose the representative of $\tilde{u}_\infty$ such that

$$\int_{\Omega_{\ell_\sigma}} \tilde{u}_\infty \, dx = 0. \tag{2.22}$$

Then on $\Omega_{\ell_\sigma+k}$ one has $\nabla u_\ell \to \nabla u_\infty, \nabla\tilde{u}_\infty$ in $L_2(\Omega_{\ell_\sigma+k})$. This implies that $\nabla u_\infty = \nabla\tilde{u}_\infty$ and $\tilde{u}_\infty = u_\infty + \text{cst}$. Now due to (2.21), (2.22) this constant vanishes and $\tilde{u}_\infty$ is an extension of u_∞ on $\Omega_{\ell_\sigma+k+1}$.

(v) u_∞ is solution to (2.12).

Indeed, for any $\ell_0 > 0$ let us set

$$W_{\ell_0} = \{v \in H^1(\Omega_{\ell_0}) \mid v = 0 \text{ on } \sigma_{\ell_0} \text{ and } \sigma_{-\ell_0}\}.$$

For any $v \in W_{\ell_0}$ (extended by 0 outside Ω_{ℓ_0}) one has by definition of u_ℓ

$$\int_{\Omega_{\ell_0}} A\nabla u_\ell \cdot \nabla v \, dx = \int_{\Omega_{\ell_0}} fv \, dx.$$

Passing to the limit in ℓ, noting that $\nabla u_\ell \to \nabla u_\infty$ in $L_2(\Omega_{\ell_0})$ we get

$$\int_{\Omega_{\ell_0}} A\nabla u_\infty \cdot \nabla v \, dx = \int_{\Omega_{\ell_0}} fv \, dx \quad \forall v \in W_{\ell_0} \tag{2.23}$$

and the two first equations of (2.12) are satisfied in the weak sense.

We know -see (2.14)- that

$$\int_{\sigma_{x_1}} A_1 \cdot \nabla u_\ell \, dX_2 = 0 \quad \text{a.e. } x_1 \in (-\ell, \ell). \tag{2.24}$$

Since $\nabla u_\ell \to \nabla u_\infty$ in $L_2(\Omega_{\ell_0})$, $\forall \ell_0$, for almost every section one has

$$\nabla u_\ell \to \nabla u_\infty \quad \text{in } L_2(\sigma_{x_1}).$$

Passing to the limit in (2.24) leads to the third equality of (2.12).
Passing now to the limit in t in (2.20) we obtain for any $\ell > 2\ell_\sigma$

$$\big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{L_2(\Omega_{\ell/2})} \le Ce^{-\alpha\ell}$$

i.e. (2.13). This leads also to -using (2.18)-

$$\begin{aligned}\big|\big|\nabla u_\infty\big|\big|_{L_2(\Omega_{\ell/2})} &\le Ce^{-\alpha\ell} + \big|\big|\nabla u_\ell\big|\big|_{L_2(\Omega_{\ell/2})} \\ &\le Ce^{-\alpha\ell} + \big|\big|\nabla u_\ell\big|\big|_{L_2(\Omega_\ell)} \\ &\le Ce^{-\alpha\ell} + C'e^{\gamma\ell} = O(e^{\gamma\ell})\end{aligned}$$

which finishes to show that u_∞ satisfies (2.12).

(vi) u_∞ is unique (up to a constant).

Let u_∞, u'_∞ be two solutions of (2.12). From (2.12) (see for instance (2.23)) (2.15) is valid with u_ℓ, $u_{\ell+r}$ replaced respectively by u_∞, u'_∞ -i.e. one has

$$\int_{\Omega_{\ell_1+1}} A\nabla(u_\infty - u'_\infty) \cdot \nabla\{(u_\infty - u'_\infty)\rho\} \, dx = 0.$$

Then following step (ii) one arrives to

$$\int_{\Omega_{\ell_1}} |\nabla(u_\infty - u'_\infty)|^2 \, dx \le \frac{\delta}{\delta + \lambda} \int_{\Omega_{\ell_1+1}} |\nabla(u_\infty - u'_\infty)|^2 \, dx.$$

Then, for every $k \in \mathbb{N}$, $\alpha' = \frac{1}{2} \ln \frac{\delta+\lambda}{\delta}$ one deduces

$$\begin{aligned}\int_{\Omega_{\ell_1}} |\nabla(u_\infty - u'_\infty)|^2 \, dx &\le 2e^{-2\alpha' k} \int_{\Omega_{\ell_1+k}} |\nabla u_\infty|^2 + |\nabla u'_\infty|^2 \, dx \\ &\le 2e^{-2\alpha' k} e^{4\gamma(\ell_1+k)} \\ &= Ce^{-(2\alpha'-4\gamma)k} e^{4\gamma\ell_1}\end{aligned}$$

by (2.12). Since we choose $2\gamma < \alpha'$ one derives letting $k \to \infty$ that

$$\nabla u_\infty = \nabla u'_\infty \quad \text{a.e. in } \Omega$$

and thus $u_\infty = u'_\infty$ up to a constant since Ω_ℓ is connected for $\ell \ge \ell_\sigma$.
This completes the proof of the theorem. □

Remark 2.2.3. *In the particular case where $n = 2$, (2.8) reduces to*

$$\sigma_{x_1} \textit{ is a non empty bounded interval } \forall x_1, |x_1| \ge \ell_\sigma. \tag{2.25}$$

Indeed if $\omega_2 \subset (-a,a)$ and I an interval contained in ω_2 one has $H^1(I) \subset C^{1/2}(I)$ where $C^{1/2}(I)$ denotes the space of Hölder continuous functions of exponent $1/2$ (see [28]). Thus

$$\overline{u} = \frac{1}{|I|}\int_I u(x)\,dx = u(y)$$

for some $y \in I$. It follows that for every $x \in I$ one has

$$|u(x)-u(y)| = |\int_x^y u'(t)\,dt| \le |y-x|^{\frac12}(\int_x^y u'(t)^2\,dt)^{\frac12}$$

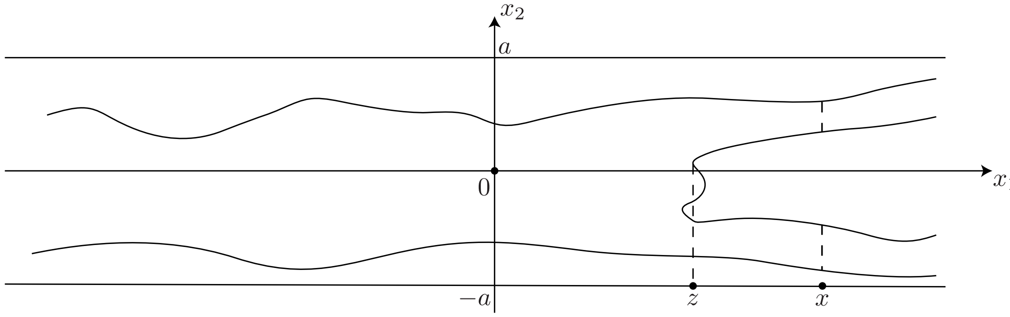
by the Cauchy-Schwarz inequality. Squaring the above inequality, we get if $|I|$ denotes the length of I,

$$(u(x)-\overline{u})^2 \le |I|\int_I (u'(t))^2\,dt$$

and by integrating in x

$$\begin{aligned}\int_I (u(x)-\overline{u})^2\,dx &\le |I|^2\int_I (u'(t))^2\,dt\\ &\le (2a)^2\int_I (u'(t))^2\,dt.\end{aligned}$$

This shows (2.8) with $\Lambda = 2a$.
In the case $n=2$ and if σ_{x_1} is a reunion of intervals -see the figure below-



one can replace (2.9) with

$$\int_{C_{x_1}} f(x_1,X_2)\,dX_2 = 0 \tag{2.26}$$

for every connected component C_{x_1} of σ_{x_1}. Then one can see easily that (2.14) is replaced by

$$\int_{C_{x_1}} A_1\cdot\nabla u_\ell\,dX_2 = 0, \quad \text{a.e. } x_1 \in (-\ell,\ell). \tag{2.27}$$

Indeed suppose that we argue for the branch of the top in the figure above. Taking a

$$v = v(x_1) \in \mathcal{D}(z,+\infty)$$

restricted to this branch in (2.10) leads to

$$\int_{C_{x_1}} A_1\cdot\nabla u_\ell\,dX_2 = \text{ cst} \qquad \text{a.e. } x_1 > z.$$

Then taking $v = (x_1 - z)^+$ restricted to the branch leads to (2.27). Then one can follow the steps of the proof of the Theorem 2.2.2 to reach with straightforward modifications the same result, the third condition of (2.12) being replaced by

$$\int_{C_{x_1}} A_1 \cdot \nabla u_\infty \, dX_2 = 0 \quad \text{a.e. } x_1 \in \mathbb{R}$$

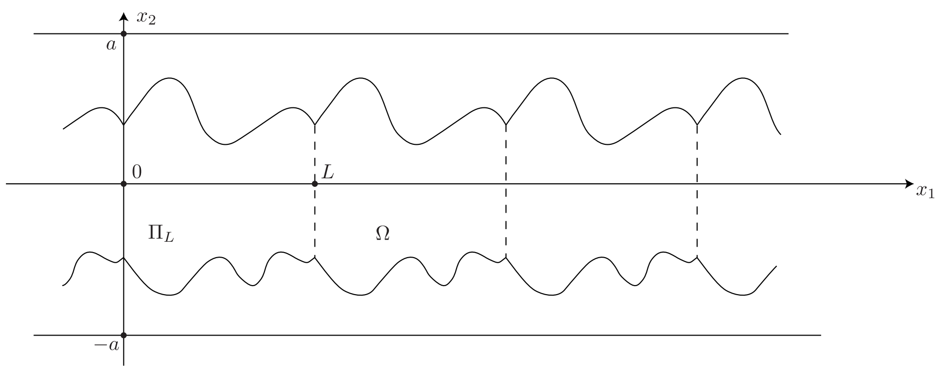
for any connected component C_{x_1} of σ_{x_1}.

Remark 2.2.4. *In the case where $n > 2$ one has (cf. [28], [36], [67]) if σ_{x_1} is convex*

$$C_{\sigma_{x_1}} \leq (d_{\sigma_{x_1}})^{n-1} (\frac{\omega_{n-1}}{|\sigma_{x_1}|})^{1-\frac{1}{n-1}}$$

where $d_{\sigma_{x_1}}$ denotes the diameter of σ_{x_1}, $|\sigma_{x_1}|$ the measure of σ_{x_1}, ω_{n-1} the volume of the unit ball in $\mathbb{R}^{n-1}$. In particular $C_{\sigma_{x_1}}$ is bounded when $(d_{\sigma_{x_1}})^{n-1}(\frac{1}{|\sigma_{x_1}|})^{1-\frac{1}{n-1}}$ is.

2.2.2 The periodic case



We suppose in this section that our data are periodic with period L in the x_1 direction. For the domain Ω this means that

$$(x_1, X_2) \in \Omega \quad \Leftrightarrow \quad (x_1 + zL, X_2) \in \Omega \quad \forall z \in \mathbb{Z} \tag{2.28}$$

and for A, f we have

$$\left.\begin{aligned} A(x_1 + zL, X_2) = A(x_1, X_2) \\ f(x_1 + zL, X_2) = f(x_1, X_2) \end{aligned}\right\} \quad \forall z \in \mathbb{Z}, \text{ a.e. } (x_1, X_2). \tag{2.29}$$

In this situation one would like to know if these periodic data force the solution to (2.12) to be periodic. Note the similarity with the case in [21] where one wanted to know if the solution to (2.12) in a strip is independent of x_1 -i.e. periodic with any period- when the data were themselves independent of x_1.
We set

$$\Pi_L = ((0, L) \times \omega_2) \cap \Omega \tag{2.30}$$

(see the picture above for the case $n = 2$) and we suppose that Π_L is a domain in $\mathbb{R}^n$. Let us denote by A^1 the first column of A and by $H^1_{per}(\Pi_L)$ the space defined as

$$H^1_{per}(\Pi_L) = \{v \in H^1(\Pi_L) \mid v(0, X_2) = v(L, X_2) \text{ a.e. } X_2 \in \sigma_0\}. \tag{2.31}$$

Then we have:

Lemma 2.2.5. *There exists a pair* $(\tilde{u}_\infty, k) \in H^1_{per}(\Pi_L) \times \mathbb{R}$ *weak solution to*

$$\begin{cases} -\mathrm{div}(A\nabla\tilde{u}_\infty) = f + k\mathrm{div}(A^1) & \text{in } \Pi_L, \\ A\nabla\tilde{u}_\infty \cdot \nu = -kA^1 \cdot \nu & \text{on } \partial\Pi_L \cap \partial\Omega, \\ \int_{\sigma_{x_1}} A_1 \cdot \nabla\tilde{u}_\infty \, dX_2 = -k \int_{\sigma_{x_1}} a_{11} \, dX_2 & \text{a.e. } x_1 \in (0, L). \end{cases} \tag{2.32}$$

(Note that $\tilde{u}_\infty$ *is defined up to a constant.)*

Proof. By a weak solution we mean a pair $(\tilde{u}_\infty, k) \in H^1_{per}(\Pi_L) \times \mathbb{R}$ such that

$$\begin{cases} \int_{\Pi_L} A\nabla\tilde{u}_\infty \cdot \nabla v \, dx = \int_{\Pi_L} fv \, dx - k \int_{\Pi_L} A^1 \cdot \nabla v \, dx & \forall v \in H^1_{per}(\Pi_L), \\ \int_{\sigma_{x_1}} A_1 \cdot \nabla\tilde{u}_\infty \, dX_2 = -k \int_{\sigma_{x_1}} a_{11} \, dX_2 & \text{a.e. } x_1 \in (0, L). \end{cases} \tag{2.33}$$

Arguing as in Theorem 2.2.1 one can show that there exist u_f and u_A solution to

$$\begin{cases} u_f \in H^1_{per}(\Pi_L), \\ \int_{\Pi_L} A\nabla u_f \cdot \nabla v \, dx = \int_{\Pi_L} fv \, dx \quad \forall v \in H^1_{per}(\Pi_L), \end{cases} \tag{2.34}$$

$$\begin{cases} u_A \in H^1_{per}(\Pi_L), \\ \int_{\Pi_L} A\nabla u_A \cdot \nabla v \, dx = \int_{\Pi_L} A^1 \cdot \nabla v \, dx \quad \forall v \in H^1_{per}(\Pi_L). \end{cases} \tag{2.35}$$

(Note that u_f, u_A are defined up to a constant.) Then, clearly

$$\tilde{u}_\infty = u_f - ku_A$$

satisfies the first equation of (2.33). Now taking $v = v(x_1) \in \mathcal{D}(0, L)$ in (2.34), (2.35) leads to

$$\int_0^L v'(x_1) \int_{\sigma_{x_1}} A_1 \cdot \nabla u_f \, dX_2 dx_1 = 0, \quad \int_0^L v'(x_1) \int_{\sigma_{x_1}} A_1 \cdot \nabla u_A - a_{11} \, dX_2 dx_1 = 0.$$

Since this holds for any v one has

$$\int_{\sigma_{x_1}} A_1 \cdot \nabla u_f \, dX_2 = \text{ cst } = C_1, \int_{\sigma_{x_1}} A_1 \cdot \nabla u_A - a_{11} \, dX_2 = \text{ cst } = C_2 \text{ a.e. } x_1 \in (0, L). \tag{2.36}$$

We claim that $C_2 \neq 0$. Indeed first note that taking $v = u_A$ in (2.35) we get

$$\int_{\Pi_L} A\nabla u_A \cdot \nabla u_A - A^1 \cdot \nabla u_A \, dx = 0.$$

If C_2 vanishes we derive that

$$\int_{\Pi_L} A\nabla u_A \cdot \nabla u_A - A^1 \cdot \nabla u_A - A_1 \cdot \nabla u_A + a_{11} \, dx = 0$$

$$\Leftrightarrow \int_{\Pi_L} A\nabla(-x_1 + u_A) \cdot \nabla(-x_1 + u_A) \, dx = 0.$$

Due to (2.1) this would imply

$$-x_1 + u_A = \text{ cst}$$

and a contradiction with $u_A \in H^1_{per}(\Pi_L)$. Now $(\tilde{u}_\infty = u_f - ku_A, k)$ satisfies the third equation of (2.32) provided

$$\int_{\sigma_{x_1}} A_1 \cdot \nabla(u_f - ku_A) \, dX_2 = -k \int_{\sigma_{x_1}} a_{11} \, dX_2$$

i.e.

$$k = \frac{\int_{\sigma_{x_1}} A_1 \cdot \nabla u_f \, dX_2}{\int_{\sigma_{x_1}} A_1 \cdot \nabla u_A - a_{11} \, dX_2} = \frac{C_1}{C_2}. \tag{2.37}$$

This completes the proof of the lemma. □

We can now show:

Theorem 2.2.6. *Under the assumptions above, if u_∞ is the solution to (2.12) and if $\tilde{u}_\infty$ is supposed to be extended by periodicity to Ω, one has*

$$u_\infty = \tilde{u}_\infty + kx_1 \tag{2.38}$$

where k is given by (2.37).

Proof. Let $\ell_0 > 0$ be fixed and

$$v \in W_{\ell_0} = \{v \in H^1(\Omega_{\ell_0}) \mid v = 0 \text{ on } \sigma_{\pm \ell_0}\}.$$

We suppose that v is extended by 0 outside Ω_{ℓ_0} and we set for $z \in \mathbb{Z}$

$$\Pi_L + zL = \{(x_1 + zL, X_2) \mid (x_1, X_2) \in \Pi_L\}.$$

Then, if u_∞ is given by (2.38) one has

$$\begin{aligned}\int_{\Omega_{\ell_0}} A\nabla u_\infty \cdot \nabla v \, dx &= \sum_{z\in\mathbb{Z}} \int_{\Pi_L + zL} A\nabla \tilde{u}_\infty \cdot \nabla v \, dx + \int_{\Omega_{\ell_0}} kA^1 \cdot \nabla v \, dx \\ &= \sum_{z\in\mathbb{Z}} \int_{\Pi_L} A(x_1 + zL, X_2)\nabla \tilde{u}_\infty(x_1 + zL, X_2) \cdot \nabla v(x_1 + zL, X_2) \, dx + \int_{\Omega_{\ell_0}} kA^1 \cdot \nabla v \, dx.\end{aligned}$$

Using the periodicity of A and $\tilde{u}_\infty$ it comes

$$\int_{\Omega_{\ell_0}} A\nabla u_\infty \cdot \nabla v \, dx = \int_{\Pi_L} A\nabla \tilde{u}_\infty \cdot \nabla \sum_{z\in\mathbb{Z}} v(x_1 + zL, X_2) \, dx + \int_{\Omega_{\ell_0}} kA^1 \cdot \nabla v \, dx.$$

(The sum above is finite since v has a compact support.) Using (2.33) we derive

$$\begin{aligned}\int_{\Omega_{\ell_0}} A\nabla u_\infty \cdot \nabla v \, dx &= \int_{\Pi_L} f \sum_{z\in\mathbb{Z}} v(x_1 + zL, X_2) \, dx \\ &- k \int_{\Pi_L} A^1 \cdot \nabla \sum_{z\in\mathbb{Z}} v(x_1 + zL, X_2) \, dx + \int_{\Omega_{\ell_0}} kA^1 \cdot \nabla v \, dx.\end{aligned}$$

Using the change of variable $x_1 \to x_1 + zL$ and the periodicity of f and A we get

$$\begin{aligned}\int_{\Omega_{\ell_0}} A\nabla u_\infty \cdot \nabla v \, dx &= \sum_{z\in\mathbb{Z}} \{\int_{\Pi_L + zL} fv \, dx - k \int_{\Pi_L + zL} A^1 \cdot \nabla v \, dx\} + \int_{\Omega_{\ell_0}} kA^1 \cdot \nabla v \, dx \\ &= \int_{\Omega_{\ell_0}} fv \, dx - k \int_{\Omega_{\ell_0}} A^1 \cdot \nabla v \, dx + k \int_{\Omega_{\ell_0}} A^1 \cdot \nabla v \, dx \\ &= \int_{\Omega_{\ell_0}} fv \, dx \quad \forall v \in W_{\ell_0}.\end{aligned}$$

This shows the two first rows of (2.12). One has also

$$\int_{\sigma_{x_1}} A_1 \cdot \nabla u_\infty \, dX_2 = \int_{\sigma_{x_1}} A_1 \cdot \nabla \tilde{u}_\infty + ka_{11} \, dX_2 = 0$$

due to the last line of (2.32).
Finally to obtain the last equality of (2.12) one notes that due to the periodicity of $\tilde{u}_\infty$ one has

$$\int_{\Omega_\ell} |\nabla u_\infty|^2\,dx = \int_{\Omega_\ell} |\nabla(\tilde{u}_\infty + kx_1)|^2\,dx \le 2\int_{\Omega_\ell} |\nabla\tilde{u}_\infty|^2\,dx + 2k^2\int_{\Omega_\ell} 1\,dx$$
$$\le 4(\frac{\ell}{L}+1)\int_{\Pi_L} \{|\nabla\tilde{u}_\infty|^2 + k^2\}\,dx = O(\ell).$$

Similarly, due to the periodicity of f one has

$$\int_{\Omega_\ell} f^2\,dx \le 2(\frac{\ell}{L}+1)\int_{\Pi_L} f^2\,dx = O(\ell).$$

This completes the proof of the theorem. □

Corollary 2.2.7. *Under the assumptions above the solution to (2.12) is periodic if and only if*

$$\int_{\sigma_{x_1}} A_1\cdot\nabla u_f\,dX_2 = 0 \qquad a.e.\ x_1 \in (0,L). \tag{2.39}$$

Proof. u_∞ is periodic if and only if $k = 0$ -i.e. (2.39) holds. □

2.3 The case $p > 1$

Suppose that ω_1 is a bounded convex domain of $\mathbb{R}^p$ containing 0. If ω_2 is a bounded domain of $\mathbb{R}^{n-p}$ we set

$$\Omega_\ell = \ell\omega_1 \times \omega_2 \tag{2.40}$$

where

$$\ell\omega_1 = \{\ell X_1 \mid X_1 \in \omega_1\}. \tag{2.41}$$

(One can think for instance to $\omega_1 = (-1,1)^p$ or to some unit ball of $\mathbb{R}^p$ corresponding to some fixed norm.)
Suppose that $f \in L_2(\Omega_\ell)$ and satisfies

$$\int_{\omega_2} f(X_1,X_2)\,dX_2 = 0 \quad \text{a.e. } X_1 \in \mathbb{R}^p. \tag{2.42}$$

Then, under the assumptions (2.1) and (2.2) there exists u_ℓ -unique up to a constant- solution to

$$\begin{cases} u_\ell \in H^1(\Omega_\ell), \\ \int_{\Omega_\ell} A\nabla u_\ell\cdot\nabla v\,dx = \int_{\Omega_\ell} fv\,dx \quad \forall v \in H^1(\Omega_\ell) \end{cases} \tag{2.43}$$

(cf. Theorem 2.2.1). We would like to study the asymptotic behaviour of u_ℓ when $\ell \to +\infty$. We will impose some conditions on the matrix A, more precisely we will assume that

$$A = \begin{pmatrix} a(X_2)A_{11}(X_1) & 0 \\ A_{21}(x) & A_{22}(x) \end{pmatrix}, \quad x = (X_1,X_2) \in \mathbb{R}^p \times \mathbb{R}^{n-p} \tag{2.44}$$

where A_{11} is a $p\times p$ matrix, a a positive function, A_{22} a $(n-p)\times(n-p)$ matrix. Then we have:

Theorem 2.3.1. *Under the assumptions above suppose that*

$$|f|_{L_2(\Omega_\ell)} = O(e^{\gamma\ell})$$

for γ small enough. Then u_ℓ is a Cauchy sequence in $H^1(\Omega_{\ell_0})/\mathbb{R}$ which converges towards the unique solution (up to a constant) of the Neumann problem

$$\begin{cases} u_\infty \in H^1_{loc}(\overline{\Omega}_\infty), \qquad -\mathrm{div}(A\nabla u_\infty) = f \quad \textit{in } \Omega_\infty = \mathbb{R}^p \times \omega_2, \\ A\nabla u_\infty \cdot \nu = 0 \quad \textit{on } \partial\Omega_\infty, \\ \int_{\omega_2} a u_\infty(X_1, X_2)\, dX_2 = \; cst \; = C_\infty \quad \textit{a.e. } X_1 \in \mathbb{R}^p, \\ \big|\big|\nabla u_\infty\big|\big|_{L_2(\Omega_\ell)} = O(e^{2\gamma\ell}). \end{cases} \tag{2.45}$$

Moreover one has

$$\big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{L_2(\Omega_{\ell/2})} \le Ce^{-\alpha\ell} \tag{2.46}$$

for some positive constants C, α.

Proof. Taking $v = v(X_1) \in H^1(\ell\omega_1)$ in (2.43) leads to ($\nabla_{X_1} = (\partial x_1, \ldots, \partial x_p)^T$)

$$\int_{\Omega_\ell} A\nabla u_\ell \cdot \begin{pmatrix} \nabla_{X_1} v \\ 0 \end{pmatrix} dx = \int_{\ell\omega_1} v(X_1) \int_{\omega_2} f(X_1, X_2)\, dX_2 dX_1 = 0.$$

Taking into account (2.44) we deduce

$$\int_{\Omega_\ell} aA_{11}\nabla_{X_1} u_\ell \cdot \nabla_{X_1} v\, dx = \int_{\ell\omega_1} A_{11}(X_1)\nabla_{X_1} \int_{\omega_2} au_\ell\, dX_2 \cdot \nabla_{X_1} v\, dX_1 = 0.$$

Taking

$$v = \int_{\omega_2} au_\ell\, dX_2$$

and using the ellipticity of A we derive

$$\int_{\omega_2} au_\ell(X_1, X_2)\, dX_2 = \; \text{cst} \; = C_\ell \quad \text{a.e. } X_1 \in \ell\omega_1. \tag{2.47}$$

Suppose that

$$B(0, R) \subset \omega_1 \tag{2.48}$$

where $B(0, R)$ denotes the euclidean ball in $\mathbb{R}^p$ with center 0 and radius R. Set for $\ell_1 \le \ell - 1$

$$\rho_{\ell_1}(X_1) = \rho(X_1) = 0 \vee (1 - \frac{\text{dist}(X_1, \ell_1\omega_1)}{R})$$

where $\vee$ stands for the maximum of two numbers. Clearly one has

$$0 \le \rho \le 1, \quad \rho = 1 \text{ on } \ell_1\omega_1, \quad \rho = 0 \text{ outside of } (\ell_1 + 1)\omega_1, \tag{2.49}$$

$$|\nabla_{X_1}\rho| \le \frac{1}{R}. \tag{2.50}$$

To prove the last claim of (2.49) it is enough to show that if $X_1 \notin (\ell_1 + 1)\omega_1$, then $\text{dist}(X_1, \ell_1\omega_1) \ge R$. If not, for some $Z_1 \in \omega_1$ one has

$$|X_1 - \ell_1 Z_1| < R.$$

This implies by (2.48) that $X_1 - \ell_1 Z_1 = Z_1'$ where $Z_1' \in \omega_1$. It follows that

$$X_1 = (\ell_1 + 1)\left(\tfrac{\ell_1}{\ell_1+1} Z_1 + \tfrac{1}{\ell_1+1} Z_1'\right) \in (\ell_1 + 1)\omega_1$$

by the convexity of ω_1. This completes the proof of the last claim of (2.49). We can then proceed as in the proof of Theorem 2.2.2. Let us indeed show that u_ℓ is a Cauchy sequence. Let us first fix $r \in [0, 1]$. Then

$$(u_\ell - u_{\ell+r})\rho \in H^1(\Omega_\ell) \cap H^1(\Omega_{\ell+r})$$

and by (2.43) one derives (cf. (2.15))

$$\int_{\Omega_{\ell_1+1}} A\nabla(u_\ell - u_{\ell+r}) \cdot \nabla\{(u_\ell - u_{\ell+r})\rho\}\, dx = 0 \tag{2.51}$$

which can be written as

$$\begin{aligned} &\int_{\Omega_{\ell_1+1}} A\nabla(u_\ell - u_{\ell+r}) \cdot \nabla(u_\ell - u_{\ell+r})\rho\, dx \\ &= -\int_{\Omega_{\ell_1+1}} aA_{11}\nabla_{X_1}(u_\ell - u_{\ell+r}) \cdot \nabla_{X_1}\rho(u_\ell - u_{\ell+r})\, dx. \end{aligned} \tag{2.52}$$

Now we claim that

$$\int_{\Omega_{\ell_1+1}} aA_{11}\nabla_{X_1}(u_\ell - u_{\ell+r}) \cdot \nabla_{X_1}\rho v(X_1)\, dx = 0$$

for any $v \in H^1((\ell_1 + 1)\omega_1)$. Indeed this follows from

$$\begin{aligned} &\int_{\Omega_{\ell_1+1}} aA_{11}\nabla_{X_1}(u_\ell - u_{\ell+r}) \cdot \nabla_{X_1}\rho v\, dx \\ &= \int_{(\ell_1+1)\omega_1}\int_{\omega_2} a(X_2)A_{11}(X_1)\nabla_{X_1}(u_\ell - u_{\ell+r}) \cdot \nabla_{X_1}\rho v\, dx \\ &= \int_{(\ell_1+1)\omega_1} A_{11}\nabla_{X_1}\{\int_{\omega_2} a(u_\ell - u_{\ell+r})\, dX_2\} \cdot \nabla_{X_1}\rho v\, dX_1 = 0 \end{aligned}$$

by (2.47). Then going back to (2.52) we derive

$$\begin{aligned} &\int_{\Omega_{\ell_1+1}} A\nabla(u_\ell - u_{\ell+r}) \cdot \nabla(u_\ell - u_{\ell+r})\rho\, dx \\ &= -\int_{\Omega_{\ell_1+1}} aA_{11}\nabla_{X_1}(u_\ell - u_{\ell+r}) \cdot \nabla_{X_1}\rho(u_\ell - u_{\ell+r} - \fint_{\omega_2} u_\ell - u_{\ell+r})\, dx \end{aligned}$$

where

$$\fint_{\omega_2} u_\ell - u_{\ell+r} = \frac{1}{|\omega_2|}\int_{\omega_2} u_\ell - u_{\ell+r}\, dX_2.$$

Using (2.1), (2.2), (2.49), (2.50) it is easy to derive -recall that $D_{\ell_1} = \Omega_{\ell_1+1}\backslash\Omega_{\ell_1}$

$$\begin{aligned} &\lambda\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2\, dx \le C\int_{D_{\ell_1}} |\nabla_{X_1}(u_\ell - u_{\ell+r})||u_\ell - u_{\ell+r} - \fint_{\omega_2}(u_\ell - u_{\ell+r})|\, dx \\ &\le \frac{C}{2}\int_{D_{\ell_1}} |\nabla_{X_1}(u_\ell - u_{\ell+r})|^2 + |u_\ell - u_{\ell+r} - \fint_{\omega_2}(u_\ell - u_{\ell+r})|^2\, dx. \end{aligned}$$

Now for a.e. X_1 one has (cf. (2.6))

$$\int_{\omega_2} |u_\ell - u_{\ell+r} - \fint_{\omega_2}(u_\ell - u_{\ell+r})|^2\, dX_2 \le C_{\omega_2}^2\int_{\omega_2} |\nabla_{X_2}(u_\ell - u_{\ell+r})|^2\, dX_2$$

and thus for some constant δ we obtain from above

$$\lambda \int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 \, dx \leq \delta \int_{\Omega_{\ell_1+1}\setminus\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 \, dx.$$

One argues then as in the proof of Theorem 2.2.2 to get (cf. (2.17))

$$\int_{\Omega_{\ell/2}} |\nabla(u_\ell - u_{\ell+r})|^2 \, dx \leq Ce^{-\alpha'\ell}\{\int_{\Omega_\ell} |\nabla u_\ell|^2 \, dx + \int_{\Omega_{\ell+r}} |\nabla u_{\ell+r}|^2 \, dx\}, \tag{2.53}$$

for some constants α' and C. Then one has to estimate the last two integrals above. For that, taking $v = u_\ell$ in (2.43) we get easily due to (2.42)

$$\begin{aligned}
&\int_{\Omega_\ell} A\nabla u_\ell \cdot \nabla u_\ell \, dx = \int_{\Omega_\ell} f u_\ell \, dx = \int_{\ell\omega_1}\int_{\omega_2} f(u_\ell - \fint_{\omega_2} u_\ell) \, dX_2 dX_1 \\
&\leq \int_{\ell\omega_1} |f|_{L_2(\omega_2)} |u_\ell - \fint_{\omega_2} u_\ell|_{L_2(\omega_2)} \, dX_1 \leq \int_{\ell\omega_1} |f|_{L_2(\omega_2)} C_{\omega_2} \big||\nabla_{X_2} u_\ell|\big|_{L_2(\omega_2)} \, dX_1 \\
&\leq C_{\omega_2} |f|_{L_2(\Omega_\ell)} \big||\nabla_{X_2} u_\ell|\big|_{L_2(\Omega_\ell)}.
\end{aligned}$$

Using (2.1) one derives easily

$$\int_{\Omega_\ell} |\nabla u_\ell|^2 \, dx \leq \frac{C_{\omega_2}}{\lambda} |f|_{L_2(\Omega_\ell)} \leq Ce^{\gamma\ell}$$

for some constant C. Then one can conclude as in the proof of Theorem 2.2.2 that u_ℓ is a Cauchy sequence in $H^1(\Omega_{\ell_0})/\mathbb{R}$.
The existence of u_∞ satisfying the conditions in the first two rows of (2.45) is obtained as in the proof of Theorem 2.2.2. Now (2.47) implies that

$$0 = \nabla_{X_1}\{\int_{\omega_2} a u_\ell \, dX_2\} = \int_{\omega_2} a\nabla_{X_1} u_\ell \, dX_2 \quad \text{a.e. } X_1 \in \ell\omega_1.$$

Since on almost every section of Ω_{ℓ_0} one has

$$\nabla_{X_1} u_\ell \to \nabla_{X_1} u_\infty \quad \text{in } L_2(\omega_2)$$

one derives that

$$\int_{\omega_2} a\nabla_{X_1} u_\infty \, dX_2 = 0 \quad \text{a.e. } X_1 \in \ell_0\omega_1$$

which implies the third line of (2.45). The last row of (2.45) and (2.46) are obtained as in the proof of Theorem 2.2.2.
Finally to prove uniqueness, if u_∞, u'_∞ are solutions to (2.45), one proves like in (2.51) that

$$\int_{\Omega_{\ell_1+1}} A\nabla(u_\infty - u'_\infty) \cdot \nabla\{(u_\infty - u'_\infty)\rho\} \, dx = 0.$$

Then, arguing like below (2.52) and using the property of row 3 in (2.45), one derives

$$\int_{\Omega_{\ell_1+1}} aA_{11}\nabla_{X_1}(u_\infty - u'_\infty) \cdot \nabla_{X_1}\rho v(X_1) \, dx = 0$$

for any $v = v(X_1) \in H^1((\ell_1 + 1)\omega_1)$. Then it is easy to obtain (2.53) with $u_\ell, u_{\ell+r}$ replaced by u_∞, u'_∞ respectively. Due to the last row of (2.45) this implies that $u_\infty = u'_\infty$ (up to a constant). This completes the proof of the theorem. □

Remark 2.3.2. *The condition in the third row of (2.45) seems to be different of the corresponding condition in the case $p = 1$. In fact when $p = 1$ and A is given by (2.44) the condition in the third row of (2.12) reads*

$$\int_{\omega_2} a(X_2)a_{11}(x_1)\partial_{x_1}u_\infty \, dX_2 = 0 \quad a.e.\ x_1$$

$$\Leftrightarrow$$

$$a_{11}(x_1)\partial_{x_1}\{\int_{\omega_2} a(X_2)u_\infty \, dX_2\} = 0 \quad a.e.\ x_1.$$

Taking into account the ellipticity condition (2.1) one has $a_{11} \geq \lambda > 0$ and one recovers the condition in the third row of (2.45).

Remark 2.3.3. *As an application, and as we did in the case where $p = 1$, we could use the solution of the pure Neumann problem in*

$$\ell\omega_1 \times \omega_2$$

to construct the solution of the pure Neumann problem in the unbounded domain $\mathbb{R}^p \times \omega_2$.

Chapter 3

Approximation of Poisson's problem in a cylindrical domain

In this chapter we focus on the approximation of the weak solution of the n-dimensional Poisson problem in cylindrical domains. The n-dimensional Poisson problem $-\Delta u_\ell(x_1, x') = f(x')$ in Ω_ℓ with Dirichlet boundary conditions converges in the center to the analogous (n-1)-dimensional Poisson problem when the domain increases to infinity in the x_1-direction. This result raises the idea to approximate the solution of this problem by the analogous weak solution of the (n-1)-dimensional Poisson problem multiplied with a cutoff function ψ, depending only on x_1. In addition we study the upper bound of the error between u_ℓ and its approximation.

3.1 Problem formulation

We analyse the following two Poisson problems with Dirichlet boundary conditions:

$$\begin{cases} -\Delta u_\ell(x_1, x') = f(x') & \text{in } \Omega_\ell, \\ u_\ell = 0 & \text{on } \partial\Omega_\ell \end{cases} \tag{3.1}$$

and

$$\begin{cases} -\Delta' u_\infty = f(x') & \text{in } \omega, \\ u_\infty = 0 & \text{on } \partial\omega \end{cases} \tag{3.2}$$

with

$$\Omega_\ell = (-\ell, \ell) \times \omega, \quad f \in L_2(\omega) \tag{3.3}$$

and ω is an open bounded connected Lipschitz domain in $\mathbb{R}^{n-1}$. We denote $x = (x_1, x') \in \mathbb{R}^n$ with $x_1 \in \mathbb{R}$, $x' = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$ and $\Delta' = \partial^2_{x_2 x_2} + \cdots + \partial^2_{x_n x_n}$. The weak formulation of the previous two problems is

$$\begin{cases} u_\ell \in H^1_0(\Omega_\ell), \\ (\nabla u_\ell, \nabla v)_{L_2(\Omega_\ell)} = (f, v)_{L_2(\Omega_\ell)} \quad \forall v \in H^1_0(\Omega_\ell), \end{cases}$$

$$\begin{cases} u_\infty \in H^1_0(\omega), \\ (\nabla' u_\infty, \nabla' v)_{L_2(\omega)} = (f, v)_{L_2(\omega)} \quad \forall v \in H^1_0(\omega). \end{cases}$$

The $L_2(\Omega)$-scalar product for real-valued functions u, v is denoted by

$$(u, v)_{L_2(\Omega)} = \int_\Omega uv \, dx$$

and

$$\nabla' = (\partial_{x_2}, \dots, \partial_{x_n})^T.$$

In addition we assume that ω is convex, so we have that $u_\infty \in H^2(\omega)$ for $f \in L_2(\omega)$ (see Satz 7.2, Kapitel 2 in [12]).

Remark 3.1.1. *We assume that $f \neq 0$, so we exclude the trivial case $u_\infty = u_\ell = 0$.*

The aim of this work is to find an optimal cutoff function $\psi(x_1)$ such that $\psi \otimes u_\infty$ is a good approximation of u_ℓ, the unique weak solution of (3.1).
This chapter is structured as follows: First we give an a priori argument why we approximate u_ℓ by $\psi \otimes u_\infty$. Next we compute u_∞ resp. an approximation $u_{\infty h}$ and study its properties, then we are looking for a cutoff function ψ_h resp. an approximation $\psi_{h,h}$. Finally we analyse the error between the approximation $\psi_h \otimes u_\infty$ resp. $\psi_{h,h} \otimes u_{\infty,h}$ and u_ℓ, i.e. we derive an a posteriori error majorant.

3.2 Idea of the approximation

From Theorem 6.3 in [15] we know for $n = 2$, $\omega = (-1, 1)$ and $0 < \ell_0 \leq \ell$ that the following asymptotic behaviour result holds:

$$\big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{L_2(\Omega_{\ell_0})} \leq \sqrt{2}\|u_\infty\|_{H^1(\omega)} e^{\sqrt{\lambda_1}(\ell_0 - \ell)},$$

i.e. u_ℓ converges to u_∞ in Ω_{ℓ_0} with an exponential rate for $\ell \to +\infty$. Note λ_1 is the first eigenvalue of the operator $-\Delta' = \partial^2_{x_2x_2} + \cdots + \partial^2_{x_nx_n}$ for the Dirichlet problem. The previous estimate can be shown analogously for any $n \in \mathbb{N}_{\geq 2}$ and bounded open connected Lipschitz domain ω. Indeed, we have:

Theorem 3.2.1. *Let $f \in L_2(\omega)$, u_ℓ being the unique weak solution of (3.1), u_∞ being the unique weak solution of (3.2) and $\Omega_\ell = (-\ell, \ell) \times \omega$ where ω is an open bounded connected Lipschitz domain. Then for any $0 < \ell_0 \leq \ell$ it holds*

$$\big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{L_2(\Omega_{\ell_0})} \leq \sqrt{2}\|u_\infty\|_{H^1(\omega)} e^{\sqrt{\lambda_1}(\ell_0 - \ell)} \tag{3.4}$$

where

$$\lambda_1 = \inf_{0 \neq u \in H^1_0(\omega)} \frac{\int_\omega |\nabla' u|^2 \, dx'}{\int_\omega u^2 \, dx'}$$

(first eigenvalue of the operator $-\Delta'$ for the Dirichlet problem).

Proof. For details we refer to the proofs of Theorems 6.1, 6.2 and 6.3 in [15]. We divide our proof into 3 steps.
(i) Suppose that $\ell_2 < \ell_1 \leq \ell$ then we have

$$\big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{L_2(\Omega_{\ell_2})} \leq e^{-\sqrt{\lambda_1}(\ell_1 - \ell_2)} \big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{L_2(\Omega_{\ell_1})}.$$

Indeed: Let $v \in H^1_0(\Omega_\ell)$ then we have for a.e. x_1

$$\int_\omega \nabla' u_\infty \cdot \nabla' v(x_1, \cdot) \, dx' = (f, v(x_1, \cdot))_{L_2(\omega)}$$

and integrating in $(-\ell, \ell)$ we obtain, since u_∞ is independent of x_1,

$$\int_{\Omega_\ell} \nabla u_\infty \cdot \nabla v \, dx = (f, v)_{L_2(\Omega_\ell)}$$

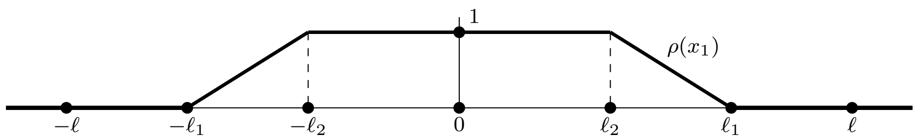
which implies

$$\int_{\Omega_\ell} \nabla(u_\ell - u_\infty) \cdot \nabla v \, dx = 0 \quad \forall v \in H^1_0(\Omega_\ell). \tag{3.5}$$

Let us introduce the function ρ such that

$$0 \leq \rho \leq 1, \quad \rho = 1 \text{ on } (-\ell_2, \ell_2), \quad \rho = 0 \text{ on } \mathbb{R}\backslash(-\ell_1, \ell_1), \quad |\rho'| \leq \frac{1}{\ell_1 - \ell_2}$$

(see graph below).



So we have $(u_\ell - u_\infty)\rho \in H_0^1(\Omega_\ell)$ and we deduce from (3.5) that

$$\int_{\Omega_{\ell_1}} \nabla(u_\ell - u_\infty) \cdot \nabla\{(u_\ell - u_\infty)\rho\}\, dx = 0.$$

Therefore we have

$$\begin{aligned}&\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 \rho\, dx \\ &\leq \frac{1}{\ell_1 - \ell_2}\{\frac{1}{\sqrt{\lambda_1}} \int_{\Omega_{\ell_1}\backslash\Omega_{\ell_2}} |\partial_{x_1}(u_\ell - u_\infty)|^2\, dx + \sqrt{\lambda_1} \int_{\Omega_{\ell_1}\backslash\Omega_{\ell_2}} (u_\ell - u_\infty)^2\, dx\}\end{aligned}$$

and due to the definition of λ_1 we derive

$$\int_{\Omega_{\ell_2}} |\nabla(u_\ell - u_\infty)|^2\, dx \leq \frac{1}{2\sqrt{\lambda_1}(\ell_1 - \ell_2)} \int_{\Omega_{\ell_1}\backslash\Omega_{\ell_2}} |\nabla(u_\ell - u_\infty)|^2\, dx. \tag{3.6}$$

We set

$$F(\ell') := \int_{\Omega_{\ell'}} |\nabla(u_\ell - u_\infty)|^2\, dx$$

so (3.6) can be written as

$$F(\ell_2) \leq \frac{1}{2\sqrt{\lambda_1}} \frac{F(\ell_1) - F(\ell_2)}{\ell_1 - \ell_2}.$$

Letting $\ell_1 \to \ell_2$ we obtain

$$F(\ell_2) \leq \frac{1}{2\sqrt{\lambda_1}} F'(\ell_2) \quad \text{ for a.e. } \ell_2$$

which is equivalent to

$$(e^{-2\sqrt{\lambda_1}\ell_2} F(\ell_2))' \geq 0.$$

From this we follow

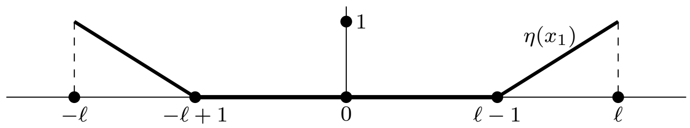
$$e^{-2\sqrt{\lambda_1}\ell_2} F(\ell_2) \leq e^{-2\sqrt{\lambda_1}\ell_1} F(\ell_1)$$

for any $\ell_2 \leq \ell_1$ and so we have shown (i).

(ii) We have

$$\big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{L_2(\Omega_\ell)} \leq \sqrt{2}\|u_\infty\|_{H^1(\omega)}$$

where $\|v\|^2_{H^1(\omega)} = \int_\omega |v|^2 + |\nabla' v|^2\, dx'$ *for* $v \in H^1(\omega)$.
Indeed: First let us introduce the function η depicted as in the graph below.



Clearly we have $u_\ell - u_\infty + \eta u_\infty \in H_0^1(\Omega_\ell)$. From (3.5) we follow

$$\int_{\Omega_\ell} \nabla(u_\ell - u_\infty) \cdot \nabla(u_\ell - u_\infty + \eta u_\infty)\, dx = 0$$

and hence

$$\big|\big|\nabla(u_\ell - u_\infty)\big|\big|^2_{L_2(\Omega_\ell)} \leq \big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{L_2(\Omega_\ell)} \big|\big|\nabla(\eta u_\infty)\big|\big|_{L_2(\Omega_\ell)}$$

by the Cauchy-Schwarz inequality. Due to η we then have

$$\big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{L_2(\Omega_\ell)} \leq \big|\big|\nabla(\eta u_\infty)\big|\big|_{L_2(\Omega_\ell \setminus \Omega_{\ell-1})}$$

and

$$\big|\big|\nabla(\eta u_\infty)\big|\big|^2_{L_2(\Omega_\ell \setminus \Omega_{\ell-1})} \leq \int_{\Omega_\ell \setminus \Omega_{\ell-1}} (\partial_{x_1} \eta u_\infty)^2 + |\eta \nabla' u_\infty|^2\, dx \leq 2\|u_\infty\|^2_{H^1(\omega)}$$

which implies (ii).

(iii) For any $\ell_0 > 0$ we have

$$\big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{L_2(\Omega_{\ell_0})} \leq \sqrt{2} e^{\sqrt{\lambda_1}(\ell_0 - \ell)} \|u_\infty\|_{H^1(\omega)}.$$

Indeed, let us apply (i) with $\ell_2 = \ell_0$, $\ell_1 = \ell$ and then the result follows from (ii) and we have proven this theorem. □

From the previous theorem we deduce that the weak solution u_ℓ of (3.1) converges exponentially to u_∞, the weak solution of (3.2), in the center of the domain when the domain increases to infinity into the x_1-direction. This asymptotic behaviour result motivates the idea to approximate u_ℓ with a cutoff function multiplied by u_∞ where the cutoff function is only depending on x_1.
We denote by $1 \otimes u_\infty$ the tensor product of the constant function 1 and u_∞ and hence

$$\big|\big|\nabla(u_\ell - 1 \otimes u_\infty)\big|\big|_{L_2(\Omega_{\ell_0})} \leq \sqrt{2} e^{\sqrt{\lambda_1}(\ell_0 - \ell)} \|u_\infty\|_{H^1(\omega)}.$$

3.3 The Poisson Equation on the cross-section ω and its disretization

3.3.1 Computing u_∞ for special cases

In this subsection we compute u_∞, the unique weak solution of the problem (3.2).
First we analyse the case $n-1 = 1$ which means that the domain ω is an interval and we set for simplicity $\omega = (a, b)$ with $a, b \in \mathbb{R}$. For the computation of the exact u_∞ we use the m-fold anti-derivative

$$f^{(-m)}(x_2) = \int_a^{x_2} \int_a^{s_{m-1}} \ldots \left(\int_a^{s_1} f(s)\, ds \right) ds_1 \ldots ds_{m-1} \tag{3.7}$$

for $s_i \in (a,b)$, $1 \le i \le m-1$. So we have for $y \in (a,b)$

$$\begin{aligned}
&-\int_a^{s_1} \partial_{ss}^2 u_\infty \, ds = \int_a^{s_1} f(s)\, ds = f^{(-1)}(s_1) \quad \Rightarrow \\
&-\partial_{s_1} u_\infty(s_1) + \tilde{c}_1 = f^{(-1)}(s_1) \quad \Rightarrow \\
&\int_a^{x_2} -\partial_{s_1} u_\infty + \tilde{c}_1 \, ds_1 = \int_a^{x_2} f^{(-1)}(s_1)\, ds_1 = f^{(-2)}(x_2) \quad \Rightarrow \\
&-u_\infty(x_2) + \tilde{c}_1(x_2 - a) + \tilde{c}_2 = f^{(-2)}(x_2).
\end{aligned}$$

Thanks to the Dirichlet boundary conditions we obtain the following two equations

$$\tilde{c}_1(a-a) + \tilde{c}_2 = f^{(-2)}(a), \quad \tilde{c}_1(b-a) + \tilde{c}_2 = f^{(-2)}(b)$$

and therefore the constants are

$$\tilde{c}_1 = (f^{(-2)}(b) - f^{(-2)}(a))/(b-a), \quad \tilde{c}_2 = f^{(-2)}(a)$$

which implies

$$u_\infty(x_2) = \frac{f^{(-2)}(b) - f^{(-2)}(a)}{b-a}(x_2 - a) - f^{(-2)}(x_2) + f^{(-2)}(a). \tag{3.8}$$

For example if we take $a = -1$, $b = 1$ and $f = 1$ then we derive $u_\infty(x_2) = 1/2 - x_2^2/2$ or if $f = x_2$ then we have $u_\infty(x_2) = -x_2^3/6 + x_2/6$.

Now we analyse the case where $n - 1 \ge 2$. From the Analysis we know that the explicit solution of u_∞ is given by

$$u_\infty(x') = \int_\omega f(y') G(x', y')\, dy'$$

where G is the Green's function (see Theorem 12 of Chapter 2.2 in [30]). For example the Green's function for a unit ball $B_{n-1}(0,1) = \{(x_2, \ldots, x_n) : x_2^2 + \cdots + x_n^2 = 1\}$ is given by

$$G(x', y') = \Phi(y' - x') - \Phi(|x'|(y' - \frac{x'}{|x'|^2}))$$

where

$$\Phi(x') = \begin{cases} -\frac{1}{2\pi} \ln(|x'|) & n-1 = 2, \\ \frac{1}{(n-1)(n-3)\omega_{n-1}} \frac{1}{|x'|^{n-3}} & n-1 \ge 3 \end{cases}$$

(the fundamental solution of the Laplace equation) defined for $x' \in \mathbb{R}^{n-1}$, $x' \neq 0$ and ω_{n-1} denotes the volume of the unit ball in $\mathbb{R}^{n-1}$.

3.3.2 Galerkin Discretization

For complicated f or the domain ω having dimension greater than 1 we approximate u_∞ by the Galerkin finite element method (FEM). We denote the approximate solution by $u_{\infty h}$ defined as the solution of

$$\begin{cases} u_{\infty h} \in S_h^{0,m}(\omega), \\ \int_\omega \nabla' u_{\infty h} \cdot \nabla' v \, dx' = \int_\omega f v \, dx' \quad \forall v \in S_h^{0,m}(\omega). \end{cases} \tag{3.9}$$

Let $S_h^m(\omega)$ be a finite dimensional subspace of $H^2(\omega)$ which has to satisfy

$$\mathbb{P}_m(K) \subseteq S_h^m(\omega)|_K \subset C^1(K), \quad \mathbb{P}_{m+1}(K) \not\subset S_h^m(\omega)|_K \tag{3.10}$$

and we set

$$S_h^{0,m}(\omega) := S_h^m(\omega) \cap H_0^1(\omega) \subset H^2(\omega) \tag{3.11}$$

where $\overline{\omega} = \cup_{K \in \mathcal{G}_{\omega,h}} K$, $\mathcal{G}_{\omega,h}$ denotes a shape-regular conforming triangulation of ω (for the definition see Appendix, Section 4.3), ω a polygonal domain and $\mathbb{P}_m$ denotes the set of polynomials of degree $\leq m$. The index m in $S_h^m(\omega)$ denotes the maximal number such that the whole polynomial domain $\mathbb{P}_m(K)$ is included in $S_h^m(K)$, but it is possible that the subspace $S_h^m(K)$ includes also polynomials of higher order. Further we set

$$h = \max_{K \in \mathcal{G}_{\omega,h}} h_K \tag{3.12}$$

where h_K denotes the diameter of the element K. Note for the FEM we can take, for instance, Splines if $n-1=1$ (see below), Argyris elements if $n-1=2$ (cf. [12] or [13]) and for $n-1=3$ compare paper [47].

The one dimensional case: $\omega = (a,b)$

Now we compute $u_{\infty h}$ when $n-1=1$ and we introduce a concrete basis for the space $S_h^{0,3}(\omega)$. In this case $\omega=(a,b)$ with $a,b \in \mathbb{R}$ is an interval. We employ the ansatz

$$u_{\infty h}(x_2) = \sum_{i=1}^{2N} u_{\infty i} b_i(x_2) \tag{3.13}$$

where the basis elements $b_i|_{K_j} \in \mathbb{P}_3$ with $K_j = [y_j, y_{j+1}] \subset \overline{\omega} = \cup_{j=0}^{N-1} K_j$ and $y_0 = a, y_N = b$ are given by

$$\begin{gathered} b_i(y) := \begin{cases} b_i^{(1)}(y) & \text{for } 1 \leq i \leq N-1, \\ b_{i-N}^{(2)}(y) & \text{for } N \leq i \leq 2N, \end{cases} \\ \begin{cases} b_i^{(1)}(y_j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases} \\ b_i^{(1)\prime}(y_j) = 0 \quad \text{for } 0 \leq j \leq N, \end{cases} \qquad \begin{cases} b_{i-N}^{(2)}(y_j) = 0 \quad \text{for } 0 \leq j \leq N, \\ b_{i-N}^{(2)\prime}(y_j) = \begin{cases} 1 & \text{for } i-N = j, \\ 0 & \text{for } i-N \neq j. \end{cases} \end{cases} \end{gathered} \tag{3.14}$$

Precisely for $0 \leq i \leq N$ we have

$$b_i^{(1)}(y) = \begin{cases} \frac{1}{(y_i - y_{i-1})^3}((-y_{i-1}^3 + 3y_i y_{i-1}^2) - 6y_i y_{i-1} y + 3(y_i + y_{i-1})y^2 - 2y^3) & \text{if } y \in K_{i-1} \text{ and } i \geq 1, \\ \frac{1}{(y_i - y_{i+1})^3}((-y_{i+1}^3 + 3y_i y_{i+1}^2) - 6y_i y_{i+1} y + 3(y_i + y_{i+1})y^2 - 2y^3) & \text{if } y \in K_i \text{ and } i \leq N-1, \\ 0 & \text{otherwise,} \end{cases}$$

$$b_i^{(2)}(y) = \begin{cases} \frac{1}{(y_i - y_{i-1})^2}(-y_i y_{i-1}^2 + (y_{i-1}^2 + 2y_i y_{i-1})y - (y_i + 2y_{i-1})y^2 + y^3) & \text{if } y \in K_{i-1} \text{ and } i \geq 1, \\ \frac{1}{(y_i - y_{i+1})^2}(-y_i y_{i+1}^2 + (y_{i+1}^2 + 2y_i y_{i+1})y - (y_i + 2y_{i+1})y^2 + y^3) & \text{if } y \in K_i \text{ and } i \leq N-1, \\ 0 & \text{otherwise.} \end{cases}$$

We set

$$S_h^3(\omega) = S_h^3(a,b) = \operatorname{span}\{b_i^{(1)}, b_j^{(2)} \mid 0 \leq i,j \leq N\}$$

and

$$S_h^{0,3}(\omega) = S_h^{0,3}(a,b) = \operatorname{span}\{b_i^{(1)}, b_j^{(2)} \mid 1 \leq i \leq N-1,\ 0 \leq j \leq N\} = \operatorname{span}\{b_i \mid 1 \leq i \leq 2N\}. \tag{3.15}$$

$S_h^3(\omega)$ has dimension $2(N+1) = 2N+2$ and $S_h^{0,3}(\omega)$ has dimension $(N-1)+(N+1) = 2N$.

Claim 3.3.1. *$S_h^m(\omega) = S_h^3(\omega)$, i.e. $m=3$ for the above chosen basis.*

Proof. For the proof we have to show that an arbitrary polynomial $p(y) = d_0 + d_1 y + d_2 y^2 + d_3 y^3 \in \mathbb{P}_3(K_j)$ for $j \in \{0, \dots, N-1\}$ can be represented in basis elements of $S_h^m(\omega)|_{K_j}$, i.e. we need the existence of constants $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that

$$d_0 + d_1 y + d_2 y^2 + d_3 y^3 = a_1 b_j^{(1)}(y) + a_2 b_{j+1}^{(1)}(y) + a_3 b_j^{(2)}(y) + a_4 b_{j+1}^{(2)}(y).$$

First let us set $h := y_{j+1} - y_j$ and then we have

$$\begin{aligned}
& a_1 b_j^{(1)}(y) + a_2 b_{j+1}^{(1)}(y) + a_3 b_j^{(2)}(y) + a_4 b_{j+1}^{(2)}(y) \\
&= -\frac{a_1}{h^3}((-y_{j+1}^3 + 3y_j y_{j+1}^2) - 6y_j y_{j+1} y + 3(y_j + y_{j+1})y^2 - 2y^3) \\
&+ \frac{a_2}{h^3}((-y_j^3 + 3y_j^2 y_{j+1}) - 6y_{j+1} y_j y + 3(y_{j+1} + y_j)y^2 - 2y^3) \\
&+ \frac{a_3}{h^2}(-y_j y_{j+1}^2 + (y_{j+1}^2 + 2y_j y_{j+1})y - (y_j + 2y_{j+1})y^2 + y^3) \\
&+ \frac{a_4}{h^2}(-y_{j+1} y_j^2 + (y_j^2 + 2y_{j+1} y_j)y - (y_{j+1} + 2y_j)y^2 + y^3) \\
&= (\frac{a_1}{h^3}(y_{j+1}^3 - 3y_j y_{j+1}^2) + \frac{a_2}{h^3}(-y_j^3 + 3y_j^2 y_{j+1}) - \frac{a_3}{h^2} y_j y_{j+1}^2 - \frac{a_4}{h^2} y_{j+1} y_j^2) \\
&+ y(\frac{a_1}{h^3} 6y_j y_{j+1} - \frac{a_2}{h^3} 6y_j y_{j+1} + \frac{a_3}{h^2}(y_{j+1}^2 + 2y_j y_{j+1}) + \frac{a_4}{h^2}(y_j^2 + 2y_{j+1} y_j)) \\
&+ y^2(-\frac{3a_1}{h^3}(y_j + y_{j+1}) + \frac{3a_2}{h^3}(y_{j+1} + y_j) - \frac{a_3}{h^2}(y_j + 2y_{j+1}) - \frac{a_4}{h^2}(y_{j+1} + 2y_j)) \\
&+ y^3(\frac{2a_1}{h^3} - \frac{2a_2}{h^3} + \frac{a_3}{h^2} + \frac{a_4}{h^2})
\end{aligned}$$

and the previous term is equal to $d_0 + d_1 y + d_2 y^2 + d_3 y^3$ when the following linear equation system

$$\mathbf{A}\mathbf{y} = \mathbf{b}$$

with

$$\mathbf{A} = \begin{pmatrix} \frac{y_{j+1}^3 - 3y_j y_{j+1}^2}{h^3} & \frac{-y_j^3 + 3y_j^2 y_{j+1}}{h^3} & \frac{-y_j y_{j+1}^2}{h^2} & \frac{-y_j^2 y_{j+1}}{h^2} \\ \frac{6y_j y_{j+1}}{h^3} & \frac{-6y_j y_{j+1}}{h^3} & \frac{y_{j+1}^2 + 2y_j y_{j+1}}{h^2} & \frac{y_j^2 + 2y_{j+1} y_j}{h^2} \\ \frac{-3(y_j + y_{j+1})}{h^3} & \frac{3(y_{j+1} + y_j)}{h^3} & \frac{-(y_j + 2y_{j+1})}{h^2} & \frac{-(y_{j+1} + 2y_j)}{h^2} \\ \frac{2}{h^3} & \frac{-2}{h^3} & \frac{1}{h^2} & \frac{1}{h^2} \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

has a solution. From the Linear Algebra we know that a linear equation system has a unique solution when the determinant of $\mathbf{A}$ is not zero and $\mathbf{y} = 0$ if and only if $\mathbf{b} = 0$ (cf. Bemerkung p.134 in [32] and Folgerung 5.4.3 in [42]). We have

$$\det(\mathbf{A}) = -(y_j^6 - 6y_j^5 y_{j+1} + 15y_j^4 y_{j+1}^2 - 20y_j^3 y_{j+1}^3 + 15y_j^2 y_{j+1}^4 - 6y_j y_{j+1}^5 + y_{j+1}^6)/h^{10}$$

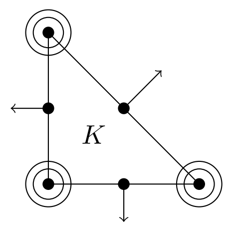
and $\det(\mathbf{A})$ is zero for $y_j = y_{j+1}$, which is never the case since $y_{j+1} = y_j + h$, $h > 0$. Hence $\det(\mathbf{A}) \neq 0$ and therefore we conclude that $m = 3$. □

We have for b_j, $1 \leq j \leq 2N$ the equation

$$\int_\omega \partial_{x_2} u_{\infty h} \partial_{x_2} b_j \, dx_2 = \int_\omega f b_j \, dx_2 \quad \Leftrightarrow \quad \sum_{i=1}^{2N} u_{\infty i} \int_\omega b_i' b_j' \, dx_2 = \int_\omega f b_j \, dx_2.$$

From the definition of the basis functions, it follows that if $|i - j| \geq 2$ implies $\int_\omega b_i^{(k)} b_j^{(l)} \, dx_2 = 0$ for $k, l \in \{1, 2\}$. The bilinear form $\tilde{a}(u, v) = \int_\omega \partial_{x_2} u \partial_{x_2} v \, dx_2$ for $u, v \in S_h^{0,3}(\omega)$ is coercive and continuous. Indeed we have $\tilde{a}(u, v) \leq \big||\nabla u|\big|_{L_2(\omega)} \big||\nabla v|\big|_{L_2(\omega)}$ (continuity) and $\tilde{a}(u, u) \geq \big||\nabla u|\big|_{L_2(\omega)}^2$ (coercivity) for $u, v \in S_h^{0,3}(\omega)$. Hence we can apply the Lax-Milgram Theorem (cf. Theorem 4.2.2) and derive the existence of a unique solution $u_{\infty h}$ of the problem $\tilde{a}(u_{\infty h}, v) = \int_\omega f v \, dx_2$ for all $v \in S_h^{0,3}(\omega)$. Therefore the coefficients $u_{\infty i}$ are computable and, in turn, the approximated solution $u_{\infty h}$ is determined.

Remark 3.3.2. *For $n = 3$ resp. ω is two-dimensional, we use for the approximation the Argyris method (see for instance [13] or [27]). For the Argyris method we need to know from the 3 nodal points of a triangle the value, first and second derivative and from the 3 middle points the normal derivative. Hence the degree of freedom is 21 and the polynomial of a basis element $b_i : \mathbb{R}^2 \to \mathbb{R}$ has degree 5.*
A possible triangle K:



3.4 Properties of u_∞ resp. $u_{\infty h}$

In this section we study properties of u_∞ resp. $u_{\infty h}$ which will be useful later.

3.4.1 Properties of u_∞

Let us set

$$c_1 := \int_\omega |u_\infty|^2 \, dx' \quad \text{and} \quad c_2 := \int_\omega f u_\infty \, dx' \overset{(3.2)}{=} \int_\omega |\nabla' u_\infty|^2 \, dx', \tag{3.16}$$

where u_∞ is the weak solution of (3.2).

Remark 3.4.1. *For $f \neq 0$, $f \in L_2(\omega)$ we have that $c_1, c_2 > 0$. This follows immediately from Remark 3.1.1 (we have: $u_\infty \neq 0$) and that c_1, c_2 are squares of norms.*

Before we continue let us define the Friedrichs constant. For any bounded connected Lipschitz domain $\Omega \subset \mathbb{R}^n$ and $v \in H_0^1(\Omega)$ we have

$$|v|_{L_2(\Omega)} \leq C_F(\Omega) \big| |\nabla v| \big|_{L_2(\Omega)} \tag{3.17}$$

where $C_F(\Omega)$ is called the Friedrichs constant. $|\cdot|_{L_2(\Omega)}$ denotes the norm of the domain $L_2(\Omega)$, i.e. $|v|_{L_2(\Omega)} = (\int_\Omega |v|^2 \, dx)^{1/2}$ for $v \in L_2(\Omega)$. For the proof of the inequality (3.17) we refer to Chapter 2 in [15] or Chapter 4 in [28]. Further we have for

$$\Omega \subset B_d(\zeta) = \{x \in \mathbb{R}^n \mid |x - \zeta| < d\}$$

that

$$C_F(\Omega) \leq \sqrt{2} d$$

(cf. [15]).

Lemma 3.4.2. *Let $f \in L_2(\omega)$, then we have the following bounds for the constants c_1, c_2:*

$$\|f\|_{H^{-2}(\omega)} \leq \sqrt{c_1} \leq C_F(\omega) \sqrt{c_2} \leq C_F(\omega)^2 |f|_{L_2(\omega)} \tag{3.18}$$

where $C_F(\omega)$ is the Friedrichs constant.

Proof. For showing an upper bound of c_1 and c_2 we put $v = u_\infty$ as a test function into the weak formulation of (3.2) and then we obtain, by applying Hölder and Friedrichs inequality, that

$$\int_\omega \nabla' u_\infty \cdot \nabla' u_\infty \, dx' = \int_\omega f u_\infty \, dx' \leq |f|_{L_2(\omega)} |u_\infty|_{L_2(\omega)} \leq C_F(\omega) |f|_{L_2(\omega)} \big| |\nabla' u_\infty| \big|_{L_2(\omega)}$$

which implies

$$\sqrt{c_1} = |u_\infty|_{L_2(\omega)} \leq C_F(\omega)\big||\nabla' u_\infty|\big|_{L_2(\omega)} = C_F(\omega)\sqrt{c_2} \leq C_F(\omega)^2|f|_{L_2(\omega)}. \tag{3.19}$$

Next we study the lower bound for c_1 and c_2. For this we define $\mathcal{N} : H^{-1}(\omega) \to H_0^1(\omega)$ the solution operator for the problem on page 43, i.e.,

$$(\nabla'(\mathcal{N}f), \nabla' v)_{L_2(\omega)} = (f, v)_{L_2(\omega)} \quad \forall v \in H_0^1(\omega).$$

There exists an eigensystem $e_m \in H_0^1(\omega)$, $m \in \mathbb{N}$, where e_m is an eigenvector corresponding to λ_m:

$$-\Delta' e_m = \lambda_m e_m \quad \text{ in } \omega, \qquad e_m = 0 \quad \text{on } \partial\omega \qquad \text{for } m \in \mathbb{N} \tag{3.20}$$

(see Theorem 1, Chapter 6.5 in [30]). Further we have

$$\mathcal{N}e_m = \frac{1}{\lambda_m} e_m.$$

It holds

$$|u_\infty|^2_{L_2(\omega)} = (\mathcal{N}f, \mathcal{N}f)_{L_2(\omega)} = \sum_{m\in\mathbb{N}} \lambda_m^{-2} f_m^2 \tag{3.21}$$

where $f = \sum_{m\in\mathbb{N}} f_m e_m$ and note $\{e_m\}_{m\in\mathbb{N}}$ is also an orthonormal basis of $L_2(\omega)$ (see Theorem 1, Chapter 6.5 in [30]). We use the standard norm for $v \in H^2(\omega)$, i.e.

$$\|v\|^2_{H^2(\omega)} = \sum_{|\alpha|\leq 2} \int_\omega |D^\alpha v|^2 \, dx.$$

Note for $v \in H_0^1(\omega) \cap H^2(\omega)$ and ω convex we have that $\|v\|_{H^2(\omega)}$ and $|\Delta' v|_{L_2(\omega)}$ are two equivalent norms (see (9.1.29) in [40] and (3.7) in [58]), i.e. it holds

$$1 \cdot |\Delta' v|_{L_2(\omega)} = C_1 |\Delta' v|_{L_2(\omega)} \leq \|v\|_{H^2(\omega)} \leq C_{reg} |\Delta' v|_{L_2(\omega)} \tag{3.22}$$

with C_{reg} being a positive constant and $C_1 = 1$. Hence we have for $v \in H_0^1(\omega) \cap H^2(\omega)$ that

$$\begin{aligned}\|v\|^2_{H^2(\omega)} &= \|\sum_{m\in\mathbb{N}} v_m e_m\|^2_{H^2(\omega)} \geq |\Delta' \sum_{m\in\mathbb{N}} v_m e_m|^2_{L_2(\omega)} \geq |\sum_{m,n\in\mathbb{N}} v_m v_n \int_\omega \Delta' e_m \Delta' e_n \, dx'| \\ &= |\sum_{m,n\in\mathbb{N}} \lambda_m \lambda_n v_m v_n \int_\omega e_m e_n \, dx'| = \sum_{m\in\mathbb{N}} v_m^2 \lambda_m^2. \end{aligned} \tag{3.23}$$

Here we used the properties that we have an $L_2(\omega)$-orthonormal basis and $-\Delta' e_m = \lambda_m e_m$. Next we analyse the H^{-2}-norm which is defined by

$$\|g\|_{H^{-2}(\omega)} = \sup_{v\in H_0^2(\omega),\, v\neq 0} \frac{(g, v)_{L_2(\omega)}}{\|v\|_{H^2(\omega)}}$$

for $g \in H^{-2}(\omega) = (H_0^2(\omega))'$. So we have

$$\begin{aligned}\|f\|_{H^{-2}(\omega)} &= \sup_{v\in H_0^2(\omega),\, v\neq 0} \frac{(f, v)_{L_2(\omega)}}{\|v\|_{H^2(\omega)}} \leq \sup_{0\neq \sum_{m\in\mathbb{N}} v_m e_m \in H_0^2(\omega)} \frac{\sum_{m\in\mathbb{N}} f_m v_m}{(\sum_{m\in\mathbb{N}} v_m^2 \lambda_m^2)^{1/2}} \\ &\leq \sup_{0\neq \sum_{m\in\mathbb{N}} v_m e_m \in H_0^2(\omega)} \frac{(\sum_{m\in\mathbb{N}} f_m^2 \lambda_m^{-2})^{1/2} (\sum_{m\in\mathbb{N}} v_m^2 \lambda_m^2)^{1/2}}{(\sum_{m\in\mathbb{N}} v_m^2 \lambda_m^2)^{1/2}} = (\sum_{m\in\mathbb{N}} f_m^2 \lambda_m^{-2})^{1/2}. \end{aligned} \tag{3.24}$$

Therefore we have

$$\frac{|u_\infty|^2_{L_2(\omega)}}{\|f\|^2_{H^{-2}(\omega)}} \geq \frac{\sum_{m\in\mathbb{N}} f_m^2 \lambda_m^{-2}}{\sum_{m\in\mathbb{N}} f_m^2 \lambda_m^{-2}} = 1$$

and, in turn,

$$\|f\|_{H^{-2}(\omega)} \leq |u_\infty|_{L_2(\omega)} = \sqrt{c_1} \leq C_F(\omega)\big||\nabla' u_\infty|\big|_{L_2(\omega)} = C_F(\omega)\sqrt{c_2} \leq C_F(\omega)^2 |f|_{L_2(\omega)}. \tag{3.25}$$

□

Remark 3.4.3. *The proof of the first inequality of (3.18) can be simplified if we use that the map* $-\Delta' : H^2(\omega) \cap H^1_0(\omega) \to L_2(\omega)$ *is a bijection (see Chapter 6.5 in [30]). Indeed: Since also* $H^2_0(\omega) \subset H^1_0(\omega) \cap H^2(\omega)$ *and* $|\Delta' v|_{L_2(\omega)} \leq \|v\|_{H^2(\omega)}$ *we have*

$$\begin{aligned}\sqrt{c_1} = |u_\infty|_{L_2(\omega)} &= \sup_{g\in L_2(\omega)\setminus\{0\}} \frac{(u_\infty, g)_{L_2(\omega)}}{|g|_{L_2(\omega)}} = \sup_{v\in H^2(\omega)\cap H^1_0(\omega)\setminus\{0\}} \frac{(u_\infty, -\Delta' v)_{L_2(\omega)}}{|\Delta' v|_{L_2(\omega)}} \\ &\geq \sup_{v\in H^2_0(\omega)\setminus\{0\}} \frac{(u_\infty, -\Delta' v)_{L_2(\omega)}}{\|v\|_{H^2(\omega)}} = \sup_{v\in H^2_0(\omega)\setminus\{0\}} \frac{(f, v)_{L_2(\omega)}}{\|v\|_{H^2(\omega)}} = \|f\|_{H^{-2}(\omega)}.\end{aligned}$$

Note we can say that we know the constants c_1, c_2 a priori and they exist only under special conditions.

3.4.2 Properties of $u_{\infty h}$

We define analogously

$$c_{1h} := \int_\omega u^2_{\infty h}\, dx', \quad c_{2h} := \int_\omega f u_{\infty h}\, dx' \tag{3.26}$$

where $u_{\infty h}$ is the unique solution of (3.9).

Remark 3.4.4. *It holds* c_{1h}, $c_{2h} > 0$ *if* $f \neq 0$, $f \in L_2(\omega)$, *see Remark 3.4.1.*

We denote by Π_h an interpolation operator from $H^2(\omega) \cap H^1_0(\omega)$ to $S^{0,m}_h(\omega)$ such that there exists a constant C_{int} with

$$\|u - \Pi_h u\|_{H^1(\omega)} \leq h C_{int} |u|_{H^2(\omega)} \tag{3.27}$$

and

$$h = \max_{K\in\mathcal{G}_{\omega,h}} h_K \tag{3.28}$$

where h_K denotes the diameter of the element K and $|\cdot|_{H^2(\omega)}$ denotes the seminorm of $H^2(\omega)$. Note the constant C_{int} depends on m (degree of polynomial) of the space $S^{0,m}_h(\omega)$ (see (3.11)).
For underlying the existence of an interpolation operator Π_h such that inequality (3.27) holds, we give a reference for $n-1=1$ and $n-1=2$. In the case $n-1=1$ we set $\Pi_h u = \sum_{i=1}^{2N} u_i b_i(x_2)$ with $u_i = u(y_i)$ for $1 \leq i \leq N-1$, $u_i = u'(y_i)$ for $N \leq i \leq 2N$ where y_i is a nodal point and for the basis b_i see (3.14). Due to our choice of the interpolation operator we have $u = \Pi_h u$ for $u \in \mathbb{P}_1(K)$ and therefore we can apply Theorem 3.1.4 in [27] and derive the existence of C_{int} such that (3.27) holds.
For $n-1=2$ we use the Argyris method and Π_h on K denotes the associated $\mathbb{P}_5(K)$-interpolation operator and then we obtain (3.27) from Theorem 6.1.1 in [27]. From (2.3.38) in [27] we know that the boundary conditions for the interpolation operator are preserved.

Lemma 3.4.5. *(a priori stability) Let $f \in L_2(\omega)$, ω being convex and h small enough such that*

$$h^2(1 + C_F(\omega)^2)C_{int}^2 C_{reg}^2 \frac{|f|_{L_2(\omega)}}{\|f\|_{H^{-2}(\omega)}} \leq \frac{1}{2} \tag{3.29}$$

holds. For the definition of the constant C_{reg} we refer to (3.22), for C_{int} see (3.27) and for h see (3.28). Then we have the following bounds for the constants c_{1h}, c_{2h}:

$$\|f\|_{H^{-2}(\omega)}/2 \leq \sqrt{c_{1h}} \leq C_F(\omega)\sqrt{c_{2h}} \leq C_F(\omega)^2 |f|_{L_2(\omega)}. \tag{3.30}$$

Proof. The upper bound of c_{1h} and c_{2h} can be shown in the same way as in the proof of Lemma 3.4.2. Next we analyse the lower bound. We have

$$\begin{aligned}\sqrt{c_{1h}} = |u_{\infty h}|_{L_2(\omega)} &\geq |u_\infty|_{L_2(\omega)} - |u_\infty - u_{\infty h}|_{L_2(\omega)} \\ &\geq \|f\|_{H^{-2}(\omega)} - |u_\infty - u_{\infty h}|_{L_2(\omega)}\end{aligned} \tag{3.31}$$

where we used for the last inequality the result of Lemma 3.4.2. From Céa's Lemma (cf. Lemma 4.3.6) it follows

$$\|u_\infty - u_{\infty h}\|_{H^1(\omega)} \leq \frac{C_{con}}{C_{coer}} \inf_{v \in S_h^{0,m}(\omega)} \|u_\infty - v\|_{H^1(\omega)}$$

where C_{con} is the continuity constant and C_{coer} the coercivity constant with respect to the norm $\|\cdot\|_{H^1(\omega)}$ of our bilinear form

$$\int_\omega \nabla' u \cdot \nabla' v \, dx'$$

for $u, v \in H_0^1(\omega)$. For the definition of the constants C_{con} and C_{coer} we refer to Definition 4.2.1. The constant C_{coer} is equal to $\frac{1}{1+C_F(\omega)^2}$ and C_{con} is 1 where $C_F(\omega)$ is the Friedrichs constant. Indeed, we have

$$\begin{aligned}C_{coer}\|u\|_{H^1(\omega)}^2 &= \frac{1}{1+C_F(\omega)^2}\|u\|_{H^1(\omega)}^2 = \frac{1}{1+C_F(\omega)^2}(|u|_{L_2(\omega)}^2 + \big||\nabla' u|\big|_{L_2(\omega)}^2) \\ &\leq \frac{1}{1+C_F(\omega)^2}(C_F(\omega)^2\big||\nabla' u|\big|_{L_2(\omega)}^2 + \big||\nabla' u|\big|_{L_2(\omega)}^2) = \big||\nabla' u|\big|_{L_2(\omega)}^2 \quad \forall u \in H_0^1(\omega)\end{aligned}$$

and

$$\begin{aligned}&\int_\omega \nabla' u \cdot \nabla' v \, dx' \leq \big||\nabla' u|\big|_{L_2(\omega)} \big||\nabla' v|\big|_{L_2(\omega)} \leq \|u\|_{H^1(\omega)}\|v\|_{H^1(\omega)} \\ &= C_{con}\|u\|_{H^1(\omega)}\|v\|_{H^1(\omega)} \quad \forall u, v \in H_0^1(\omega).\end{aligned}$$

This leads to

$$\|u_\infty - u_{\infty h}\|_{H^1(\omega)} \leq (1 + C_F(\omega)^2) \inf_{v \in S_h^{0,m}(\omega)} \|u_\infty - v\|_{H^1(\omega)}. \tag{3.32}$$

Let Π_h be the above introduced interpolation operator, then we have (cf. (3.27))

$$\|u_\infty - \Pi_h u_\infty\|_{H^1(\omega)} \leq h C_{int} |u_\infty|_{H^2(\omega)}$$

and therefore, in combination with inequality (3.32), we derive

$$\begin{aligned}\|u_\infty - u_{\infty h}\|_{H^1(\omega)} &\leq (1 + C_F(\omega)^2) h C_{int} |u_\infty|_{H^2(\omega)} \\ &\leq (1 + C_F(\omega)^2) h C_{int} C_{reg} |\Delta' u_\infty|_{L_2(\omega)} \\ &\leq (1 + C_F(\omega)^2) h C_{int} C_{reg} |f|_{L_2(\omega)}.\end{aligned} \tag{3.33}$$

Further from Aubin-Nitsche Lemma (cf. Lemma 7.6, Kapitel 2 in [12]) we obtain

$$|u_\infty - u_{\infty h}|_{L_2(\omega)} \leq C_{con} \|u_\infty - u_{\infty h}\|_{H^1(\omega)} \sup_{\phi \in L_2(\omega)\setminus\{0\}} \inf_{v \in S_h^{0,m}(\omega)} \frac{\|u_\phi - v\|_{H^1(\omega)}}{|\phi|_{L_2(\omega)}}, \tag{3.34}$$

where $u_\phi \in H_0^1(\omega)$ is the solution of the problem

$$\int_\omega \nabla' v \cdot \nabla' u_\phi \, dx' = \int_\omega \phi v \, dx' \quad \forall v \in H_0^1(\omega)$$

($u_\phi \in H^2(\omega)$ since ω is convex). From (3.27) we derive

$$\begin{aligned} \inf_{v \in S_h^{0,m}(\omega)} \|u_\phi - v\|_{H^1(\omega)} \leq \|u_\phi - \Pi_h u_\phi\|_{H^1(\omega)} \leq h C_{int} |u_\phi|_{H^2(\omega)} &\overset{(3.22)}{\leq} h C_{int} C_{reg} |\Delta' u_\phi|_{L_2(\omega)} \\ &\leq h C_{int} C_{reg} |\phi|_{L_2(\omega)}. \end{aligned}$$

Going back to (3.34) we deduce

$$|u_\infty - u_{\infty h}|_{L_2(\omega)} \leq h C_{int} C_{reg} \|u_\infty - u_{\infty h}\|_{H^1(\omega)}.$$

Combining this result with (3.33) we obtain

$$|u_\infty - u_{\infty h}|_{L_2(\omega)} \leq h^2 (1 + C_F(\omega)^2) C_{int}^2 C_{reg}^2 |f|_{L_2(\omega)}.$$

Due to the assumption (3.29) we then have that

$$|u_\infty - u_{\infty h}|_{L_2(\omega)} \leq \frac{\|f\|_{H^{-2}(\omega)}}{2}.$$

So going back to the inequality (3.31) we derive

$$\sqrt{c_{1h}} \geq \frac{\|f\|_{H^{-2}(\omega)}}{2}$$

and therefore we have shown this lemma. □

Remark 3.4.6. *(a posteriori stability) Instead of estimating the term $|u_\infty - u_{\infty h}|_{L_2(\omega)}$ a priori as in the previous proof one can also estimate this term a posteriori. For the a posteriori estimate of our Poisson problem we refer to Chapter 2.6.1 in [57] or Chapter 4.1 in [72].*

3.5 Computation and Approximation of ψ

The first goal of this section is to find the solution $\psi \in H_0^1(-\ell, \ell)$ of

$$\int_{\Omega_\ell} \nabla(\psi \otimes u_\infty) \cdot \nabla v \, dx = \int_{\Omega_\ell} f v \, dx \tag{3.35}$$

for all $v = \varphi \otimes u_\infty$, $\varphi \in H_0^1(-\ell, \ell)$. Equation (3.35) can be rewritten as

$$\int_{-\ell}^{\ell} \int_\omega \begin{pmatrix} \partial_{x_1} \psi \otimes u_\infty \\ \psi \otimes \nabla' u_\infty \end{pmatrix} \cdot \begin{pmatrix} \partial_{x_1} \varphi \otimes u_\infty \\ \varphi \otimes \nabla' u_\infty \end{pmatrix} dx' dx_1 = \int_{-\ell}^{\ell} \int_\omega f(\varphi \otimes u_\infty) \, dx' dx_1$$

and $\psi \otimes \nabla' u_\infty$ resp. $\varphi \otimes \nabla' u_\infty$ is a short hand for

$$\begin{pmatrix} \psi \otimes \partial_{x_2} u_\infty \\ \vdots \\ \psi \otimes \partial_{x_n} u_\infty \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} \varphi \otimes \partial_{x_2} u_\infty \\ \vdots \\ \varphi \otimes \partial_{x_n} u_\infty \end{pmatrix}.$$

Componentwise we have

$$\int_{-\ell}^{\ell} (\partial_{x_1}\psi)(\partial_{x_1}\varphi)\,dx_1 \int_\omega u_\infty^2\,dx' + \int_{-\ell}^{\ell} \psi\varphi\,dx_1 \int_\omega |\nabla' u_\infty|^2\,dx' = \int_{-\ell}^{\ell} \varphi\,dx_1 \int_\omega f u_\infty\,dx'$$

$$\Leftrightarrow$$

$$\int_{-\ell}^{\ell} c_1\partial_{x_1}\psi\partial_{x_1}\varphi + c_2\psi\varphi\,dx_1 = c_2\int_{-\ell}^{\ell}\varphi\,dx_1 \tag{3.36}$$

with c_1, c_2 as in (3.16). So we are interested in the problem

$$\begin{cases} \psi \in H_0^1(-\ell,\ell), \\ \int_{-\ell}^{\ell} c_1\partial_{x_1}\psi\partial_{x_1}\varphi + c_2\psi\varphi\,dx_1 = c_2\int_{-\ell}^{\ell}\varphi\,dx_1 \quad \forall\varphi\in H_0^1(-\ell,\ell). \end{cases} \tag{3.37}$$

In view of Lemma 3.4.2 the bilinear form $\hat{a}(u,v) = \int_{-\ell}^{\ell} c_1\partial_{x_1}u\partial_{x_1}v + c_2uv\,dx_1$ for $u, v \in H_0^1(-\ell,\ell)$ is continuous (continuity constant=max$\{c_1,c_2\}$) and coercive (coercivity constant = min$\{c_1,c_2\}$) with respect to the H^1-norm for $f \neq 0$. So we can apply the Lax-Milgram Theorem (cf. Theorem 4.2.2) and derive that there exists a unique $\psi \in H_0^1(-\ell,\ell)$ solution of the problem (3.37). Note since $(-\ell,\ell)$ is a convex domain we have as well $\psi \in H^2(-\ell,\ell)$ (see Satz 7.2, Kapitel 2 in [12]).

Since we are in the one-dimensional case and the coefficients c_i are constants, we can compute ψ exactly. Indeed, the strong formulation of (3.37) is

$$\begin{cases} -c_1\psi'' + c_2\psi = c_2 & \text{in } (-\ell,\ell), \\ \psi = 0 & \text{on } \partial(-\ell,\ell). \end{cases}$$

Let us take the ansatz

$$\psi(x_1) = 1 + \tilde{\psi}(x_1)$$

which implies

$$\begin{cases} -c_1\tilde{\psi}'' + c_2\tilde{\psi} = 0 & \text{in } (-\ell,\ell), \\ \tilde{\psi} = -1 & \text{on } \partial(-\ell,\ell). \end{cases} \tag{3.38}$$

For $\tilde{\psi}$ we take the ansatz

$$\tilde{\psi}(x_1) = ae^{\lambda x_1}$$

with $a, \lambda \in \mathbb{R}$ constants and then, when we plug in this ansatz in the first equation of (3.38), we get

$$(-c_1\lambda^2 + c_2)\tilde{\psi} = 0.$$

So we derive the equation $-c_1\lambda^2 + c_2 = 0$ which implies $\lambda^2 = c_2/c_1$ and therefore $\lambda_1 = \sqrt{c_2/c_1}$, $\lambda_2 = -\sqrt{c_2/c_1}$. Hence the general solution of the problem (3.38) is

$$\tilde{\psi}(x_1) = a_1e^{\sqrt{c_2/c_1}x_1} + a_2e^{-\sqrt{c_2/c_1}x_1}.$$

The constants a_1, a_2 we derive from the boundary conditions, i.e. we have the equations

$$-1 = a_1e^{\sqrt{c_2/c_1}\ell} + a_2e^{-\sqrt{c_2/c_1}\ell}, \quad -1 = a_1e^{-\sqrt{c_2/c_1}\ell} + a_2e^{\sqrt{c_2/c_1}\ell}.$$

We compute the constants a_1, a_2 by applying Cramer's rule (cf. p.205 in [32]) to solve the previous equation system, i.e. we have

$$a_1 = \frac{\det\begin{pmatrix} -1 & e^{\sqrt{c_2/c_1}\ell} \\ -1 & e^{-\sqrt{c_2/c_1}\ell} \end{pmatrix}}{\det\begin{pmatrix} e^{-\sqrt{c_2/c_1}\ell} & e^{\sqrt{c_2/c_1}\ell} \\ e^{\sqrt{c_2/c_1}\ell} & e^{-\sqrt{c_2/c_1}\ell} \end{pmatrix}} = \frac{-e^{-\sqrt{c_2/c_1}\ell} + e^{\sqrt{c_2/c_1}\ell}}{e^{-2\sqrt{c_2/c_1}\ell} - e^{2\sqrt{c_2/c_1}\ell}},$$

$$a_2 = \frac{\det\begin{pmatrix} e^{-\sqrt{c_2/c_1}\ell} & -1 \\ e^{\sqrt{c_2/c_1}\ell} & -1 \end{pmatrix}}{\det\begin{pmatrix} e^{-\sqrt{c_2/c_1}\ell} & e^{\sqrt{c_2/c_1}\ell} \\ e^{\sqrt{c_2/c_1}\ell} & e^{-\sqrt{c_2/c_1}\ell} \end{pmatrix}} = \frac{-e^{-\sqrt{c_2/c_1}\ell} + e^{\sqrt{c_2/c_1}\ell}}{e^{-2\sqrt{c_2/c_1}\ell} - e^{2\sqrt{c_2/c_1}\ell}} = a_1.$$

Hence we have

$$\tilde{\psi}(x_1) = \left(\frac{-e^{-\sqrt{c_2/c_1}\ell} + e^{\sqrt{c_2/c_1}\ell}}{e^{-2\sqrt{c_2/c_1}\ell} - e^{2\sqrt{c_2/c_1}\ell}} \right) (e^{\sqrt{c_2/c_1}x_1} + e^{-\sqrt{c_2/c_1}x_1})$$

and therefore

$$\begin{aligned} \psi(x_1) &= 1 + \left(\frac{-e^{-\sqrt{c_2/c_1}\ell} + e^{\sqrt{c_2/c_1}\ell}}{e^{-2\sqrt{c_2/c_1}\ell} - e^{2\sqrt{c_2/c_1}\ell}} \right) (e^{\sqrt{c_2/c_1}x_1} + e^{-\sqrt{c_2/c_1}x_1}) \\ &= 1 - \frac{\cosh(\sqrt{c_2/c_1}x_1)}{\cosh(\sqrt{c_2/c_1}\ell)}. \end{aligned}$$

Note the function ψ is depending on ℓ. For more complicated situations, i.e. when we have, for instance $-a(x_1)b(x')\Delta u_\ell = f(x')$ in Ω_ℓ instead of $-\Delta u_\ell = f(x')$ in Ω_ℓ, then we approximate ψ. Note in this complicated situation we do not have constant coefficients c_i in the problem (3.37) and it is therefore not anymore so easy to compute the solution ψ.

For the approximation of ψ we use the Galerkin finite element method. For a finite dimensional subspace of $H_0^1(-\ell,\ell)$ we choose $S_h^{0,m}(-\ell,\ell)$. For the general definition of the domain $S_h^{0,m}(-\ell,\ell)$ we refer to (3.11) and for the concrete basis elements see (3.15). Then we are analysing the problem

$$\begin{cases} \psi_h \in S_h^{0,m}(-\ell,\ell), \\ \int_{\Omega_\ell} \nabla(\psi_h \otimes u_\infty) \cdot \nabla v \, dx = \int_{\Omega_\ell} f v \, dx \quad \forall v = \varphi \otimes u_\infty, \, \varphi \in S_h^{0,m}(-\ell,\ell) \end{cases}$$

resp.

$$\begin{cases} \psi_h \in S_h^{0,m}(-\ell,\ell), \\ \int_{-\ell}^{\ell} c_1 \partial_{x_1}\psi_h \partial_{x_1}\varphi + c_2 \psi_h \varphi \, dx_1 = c_2 \int_{-\ell}^{\ell} \varphi \, dx_1 \quad \forall \varphi \in S_h^{0,m}(-\ell,\ell). \end{cases} \tag{3.39}$$

From the Lax-Milgram Theorem (cf. Theorem 4.2.2) we conclude that a unique solution $\psi_h \in S_h^{0,m}(-\ell,\ell)$ of the problem (3.39) exists. For the properties of c_i we refer to Subsection 3.4.1.

In the case where we can not explicitly compute u_∞ resp. c_1, c_2 we work with the approximation $u_{\infty h}$ resp. with c_{1h}, c_{2h} for finding the cutoff function. Concretely we then analyse the analogous problem

$$\begin{cases} \psi_{h,h} \in S_h^{0,m}(-\ell,\ell), \\ \int_{\Omega_\ell} \nabla(\psi_{h,h} \otimes u_{\infty h}) \cdot \nabla v \, dx = \int_{\Omega_\ell} f v \, dx \quad \forall v = \varphi \otimes u_{\infty h}, \, \varphi \in S_h^{0,m}(-\ell,\ell) \end{cases}$$

resp.

$$\begin{cases} \psi_{h,h} \in S_h^{0,m}(-\ell,\ell), \\ \int_{-\ell}^{\ell} c_{1h}\partial_{x_1}\psi_{h,h}\partial_{x_1}\varphi + c_{2h}\psi_{h,h}\varphi\,dx_1 = c_{2h}\int_{-\ell}^{\ell}\varphi\,dx_1 \quad \forall\varphi \in S_h^{0,m}(-\ell,\ell). \end{cases} \tag{3.40}$$

The existence of $\psi_{h,h}$ we get from the Lax-Milgram Theorem (cf. Theorem 4.2.2) and for the properties of the constants c_{ih} we refer to Subsection 3.4.2.

3.6 Upper bound

In this section we analyse the error between u_ℓ, the weak solution of (3.1) and the approximation $\psi_{h,h}\otimes u_{\infty h}$ resp. $\psi_h \otimes u_\infty$. More precisely we will derive an upper bound for the error.

3.6.1 The term $M^2(v;\beta,\mathbf{Y},\Omega)$

In this subsection we introduce an a posteriori error majorant for the difference between $u \in H$, the weak solution of a Dirichlet Poisson problem, and a function $v \in H$ where $H \subseteq H_0^1(\Omega)$ is a Hilbert domain. We will use this technique later for computing an upper bound of the error between u_ℓ and its approximation.

Let Ω be an open bounded connected Lipschitz domain and we will apply the next theorem for $\Omega = \omega$ and $\Omega = \Omega_\ell$. Let $H \subseteq H_0^1(\Omega)$ be a Hilbert space and we will apply the next theorem for $H = H_0^1(\omega)$ when $\Omega = \omega$ and for $H = H_0^1(-\ell,\ell)\otimes u_{\infty h}$ or $H = H_0^1(-\ell,\ell)\otimes u_\infty$ when $\Omega = \Omega_\ell$.

Remark 3.6.1. $H_0^1(-\ell,\ell)\otimes u_\infty$ *and* $H_0^1(-\ell,\ell)\otimes u_{\infty h}$ *are closed in* $H_0^1(\Omega_\ell)$*. Indeed: Since* $u_{\infty h}$ *resp.* u_∞ *have zero boundary conditions we have for any converging sequence* $\{w_n = v_n \otimes u_\infty\}$ *in* $H_0^1(-\ell,\ell)\otimes u_\infty$ *resp.* $\{w_n = v_n \otimes u_{\infty h}\}$ *in* $H_0^1(-\ell,\ell)\otimes u_{\infty h}$ *that the limit belongs to* $H_0^1(\Omega_\ell)$*. Therefore* $H_0^1(-\ell,\ell)\otimes u_\infty$ *and* $H_0^1(-\ell,\ell)\otimes u_{\infty h}$ *are closed in* $H_0^1(\Omega_\ell)$*.*

Theorem 3.6.2. *Let* $f \in L_2(\Omega)$*,* $H \subseteq H_0^1(\Omega)$ *be a Hilbert space and* $u \in H$ *be the unique weak solution of the problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.41}$$

Then we have the estimate

$$\big|\big|\nabla(u-v)\big|\big|_{L_2(\Omega)} \le \big|\big|\mathbf{Y} - \nabla v\big|\big|_{L_2(\Omega)} + C_F(\Omega)|\mathrm{div}(\mathbf{Y}) + f|_{L_2(\Omega)} \tag{3.42}$$

with $v \in H$, $\mathbf{Y} \in \mathbf{H}(\Omega;\mathrm{div}) = \{\mathbf{Y} \mid \mathbf{Y} \in \mathbf{L}_2(\Omega), \mathrm{div}(\mathbf{Y}) \in L_2(\Omega)\}$ *equipped with the norm* $\|\mathbf{V}\|_{\mathbf{H}(\Omega;\mathrm{div})} = (\int_\Omega \mathbf{V}\cdot\mathbf{V} + \mathrm{div}(\mathbf{V})\mathrm{div}(\mathbf{V})\,dx)^{1/2}$*. In the quadratic form we have*

$$\begin{aligned} \big|\big|\nabla(u-v)\big|\big|^2_{L_2(\Omega)} &\le (1+\beta)\big|\big|\mathbf{Y} - \nabla v\big|\big|^2_{L_2(\Omega)} + (1+\frac{1}{\beta})C_F(\Omega)^2|\mathrm{div}(\mathbf{Y}) + f|^2_{L_2(\Omega)} \\ &=: M^2(v;\beta,\mathbf{Y},\Omega) \end{aligned} \tag{3.43}$$

for all $\beta > 0$, $\mathbf{Y} \in \mathbf{H}(\Omega,\mathrm{div})$.

For the constant $C_F(\Omega)$ we refer to (3.17).

Proof. We will only outline the main steps of the proof and we refer for details to Chapter 3 in the book [57]. First we show the inequality (3.42). From the weak formulation of the Dirichlet problem (3.41) we get that

$$\int_\Omega \nabla(u-v)\cdot\nabla w\,dx = \int_\Omega fw - \nabla v\cdot\nabla w\,dx$$

for $w, v \in H$ and for $\mathbf{Y} \in \mathbf{H}(\Omega;\mathrm{div})$ we have

$$\int_\Omega w\mathrm{div}(\mathbf{Y}) + \nabla w \cdot \mathbf{Y}\,dx = 0$$

since w is zero at the boundary. Combining this result/equation with the previous one, we derive for $w = u - v \in H$ that

$$\big||\nabla(u-v)|\big|^2_{L_2(\Omega)} \leq \big||\mathbf{Y} - \nabla v|\big|_{L_2(\Omega)}\big||\nabla(u-v)|\big|_{L_2(\Omega)} + |\mathrm{div}(\mathbf{Y}) + f|_{L_2(\Omega)}|u-v|_{L_2(\Omega)}.$$

From this inequality and applying on $|u-v|_{L_2(\Omega)}$ the Friedrichs inequality ($u, v \in H \subseteq H^1_0(\Omega)$) we can conclude the inequality (3.42).
The estimate (3.43), the quadratic form, is a trivial consequence of (3.42) and the Young inequality. $\square$

Lemma 3.6.3. *$M^2(v;\beta,\cdot,\Omega)$ is continuous, strictly convex and coercive on $\boldsymbol{H}(\Omega,\mathrm{div})$.*

Proof.

$$M^2(v;\beta,\mathbf{Y},\Omega) = (1+\beta)\big||\mathbf{Y} - \nabla v|\big|^2_{L_2(\Omega)} + (1+\frac{1}{\beta})C_F(\Omega)^2|\mathrm{div}(\mathbf{Y}) + f|^2_{L_2(\Omega)}$$

is continuous because (squares of) norms are continuous and the norms are bounded for all $\mathbf{Y} \in \mathbf{H}(\Omega,\mathrm{div})$. Next we show the strict convexity. Let us take $\mathbf{Y}_1$, $\mathbf{Y}_2 \in \mathbf{H}(\Omega,\mathrm{div})$ with $\mathbf{Y}_1 \neq \mathbf{Y}_2$ and $t \in (0,1)$. In the next computation we use the inequality

$$\int_\Omega |t\mathbf{u}_1 + (1-t)\mathbf{u}_2|^2\,dx < \int_\Omega t|\mathbf{u}_1|^2 + (1-t)|\mathbf{u}_2|^2\,dx$$

for $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{L}_2(\Omega)$ which follows from the strict convexity of the quadratic function (see [5]). So we have

$$\begin{aligned}
&M^2(v;\beta,t\mathbf{Y}_1 + (1-t)\mathbf{Y}_2,\Omega) = (1+\beta)\big||t\mathbf{Y}_1 + (1-t)\mathbf{Y}_2 - \nabla v|\big|^2_{L_2(\Omega)}\\
&+ (1+\frac{1}{\beta})C_F(\Omega)^2|\mathrm{div}(t\mathbf{Y}_1 + (1-t)\mathbf{Y}_2) + f|^2_{L_2(\Omega)}\\
&= (1+\beta)\int_\Omega |\nabla v|^2 - 2\nabla v\cdot(t\mathbf{Y}_1 + (1-t)\mathbf{Y}_2) + |t\mathbf{Y}_1 + (1-t)\mathbf{Y}_2|^2\,dx\\
&+ (1+\frac{1}{\beta})C_F(\Omega)^2\int_\Omega f^2 + \mathrm{div}(t\mathbf{Y}_1 + (1-t)\mathbf{Y}_2)^2 + 2f\mathrm{div}(t\mathbf{Y}_1 + (1-t)\mathbf{Y}_2)\,dx\\
&< (1+\beta)\int_\Omega t|\nabla v|^2 + (1-t)|\nabla v|^2 - 2t\nabla v\cdot\mathbf{Y}_1 - 2(1-t)\nabla v\cdot\mathbf{Y}_2 + t|\mathbf{Y}_1|^2 + (1-t)|\mathbf{Y}_2|^2\,dx\\
&+ (1+\frac{1}{\beta})C_F(\Omega)^2\int_\Omega tf^2 + (1-t)f^2 + 2tf\mathrm{div}(\mathbf{Y}_1) + 2(1-t)f\mathrm{div}(\mathbf{Y}_2) + t\mathrm{div}(\mathbf{Y}_1)^2 + (1-t)\mathrm{div}(\mathbf{Y}_2)^2\,dx\\
&= tM^2(v;\beta,\mathbf{Y}_1,\Omega) + (1-t)M^2(v;\beta,\mathbf{Y}_2,\Omega)
\end{aligned}$$

which implies the strict convexity of M^2 concerning $\mathbf{Y}$.
Now it remains to show the coercivity of M^2 w.r.t $\mathbf{Y}$. For this we need to show that for any sequence $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ in $\mathbf{H}(\Omega,\mathrm{div})$ with $\|\mathbf{Y}_n\|_{\mathbf{H}(\Omega,\mathrm{div})} \to \infty$ as $n \to \infty$ we have $\lim_{n\to\infty} M^2(v;\beta,\mathbf{Y}_n,\Omega) = \infty$ (see Chapter 3 in [73]). Let $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ be any sequence with $\|\mathbf{Y}_n\|_{\mathbf{H}(\Omega,\mathrm{div})} \to \infty$ as $n \to \infty$. We have

$$\begin{aligned}
M^2(v;\beta,\mathbf{Y}_n,\Omega) &= (1+\beta)\big||\nabla v - \mathbf{Y}_n|\big|^2_{L_2(\Omega)} + (1+\frac{1}{\beta})C_F(\Omega)^2|\mathrm{div}(\mathbf{Y}_n) + f|^2_{L_2(\Omega)}\\
&\geq (1+\beta)\int_\Omega |\mathbf{Y}_n|^2 - 2|\nabla v||\mathbf{Y}_n|\,dx + (1+\frac{1}{\beta})C_F(\Omega)^2\int_\Omega |\mathrm{div}(\mathbf{Y}_n)|^2 - 2|f||\mathrm{div}(\mathbf{Y}_n)|\,dx\\
&\geq \min\{(1+\beta),(1+\frac{1}{\beta})C_F(\Omega)^2\}(\|\mathbf{Y}_n\|^2_{\mathbf{H}(\Omega,\mathrm{div})} - 2\|\mathbf{Y}_n\|_{\mathbf{H}(\Omega,\mathrm{div})}(\int_\Omega |f|^2 + |\nabla v|^2\,dx)^{1/2})\\
&= \min\{(1+\beta),(1+\frac{1}{\beta})C_F(\Omega)^2\}\|\mathbf{Y}_n\|_{\mathbf{H}(\Omega,\mathrm{div})}(\|\mathbf{Y}_n\|_{\mathbf{H}(\Omega,\mathrm{div})} - 2(\int_\Omega |f|^2 + |\nabla v|^2\,dx)^{1/2})
\end{aligned}$$

which implies

$$\lim_{n\to\infty} M^2(v;\beta,\mathbf{Y}_n,\Omega) = \infty$$

due to the property of $\mathbf{Y}_n$. So M^2 is coercive and this completes the proof of this lemma. $\square$

3.6.2 General upper bound

We apply in this subsection the previous introduced technique for finding an upper bound of the difference between u_ℓ, the unique weak solution of (3.1) and its approximation.
First we analyse the case which includes the approximation $u_{\infty h}$.

Theorem 3.6.4. *Let u_ℓ being the unique weak solution of (3.1), $u_{\infty h}$ being the solution of (3.9), $\psi_{h,h}$ being the solution of the problem (3.40) and $f \in L_2(\omega)$. Then we have the estimate*

$$\big|\big|\nabla(u_\ell - \psi_{h,h} \otimes u_{\infty h})\big|\big|_{L_2(\Omega_\ell)} \le M_{Mod} + M_{Disc} \tag{3.44}$$

with

$$M_{Mod} = \big|\big|\nabla(u_\ell - \psi \otimes u_\infty)\big|\big|_{L_2(\Omega_\ell)}, \quad M_{Disc} = \big|\big|\nabla(\psi \otimes u_\infty - \psi_{h,h} \otimes u_{\infty h})\big|\big|_{L_2(\Omega_\ell)}$$

and it holds

$$M_{Disc} \le M^{ub}_{Disc}, \quad M_{Mod} \le M^{ub}_{Mod} \tag{3.45}$$

where

$$\begin{aligned}(M^{ub}_{Disc})^2 &= (1+\delta)C_{\omega,\ell,f}M^2(u_{\infty h};\gamma,\widetilde{\boldsymbol{Y}},\omega) + (1+1/\delta)M^2(\psi_{h,h}\otimes u_{\infty h};\alpha,\widehat{\boldsymbol{Y}},\Omega_\ell)\\ (M^{ub}_{Mod})^2 &= (1+\rho)M^2(\psi_{h,h}\otimes u_{\infty h};\alpha,\widehat{\boldsymbol{Y}},\Omega_\ell) + (1+1/\rho)(M^{ub}_{Disc})^2\end{aligned} \tag{3.46}$$

for any $\alpha,\gamma,\delta,\rho > 0$, any $\widehat{\boldsymbol{Y}} \in \boldsymbol{H}(\Omega_\ell, \mathrm{div})$ and any $\widetilde{\boldsymbol{Y}} \in \boldsymbol{H}(\omega, \mathrm{div}')$ and $C_{\omega,\ell,f}$ is a constant depending on ω,ℓ,f, see (3.52).

Proof. The estimate (3.44) is a simple consequence of the triangle inequality.
Next we verify (3.45) resp. (3.46). Let us start with M_{Disc}, i.e. we have

$$\begin{aligned}M_{Disc} &= \big|\big|\nabla(\psi_{h,h}\otimes u_{\infty h} - \psi\otimes u_\infty)\big|\big|_{L_2(\Omega_\ell)}\\ &\le \big|\big|\nabla(\psi_{h,h}\otimes u_{\infty h} - \psi\otimes u_{\infty h})\big|\big|_{L_2(\Omega_\ell)} + \big|\big|\nabla(\psi\otimes u_{\infty h} - \psi\otimes u_\infty)\big|\big|_{L_2(\Omega_\ell)}\end{aligned}$$

respectively (Young inequality)

$$\begin{aligned}M^2_{Disc} \le &(1+1/\delta)\big|\big|\nabla(\psi_{h,h}\otimes u_{\infty h} - \psi\otimes u_{\infty h})\big|\big|^2_{L_2(\Omega_\ell)}\\ &+ (1+\delta)\big|\big|\nabla(\psi\otimes u_{\infty h} - \psi\otimes u_\infty)\big|\big|^2_{L_2(\Omega_\ell)}\end{aligned} \tag{3.47}$$

for any $\delta > 0$. For the first term on the right hand side of the inequality (3.47) we have (see Theorem 3.6.2)

$$\big|\big|\nabla(\psi_{h,h}\otimes u_{\infty h} - \psi\otimes u_{\infty h})\big|\big|^2_{L_2(\Omega_\ell)} \le M^2(\psi_{h,h}\otimes u_{\infty h};\alpha,\widehat{\mathbf{Y}},\Omega_\ell) \tag{3.48}$$

with $\widehat{\mathbf{Y}} \in \mathbf{H}(\Omega_\ell, \mathrm{div})$ and any $\alpha > 0$. Note in the case where we work with ψ exactly, the term $\big|\big|\nabla(\psi_{h,h}\otimes u_{\infty h} - \psi\otimes u_{\infty h})\big|\big|^2_{L_2(\Omega_\ell)}$ drops out.

Now we analyse the second term on the right hand side of the inequality (3.47). We have

$$
\begin{aligned}
&\big|\big|\nabla(\psi\otimes u_{\infty h}-\psi\otimes u_\infty)\big|\big|^2_{L_2(\Omega_\ell)} = \int_{-\ell}^{\ell}(\partial_{x_1}\psi)^2\,dx_1\int_\omega (u_{\infty h}-u_\infty)^2\,dx' \\
&+\int_{-\ell}^{\ell}\psi^2\,dx_1\int_\omega |\nabla'(u_{\infty h}-u_\infty)|^2\,dx' \\
&\le |\partial_{x_1}\psi|^2_{L_2(-\ell,\ell)}\big|\big|\nabla'(u_{\infty h}-u_\infty)\big|\big|^2_{L_2(\omega)}(C_F(\omega)^2+C_F(-\ell,\ell)^2).
\end{aligned}
\tag{3.49}
$$

For computing the upper bound of $|\partial_{x_1}\psi|_{L_2(-\ell,\ell)}$ we put ψ as a test function into the equation

$$
\int_{-\ell}^{\ell} c_1\partial_{x_1}\psi\partial_{x_1}\varphi + c_2\psi\varphi\,dx_1 = \int_{-\ell}^{\ell} c_2\varphi\,dx_1 \quad \forall\varphi\in H^1_0(-\ell,\ell)
$$

and we derive

$$
|\partial_{x_1}\psi|_{L_2(-\ell,\ell)} \le \frac{c_2}{c_1}\sqrt{2\ell}C_F(-\ell,\ell) \le \frac{|f|^2_{L_2(\omega)}}{\|f\|^2_{H^{-2}(\omega)}}\sqrt{2\ell}C_F(-\ell,\ell)C_F(\omega)^2 \tag{3.50}
$$

(for the last inequality see Lemma 3.4.2). Going back to (3.49) we get

$$
\big|\big|\nabla(\psi\otimes u_{\infty h}-\psi\otimes u_\infty)\big|\big|^2_{L_2(\Omega_\ell)} \le C_{\omega,\ell,f}\big|\big|\nabla'(u_{\infty h}-u_\infty)\big|\big|^2_{L_2(\omega)} \tag{3.51}
$$

with

$$
C_{\omega,\ell,f} := (C_F(\omega)^2+C_F(-\ell,\ell)^2)\frac{|f|^2_{L_2(\omega)}}{\|f\|^2_{H^{-2}(\omega)}}\sqrt{2\ell}C_F(-\ell,\ell)C_F(\omega)^2. \tag{3.52}
$$

For the term $\big|\big|\nabla'(u_\infty-u_{\infty h})\big|\big|_{L_2(\omega)}$ we have the estimate (see Theorem 3.6.2)

$$
\big|\big|\nabla'(u_\infty-u_{\infty h})\big|\big|^2_{L_2(\omega)} \le M^2(u_{\infty h};\gamma,\widetilde{\mathbf{Y}},\omega) \tag{3.53}
$$

for any $\widetilde{\mathbf{Y}}\in\mathbf{H}(\omega;\mathrm{div}')$ and any $\gamma>0$. This implies the estimate

$$
\big|\big|\nabla(\psi\otimes u_{\infty h}-\psi\otimes u_\infty)\big|\big|^2_{L_2(\Omega_\ell)} \le C_{\omega,\ell,f}M^2(u_{\infty h};\gamma,\widetilde{\mathbf{Y}},\omega). \tag{3.54}
$$

Putting now all the obtained results/estimates together (see (3.47), (3.48), (3.54)) we finally obtain

$$
\begin{aligned}
M^2_{Disc} &\le (1+\delta)C_{\omega,\ell,f}M^2(u_{\infty h};\gamma,\widetilde{\mathbf{Y}},\omega) + (1+1/\delta)M^2(\psi_{h,h}\otimes u_{\infty h};\alpha,\widehat{\mathbf{Y}},\Omega_\ell) \\
&=: (M^{ub}_{Disc})^2.
\end{aligned}
\tag{3.55}
$$

Next we analyse M_{Mod} and we have (using Young inequality and applying Theorem 3.6.2)

$$
\begin{aligned}
&M^2_{Mod} = \big|\big|\nabla(u_\ell-\psi\otimes u_\infty)\big|\big|^2_{L_2(\Omega_\ell)} \\
&\le (1+\rho)\big|\big|\nabla(u_\ell-\psi_{h,h}\otimes u_{\infty h})\big|\big|^2_{L_2(\Omega_\ell)} + (1+1/\rho)\big|\big|\nabla(\psi_{h,h}\otimes u_{\infty h}-\psi\otimes u_\infty)\big|\big|^2_{L_2(\Omega_\ell)} \\
&\le (1+\rho)M^2(\psi_{h,h}\otimes u_{\infty h};\alpha,\widehat{\mathbf{Y}},\Omega_\ell) + (1+1/\rho)(M^{ub}_{Disc})^2 =: (M^{ub}_{Mod})^2
\end{aligned}
\tag{3.56}
$$

for any positive α,ρ and $\widehat{\mathbf{Y}}\in\mathbf{H}(\Omega_\ell;\mathrm{div})$. This completes the proof of the theorem. □

If we can compute u_∞ exactly we have the following analogous theorem:

Theorem 3.6.5. *Let u_ℓ be the unique weak solution of (3.1), u_∞ be the unique weak solution of (3.2), ψ_h be the solution of the problem (3.39) and $f \in L_2(\omega)$. Then we have the estimate*

$$\left||\nabla(u_\ell - \psi_h \otimes u_\infty)|\right|_{L_2(\Omega_\ell)} \leq M_{Mod} + M_{Disc}$$

with

$$M_{Mod} = \left||\nabla(u_\ell - \psi \otimes u_\infty)|\right|_{L_2(\Omega_\ell)}, \quad M_{Disc} = \left||\nabla(\psi \otimes u_\infty - \psi_h \otimes u_\infty)|\right|_{L_2(\Omega_\ell)}$$

and it holds

$$M_{Disc} \leq M_{Disc}^{ub}, \quad M_{Mod} \leq M_{Mod}^{ub} \tag{3.57}$$

where

$$\begin{aligned} (M_{Disc}^{ub})^2 &= M^2(\psi_h \otimes u_\infty; \alpha, \widehat{\boldsymbol{Y}}, \Omega_\ell) \\ (M_{Mod}^{ub})^2 &= (1+\rho)M^2(\psi_h \otimes u_\infty; \alpha, \widehat{\boldsymbol{Y}}, \Omega_\ell) + (1+1/\rho)(M_{Disc}^{ub})^2 \\ &= (1+\rho)(M_{Disc}^{ub})^2 + (1+1/\rho)(M_{Disc}^{ub})^2 \end{aligned} \tag{3.58}$$

for any $\alpha, \rho > 0$ and any $\widehat{\boldsymbol{Y}} \in \boldsymbol{H}(\Omega_\ell, \mathrm{div})$.

The proof of this theorem follows the same lines as the proof of Theorem 3.6.4 and is skipped. We see that the majorant in $(M_{disc}^{ub})^2$ is in this case simplified.

3.7 Ansatz for $\widehat{\mathbf{Y}}$

In this section we introduce the choice of $\widehat{\mathbf{Y}}$ which appears in $M^2(\psi_{h,h} \otimes u_{\infty h}; \alpha, \widehat{\mathbf{Y}}, \Omega_\ell)$ of Theorem 3.6.4. The goal is to compute $\widehat{\mathbf{Y}}$ without solving a n-dimensional problem and so we take a tensor ansatz. In our case one function is depending on x' and the other one on x_1, i.e. we have to solve a $(n-1)$-dimensional and a one dimensional problem. For gaining more accuracy we add an additional term and in our case of the same structure. This kind of strategy has already been used in other works, see for instance [2], [60], [61] or [63].
First let us set

$$\psi_{h,h}^0 := \psi_{h,h}, \quad u_{\infty h}^0 := u_{\infty h}$$

where $\psi_{h,h}$ is the solution of (3.40) and $u_{\infty h}$ is the solution of (3.9). We employ the ansatz

$$u_\ell \approx \psi_{h,h}^0 \otimes u_{\infty h}^0 + \psi_{h,h}^0 \otimes u_{\infty h}^1 \tag{3.59}$$

where $u_{\infty h}^1 \in S_h^{0,m}(\omega)$ is the solution of

$$\int_{\Omega_\ell} \nabla(\psi_{h,h}^0 \otimes u_{\infty h}^1) \cdot \nabla(\psi_{h,h}^0 \otimes v)\, dx = \int_{\Omega_\ell} f(\psi_{h,h}^0 \otimes v) - \nabla(\psi_{h,h}^0 \otimes u_{\infty h}^0) \cdot \nabla(\psi_{h,h}^0 \otimes v)\, dx \quad \forall v \in S_h^{0,m}(\omega)$$

or equivalently

$$\int_\omega d_{1h} \nabla' u_{\infty h}^1 \cdot \nabla' v + d_{2h} u_{\infty h}^1 v\, dx' = \int_\omega d_{3h} f v - d_{1h} \nabla' u_{\infty h}^0 \cdot \nabla' v - d_{2h} u_{\infty h}^0 v\, dx' \quad \forall v \in S_h^{0,m}(\omega) \tag{3.60}$$

with

$$d_{1h} := \int_{-\ell}^{\ell} (\psi_{h,h}^0)^2\, dx_1, \quad d_{2h} := \int_{-\ell}^{\ell} (\partial_{x_1} \psi_{h,h}^0)^2\, dx_1, \quad d_{3h} := \int_{-\ell}^{\ell} \psi_{h,h}^0\, dx_1.$$

In principle we can iterate the ansatz (3.59) and perform a further step, i.e. we take $u_\ell \approx \psi^0_{h,h} \otimes u^0_{\infty h} + \psi^1_{h,h} \otimes u^1_{\infty h}$. We know $\psi^0_{h,h}$, $u^0_{\infty h}$ and $u^1_{\infty h}$ and then we can compute in an analogous way as $u^1_{\infty h}$ the cutoff function $\psi^1_{h,h} \in S^{0,m}_h(-\ell,\ell)$. For the choice of $\hat{\mathbf{Y}}$ we restrict to the ansatz $\psi^0_{h,h} \otimes u^0_{\infty h} + \psi^0_{h,h} \otimes u^1_{\infty h}$ and set

$$\widehat{\mathbf{Y}} := \nabla(\psi^0_{h,h} \otimes u^0_{\infty h} + \psi^0_{h,h} \otimes u^1_{\infty h}) \tag{3.61}$$

and $\widehat{\mathbf{Y}} \in \mathbf{H}(\Omega_\ell, \mathrm{div})$. Indeed since $\psi^0_{h,h} \in S^{0,m}_h(-\ell,\ell)$ and $u^0_{\infty h}$, $u^1_{\infty h} \in S^{0,m}_h(\omega)$ we have that $\psi^0_{h,h} \in C^1(-\ell,\ell)$ and $u^0_{\infty h}$, $u^1_{\infty h} \in C^1(\omega)$ and therefore $\widehat{\mathbf{Y}} \in \mathbf{H}(\Omega_\ell, \mathrm{div})$.

Remark 3.7.1. *If u_∞ is known explicitly we choose $\widehat{\boldsymbol{Y}}$ in $M^2(\psi_h \otimes u_\infty; \alpha, \widehat{\boldsymbol{Y}}, \Omega_\ell)$ of Theorem 3.6.5 as $\widehat{\boldsymbol{Y}}$ in $M^2(\psi_{h,h} \otimes u_{\infty h}; \alpha, \widehat{\boldsymbol{Y}}, \Omega_\ell)$ of Theorem 3.6.4, i.e. we take*

$$\widehat{\boldsymbol{Y}} := \nabla(\psi^0_h \otimes u^0_\infty + \psi^0_h \otimes u^1_\infty) \tag{3.62}$$

with

$$\psi^0_h := \psi_h, \quad u^0_\infty := u_\infty$$

where $\psi_h \in S^{0,m}_h(-\ell,\ell)$ is the solution of (3.39), $u_\infty \in H^1_0(\omega)$ is the unique weak solution of (3.2) and $u^1_\infty \in S^{0,m}_h(\omega)$ is the solution of the analogous problem of (3.60). Further we assume that $u_\infty \in H^2(\omega)$, which is true for instance if ω is convex.

3.8 Minimization processes

The estimate (3.45) resp. (3.57) holds for any $\alpha, \gamma, \delta, \rho > 0$ and $\widehat{\mathbf{Y}} \in \mathbf{H}(\Omega_\ell, \mathrm{div})$, $\widetilde{\mathbf{Y}} \in \mathbf{H}(\omega, \mathrm{div}')$ resp. for any $\alpha, \rho > 0$ and $\widehat{\mathbf{Y}} \in \mathbf{H}(\Omega_\ell, \mathrm{div})$. The goal of this section is to find optimal $\alpha, \gamma, \delta, \rho$ and $\widetilde{\mathbf{Y}}$ resp. α, ρ such that the upper bound of the error becomes minimal. For the choice of $\widehat{\mathbf{Y}}$ we refer to the previous section.

3.8.1 Minimization of $(1+\alpha)a + (1+1/\alpha)b$ w.r.t. α for $a, b \geq 0$

In this subsection we study the minimization of the function $(1+\alpha)a+(1+1/\alpha)b$ with positive α concerning α for a, b being non-negative constants. Note by applying Young inequality we have

$$\sqrt{a} + \sqrt{b} \leq (1+\alpha)a + (1+1/\alpha)b =: h(\alpha) \tag{3.63}$$

for any $\alpha > 0$. Then we have

$$\frac{\partial h}{\partial \alpha} = a - \alpha^{-2} b.$$

So

$$\frac{\partial h}{\partial \alpha} = 0 \quad \Rightarrow \alpha^2 = \frac{b}{a} \quad \Rightarrow \alpha = \sqrt{\frac{b}{a}} \geq 0.$$

Since $\frac{\partial^2 h}{\partial \alpha^2} = 2\alpha^{-3} b > 0$ for $\alpha, b > 0$ we have that $\alpha = \sqrt{\frac{b}{a}}$ is a minimum of the function h. Note in the case $a = 0$ we have $h(\alpha) = (1+1/\alpha)b$ and so there is no problem with $\alpha = \sqrt{\frac{b}{a}}$. If we plug in our minimal α in (3.63) we derive

$$h(\sqrt{\frac{b}{a}}) = (\sqrt{a} + \sqrt{b})^2$$

which is equivalent to

$$\inf_{\alpha>0} h(\alpha).$$

Note for the special case $a = b$ we have that

$$\inf_{\alpha>0} h(\alpha) = 4a.$$

This special case appears in $(M^{ub}_{Mod})^2$ of Theorem 3.6.5 and therefore we have

$$\left|\left|\nabla(u_\ell - \psi_h \otimes u_\infty)\right|\right|_{L_2(\Omega_\ell)} \leq \sqrt{5} M^{ub}_{Disc}.$$

This minimization process for $a \neq b$ we apply on $(M^{ub}_{Disc})^2$ and $(M^{ub}_{Mod})^2$ of Theorem 3.6.4 and obtain

$$\inf_{\delta\in\mathbb{R}^+} (M^{ub}_{Disc})^2 = \left(\sqrt{C_{\omega,\ell,f} M^2(u_{\infty h}; \gamma, \widetilde{\mathbf{Y}}, \omega)} + \sqrt{M^2(\psi_{h,h} \otimes u_{\infty h}; \alpha, \widehat{\mathbf{Y}}, \Omega_\ell)}\right)^2,$$
$$\inf_{\rho\in\mathbb{R}^+} (M^{ub}_{Mod})^2 = \left(\sqrt{M^2(\psi_{h,h} \otimes u_{\infty h}; \alpha, \widehat{\mathbf{Y}}, \Omega_\ell)} + M^{ub}_{Disc}\right)^2.$$

A term such as (3.63) also appears in

$$M^2(v; \beta, \mathbf{Y}, \Omega) = (1+\beta)\left|\left|\mathbf{Y} - \nabla v\right|\right|^2_{L_2(\Omega)} + (1 + \frac{1}{\beta}) C_F(\Omega)^2 |\mathrm{div}(\mathbf{Y}) + f|^2_{L_2(\Omega)} \tag{3.64}$$

and we have

$$\inf_{\beta\in\mathbb{R}^+} M^2(v; \beta, \mathbf{Y}, \Omega) = (\left|\left|\mathbf{Y} - \nabla v\right|\right|_{L_2(\Omega)} + C_F(\Omega)|\mathrm{div}(\mathbf{Y}) + f|_{L_2(\Omega)})^2 \tag{3.65}$$

which is equivalent to $M^2(v; \beta_{min}, \mathbf{Y}, \Omega)$ with

$$\beta_{min} = \frac{C_F(\Omega)|\mathrm{div}(\mathbf{Y}) + f|_{L_2(\Omega)}}{\left|\left|\mathbf{Y} - \nabla v\right|\right|_{L_2(\Omega)}}. \tag{3.66}$$

Note for the later computations concerning the minimization of $M^2(v; \beta, \mathbf{Y}, \Omega)$ with respect to $\mathbf{Y}$ we will continue working with the form (3.64) for $M^2(v; \beta, \mathbf{Y}, \Omega)$ since it is a quadratic form with respect to each norm. With this form it is easier to compute the minimum w.r.t. $\mathbf{Y}$ which we will see later.
Recall in Theorem 3.6.4 resp. Theorem 3.6.5 we have the terms $M^2(u_{\infty h}; \gamma, \widetilde{\mathbf{Y}}, \omega)$, $M^2(\psi_{h,h} \otimes u_{\infty h}; \alpha, \widehat{\mathbf{Y}}, \Omega_\ell)$ resp. $M^2(\psi_h \otimes u_\infty; \alpha, \widehat{\mathbf{Y}}, \Omega_\ell)$. Since we fixed $\widehat{\mathbf{Y}}$ with the ansatz (cf. (3.61), (3.62)), the minimization process of $M^2(\psi_{h,h} \otimes u_{\infty h}; \alpha, \widehat{\mathbf{Y}}, \Omega_\ell)$ and $M^2(\psi_h \otimes u_\infty; \alpha, \widehat{\mathbf{Y}}, \Omega_\ell)$ stops here.

3.8.2 Minimization of $M^2(v; \beta, \mathbf{Y}, \Omega)$ w.r.t. $\mathbf{Y}$

In this subsection we are interested in the minimization process of

$$M^2(v; \beta, \mathbf{Y}, \Omega) = (1+\beta)\left|\left|\mathbf{Y} - \nabla v\right|\right|^2_{L_2(\Omega)} + (1 + \frac{1}{\beta}) C_F(\Omega)^2 |\mathrm{div}(\mathbf{Y}) + f|^2_{L_2(\Omega)}$$

with respect to $\mathbf{Y} \in \mathbf{H}(\Omega, \mathrm{div})$. Note from Lemma 3.6.3 we know that there exists a unique minimizer $\mathbf{Y}_{opt} \in \mathbf{H}(\Omega, \mathrm{div})$ of $M^2(v; \beta, \mathbf{Y}, \Omega)$. For the minimization process we assume that β is a fixed positive constant. Now let us define

$$J(\mathbf{Y}) := (1+\beta)\left|\left|\mathbf{Y} - \nabla v\right|\right|^2_{L_2(\Omega)} + (1 + 1/\beta) C_F(\Omega)^2 |\mathrm{div}(\mathbf{Y}) + f|^2_{L_2(\Omega)}$$

and let us set $\mathbf{Y} = \mathbf{Y}_{opt} + \epsilon\mathbf{V}$ with $\mathbf{Y}_{opt}, \mathbf{V} \in \mathbf{H}(\Omega, \mathrm{div})$ and $\epsilon \in \mathbb{R}$. The minimum is given by computing

$$\begin{aligned}
&\lim_{\epsilon\to 0} \frac{J(\mathbf{Y}_{opt} + \epsilon\mathbf{V}) - J(\mathbf{Y}_{opt})}{\epsilon} = 0 \quad \Leftrightarrow \\
&\lim_{\epsilon\to 0} \frac{(1+\beta)(\big|\big|\mathbf{Y}_{opt} + \epsilon\mathbf{V} - \nabla v\big|\big|^2_{L_2(\Omega)} + \big|\big|\mathbf{Y}_{opt} - \nabla v\big|\big|^2_{L_2(\Omega)})}{\epsilon} \\
&+ \lim_{\epsilon\to 0} \frac{(1+\frac{1}{\beta})C_F(\Omega)^2(|\mathrm{div}(\mathbf{Y}_{opt} + \epsilon\mathbf{V}) + f|^2_{L_2(\Omega)} - |\mathrm{div}(\mathbf{Y}_{opt}) + f|^2_{L_2(\Omega)})}{\epsilon} = 0 \quad \Leftrightarrow \\
&(1+\beta)(\mathbf{Y}_{opt} - \nabla v, \mathbf{V})_{L_2(\Omega)} + (1+1/\beta)C_F(\Omega)^2(\mathrm{div}(\mathbf{Y}_{opt}) + f, \mathbf{V})_{L_2(\Omega)} = 0 \quad \forall \mathbf{V} \in \mathbf{H}(\Omega, \mathrm{div}).
\end{aligned}$$

So we would like to find $\mathbf{Y}_{opt} \in \mathbf{H}(\Omega, \mathrm{div})$ such that

$$\begin{aligned}
&(\mathbf{Y}_{opt}, \mathbf{V})_{L_2(\Omega)} + \frac{C_F(\Omega)^2}{\beta}(\mathrm{div}(\mathbf{Y}_{opt}), \mathrm{div}(\mathbf{V}))_{L_2(\Omega)} \\
&= (\nabla v, \mathbf{V})_{L_2(\Omega)} - \frac{C_F(\Omega)^2}{\beta}(f, \mathrm{div}(\mathbf{V}))_{L_2(\Omega)} \quad \forall \mathbf{V} \in \mathbf{H}(\Omega, \mathrm{div}).
\end{aligned} \tag{3.67}$$

The bilinear form

$$a(\mathbf{u}, \mathbf{v}) := (\mathbf{u}, \mathbf{v})_{L_2(\Omega)} + \frac{C_F(\Omega)^2}{\beta}(\mathrm{div}(\mathbf{u}), \mathrm{div}(\mathbf{v}))_{L_2(\Omega)} \tag{3.68}$$

for $\mathbf{u}$, $\mathbf{v} \in \mathbf{H}(\Omega, \mathrm{div})$ is continuous and coercive with respect to the norm $\|\cdot\|_{\mathbf{H}(\Omega,\mathrm{div})} = (|\cdot|^2_{L_2(\Omega)} + |\mathrm{div}(\cdot)|^2_{L_2(\Omega)})^{1/2}$. The continuity constant of $a(\cdot,\cdot)$ is $\max\{1, \frac{C_F(\Omega)^2}{\beta}\}$ and the coercivity constant of $a(\cdot,\cdot)$ is $\min\{1, \frac{C_F(\Omega)^2}{\beta}\}$. To avoid ill-posedness of $a(\cdot,\cdot)$ due to division by 0 or ∞ by the coefficient $\frac{C_F(\Omega)}{\beta}$ we select $\beta_0 \in (0,1)$ and restrict

$$\beta \in [\beta_0, \frac{1}{\beta_0}] \tag{3.69}$$

and then the constants are $\max\{1, \frac{C_F(\Omega)^2}{\beta_0}\}$, $\min\{1, \frac{C_F(\Omega)^2}{\beta_0^{-1}}\}$.

The map $\mathbf{V} \mapsto (\nabla v, \mathbf{V})_{L_2(\Omega)} - \frac{C_F(\Omega)^2}{\beta}(f, \mathrm{div}(\mathbf{V}))_{L_2(\Omega)}$ for $\mathbf{V} \in \mathbf{H}(\Omega, \mathrm{div})$ is a continuous linear map and we have f, ∇v belonging to $L_2(\Omega_\ell)$. Hence we can apply the Lax-Milgram Theorem (cf. Theorem 4.2.2) for any fixed positive β and we derive that there exists a solution $\mathbf{Y}_{opt} \in \mathbf{H}(\Omega, \mathrm{div})$ of the problem

$$a(\mathbf{Y}_{opt}, \mathbf{V}) = (\nabla v, \mathbf{V})_{L_2(\Omega)} - \frac{C_F(\Omega)^2}{\beta}(f, \mathrm{div}(\mathbf{V}))_{L_2(\Omega)} \quad \forall \mathbf{V} \in \mathbf{H}(\Omega, \mathrm{div}). \tag{3.70}$$

3.8.3 Regularity of $\mathbf{Y}_{opt}$

First let us define

$$\mathbf{H}^m(\Omega, \mathrm{div}) := \{\mathbf{q} \mid \mathbf{q} \in \mathbf{H}^m(\Omega) = (H^m(\Omega))^n,\ \mathrm{div}(\mathbf{q}) \in H^m(\Omega)\}.$$

Lemma 3.8.1. *Let us take $f, v \in H^1(\Omega)$ with $\Delta v \in L_2(\Omega)$. Then $\mathbf{Y}_{opt}$, the solution of the problem (3.70), belongs to*

(i) $\boldsymbol{H}^1(\Omega, \mathrm{div})$, when Ω is convex and $v, f \in H^2(\Omega)$,

(ii) $\boldsymbol{H}^{m-1}(\Omega, \mathrm{div})$, when $\partial\Omega$ belongs to C^m, $m \geq 2$ and $v, f \in H^m(\Omega)$.

Proof. We have that $\mathbf{Y}_{opt} \in \mathbf{H}(\Omega, \mathrm{div})$ and (cf. (3.70))

$$a(\mathbf{Y}_{opt}, \mathbf{V}) = (\nabla v, \mathbf{V})_{L_2(\Omega)} - \frac{C_F(\Omega)^2}{\beta}(f, \mathrm{div}(\mathbf{V}))_{L_2(\Omega)} \quad \forall \mathbf{V} \in \mathbf{H}(\Omega, \mathrm{div}) \qquad \Leftrightarrow$$

$$(\mathbf{Y}_{opt}, \mathbf{V})_{L_2(\Omega)} + \frac{C_F(\Omega)^2}{\beta}(\mathrm{div}(\mathbf{Y}_{opt}), \mathrm{div}(\mathbf{V}))_{L_2(\Omega)} = (\nabla v, \mathbf{V})_{L_2(\Omega)} - \frac{C_F(\Omega)^2}{\beta}(f, \mathrm{div}(\mathbf{V}))_{L_2(\Omega)} \tag{3.71}$$

$\forall \mathbf{V} \in \mathbf{H}(\Omega, \mathrm{div})$.

By partial integration we derive

$$\begin{aligned} &\int_\Omega \mathbf{Y}_{opt} \cdot \mathbf{V}\, dx + \frac{C_F(\Omega)^2}{\beta}\left(\int_\Omega -\nabla(\mathrm{div}(\mathbf{Y}_{opt})) \cdot \mathbf{V} + \mathrm{div}(\mathrm{div}(\mathbf{Y}_{opt})\mathbf{V})\, dx\right) \\ &= \int_\Omega \nabla v \cdot \mathbf{V}\, dx - \frac{C_F(\Omega)^2}{\beta}\left(\int_\Omega \mathrm{div}(f\mathbf{V}) - \nabla f \cdot \mathbf{V}\, dx\right) \quad \forall \mathbf{V} \in \mathbf{H}(\Omega, \mathrm{div}). \end{aligned}$$

Next, by applying the divergence formula we derive

$$\begin{aligned} &\int_\Omega \mathbf{Y}_{opt} \cdot \mathbf{V}\, dx + \frac{C_F(\Omega)^2}{\beta}\left(\int_\Omega -\nabla(\mathrm{div}(\mathbf{Y}_{opt})) \cdot \mathbf{V}\, dx + \int_{\partial\Omega} \mathrm{div}(\mathbf{Y}_{opt})\mathbf{V} \cdot \nu\, d\sigma(x)\right) \\ &= \int_\Omega \nabla v \cdot \mathbf{V}\, dx - \frac{C_F(\Omega)^2}{\beta}\left(\int_{\partial\Omega} f\mathbf{V} \cdot \nu\, d\sigma(x) - \int_\Omega \nabla f \cdot \mathbf{V}\, dx\right) \quad \forall \mathbf{V} \in \mathbf{H}(\Omega, \mathrm{div}), \end{aligned} \tag{3.72}$$

where ν denotes the unit outward normal to $\partial\Omega$. Now let us set

$$\mathbf{H}_0(\Omega, \mathrm{div}) = \{\mathbf{v} \in \mathbf{H}(\Omega, \mathrm{div}) \mid \mathbf{v} \cdot \nu|_{\partial\Omega} = 0\}.$$

From (3.72) we then derive

$$\begin{aligned} &\int_\Omega \mathbf{Y}_{opt} \cdot \mathbf{V}\, dx - \frac{C_F(\Omega)^2}{\beta}\int_\Omega \nabla(\mathrm{div}(\mathbf{Y}_{opt})) \cdot \mathbf{V}\, dx \\ &= \int_\Omega \nabla v \cdot \mathbf{V}\, dx + \frac{C_F(\Omega)^2}{\beta}\int_\Omega \nabla f \cdot \mathbf{V}\, dx \quad \forall \mathbf{V} \in \mathbf{H}_0(\Omega, \mathrm{div}). \end{aligned} \tag{3.73}$$

The next step is to find the strong formulation of the problem (3.71).
Claim: Y_{opt} is the solution of

$$\begin{cases} \mathbf{Y}_{opt} - \frac{C_F(\Omega)^2}{\beta}\nabla(\mathrm{div}(\mathbf{Y}_{opt})) = \nabla v + \frac{C_F(\Omega)^2}{\beta}\nabla f & \text{a.e. in } \Omega, \\ \mathrm{div}(\mathbf{Y}_{opt}) = -f & \text{a.e. on } \partial\Omega. \end{cases} \tag{3.74}$$

Proof. From (3.73) we deduce

$$(\mathbf{Y}_{opt} - \frac{C_F(\Omega)^2}{\beta}\nabla(\mathrm{div}(\mathbf{Y}_{opt})) - \nabla v - \frac{C_F(\Omega)^2}{\beta}\nabla f, \mathbf{V})_{L_2(\Omega)} = 0 \tag{3.75}$$

for all $\mathbf{V} \in \mathbf{H}_0(\Omega, \mathrm{div})$. From this we conclude

$$\mathbf{Y}_{opt} - \frac{C_F(\Omega)^2}{\beta}\nabla(\mathrm{div}(\mathbf{Y}_{opt})) - \nabla v - \frac{C_F(\Omega)^2}{\beta}\nabla f = 0 \quad \text{a.e. in } \Omega \tag{3.76}$$

(see Corollary 4.24 in [11]). From this and (3.72) we get

$$\begin{aligned} &\int_{\partial\Omega} \mathrm{div}(\mathbf{Y}_{opt})\mathbf{V} \cdot \nu\, d\sigma(x) = -\int_{\partial\Omega} f\mathbf{V} \cdot \nu\, d\sigma(x) \quad \forall \mathbf{V} \in \mathbf{H}(\Omega, \mathrm{div}) \quad \Leftrightarrow \\ &\int_{\partial\Omega} (\mathrm{div}(\mathbf{Y}_{opt}) + f)\mathbf{V} \cdot \nu\, d\sigma(x) = 0 \quad \forall \mathbf{V} \in \mathbf{H}(\Omega, \mathrm{div}) \end{aligned}$$

which implies (cf. Corollary 4.24 in [11])

$$\operatorname{div}(\mathbf{Y}_{opt}) = -f \quad \text{a.e. on } \partial\Omega.$$

Note if $\Omega = (a,b) \subset \mathbb{R}$, $n = 1$ so the boundary is a finite number of points, then we can show previous equality also in a different way. Indeed: We have

$$\int_{\partial\Omega} (\operatorname{div}(\mathbf{Y}_{opt}) + f)\mathbf{V} \cdot \nu \, d\sigma(x) = 0 \quad \forall \mathbf{V} \in \mathbf{H}(\Omega, \operatorname{div}) \quad \Rightarrow$$
$$(\operatorname{div}(\mathbf{Y}_{opt}) + f)\mathbf{V}(a) - (\operatorname{div}(\mathbf{Y}_{opt}) + f)\mathbf{V}(b) = 0 \quad \forall \mathbf{V} \in \mathbf{H}(\Omega, \operatorname{div})$$

and let us take once a test function $\mathbf{V}_1 \in \mathbf{H}(\Omega, \operatorname{div})$ with $\mathbf{V}_1(a) = 0$ and $\mathbf{V}_1(b) = 1$ and the other time $\mathbf{V}_2 \in \mathbf{H}(\Omega, \operatorname{div})$ with $\mathbf{V}_2(a) = 1$ and $\mathbf{V}_2(b) = 0$. Then from $\mathbf{V}_1$ we derive $(\operatorname{div}(\mathbf{Y}_{opt}) + f)(b) = 0$ and from $\mathbf{V}_2$ we derive $(\operatorname{div}(\mathbf{Y}_{opt}) + f)(a) = 0$ which implies

$$\operatorname{div}(\mathbf{Y}_{opt}) + f = 0 \quad \text{a.e. on } \partial\Omega.$$

□

From equation (3.76) and $\nabla f, \mathbf{Y}_{opt}, \nabla v \in \mathbf{L}_2(\Omega)$ we deduce $\nabla(\operatorname{div}(\mathbf{Y}_{opt})) \in \mathbf{L}_2(\Omega)$. Taking the divergence of (3.76) and let us set $\operatorname{div}(\mathbf{Y}_{opt}) = \tilde{u} \in H^1(\Omega)$, we then derive the following non-homogeneous elliptic Dirichlet problem

$$\begin{cases} \tilde{u} - \frac{C_F(\Omega)^2}{\beta}\Delta\tilde{u} = \Delta v + \frac{C_F(\Omega)^2}{\beta}\Delta f & \text{a.e. in } \Omega, \\ \tilde{u} = -f & \text{a.e. on } \partial\Omega. \end{cases}$$

Let us set $U = \tilde{u} + f \in H_0^1(\Omega)$ and then we have in the weak form

$$\begin{cases} U \in H_0^1(\Omega), \\ \int_\Omega Ug + \frac{C_F(\Omega)^2}{\beta}\nabla U \cdot \nabla g \, dx = \int_\Omega (\Delta v + f) g \, dx \quad \forall g \in H_0^1(\Omega). \end{cases}$$

So if Ω is convex we get the regularity result $U \in H^2(\Omega)$ and if Ω has a C^m-boundary, $m \geq 2$ then $U \in H^m(\Omega)$ (see Satz 7.2, Kapitel 2 in [12]). Now for (i): Due to our assumption we have that Ω is convex and so $U \in H^2(\Omega)$ which also implies $\tilde{u} = \operatorname{div}(\mathbf{Y}_{opt}) \in H^2(\Omega)$ respectively $\nabla(\operatorname{div}(\mathbf{Y}_{opt})) \in H^1(\Omega)$ for $f \in H^2(\Omega)$. Indeed $\|\tilde{u}\|_{H^2(\Omega)} \leq \|U\|_{H^2(\Omega)} + \|f\|_{H^2(\Omega)}$. Now going back to the equation (3.76) and knowing that $\nabla(\operatorname{div}(\mathbf{Y}_{opt})), \nabla v, \nabla f \in \mathbf{H}^1(\Omega)$ we get $\mathbf{Y}_{opt} \in \mathbf{H}^1(\Omega)$.
Next for (ii): We have that U belongs to $H^m(\Omega)$ which implies that $\operatorname{div}(\mathbf{Y}_{\text{opt}}) \in H^m(\Omega)$ because $\|\tilde{u}\|_{H^m(\Omega)} \leq \|U\|_{H^m(\Omega)} + \|f\|_{H^m(\Omega)}$. Going back to equation (3.76), we derive that $\mathbf{Y}_{opt} \in \mathbf{H}^{m-1}(\Omega)$ due to $\nabla(\operatorname{div}(\mathbf{Y}_{opt})), \nabla v, \nabla f \in \mathbf{H}^{m-1}(\Omega)$ and therefore we have proven this lemma. □

3.8.4 Convergence of $\mathbf{Y}_{opt}$ to ∇u

The exact minimizer of $M^2(v; \cdot, \cdot, \Omega)$ over $\mathbb{R}_{\geq 0} \times \mathbf{H}(\Omega, \operatorname{div})$ is $(\beta_{opt}, \mathbf{Y}_{opt}) = (0, \nabla u)$. Indeed, if we take $\mathbf{Y} = \nabla u$ we derive, since $\operatorname{div}(\nabla \mathrm{u}) = \Delta u = -f$

$$\left|\left|\nabla(u - v)\right|\right|^2_{L_2(\Omega)} \leq M^2(v; \beta, \nabla u, \Omega) = (1 + \beta)\left|\left|\nabla(u - v)\right|\right|^2_{L_2(\Omega)}$$

so it is clear that the exact minimizer of $M^2(v; \cdot, \cdot, \Omega)$ is $(\beta_{opt}, \mathbf{Y}_{opt}) = (0, \nabla u)$.
Now we estimate $\left|\left|\mathbf{Y}_{opt} - \nabla u\right|\right|_{\mathbf{H}(\Omega, \operatorname{div})}$.

Theorem 3.8.2. *Let $f \in L_2(\Omega)$, Ω being convex, u being the weak solution of the problem (3.41) and $\boldsymbol{Y}_{opt} \in \boldsymbol{H}(\Omega, \operatorname{div})$ being the minimizer of $M^2(v; \beta, \cdot, \Omega)$ for any fixed $\beta > 0$ (see (3.43) for the formula of M^2), then $\boldsymbol{Y}_{opt}$ converges to ∇u in $\boldsymbol{H}(\Omega, \operatorname{div})$ as $\beta \to 0$, i.e. we have*

$$\left|\left| \boldsymbol{Y}_{opt} - \nabla u\right|\right|_{L_2(\Omega)} \leq C_1\beta^{1/2}, \quad |\operatorname{div}(\boldsymbol{Y}_{opt} - \nabla u)|_{L_2(\Omega)} \leq C_2\beta \tag{3.77}$$

where the constants C_1, C_2 depend on f, v and $C_F(\Omega)$.

Proof. Since we assumed that the domain Ω is convex we have $u \in H^2(\Omega)$ which implies $\nabla u \in \mathbf{H}(\Omega, \mathrm{div})$ and therefore $\mathbf{Y}_{opt} - \nabla u \in \mathbf{H}(\Omega, \mathrm{div})$ is a possible test function for the problem (3.67). Next we use the Helmholtz orthogonal decomposition for $\mathbf{Y}_{opt} - \nabla u$, i.e. we have $\mathbf{Y}_{opt} - \nabla u = \mathbf{q} + \nabla w$ where $\mathbf{q} \in \mathbf{Q}_0(\Omega) = \{\mathbf{q} \in \mathbf{L}_2(\Omega) \mid \int_\Omega \mathbf{q} \cdot \nabla w \, dx = 0 \, \forall w \in H_0^1(\Omega)\}$, $w \in H_0^1(\Omega)$, $\Delta w = \mathrm{div}(\mathbf{Y}_{opt} - \nabla u) \in L_2(\Omega)$. (For this particular variant of the Helmholtz decomposition we refer to [46].) ∇u is the minimizer of the functional $F(\mathbf{q}) = \|\nabla v - \mathbf{q}\|_{L_2(\Omega)}$ on the set $\mathbf{Q}_f(\Omega) = \{\mathbf{q} \in \mathbf{L}_2(\Omega) \mid \int_\Omega \mathbf{q} \cdot \nabla w \, dx = \int_\Omega f w \, dx \quad \forall w \in H_0^1(\Omega)\}$ (see (2.7) in [58]). The corresponding Euler equation for the functional F reads

$$\int_\Omega \nabla u \cdot \mathbf{q} \, dx = \int_\Omega \nabla v \cdot \mathbf{q} \, dx \quad \forall \mathbf{q} \in \mathbf{Q}_0(\Omega)$$

and this implies

$$\int_\Omega \nabla u \cdot (\mathbf{Y}_{opt} - \nabla u) \, dx - \int_\Omega \nabla u \cdot \nabla w \, dx = \int_\Omega \nabla v \cdot (\mathbf{Y}_{opt} - \nabla u) \, dx - \int_\Omega \nabla v \cdot \nabla w \, dx.$$

If we put $\mathbf{Y}_{opt} - \nabla u$ in (3.67) as a test function we derive

$$\begin{aligned}
&\|\mathbf{Y}_{opt} - \nabla u\|^2_{L_2(\Omega)} + \frac{C_F(\Omega)^2}{\beta} |\mathrm{div}(\mathbf{Y}_{opt} - \nabla u)|^2_{L_2(\Omega)} \\
&\le C_F(\Omega)^2 |f|_{L_2(\Omega)} |\Delta w|_{L_2(\Omega)} + C_F(\Omega) \|\nabla v\|_{L_2(\Omega)} |\Delta w|_{L_2(\Omega)} \\
&= (C_F(\Omega)^2 |f|_{L_2(\Omega)} + C_F(\Omega) \|\nabla v\|_{L_2(\Omega)}) |\mathrm{div}(\mathbf{Y}_{opt} - \nabla u)|^2_{L_2(\Omega)}
\end{aligned}$$

where we used above that $f = -\Delta u = -\mathrm{div}(\nabla u)$.
Therefore we follow

$$|\mathrm{div}(\mathbf{Y}_{opt} - \nabla u)|_{L_2(\Omega)} \le (|f|_{L_2(\Omega)} + \frac{1}{C_F(\Omega)} \|\nabla v\|_{L_2(\Omega)}) \beta$$

and so

$$\begin{aligned}
&\|\mathbf{Y}_{opt} - \nabla u\|^2_{L_2(\Omega)} \le C_F(\Omega)(\|\nabla v\|_{L_2(\Omega)} + C_F(\Omega)|f|_{L_2(\Omega)})(|f|_{L_2(\Omega)} + \frac{1}{C_F(\Omega)} \|\nabla v\|_{L_2(\Omega)}) \beta \\
&\le \max\{\frac{1}{C_F(\Omega)}, C_F(\Omega)^2\} (|f|_{L_2(\Omega)} + \|\nabla v\|_{L_2(\Omega)})^2 \beta.
\end{aligned}$$

Hence we have shown the theorem. □

3.8.5 Minimization of $M^2(v; \beta, \mathbf{Y}, \Omega)$ w.r.t. $\mathbf{Y}$ and β

In this subsection we minimize the problem $M^2(v; \beta, \mathbf{Y}, \Omega)$ with respect to the two variables β and $\mathbf{Y}$. For this we need $\frac{\partial}{\partial \beta} M^2(v; \beta, \mathbf{Y}, \Omega) = 0$ and $\frac{\partial}{\partial \mathbf{Y}} M^2(v; \beta, \mathbf{Y}, \Omega) = 0$ which we computed each separately above. Basically we are interested in

$$\inf_{\beta \in \mathbb{R}^+, \, \mathbf{Y} \in \mathbf{H}(\Omega, \mathrm{div})} M^2(v; \beta, \mathbf{Y}, \Omega) = M^2(v; \beta_{min}(\mathbf{Y}_{opt}), \mathbf{Y}_{opt}, \Omega).$$

We analyse an alternating iteration process between β_{min} and $\mathbf{Y}_{opt}$. We have the following minimization algorithm:

Set $\beta_{min}^{(0)} = 1/2$

For $n = 1, 2, \ldots$

Minimize $M^2(v; \beta_{min}^{(n-1)}, \mathbf{Y}, \Omega)$ *w.r.t.* $\mathbf{Y} \in \mathbf{H}(\Omega, \mathrm{div})$ *(see equation (3.70))* $\Rightarrow \mathbf{Y}_{opt}^{(n)}$

Using $\mathbf{Y}_{opt}^{(n)}$ and update $\beta_{min}^{(n)}$ with the formula (3.66)

Then we have that $M^2(v;\beta_{min}^{(i)},\mathbf{Y}_{opt}^{(i)},\Omega) \le M^2(v;\beta_{min}^{(j)},\mathbf{Y}_{opt}^{(j)},\Omega)$ for all $i \ge j$, if the algorithm does not stop because of the degenerate situation when $\beta_{min}^{(n)} = 0$. So we derive

$$\begin{aligned}&\inf_{\mathbf{Y}\in\mathbf{H}(\Omega,\mathrm{div}),\beta\in\mathbb{R}^+} M^2(v;\beta,\mathbf{Y},\Omega)\\ &\le (1+\beta_{min}^{(i)})||\mathbf{Y}_{opt}^{(i)} - \nabla v||_{L_2(\Omega)}^2 + (1+1/\beta_{min}^{(i)})C_F(\Omega)^2|\mathrm{div}(\mathbf{Y}_{opt}^{(i)}) + f|_{L_2(\Omega)}^2.\end{aligned}$$

For numerical examples we refer to paper [58].

Remark 3.8.3. *Since one is more interested in the accuracy of u instead of $\mathbf{Y}_{opt}$ one stops the iteration process already after a few iterations, see for instance [58].*
For numerical examples of the error majorant and the property of asymptotic exactness, i.e. when the majorant tends to the exact error, we refer to [58],[59] and [62].

3.8.6 Concrete choice of $\tilde{\mathbf{Y}}_{opt,h}$

For the approximation of $\tilde{\mathbf{Y}}_{opt}$, the minimizer of $M^2(u_{\infty h};\gamma,\tilde{\mathbf{Y}},\omega)$, we apply the Galerkin finite element method.
For this let us take $\mathbf{S}_h(\omega)$, a finite dimensional subspace of $\mathbf{H}(\omega,\mathrm{div}')$ and then we are analysing the problem:

$$\begin{cases}\tilde{\mathbf{Y}}_{opt,h} \in \mathbf{S}_h(\omega),\\ a(\tilde{\mathbf{Y}}_{opt,h},\mathbf{V}) = (\nabla' v,\mathbf{V})_{L_2(\omega)} - \frac{C_F(\omega)^2}{\gamma}(f,\mathrm{div}'(\mathbf{V}))_{L_2(\omega)} \quad \forall \mathbf{V}\in\mathbf{S}_h(\omega)\end{cases} \tag{3.78}$$

and for the definition of $a(\cdot,\cdot)$ see (3.68). Note we still have the analogous restriction (3.69) for γ. Let us set

$$\mathbf{S}_h(\omega) = \mathrm{span}\{\mathbf{b}_1,\ldots,\mathbf{b}_{\widetilde{N}}\}$$

then the previous problem is equivalent to

$$\begin{cases}\tilde{\mathbf{Y}}_{opt,h} \in \mathbf{S}_h(\omega),\\ a(\tilde{\mathbf{Y}}_{opt,h},\mathbf{b}_i) = (\nabla' v,\mathbf{b}_i)_{L_2(\omega)} - \frac{C_F(\omega)^2}{\gamma}(f,\mathrm{div}'(\mathbf{b}_i))_{L_2(\omega)} \quad \forall i\in\{1,\ldots,\widetilde{N}\}.\end{cases} \tag{3.79}$$

From the Lax-Milgram Theorem (cf. Theorem 4.2.2) we conclude that there exists a unique $\tilde{\mathbf{Y}}_{opt,h} \in \mathbf{S}_h(\omega)$ solution of the problem (3.79). We obtain the coefficients $Y_{opt,h,i}$ of $\tilde{\mathbf{Y}}_{opt,h} = \sum_{i=1}^{\widetilde{N}} Y_{opt,h,i}\mathbf{b}_i$ by solving the following matrix system of linear equations

$$\mathbf{L}\mathbf{Y}_{opt,h}^{const} = \mathbf{F}$$

where

$$\begin{aligned}&(\mathbf{Y}_{opt,h}^{const})_i = Y_{opt,h,i}, \quad (\mathbf{F})_i = (\nabla' v,\mathbf{b}_i)_{L_2(\omega)} - \frac{C_F(\omega)^2}{\gamma}(f,\mathrm{div}'(\mathbf{b}_i))_{L_2(\omega)},\\ &(\mathbf{L})_{ij} = (\mathbf{b}_j,\mathbf{b}_i)_{L_2(\omega)} + \frac{C_F(\omega)^2}{\gamma}(\mathrm{div}'(\mathbf{b}_j),\mathrm{div}'(\mathbf{b}_i))_{L_2(\omega)}.\end{aligned}$$

3.9 Discussion

At the end few words about the convergence of our approximation to u_ℓ.
We know from Theorem 3.2.1 that $1\otimes u_\infty$ is a quasi optimal approximation of u_ℓ, the weak solution of (3.1), in Ω_{ℓ_0} when ℓ converges to infinity. We say that $1\otimes u_\infty$ is a quasi optimal approximation because

it does not fulfil the zero boundary condition but it is a good approximation in the center of the domain. From this result we can conclude that $\psi_{h,h} \otimes u_{\infty h}$ (cf. (3.35), (3.40)) is a good approximation of u_ℓ in $\Omega_{\ell_0} = (-\ell_0, \ell_0) \times \omega$. Therefore when we take $\psi^0_{h,h} \otimes u^0_{\infty h} + \psi^0_{0,0} \otimes u^1_{\infty h}$ (cf. (3.61)) for the error estimate we derive at most a better upper bound of the error.

Note that upper and lower bounds for the constants d_{1h}, d_{2h}, d_{3h} which appear in the equation (3.60) for computing $u^1_{\infty h}$ can be shown in an analogous way as the bounds for the constants c_{1h}, c_{2h}.

Chapter 4

Appendix

In this chapter we introduce various mathematical tools which were used in this thesis. First we concentrate on spaces of functions, then we mention some useful results in Analysis as the Lax Milgram Theorem and finally we explain the finite element method (FEM). We mention the crucial points and for more details/proofs we refer to some literature.

4.1 Spaces of functions

The goal of this section is to introduce various functional spaces and some well known results about these spaces as the Hölder inequality. Note we work only with real valued functions and in the space of real numbers where we denote by Ω an open domain in $\mathbb{R}^n$ with $n \in \mathbb{N}^+$.

4.1.1 L_p-Spaces

In this section we introduce the L_p-spaces (also called Lebesgue spaces) with their norm and in addition we mention important results concerning the L_p-spaces. For further details and proofs see for instance [1], [44] or [73].

Definition 4.1.1 (L_p-Space)**.** *Let $1 \leq p < \infty$, then we define*

$$L_p(\Omega) := \{v : \Omega \to \mathbb{R} \mid v \textit{ is Lebesgue measurable and } \int_\Omega |v(x)|^p \, dx < +\infty\}$$

and if $p = \infty$ we set

$$L_\infty(\Omega) := \{v : \Omega \to \mathbb{R} \mid v \textit{ is Lebesgue measurable and } \exists\, C \textit{ s.t. } |v(x)| \leq C \textit{ a.e. } x \in \Omega\}.$$

The spaces are equipped with the norm $|\cdot|_{L_p(\Omega)}$ for $1 \leq p < \infty$

$$|v|_{L_p(\Omega)} = \left(\int_\Omega |v(x)|^p \, dx\right)^{1/p} \tag{4.1}$$

and for $p = \infty$ we have the norm

$$|v|_{L_\infty(\Omega)} = \inf\{C \textit{ s.t. } |v(x)| \leq C \textit{ a.e. } x \in \Omega\}. \tag{4.2}$$

Theorem 4.1.2. *We have that $L_p(\Omega)$ for $1 \leq p \leq \infty$ is a Banach space and for $p = 2$ we have that $L_2(\Omega)$ is a Hilbert space with the scalar product*

$$(u, v)_{L_2(\Omega)} = \int_\Omega u(x)v(x) \, dx.$$

Theorem 4.1.3 (Hölder's inequality)**.** *Let $1 \leq p, q \leq +\infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $u \in L_p(\Omega)$, $v \in L_q(\Omega)$, then we have the so called Hölder inequality*

$$\int_\Omega |u(x)v(x)|\,dx \leq |u|_{L_p(\Omega)} |v|_{L_q(\Omega)} \tag{4.3}$$

and that $uv \in L_1(\Omega)$. If $p = 2$, then the previous inequality is called Cauchy-Schwarz inequality.

Theorem 4.1.4 (Minkowski's inequality)**.** *Let us assume $1 \leq p \leq \infty$ and $u, v \in L_p(\Omega)$, then we have the so called Minkowski inequality*

$$|u + v|_{L_p(\Omega)} \leq |u|_{L_p(\Omega)} + |v|_{L_p(\Omega)}.$$

Definition 4.1.5. *We set*

$$\boldsymbol{L}_2(\Omega) := (L_2(\Omega))^n$$

which is equipped with the norm

$$|\boldsymbol{v}|_{\boldsymbol{L}_2(\Omega)} = \left(\int_\Omega \boldsymbol{v} \cdot \boldsymbol{v}\,dx \right)^{1/2} \tag{4.4}$$

where $\cdot$ denotes the usual scalar product in $\mathbb{R}^n$.

4.1.2 Sobolev spaces

Sobolev spaces are useful instruments to solve partial differential equations. In this section we introduce various classical results of the Sobolev spaces. We refer the reader for instance to [1], [28], [36], [40] or [45] for proofs and additional information.

Before we define the Sobolev spaces we introduce some notation and the weak derivative.
For any multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we denote by D^α the partial derivative given by

$$D^\alpha = \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}}$$

and $|\alpha| = \sum_{i=1}^n \alpha_i$. We set

$$\begin{aligned}
C^0(\Omega) &:= \{v : \Omega \to \mathbb{R} \mid v \text{ is continuous on } \Omega\}, \\
C^m(\Omega) &:= \{v : \Omega \to \mathbb{R} \mid D^\alpha v \text{ exists and } D^\alpha v \in C^0(\Omega)\, \forall |\alpha| \leq m\}, \\
C^\infty(\Omega) &:= \{v : \Omega \to \mathbb{R} \mid D^\alpha v \text{ exists and } D^\alpha v \in C^0(\Omega)\, \forall |\alpha| \in \mathbb{N}\}, \\
C_0^\infty(\Omega) &= \mathcal{D}(\Omega) := \{v \in C^\infty(\Omega) \mid \text{supp } v \subset\subset \Omega\},
\end{aligned}$$

where $\text{supp } v := \overline{\{x \in \Omega : v(x) \neq 0\}}$ is the support of v and $K \subset\subset \Omega :\Leftrightarrow K$ is a compact subset of Ω.

Definition 4.1.6 (Weak derivative)**.** *Suppose $u, v \in L_1(\Omega)$ and α is a multiindex. We say that v is the α^{th}-weak partial derivative of u, written $D^\alpha u = v$, provided*

$$\int_\Omega u D^\alpha \psi\,dx = (-1)^{|\alpha|} \int_\Omega v \psi\,dx$$

for all test functions $\psi \in C_0^\infty(\Omega)$.

Remark 4.1.7. *If $u \in C^{|\alpha|}(\Omega)$, then we have that the weak $|\alpha|-$derivative of u is the same as the partial derivative in the classical sense.*

Definition 4.1.8 (Sobolev space)**.** *For $m \in \mathbb{N}$ and $1 \le p < +\infty$ we set*

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega) \mid D^\alpha v \in L^p(\Omega)\ \forall \alpha,\ |\alpha| \le m\}. \tag{4.5}$$

In the above definition $D^\alpha v$ denotes the weak derivative. We call $W^{m,p}(\Omega)$ a Sobolev space. The standard norm of the space $W^{m,p}(\Omega)$ is

$$\|v\|_{W^{m,p}(\Omega)} := \left(\sum_{|\alpha| \le m} |D^\alpha v|^p_{L_p(\Omega)} \right)^{1/p}. \tag{4.6}$$

By convention we set

$$W^{0,p}(\Omega) = L_p(\Omega), \quad D^0 v = v.$$

Remark 4.1.9. *We have that*

$$|v|_{W^{m,p}(\Omega)} := \left(\sum_{|\alpha| = m} |D^\alpha v|^p_{L_p(\Omega)} \right)^{1/p} \tag{4.7}$$

is a seminorm of $W^{m,p}(\Omega)$.

Theorem 4.1.10. *The space $W^{m,p}(\Omega)$ is a Banach space for the norm (4.6).*

Lemma 4.1.11. *$C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.*

Theorem 4.1.12. *If $p = 2$, then the space $W^{m,2}(\Omega)$ is a Hilbert space with the scalar product*

$$(u, v)_{W^{m,2}(\Omega)} = \sum_{|\alpha| \le m} (D^\alpha u, D^\alpha v)_{L_2(\Omega)}, \tag{4.8}$$

where $(\cdot,\cdot)_{L_2(\Omega)}$ denotes the scalar product in $L^2(\Omega)$ defined by

$$(u, v)_{L_2(\Omega)} = \int_\Omega u(x) v(x)\, dx.$$

One denotes the space $W^{m,2}(\Omega)$ by $H^m(\Omega)$ which is equipped with the norm

$$\|v\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \le m} |D^\alpha v|^2_{L_2(\Omega)} \right)^{1/2}. \tag{4.9}$$

Definition 4.1.13. *We set*

$$\boldsymbol{H}^1(\Omega) := (H^1(\Omega))^n$$

which is equipped with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} = \left(\int_\Omega \boldsymbol{v} \cdot \boldsymbol{v} + \nabla \boldsymbol{v} \cdot \nabla \boldsymbol{v}\, dx \right)^{1/2} \tag{4.10}$$

where

$$\nabla \boldsymbol{v} \cdot \nabla \boldsymbol{v} := \sum_{i=1}^{n} \nabla v_i \cdot \nabla v_i$$

and $\cdot$ denotes the usual scalar product in $\mathbb{R}^n$.

Definition 4.1.14 (Lipschitz domain). *A domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ has Lipschitz boundary or Ω is a Lipschitz domain if there exists a constant $M \in \mathbb{N}^+$ and a collection of open sets $O_1, \dots, O_M \subset \mathbb{R}^n$ with the following two properties:*

1. $\partial\Omega \subset \cup_{i=1}^M O_i$
2. $\partial\Omega \cap O_i$ *can be represented as graph of a Lipschitz continuous function for all* $1 \leq i \leq M$.

Remark 4.1.15. *Let Ω be a Lipschitz domain, then the unit outward normal ν to Ω exists a.e. on $\partial\Omega$.*

Definition 4.1.16. *For $m \in \mathbb{N}$ and $1 \leq p < +\infty$ we set*

$$W_0^{m,p}(\Omega) := \text{the closure of } \mathcal{D}(\Omega) \text{ in } W^{m,p}(\Omega). \tag{4.11}$$

In the above definition the closure is understood in the sense of the topology defined by the norm (4.6). In the case $p = 2$ we adapt the notation

$$W_0^{m,2}(\Omega) = H_0^m(\Omega).$$

Theorem 4.1.17. *Assume that Ω is a Lipschitz domain of $\mathbb{R}^n$, then*

$$H_0^1(\Omega) = \{u \in H^1(\Omega) \mid u|_{\partial\Omega} = 0\}.$$

Theorem 4.1.18. *The two norms $\|\cdot\|_{W^{m,p}(\Omega)}$ and $|\cdot|_{W^{m,p}(\Omega)}$ are equivalent on $W_0^{m,p}(\Omega)$.*

Remark 4.1.19. *From the previous theorem we derive that on $H_0^1(\Omega)$ the norms $\|\cdot\|_{H^1(\Omega)} = (|\cdot|^2_{L_2(\Omega)} + \big||\nabla\cdot|\big|^2_{L_2(\Omega)})^{1/2}$ and $\big||\nabla\cdot|\big|_{L_2(\Omega)}$ are equivalent.*

Theorem 4.1.20 (Friedrichs inequality). *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected Lipschitz domain and $v \in H_0^1(\Omega)$, then we call the inequality*

$$|v|_{L_2(\Omega)} \leq C_F(\Omega)\big||\nabla v|\big|_{L_2(\Omega)} \tag{4.12}$$

Friedrichs inequality and $C_F(\Omega)$ is the Friedrichs constant. Further we have for

$$\Omega \subset B_d(\zeta) = \{x \in \mathbb{R}^n \mid |x - \zeta| < d\}$$

that

$$C_F(\Omega) \leq \sqrt{2}d. \tag{4.13}$$

Note in some literature the above inequality is also called Poincaré inequality or Poincaré-Friedrichs inequality.

Theorem 4.1.21 (Poincaré-Wirtinger inequality). *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected Lipschitz domain and we set for $u \in H^1(\Omega)$*

$$\overline{u} = \frac{1}{|\Omega|}\int_\Omega u(x)\,dx$$

where $|\Omega| = \int_\Omega 1\,dx$ denotes the measure of Ω. Then we have for $u \in H^1(\Omega)$ the estimate

$$|u - \overline{u}|_{L_2(\Omega)} \leq C_{PW}\big||\nabla u|\big|_{L_2(\Omega)}$$

where C_{PW} is the so called Poincaré-Wirtinger constant which is depending on n, Ω. If we have in addition that Ω is convex, then it holds

$$|u - \overline{u}|_{L_2(\Omega)} \leq d_\Omega^n \left(\frac{\omega_n}{|\Omega|}\right)^{1-1/n} \big||\nabla u|\big|_{L_2(\Omega)}$$

where d_Ω denotes the diameter of Ω and ω_n the volume of the unit ball in $\mathbb{R}^n$.

For the proof of the previous two theorems we refer to [28].

Definition 4.1.22 (Dual Sobolev space)**.** *Let $0 < m < \infty$, $1 \leq p < +\infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the dual Sobolev space is defined by*

$$W^{-m,q}(\Omega) := (W_0^{m,p}(\Omega))'$$

equipped with the norm

$$\|T\|_{W^{-m,q}(\Omega)} = \sup_{v \in W_0^{m,p}(\Omega), v \neq 0} \frac{|T(v)|}{\|v\|_{W^{m,p}(\Omega)}}$$

where T is a linear continuous function from $W_0^{m,p}$ to $\mathbb{R}$.

4.1.3 $\mathbf{H}(\Omega, \mathrm{div})$ and $\mathbf{H}^m(\Omega, \mathrm{div})$

Note the spaces $\mathbf{H}(\Omega, \mathrm{div})$ and $\mathbf{H}^m(\Omega, \mathrm{div})$ appear in the third chapter. For more information we refer to [10], [14], [37] or [57].

Definition 4.1.23. *Let us define*

$$\boldsymbol{H}(\Omega, \mathrm{div}) := \{\boldsymbol{v} \in \boldsymbol{L}_2(\Omega) \mid \mathrm{div}(\boldsymbol{v}) \in L_2(\Omega)\}.$$

Remark 4.1.24. *The space $\boldsymbol{H}(\Omega, \mathrm{div})$ is a Hilbert space endowed with the scalar product*

$$(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{H}(\Omega,\mathrm{div})} := \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} + \mathrm{div}(\boldsymbol{u})\mathrm{div}(\boldsymbol{v})\, dx \tag{4.14}$$

where $\cdot$ denotes the usual scalar product in $\mathbb{R}^n$ and for the norm we have

$$\|\boldsymbol{v}\|_{\boldsymbol{H}(\Omega,\mathrm{div})} = \sqrt{(\boldsymbol{v}, \boldsymbol{v})_{\boldsymbol{H}(\Omega,\mathrm{div})}}.$$

Definition 4.1.25. *Let us define*

$$\boldsymbol{H}^m(\Omega, \mathrm{div}) := \{\boldsymbol{v} \in \boldsymbol{H}^m(\Omega) \mid \mathrm{div}(\boldsymbol{v}) \in H^m(\Omega)\}$$

for $m \in \mathbb{N}^+$.

4.2 Useful results in Analysis

4.2.1 Useful theorems

In this section we focus on results which we used in the previous chapters concerning partial differential equations (pde), for instance the well known Lax-Milgram Theorem.

Let us denote by V a real Hilbert space with the norm $\|\cdot\|_V$ and V' denotes the dual space of V.

Definition 4.2.1. *Let $a(u, v)$ be a bilinear form on V, then*

- *$a(\cdot, \cdot)$ is continuous, if it exists a constant $\Lambda > 0$ such that $|a(u, v)| \leq \Lambda \|u\|_V \|v\|_V \quad \forall u, v \in V$,*
- *$a(\cdot, \cdot)$ is coercive, if it exists a constant $\lambda > 0$ such that $a(u, u) \geq \lambda \|u\|_V^2 \quad \forall u \in V$.*

Theorem 4.2.2 (Lax-Milgram Theorem)**.** *Let $a(u, v)$ be a continuous coercive bilinear form on V, then for every $f \in V'$ there exists a unique $u \in V$ such that*

$$a(u, v) = \langle f, v \rangle \quad \forall v \in V.$$

In the case where a is symmetric that is to say

$$a(u,v) = a(v,u) \quad \forall u,v \in V$$

then u is the unique minimizer on V of the functional

$$J(v) = \frac{1}{2}a(v,v) - \langle f, v\rangle.$$

$\langle \cdot, \cdot \rangle$ denotes the duality bracket between V' and V.

For a proof see for instance [15] (Chapter 1), [28] or [30] (Chapter 6). This theorem is useful for showing unique weak existence of partial differential equations if all the assumptions for the Lax-Milgram Theorem are fulfilled. For instance from the Lax-Milgram Theorem we obtain the existence of a unique weak solution of the Poisson problem with Dirichlet boundary conditions, i.e. we know that there exists a unique weak solution $u \in H_0^1(\Omega)$ of the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for $f \in L_2(\Omega)$ and Ω being an open bounded connected Lipschitz domain of $\mathbb{R}^n$.

Theorem 4.2.3 (Divergence Theorem/Formula)**.** *Suppose that Ω is a "smooth" open subset of $\mathbb{R}^n$ with outward unit normal ν. For any "smooth" vector field $\boldsymbol{v}$ in Ω we have*

$$\int_\Omega \operatorname{div}(\boldsymbol{v})\,dx = \int_{\partial\Omega} \boldsymbol{v}\cdot\nu\,d\sigma(x)$$

where $d\sigma(x)$ is the measure area on $\partial\Omega$.

For the proof see Chapter 4 in [15].

4.2.2 Useful inequalities

Lemma 4.2.4 (Young inequality)**.** *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have the so called Young inequality*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

for $a, b > 0$.

Remark 4.2.5. *From the Young inequality one can easily follow that*

$$(a+b)^2 \le (1+\beta)a^2 + (1+\frac{1}{\beta})b^2$$

for $a, b > 0$ and any $\beta > 0$.

Indeed, we have

$$(a+b)^2 = a^2 + 2ab\frac{\sqrt{\beta}}{\sqrt{\beta}} + b^2 \le a^2 + b^2 + 2(\frac{\beta a^2}{2} + \frac{b^2}{2\beta}) = (1+\beta)a^2 + (1+\frac{1}{\beta})b^2.$$

Theorem 4.2.6 (Gronwall's inequality)**.** *Let $v(\cdot)$ be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies for a.e. t the differential inequality*

$$\frac{d}{dt}v(t) \le h(t)v(t) + g(t)$$

for $h(t)$ and $g(t)$ being nonnegative, summable functions on $[0,T]$. Then it holds

$$v(t) \le e^{\int_0^t h(s)\,ds}\left(v(0) + \int_0^t g(s)\,ds\right)$$

for all $0 \le t \le T$.

For the proof of these two previous inequalities we refer to [30] (Appendix B).

4.2.3 Trilinear term

Let us define the trilinear term

$$t(\mathbf{w};\mathbf{u},\mathbf{v}) := \int_\Omega (\mathbf{w}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,dx = \int_\Omega w^i \partial_{x_i} u^j v^j\,dx$$

(with the summation convention in i and j) for $\mathbf{w},\mathbf{u},\mathbf{v}\in\mathbf{H}^1(\Omega)$.

Remark 4.2.7. *Such a trilinear term appears for instance in the weak form of the stationary Navier-Stokes problem for $n = 2$ or 3, i.e. we have for $\boldsymbol{u}:\mathbb{R}^n\to\mathbb{R}^n$*

$$\begin{cases} \boldsymbol{u}\in\widehat{\boldsymbol{H}}(\Omega), \\ \mu\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,dx + \int_\Omega(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v}\,dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx \quad \forall\boldsymbol{v}\in\widehat{\boldsymbol{H}}(\Omega) \end{cases}$$

where

$$\widehat{\boldsymbol{H}}(\Omega) = \{\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega) = (H_0^1(\Omega))^n \mid \operatorname{div}(\boldsymbol{v}) = 0\}. \tag{4.15}$$

Note this trilinear term appears in Chapter 1, where we work with a linearized Navier-Stokes equation.

Proposition 4.2.8. *Let us take $\boldsymbol{w}$, $\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)$ and $\boldsymbol{u}\in\boldsymbol{H}^1(\Omega)$, then there exists a constant $C = C(\Omega)$ such that*

$$|t(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})| \le C\big|\big|\nabla\boldsymbol{w}\big|\big|_{L_2(\Omega)}\big|\big|\nabla\boldsymbol{u}\big|\big|_{L_2(\Omega)}\big|\big|\nabla\boldsymbol{v}\big|\big|_{L_2(\Omega)}.$$

Proposition 4.2.9. *Let $\boldsymbol{w}\in\boldsymbol{H}^1(\Omega)$ with $\operatorname{div}(\boldsymbol{w}) = 0$, then we have the following two properties:*

$$\begin{aligned} &t(\boldsymbol{w};\boldsymbol{v},\boldsymbol{v}) = 0 \quad \forall\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega), \\ &t(\boldsymbol{w};\boldsymbol{v},\boldsymbol{u}) = -t(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) \quad \forall\boldsymbol{v},\boldsymbol{u}\in\boldsymbol{H}_0^1(\Omega). \end{aligned}$$

For the proof of the previous two propositions see [15] (Chapter 14).

4.3 Finite element method (FEM)

The finite element method (FEM) is a method for solving elliptic equations, i.e. for finding an approximation of the solution u of a problem. The idea is to replace the Hilbert domain V with $u\in V$ by a finite dimensional space V_h and then to analyse a similar problem the so called discret problem. This method of approximation is called the Galerkin method. Note the choice of such a finite dimensional subspace is not unique, it is for instance depending on the choice of the triangulation and the basis functions. For more information about FEM and proofs see for instance in the books [27], [38] or [40].

We start with the basic aspects of the FEM. Let V be a Hilbert space equipped with the norm $\|\cdot\|_V$, $a: V\times V\to\mathbb{R}$ a continuous coercive bilinear form and $F\in V'$, the dual of V. So we can apply the Lax-Milgram Theorem (cf. Theorem 4.2.2) and derive that there exists a unique solution $u\in V$ of the problem

$$\begin{cases} u\in V, \\ a(u,v) = F(v) \quad \forall v\in V. \end{cases} \tag{4.16}$$

The Galerkin method for approximating the solution u consists in defining a finite dimensional subspace V_h of V and solving the analogous discrete problem

$$\begin{cases} u_h\in V_h, \\ a(u_h,v) = F(v) \quad \forall v\in V_h. \end{cases} \tag{4.17}$$

As well from the Lax-Milgram Theorem we know that a unique solution $u_h \in V_h$ exists. Typical Hilbert spaces for V are for instance $H^1(\Omega)$ or $H_0^1(\Omega)$ where Ω is a Lipschitz domain in $\mathbb{R}^n$. Let us set

$$V_h = \text{span}\{b_1, \ldots, b_N\},$$

i.e. b_i, $i \in \{1, \ldots, N\}$ are the basis elements of the domain V_h. Then the problem (4.17) is equivalent to

$$\begin{cases} u_h \in V_h, \\ a(u_h, b_i) = F(b_i) \quad \forall i \in \{1, \ldots, N\}. \end{cases}$$

In matrix notation we have the linear equation system

$$\mathbf{Lu} = \mathbf{F} \tag{4.18}$$

where

$$L_{i,j} := a(b_j, b_i) \quad F_i = F(b_i) \quad i, j \in \{1, \ldots, N\} \tag{4.19}$$

and

$$u_h(x) = \sum_{i=1}^{N} u_i b_i(x).$$

For defining a basis over the finite space V_h it is important to know the triangulation $\mathcal{G}_{\Omega,h}$ of the domain.

Definition 4.3.1 (Triangulation). *The triangulation $\mathcal{G}_{\Omega,h}$ is established over the set $\overline{\Omega}$, i.e. the set $\overline{\Omega}$ is subdivided into a finite number of sets K, called finite elements, in such a way that the following properties are satisfied:*

1. $\overline{\Omega} = \cup_{K \in \mathcal{G}_{\Omega,h}} K$
2. *For each $K \in \mathcal{G}_{\Omega,h}$ the set K is closed and the interior $\overset{\circ}{K}$ is non empty.*
3. *For each distinct K_1, $K_2 \in \mathcal{G}_{\Omega,h}$ one has $\overset{\circ}{K}_1 \cap \overset{\circ}{K}_2 = \emptyset$.*
4. *For each $K \in \mathcal{G}_{\Omega,h}$ the boundary ∂K is Lipschitz continuous.*
5. *Any face of any simplex K_1 in the triangulation is either a subset of the boundary $\partial\Omega$ or a face of another simplex K_2 in the triangulation.*

Definition 4.3.2 (Conforming finite element mesh). *Let $\Omega \subset \mathbb{R}^n, n \in \{1, 2, 3\}$, be a bounded domain. A conforming finite element mesh $\mathcal{G}_{\Omega,h}$ on Ω is a partition of Ω into n-dimensional simplices satisfying the following properties:*

1. *For any two elements $K_1, K_2 \in \mathcal{G}_{\Omega,h}$ with $K_1 \neq K_2$ the intersection $K_1 \cap K_2$ is either empty, a common vertex (an interval endpoint if $n = 1$), a common edge (for $n = 2$), or a common face (for $n = 3$).*
2. $\overline{\Omega} = \cup_{K \in \mathcal{G}_{\Omega,h}} K$.

Next let us denote

$$h_K = \text{diameter of } K, \quad h = \max_{K \in \mathcal{G}_{\Omega,h}} h_K,$$

$$\rho_K = \text{diameter of maximal inscribed ball in } K.$$

Definition 4.3.3 (Shape regular, uniform mesh). *Let $\mathcal{G}_{\Omega,h} = \{K_i \mid 1 \leq i \leq N\}$ be a finite element mesh on $\Omega \subset \mathbb{R}^n$.*

(i) $\mathcal{G}_{\Omega,h}$ *is shape regular if there exists a constant* $C > 0$ *such that*

$$\frac{h_K}{\rho_K} \leq C \quad \forall K \in \mathcal{G}_{\Omega,h}$$

and κ *is called the shape regularity which is defined as*

$$\kappa = \max\{\frac{h_K}{\rho_K} \mid K \in \mathcal{G}_{\Omega,h}\}.$$

(ii) $\mathcal{G}_{\Omega,h}$ *is uniform if there exists a constant* C *such that*

$$\max_{1\leq i\leq N} h_{K_i} \leq C\rho_K \quad \forall K \in \mathcal{G}_{\Omega,h}.$$

Definition 4.3.4 (Regular family)**.** $\mathcal{G}_{\Omega,h}$ *is a regular family of the decomposition if:*

1. *There exists a constant* c *such that*

$$\frac{h_K}{\rho_K} \leq c \quad \forall K \in \cup_h \mathcal{G}_{\Omega,h}.$$

2. *The quantity* h *approaches zero.*

The finite dimensional space V_h contains basis functions which are defined over the set $\overline{\Omega}$ and depending on the triangulation. Most frequently the basis functions of V_h are continuous piecewise polynomials of fixed degree (continuous on a finite element K). The basis functions are depending on the choice of the finite elements, examples are the Lagrange or Argyris elements. For other choices and more details we refer to [12] and [27].

We will study an example for the case $H_0^1(\Omega)$ and for this we need the theorem below. Let us first define for a given finite element space X_h the finite dimensional space

$$\mathcal{P}_K := \{v_h|_K : v_h \in X_h\}. \tag{4.20}$$

Theorem 4.3.5. *Suppose that* $\mathcal{P}_K \subset H^1(K)$ *for all* $K \in \mathcal{G}_{\Omega,h}$ *and* $X_h \subset \mathcal{C}^0(\overline{\Omega})$ *hold. Then the inclusions*

$$X_h \subset H^1(\Omega),$$
$$X_{h,0} = \{v_h \in X_h \mid v_h = 0 \text{ on } \partial\Omega\} \subset H_0^1(\Omega)$$

hold.

An Example for $H_0^1(\Omega)$ with C^0 triangular finite elements: Let $\mathcal{G}_{\Omega,h}$ be a conforming triangulation with the finite elements K. Let x_i be a nodal point which is a vertex of K and N denotes the number of all inner nodal points. $b_i \in \mathbb{P}_k$, $1 \leq i \leq N$ are the Lagrangian polynomials such that

$$b_i(x_j) = \delta_{ij}, \quad i,j \in \{1,\ldots,N\}$$

where $\mathbb{P}_k$ denotes the polynomials of degree smaller equal k, $k \geq 1$ and δ_{ij} denotes the Kronecker delta. The finite element space V_h is given by

$$V_h = \text{span}\{b_1,\ldots,b_N\} = \{v \in C^0(\overline{\Omega}) \mid v|_K \in \mathbb{P}_k \forall K \in \mathcal{G}_{\Omega,h}\} \cap H_0^1(\Omega)$$

and from the previous theorem we know that $V_h \subset H_0^1(\Omega)$.

An important result in the FE theory is Céa's Lemma.

Lemma 4.3.6 (Céa's Lemma)**.** *Let* $a(\cdot,\cdot)$ *be a continuous, coercive bilinear form on* V *with* Λ *the continuity constant and* λ *the coercivity constant (see Definition 4.2.1). Let* u *resp.* u_h *be the solution of the problem (4.16) resp. (4.17). Then we have the estimate*

$$\|u - u_h\|_V \leq \frac{\Lambda}{\lambda} \inf_{v\in V_h} \|u - v\|_V. \tag{4.21}$$

Chapter 5

List of Symbols

In this chapter one can find for the thesis important resp. most frequent used symbols for domains, spaces of functions, operators, norms, seminorms, scalar products and other often used symbols.

Domains

ω	Open set in $\mathbb{R}^{(n-p)}$, $1 \leq p < n$
Ω	Open set in $\mathbb{R}^n$
$\mathring{\Omega}$	Interior of Ω
$\overline{\Omega}$	Closure of Ω
$\partial\Omega$	Boundary of Ω
σ	Cross-section
σ_x	Cross-section depening on x

Spaces of functions

$C^0(\Omega)$	Space of real-valued continuous functions on Ω, p.73
$C^m(\Omega)$	Space of real-valued functions on Ω having continuous derivatives up to order m, p.73
$C^\infty(\Omega)$	Space of real-valued functions on Ω having continuous derivatives up to any order, p.73
$C_0^\infty(\Omega)$	Space of real-valued functions on Ω which are contained in $C^\infty(\Omega)$ and have compact support on Ω, p.73
$\mathcal{D}(\Omega)$	$= C_0^\infty(\Omega)$, p.73
$\mathbf{H}(\Omega, \mathrm{div})$	$= \{\mathbf{v} \in \mathbf{L}_2(\Omega) \mid \mathrm{div}(\mathbf{v}) \in L_2(\Omega)\}$, p.75
$H^1(\Omega)$	Space of real-valued L_2-functions on Ω whose weak derivatives up to order 1 are in $L_2(\Omega)$, p.73
$\mathbf{H}^1(\Omega)$	$= (H^1(\Omega))^n$, p.73

$H^1_{loc}(\Omega)$	Set of functions v defined on Ω such that for any Ω' bounded with $\overline{\Omega'} \subset \Omega$ one has $v \in H^1(\Omega')$
$H^1(\Omega)/\mathbb{R}$	Quotient space of $H^1(\Omega)$ by $\mathbb{R}$
$H^1_0(\Omega)$	Closure of $C^\infty_0(\Omega)$ w.r.t. the H^1-norm, p.74
$H^{-1}(\Omega)$	Dual space of $H^1_0(\Omega)$
$H^m(\Omega)$	Space of real-valued L_2-functions on Ω whose weak derivatives up to order m are in $L_2(\Omega)$, p.73
$H^m_0(\Omega)$	Closure of $C^\infty_0(\Omega)$ w.r.t. the H^m-norm, p.74
$\mathbf{H}^m(\Omega, \mathrm{div})$	$= \{\mathbf{v} \in \mathbf{H}^m(\Omega) \mid \mathrm{div}(\mathbf{v}) \in H^m(\Omega)\}$, p.75
$L_2(\Omega)$	Space of real-valued functions integrable in Ω with power 2, p.71
$\mathbf{L}_2(\Omega)$	$= (L_2(\Omega))^n$, p.72
$L_p(\Omega)$	Space of real-valued functions integrable in Ω with power p, p.71
$L_\infty(\Omega)$	Space of real-valued functions which are essentially bounded in Ω, p.71
$\mathbb{P}_k$	Space of plynomials of degree $\leq k$
$W^{m,p}(\Omega)$	Space of real-valued L_p-functions on Ω whose weak derivatives up to order m are in $L_p(\Omega)$, p.73
$W^{m,p}_0(\Omega)$	Closure of $C^\infty_0(\Omega)$ w.r.t. the $W^{m,p}$-norm, p.74
$W^{-m,p'}(\Omega)$	Dual space of $W^{m,p}_0(\Omega)$, p.75

Operators

D^α	Differential operator of order $\vert\alpha\vert$
div	Divergence operator
∇	Gradient
Δ	Laplace operator

Norms, seminorms and scalar products

$\cdot$	Euclidian scalar product in $\mathbb{R}^n$
$\vert \cdot \vert$	Absolute value
$\Vert \cdot \Vert$	Operator norm for the matrix
$\Vert \cdot \Vert_{\mathbf{H}(\Omega,\mathrm{div})}$	Norm of the space $\mathbf{H}(\Omega, \mathrm{div})$, p.75
$(\cdot,\cdot)_{\mathbf{H}(\Omega,\mathrm{div})}$	Scalar product of the space $\mathbf{H}(\Omega, \mathrm{div})$, p.75
$\Vert \cdot \Vert_{H^1(\Omega)}$	H^1-Norm, p.73

$\|\cdot\|_{\mathbf{H}^1(\Omega)}$	$\mathbf{H}^1$-Norm, p.73
$\|\cdot\|_{H^m(\Omega)}$	H^m-Norm, p.73
$(\cdot,\cdot)_{L_2(\Omega)}$	L_2-Scalar product, p.71
$\|\cdot\|_{L_2(\Omega)}$	L_2-Norm, p.71
$\|\cdot\|_{\mathbf{L}_2(\Omega)}$	$\mathbf{L}_2$-Norm, p.72
$\|\cdot\|_{L_p(\Omega)}$	L_p-Norm, p.71
$\|\cdot\|_{L_\infty(\Omega)}$	L_∞-Norm, p.71
$\|\cdot\|_{W^{m,p}(\Omega)}$	$W^{m,p}$-Norm ($m \in \mathbb{N}$), p.73
$\|\cdot\|_{W^{m,p}(\Omega)}$	$W^{m,p}$-Seminorm ($m \in \mathbb{N}$), p.73
$(\cdot,\cdot)_{W^{m,2}(\Omega)}$	$W^{m,2}$-Scalar product, p.73
$\|\cdot\|_{W^{-m,q}(\Omega)}$	$W^{-m,q}$-Norm, p.75

Various other symbols

$\vee$	Maximum
$\wedge$	Minimum
d_Ω	Diameter of Ω
det	Determinant
diag	Diagonal matrix
$[\cdot]$	Integer part
$\perp$	Orthogonal
$\otimes$	Tensor product
$x = (X_1, X_2)$	Variable in $\mathbb{R}^n$ with $X_1 \in \mathbb{R}^p$, $X_2 \in \mathbb{R}^{n-p}$
$C_{PW}(\Omega)$	Poincaré-Wirtinger constant on Ω, p.74
$C_F(\Omega)$	Friedrichs constant on Ω, p.74
δ_{ij}	Kronecker delta
$\mathbf{e}_j$	Unit vector in x_j-direction
inf	Infimum
$\|\Omega\|$	Measure of Ω
$\mathbb{N}$	Natural numbers
$\mathbb{N}^+$	Positive natural numbers
ν	Unit outward normal
$\mathbb{R}$	Real numbers
$\mathbb{R}^+$	Positive real numbers

$\mathbb{R}^n$	Space of real n-vectors
sup	Supremum
supp	Support of a function

Chapter 6

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