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THE KONTSEVICH–ZORICH COCYCLE OVER VEECH–MCMULLEN FAMILY OF SYMMETRIC TRANSLATION SURFACES

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ABSTRACT. We describe the Kontsevich–Zorich cocycle over an affine invariant orbifold coming from a (cyclic) covering construction inspired by works of Veech and McMullen. In particular, using the terminology in a recent paper of Filip, we show that all cases of Kontsevich–Zorich monodromies of $SU(p,q)$ type are realized by appropriate covering constructions.

1. THE VEECH–MCMULLEN FAMILY OF SYMMETRIC TRANSLATION SURFACES

1.1. **Definition and notations.** Let k be a positive integer, and let ℓ be an integer at least equal to 3. We denote by R the rotation of $\mathbb{C}$ centered at 0 of angle $2\pi/\ell$, by S the symmetry $z \mapsto -\bar{z}$ with respect to the imaginary axis.

We write $\mathbb{Z}_m$ for the standard cyclic group with m elements and $\check{\mathbb{Z}}_m$ for the $\mathbb{Z}_m$-homogeneous space of pairs of consecutive elements of $\mathbb{Z}_m$.

Let $Q_{even} \subset \mathbb{C}$ be the closed regular polygon whose vertices are the roots of unity of order ℓ. Let $Q_{odd} := S(Q_{even}) = -Q_{even}$.

Consider k copies of Q_{even}, indexed by the even elements of $\mathbb{Z}_{2k}$, and k copies of Q_{odd}, indexed by the odd elements of $\mathbb{Z}_{2k}$.

The vertices of Q_{even} and Q_{odd} are indexed by

$$A(even, j) = R^j(1), \quad A(odd, j) = -A(even, j), \quad \forall\, j \in \mathbb{Z}_\ell.$$

For $j' = (j, j+1) \in \check{\mathbb{Z}}_\ell$, $\epsilon \in \{even, odd\}$, we denote by $M(\epsilon, j')$ the midpoint between $A(\epsilon, j)$ and $A(\epsilon, j+1)$, by $\aleph^+(\epsilon, j)$ the oriented segment from $A(\epsilon, j)$ to $M(\epsilon, j')$, and by $\aleph^-(\epsilon, j+1)$ the oriented segment from $A(\epsilon, j+1)$ to $M(\epsilon, j')$.

For $j \in \mathbb{Z}_\ell$ and $i \in \mathbb{Z}_{2k}$, with parity ϵ, we denote by $A(i, j)$, $M(i,(j, j+1))$, $\aleph^{\pm}(i, j)$ the copies in Q_i of $A(\epsilon, j)$, $M(\epsilon,(j, j+1)$, $\aleph^{\pm}(\epsilon, j)$.

DEFINITION 1.1. The translation surface $\mathcal{M}_{k,\ell}$ is obtained from the disjoint union of the Q_i, $i \in \mathbb{Z}_{2k}$ by identifying through the appropriate translation, for each $i \in \mathbb{Z}_{2k}, j \in \mathbb{Z}_\ell$, the segment $\aleph^+(i, j)$ with the segment $\aleph^-(i+1, j+1)$. We denote by $\aleph((i, i+1),(j, j+1))$ the image of these segments in $\mathcal{M}_{k,\ell}$.

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1.2. **Basic properties and symmetries of the translation surface $\mathcal{M}_{k,\ell}$.** We start by computing the ramification at the singular set, associated to the $M(i,(j,j+1))$ and $A(i,j)$.

For each $j' \in \check{\mathbb{Z}}_\ell$, the points $M(i,j'), i \in \mathbb{Z}_{2k}$ are identified into a single point $M(j')$ on $\mathcal{M}_{k,\ell}$ where the total angle is $2\pi k$.

On the other hand, when rotating counterclockwise around $A(i,j)$, a sector of angle $\frac{\pi(\ell-2)}{\ell}$ in Q_i is followed by a sector of the same angle at $A(i-1,j-1)$ in Q_{i-1}. The points of $\mathcal{M}_{k,\ell}$ corresponding to the $A(i,j)$ are therefore naturally indexed by the orbits of the transformation $(i,j) \to (i-1,j-1)$ on $\mathbb{Z}_{2k} \times \mathbb{Z}_\ell$. We denote by $A(\Delta)$ the point of $\mathcal{M}_{k,\ell}$ associated to an orbit Δ. The number of such orbits is the greatest common divisor ϖ of $2k$ and ℓ. The total angle at such a point $A(\Delta)$ is $\frac{2\pi k(\ell-2)}{\varpi}$.

We denote by Σ the set of marked points $M(j'), A(\Delta)$ of $\mathcal{M}_{k,\ell}$, of cardinality $\ell+\varpi$.

The genus of $\mathcal{M}_{k,\ell}$ is thus given by

$$g = \ell k + 1 - k - \frac{1}{2}(\ell + \varpi).$$

Observe that ℓ and ϖ have the same parity.

REMARK 1.2. The translation surface $\mathcal{M}_{1,\ell}$ has been first studied by Veech [9]. He shows that it is a Veech surface and that the image of the Veech group in $PSL(2,\mathbb{R})$ is the lattice generated by R and the parabolic element

$$\begin{pmatrix} 1 & 0 \\ 2\cot\frac{\pi}{\ell} & 1 \end{pmatrix}.$$

It follows easily that the the subset $\Sigma_M \subset \mathcal{M}_{1,\ell}$ consisting of the ℓ points $M_{j'}$, $j' \in \check{\mathbb{Z}}_\ell$ is invariant under the group of affine homeomorphisms of $\mathcal{M}_{1,\ell}$. The same is true of the the subset Σ_A consisting of the one (if ℓ is odd) or two (if ℓ is even) points $A(\Delta)$.

REMARK 1.3. By the previous remark, the image in $\mathcal{M}_{1,\ell}$ of the ramification set of the natural projection $\mathcal{M}_{k,\ell} \to \mathcal{M}_{1,\ell}$ is invariant under the group of affine homeomorphisms of $\mathcal{M}_{1,\ell}$. It follows from a result of Gutkin–Judge [4] that $\mathcal{M}_{k,\ell}$ is a Veech surface.

1.3. **Covers of hyperelliptic components of strata.** Algebraically, the translation surface $\mathcal{M}_{1,\ell}$ corresponds to the Riemann surface $y^2 = (x-x_1)\dots(x-x_\ell)$ together with the holomorphic one-form $c\,dx/y$ for *appropriate* choices of ℓ distinct points $x_1,\dots,x_\ell \in \mathbb{C}$ and a constant $c \in \mathbb{C}^*$: see Veech [9]. More generally, $\mathcal{M}_{k,\ell}$ is the covering of $\mathcal{M}_{1,\ell}$ given by the Riemann surface $y^{2k} = (x-x_1)\dots(x-x_\ell)$ and the holomorphic one-form $c\,dx/y^k$: see McMullen [7]. For this reason, we define the Veech–McMullen family $\mathcal{F}_{k,\ell}$ of translation surfaces the Riemann surfaces $y^{2k} = (x-x_1)\dots(x-x_\ell)$ equipped with $c\,dx/y^k$ for *arbitrary* choices of ℓ distinct points $x_1,\dots,x_\ell \in \mathbb{C}$ and constants $c \in \mathbb{C}^*$.

The Veech–McMullen family $\mathcal{F}_{1,\ell}$ for ℓ odd, resp. ℓ even, is the hyperelliptic component of the stratum $\mathcal{H}(\underbrace{(\ell-2-\varpi)/\varpi,\dots,(\ell-2-\varpi)/\varpi}_{\varpi})$ of translation surfaces with ϖ conical singularities with total angle $\frac{2\pi k(\ell-2)}{\varpi}$; see [5].

In general, the Veech–McMullen family $\mathcal{F}_{k,\ell}$ is an affine suborbifold of the stratum $\mathcal{H}(\underbrace{k-1,\dots,k-1}_{\ell},\underbrace{k(\ell-2-\varpi)/\varpi,\dots,k(\ell-2-\varpi)/\varpi}_{\varpi})$ given by a covering construction. Therefore, the Kontsevich–Zorich cocycle over the Teichmüller geodesic flow on $\mathcal{F}_{k,\ell}$ is coded by the hyperelliptic Rauzy diagrams with arrows decorated by certain matrices (describing actions on the homology of canonical translation surfaces in $\mathcal{F}_{k,l}$).

In particular, following the discussion in our previous paper [1], one can associate a *Rauzy–Veech group* $RV(k,\ell)$ to $\mathcal{F}_{k,\ell}$. By definition, the matrices in $RV(k,\ell)$ preserve the natural symplectic intersection form on the absolute homology of the translation surfaces in $\mathcal{F}_{k,\ell}$.

REMARK 1.4. In this notation, the main result from our previous paper [1] asserts that $RV(1,k)$ is naturally isomorphic to an explicit finite-index subgroup of the integral symplectic group $Sp(2g,\mathbb{Z})$ (where g is the genus of $\mathcal{M}_{k,\ell}$).

1.4. **Statement of the main result.** In this paper, we study the structure of the Rauzy–Veech groups of $\mathcal{F}_{k,\ell}$.

THEOREM 1.5. *The real Hodge bundle over $\mathcal{F}_{k,\ell}$ decomposes into a direct sum $H_1\oplus\cdots\oplus H_k$ of flat subbundles H_r associated to the eigenspaces of the generator of the deck group of $\mathcal{M}_{k,\ell}\to\mathcal{M}_{1,\ell}$ (cf. (2) and (3) below). The Rauzy–Veech group of $\mathcal{F}_{k,\ell}$ respects this decomposition. Moreover, if one denotes by $RV(k,\ell)|_{H_r}$ the group associated to the restrictions to H_r of the matrices in $RV(k,\ell)$, then:*

(a) *$RV(k,\ell)|_{H_k}$ is naturally isomorphic to $RV(1,\ell)$; thus, $RV(k,\ell)|_{H_k}$ is isomorphic to a finite-index subgroup of $Sp(2g,\mathbb{Z})$;*

(b) *for each $0<r<k$, the symplectic intersection form on H_r induces a Hermitian form $Q_{r/2k}$ of signature $(\lceil\ell(r/2k)-1\rceil,\lceil\ell(1-(r/2k))-1\rceil)$ which is preserved by $RV(k,\ell)|_{H_r}$; furthermore,*
  - *if $\ell(r/2k)<1$ and $r/2k\neq 1/6,1/4$, then $RV(k,\ell)|_{H_r}\cap SU(Q_{r/2k})$ is dense in $SU(Q_{r/2k})$ (for the usual topology);*
  - *if $\ell(r/2k)\notin\mathbb{Z}$ and $r/2k\neq 1/6,1/4,1/3$, then $RV(k,\ell)|_{H_r}\cap SU(Q_{r/2k})$ is Zariski dense in $SU(Q_{r/2k})$.*

REMARK 1.6. Actually, our discussion in Section 4 provides a precise version of Theorem 1.5: in particular, we compute the Zariski closure of $RV(\ell,k)|_{H_r}\cap SU(Q_{r/2k})$ in the exceptional cases $r/2k=1/6,1/4,1/3$. However, we have not included all possibilities in Theorem 1.5 in order to get a "cleaner" statement.

This result provides explicit examples showing that all cases of $SU(p,q)$ Kontsevich–Zorich monodromies discussed in Filip's paper [3] actually occur.

A direct consequence of Theorem 1.5 and the simplicity criterion of Avila–Viana [2] (as stated in Subsection 2.5 of [6]) is the following.

COROLLARY 1.7. *The Lyapunov exponents of the restriction of the Kontsevich–Zorich cocycle over $\mathcal{F}_{k,\ell}$ to H_r are "simple" in the sense that:*

- *they have multiplicity one when $r = k$;*
- *for $0 < r < k$, $\ell(r/2k) \notin \mathbb{Z}$ and $r/2k \neq 1/6, 1/4, 1/3$, the multiplicity of all non-zero Lyapunov exponents is one and there are exactly $\lceil \ell(1-(r/2k)) - 1\rceil - \lceil \ell(r/2k) - 1\rceil$ vanishing Lyapunov exponents.*

Proof. As it is explained in [2, §7.3], the Kontsevich–Zorich cocycle over $\mathcal{F}_{k,\ell}$ is coded by a locally constant cocycle whose supporting monoid $\mathcal{B}$ contains all matrices of the form $B_\gamma B B_\gamma$, where B_γ is a fixed Kontsevich–Zorich matrix and B are the Kontsevich–Zorich matrices associated to all oriented loops based at a fixed vertex of the Rauzy diagram. Since the group generated by all such matrices B is $RV(k,\ell)$, it follows from Theorem 1.5 that the restriction of the supporting monoid $\mathcal{B}$ to each H_r is pinching and twisting in the sense of [6, §2.5]. The desired result now follows from [6, Theorem 2.17]. □

1.5. **Organization of the paper.** In Section 2, we study a decomposition $H_1 \oplus \cdots \oplus H_k$ of the first absolute homology group of $\mathcal{M}_{k,\ell}$. In Section 3, we describe the restrictions $RV(k,\ell)|_{H_r}$ of the Rauzy–Veech group $R(k,\ell)$ to the summands of the decomposition $H_1(\mathcal{M}_{k,\ell},\mathbb{R}) = H_1 \oplus \cdots \oplus H_k$ in terms of complex matrices on the vector space $\mathbb{C}^\ell$ equipped with adequate hermitian forms. In Section 3, we reduce the proof of Theorem 1.5 to the investigation of certain groups of complex matrices. Finally, we analyze in Section 4 the relevant groups of matrices in order to establish Theorem 1.5.

REMARK 1.8. We hope that the arguments in this paper might be useful to study the question of non-continuity of the central Oseledets subspaces of the Kontsevich–Zorich cocycle.

2. HOMOLOGY GROUPS

2.1. **Subgroups of affine diffeomorphisms.** We now define a finite subgroup G of order $4k\ell$ of the group of affine diffeomorphisms of $\mathcal{M}_{k,\ell}$.

The group $\mathbb{Z}_{2k}$ acts by direct affine diffeomorphisms on $\mathcal{M}_{k,\ell}$: the element $r \in \mathbb{Z}_{2k}$ sends Q_i onto Q_{i+r} with derivative id if r is even, $-$id if r is odd. It sends $A(i,j)$ to $A(i+r,j)$, $\aleph((i,i+1),(j,j+1))$ to $\aleph((i+r,i+r+1),(j,j+1))$ and fixes each $M(j')$,

The group $\mathbb{Z}_\ell$ acts on $\mathcal{M}_{k,\ell}$ by direct affine diffeomorphisms, the derivative of the action of $s \in \mathbb{Z}_\ell$ is the rotation R^s. This action preserves each Q_i, sending $A(i,j)$ to $A(i,j+s)$, $M((j,j+1))$ to $M((j+s,j+s+1))$, $\aleph((i,i+1),(j,j+1))$ to $\aleph((i,i+1),(j+s,j+s+1))$.

This two actions commute and they combine to define an action of the product group $\mathbb{Z}_{2k} \times \mathbb{Z}_\ell$ on $\mathcal{M}_{k,\ell}$.

We now define an affine involution σ of $\mathcal{M}_{k,\ell}$ whose derivative is S. For $i \in \mathbb{Z}_{2k}$, σ sends Q_i onto Q_{1-i}, $M((j,j+1))$ to $M((-j-1,-j))$, $A(i,j)$ to $A(1-i,-j)$, $\aleph((i,i+1),(j,j+1))$ to $\aleph((-i,-i+1),(-j-1,-j))$.

The involution σ conjugates the action of every element $(r,s) \in \mathbb{Z}_{2k} \times \mathbb{Z}_\ell$ to the action of $(-r,-s)$. Combining the action of σ and the action of $\mathbb{Z}_{2k} \times \mathbb{Z}_\ell$ defines an action of a group G of order $4k\ell$. It is the usual dihedral group of order $4k\ell$ when $\varpi = 1$. The group G is the group of permutations of $\check{\mathbb{Z}}_{2k} \times \mathbb{Z}_\ell$ of the form $(i',j) \to (r,s) \pm (i',j)$, for $r \in \mathbb{Z}_{2k}$, $s \in \mathbb{Z}_\ell$.

2.2. **Conjugacy classes in** G**.** Let 1_{2k}, 1_ℓ be the standard generators of the corresponding cyclic groups.

- Assume first that ℓ is odd. There are $k\ell+1$ conjugacy classes of G contained in $\mathbb{Z}_{2k} \times \mathbb{Z}_\ell$, more precisely 2 elements of order 2 and $k\ell-1$ pairs of distinct elements inverse to each other. The other 2 conjugacy classes are those of σ and $\sigma 1_{2k}$ and have size $k\ell$.
- Assume now that ℓ is even. There are $k\ell+2$ conjugacy classes of G contained in $\mathbb{Z}_{2k} \times \mathbb{Z}_\ell$, more precisely 4 elements of order 2 and $k\ell-2$ pairs of distinct elements inverse to each other. The other 4 conjugacy classes are those of σ, $\sigma 1_{2k}$, $\sigma 1_\ell$, $\sigma 1_{2k} 1_\ell$ and have size $\frac{1}{2}k\ell$.

2.3. **Irreducible representations of** G **over** $\mathbb{R}$ **or** $\mathbb{C}$**.** The irreducible representations of G over $\mathbb{C}$ are all defined over $\mathbb{R}$ and have dimension 1 or 2.

When ℓ is odd, there are 4 nonequivalent 1-dimensional representations of G, sending 1_ℓ to 1 and 1_{2k} and σ to ± 1 (independently).

When ℓ is even, there are 8 nonequivalent 1-dimensional representations of G, sending 1_ℓ , 1_{2k} and σ to ± 1 (independently).

To parametrize the 2-dimensional representations, we define $\mathfrak{R}(2k,\ell)$ to be the quotient of $\mathbb{Z}_{2k} \times \mathbb{Z}_\ell$ by $\{\pm 1\}$ and denote by $\mathfrak{R}^*(2k,\ell)$ the subset of $\mathfrak{R}(2k,\ell)$ associated to elements $(r,s) \in \mathbb{Z}_{2k} \times \mathbb{Z}_\ell$ of order > 2 . The cardinality of $\mathfrak{R}(2k,\ell)$ (resp. $\mathfrak{R}^*(2k,\ell)$) is equal to $k\ell+1$ (resp. $k\ell-1$) if ℓ is odd, to $k\ell+2$ (resp. $k\ell-2$) if ℓ is even.

For $(r,s) \in \mathbb{Z}_{2k} \times \mathbb{Z}_\ell$, the representation $\pi_{r,s}$ sends σ to S, 1_{2k} to a rotation of angle $\pi\frac{r}{k}$ and 1_ℓ to a rotation of angle $2\pi\frac{s}{\ell}$.

The character $\chi_{r,s}$ of $\pi_{r,s}$ vanishes on $G-(\mathbb{Z}_{2k} \times \mathbb{Z}_\ell)$ and its values on $\mathbb{Z}_{2k} \times \mathbb{Z}_\ell$ are given by

$$\chi_{r,s}(i,j) = 2\cos 2\pi\left(\frac{ri}{2k} + \frac{sj}{\ell}\right).$$

When (r,s) has order > 2, the representation $\pi_{r,s}$ is irreducible and is conjugated by S to $\pi_{-r,-s}$.

When (r,s) has order 2, the representation $\pi_{r,s}$ split into two representations $\pi^+_{r,s}$ (with $\chi^+_{r,s}(\sigma) = 1$) and $\pi^-_{r,s}$ (with $\chi^-_{r,s}(\sigma) = -1$) of dimension 1. The isomorphism classes of irreducible representations are thus parametrized by $\mathfrak{R}^*(2k,\ell)$.

2.4. **Irreducible representations of** G **over** $\mathbb{Q}$**.** Let $\Pi := 2k\ell/\varpi$ be the least common multiple of $2k$ and ℓ, which is also the least common multiple of the elements of G. By a general result of Brauer (see [8, Theorem 24, Section 12.3]), which is easily checked in the case of G, all irreducible representations of G over $\mathbb{C}$ are actually defined over the cyclotomic field $\mathbb{Q}(\Pi)$. To see how to group together the $\pi_{r,s}$ in order to obtain the irreducible representations of G over

$\mathbb{Q}$, consider the action of the Galois group $(\mathbb{Z}_\Pi)^*$ of $\mathbb{Q}(\Pi)$ over $\mathbb{Q}$ on $\mathfrak{R}(2k,\ell)$ defined by

$$t.(r,s) = (tr, ts).$$

The group $(\mathbb{Z}_\Pi)^*$ also acts on G through $t.g := g^t$. Observe that we have, for all $(r,s) \in \mathfrak{R}(2k,\ell)$, $g \in G$, $t \in (\mathbb{Z}_\Pi)^*$

$$t.\chi_{r,s}(g) = \chi_{r,s}(t.g) = \chi_{t.(r,s)}(g).$$

The 1-dimensional representations of G are all defined over $\mathbb{Q}$. Similarly, each of the points in $\mathfrak{R}(2k,\ell) \smallsetminus \mathfrak{R}^*(2k,\ell)$ is fixed by $(\mathbb{Z}_\Pi)^*$.

By [8, Theorem 29, section 13.1], we conclude that the irreducible representations of G over $\mathbb{Q}$ of dimension > 1 are parametrized by the orbits of the action of $(\mathbb{Z}_\Pi)^*$ on $\mathfrak{R}^*(2k,\ell)$: for an orbit $\mathcal{O}$, the associated representation is

$$\pi_{\mathcal{O}} := \bigoplus_{(r,s)\in\mathcal{O}} \pi_{r,s}.$$

REMARK 2.1. To compute the orbits of the action of $(\mathbb{Z}_\Pi)^*$ on $\mathfrak{R}(2k,\ell)$, it is sufficient to consider the action over $\mathbb{Z}_{2k} \times \mathbb{Z}_\ell$, as $-1 \in (\mathbb{Z}_\Pi)^*$. Then one uses the Chinese remainder theorem to split Π into prime powers. Write $\Pi = \prod_p p^{C_p}$, $2k = \prod_p p^{A_p}$, $\ell = \prod_p p^{B_p}$, with $C_p = \max(A_p, B_p)$ for each prime p. The action of $(\mathbb{Z}_\Pi)^*$ on $\mathbb{Z}_{2k} \times \mathbb{Z}_\ell$ is the product of the actions of $(\mathbb{Z}_{p^C})^*$ over $\mathbb{Z}_{p^A} \times \mathbb{Z}_{p^B}$ (with $A = A_p, B = B_p, C = C_p$). Let $(r,s) \in \mathbb{Z}_{p^A} \times \mathbb{Z}_{p^B}$. Let p^a (resp. p^b) be the order of r in $\mathbb{Z}_{p^A}$ (resp. of s in $\mathbb{Z}_{p^B}$), with $0 \leqslant a \leqslant A$, $0 \leqslant b \leqslant B$. For any (r', s') in the orbit of (r,s), the order of r' (resp. s') is also p^a (resp. p^b). When $\min(a,b) = 0$, there is exactly one orbit with these orders. When $\min(a,b) > 0$, the stabilizer of (r,s) as above has cardinality equal to $p^{C-\max(a,b)}$, hence the corresponding orbit has cardinality equal to $p^{\max(a,b)-1}(p-1)$. As the number of pairs (r,s) with orders (p^a, p^b) is equal to $p^{a+b-2}(p-1)^2$, the number of orbits associated to these values of a, b is equal to $p^{\min(a,b)-1}(p-1)$.

2.5. **Decomposition of the first relative homology group.** The classes of the oriented segments $\aleph(i', j')$ ($i' \in \check{\mathbb{Z}}_{2k}$, $j' \in \check{\mathbb{Z}}_\ell$) obviously span the first relative homology group $H_1(\mathcal{M}_{k,\ell}, \Sigma, \mathbb{Q})$.

Going around the boundary of Q_i gives the relation

$$\sum_{\check{\mathbb{Z}}_\ell} \aleph((i-1,i), j') = \sum_{\check{\mathbb{Z}}_\ell} \aleph((i+1,i)), j'), \quad \forall i \in \mathbb{Z}_{2k},$$

providing $2k-1$ independent relations between the $\aleph(i', j')$. As

$$2k\ell - (2k-1) = 2g + (\#\Sigma - 1) = \dim H_1(\mathcal{M}_{k,\ell}, \Sigma, \mathbb{Q}),$$

these are the only relations between the $\aleph(i', j')$.

Denote by (e_i), $(E_{i',j'})$ the canonical bases of $\mathbb{Q}^{\mathbb{Z}_{2k}}$, $\mathbb{Q}^{\check{\mathbb{Z}}_{2k} \times \check{\mathbb{Z}}_\ell}$ respectively. We equip $\mathbb{Q}$, $\mathbb{Q}^{\mathbb{Z}_{2k}}$, $\mathbb{Q}^{\check{\mathbb{Z}}_{2k} \times \check{\mathbb{Z}}_\ell}$ with structures of G-module by defining

$$1_k.x = 1_\ell.x = x, \quad \sigma.x = -x, \quad \forall x \in \mathbb{Q},$$

$$1_k.e_i = e_{i+1}, \quad 1_\ell.e_i = e_i, \quad \sigma.e_i = -e_{1-i}, \quad \forall i \in \mathbb{Z}_{2k},$$

$$1_k.E_{i',j'} = E_{1+i',j'},\ 1_\ell.E_{i',j'} = E_{i',1+j'},\ \sigma.E_{i',j'} = E_{1-i',-j'},\ \forall (i',j') \in \check{\mathbb{Z}}_{2k} \times \check{\mathbb{Z}}_\ell.$$

We have then an exact sequence of G-modules.

$$0 \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}^{\mathbb{Z}_{2k}} \longrightarrow \mathbb{Q}^{\check{\mathbb{Z}}_{2k}\times\check{\mathbb{Z}}_{\ell}} \longrightarrow H_1(\mathcal{M}_{k,\ell},\Sigma,\mathbb{Q}) \longrightarrow 0. \tag{1}$$

The maps in this exact sequence are as follows. The map from $\mathbb{Q}$ to $\mathbb{Q}^{\mathbb{Z}_{2k}}$ sends 1 to $\sum_i e_i$. The map from $\mathbb{Q}^{\mathbb{Z}_{2k}}$ to $\mathbb{Q}^{\check{\mathbb{Z}}_{2k}\times\check{\mathbb{Z}}_{\ell}}$ sends e_i to $\sum_{j'}(E_{\{i,i+1\},j'} - E_{\{i-1,i\},j'})$. The map from $\mathbb{Q}^{\check{\mathbb{Z}}_{2k}\times\check{\mathbb{Z}}_{\ell}}$ to $H_1(\mathcal{M}_{k,\ell},\Sigma,\mathbb{Q})$ sends $E_{i',j'}$ to the class of $\aleph(i',j')$.

From (1), we deduce the character χ_{rel} of the G-module $H_1(\mathcal{M}_{k,\ell},\Sigma,\mathbb{Q})$. Firstly, for $(i,j)\in\mathbb{Z}_{2k}\times\mathbb{Z}_{\ell}$, we have

$$\chi_{rel}(i,j) = 1 - 2k\delta_{0i} + 2k\ell\delta_{0i}\delta_{0j}.$$

- When ℓ is odd, χ_{rel} is equal to 1 everywhere on $G-(\mathbb{Z}_{2k}\times\mathbb{Z}_{\ell})$. One obtains
$$\chi_{rel} = 1 + \sum_{(r,s)\in\mathfrak{R}^*(2k,\ell),\, s\neq 0} \chi_{r,s}.$$
- When ℓ is even, the values of χ_{rel} on the conjugacy classes of σ, $\sigma 1_{2k}$, $\sigma 1_{\ell}$, $\sigma 1_{2k}1_{\ell}$ are respectively -1 ,$+1$,$+3$,$+1$. One obtains
$$\chi_{rel} = 1 + \chi_+ + \chi_- + \sum_{(r,s)\in\mathfrak{R}^*(2k,\ell),\, s\neq 0} \chi_{r,s},$$
where χ_+ (resp. χ_-) is the 1-dimensional character with value -1 at σ and 1_{ℓ} and value $+1$ (resp. -1) at 1_{2k}.

2.6. **Decomposition of the first absolute homology group.** The exact sequence of G-modules for relative homology reads

$$0 \to H_1(\mathcal{M}_{k,\ell},\mathbb{Q}) \to H_1(\mathcal{M}_{k,\ell},\Sigma,\mathbb{Q}) \to H_0(\Sigma,\mathbb{Q}) \to \mathbb{Q} \to 0.$$

This gives

$$\chi_{ab} = \chi_{rel} - \chi_{\Sigma} + 1,$$

where χ_{ab}, χ_{Σ} are the characters of $H_1(\mathcal{M}_{k,\ell},\mathbb{Q})$, $H_0(\Sigma,\mathbb{Q})$ respectively.

Each of the subsets $\Sigma_M := \left\{M_{j'} \,\middle|\, j'\in\check{\mathbb{Z}}_{\ell}\right\}$ and $\Sigma_A := \{A(\Delta) \mid \Delta\in\mathbb{Z}_{\varpi}\}$ is invariant under G, hence the character χ_{Σ} splits as $\chi_{\Sigma_M} + \chi_{\Sigma_A}$.

The character χ_{Σ_M} satisfies $\chi_{\Sigma_M}(i,j) = \ell\delta_{0j}$.

When ℓ is odd, it is equal to 1 on all of $G-(\mathbb{Z}_{2k}\times\mathbb{Z}_{\ell})$. This gives in this case

$$\chi_{\Sigma_M} = 1 + \sum_{(0,s)\in\mathfrak{R}^*(2k,\ell)} \chi_{0,s}.$$

When ℓ is even, χ_{Σ_M} takes the value 0 on the conjugacy classes of σ and $\sigma 1_{2k}$, and the value 2 on the conjugacy classes of $\sigma 1_{\ell}$ and $\sigma 1_{2k}1_{\ell}$. This gives in this case

$$\chi_{\Sigma_M} = 1 + \chi_+ + \sum_{(0,s)\in\mathfrak{R}^*(2k,\ell)} \chi_{0,s}.$$

The character χ_{Σ_A} satisfies $\chi_{\Sigma_A}(i,j) = \varpi$ if i,j are congruent modulo ϖ, to $\chi_{\Sigma_A}(i,j) = 0$ otherwise.

When ℓ is odd, it is equal to 1 on all of $G-(\mathbb{Z}_{2k}\times\mathbb{Z}_{\ell})$. This gives

$$\chi_{\Sigma_A} = 1 + \sum_{(r,s)\in\mathfrak{R}^*(2k,\ell),\, \frac{r}{2k}+\frac{s}{\ell}\in\mathbb{Z}} \chi_{r,s}.$$

When ℓ is even, χ_{σ_A} takes the value 0 on the conjugacy classes of σ and $\sigma 1_{2k} 1_\ell$, and the value 2 on the conjugacy classes of $\sigma 1_\ell$ and $\sigma 1_{2k}$. This gives

$$\chi_{\Sigma_A} = 1 + \chi_- + \sum_{(r,s)\in\mathfrak{R}^*(2k,\ell),\, \frac{r}{2k}+\frac{s}{\ell}\in\mathbb{Z}} \chi_{r,s}.$$

In the formula $\chi_{ab} = \chi_{rel} - \chi_{\Sigma_M} - \chi_{\Sigma_A} + 1$ we have now computed all the terms in the right-hand side. We obtain

$$\chi_{ab} = \sum_{(r,s)\in\mathfrak{R}(2k,\ell),\, r\neq 0, s\neq 0,\, \frac{r}{2k}+\frac{s}{\ell}\notin\mathbb{Z}} \chi_{r,s}.$$

REMARK 2.2. With the conditions $r \neq 0, s \neq 0, \frac{r}{2k} + \frac{s}{\ell} \notin \mathbb{Z}$, there is no difference between $\mathfrak{R}(2k,\ell)$ and $\mathfrak{R}^*(2k,\ell)$.

From this character formula, we get

$$H_1(\mathcal{M}_{k,\ell},\mathbb{R}) = \bigoplus_{(r,s)\in\mathfrak{R}(2k,\ell),\, r\neq 0, s\neq 0,\, \frac{r}{2k}+\frac{s}{\ell}\notin\mathbb{Z}} \pi_{r,s} = \bigoplus_{0<r\leqslant k} H_r,$$

with

$$H_r := \bigoplus_{0<s<\ell,\, \frac{r}{2k}+\frac{s}{\ell}\neq 1} \pi_{r,s} \tag{2}$$

for $0 < r < k$ and

$$H_k := \bigoplus_{0<s<\ell/2} \pi_{k,s}. \tag{3}$$

PROPOSITION 2.3. *The subrepresentation H_r is defined over $\mathbb{Q}$ if and only if $r/2k$ is equal to $1/6$, $1/4$, $1/3$, or $1/2$.*

Proof. Indeed, for $0 < r \leqslant k$, let $\mathfrak{R}_r$ be the image in $\mathfrak{R}^*(2k,\ell)$ of the set

$$\left\{ (r,s) \,\middle|\, 0 < s < \ell, \frac{r}{2k} + \frac{s}{\ell} \neq 1 \right\}.$$

By subsection 2.4, the subrepresentation H_r is defined over $\mathbb{Q}$ iff $\mathfrak{R}_r$ is invariant under the action of $(\mathbb{Z}_\Pi)^*$. As the subsets $\mathbb{Z}_{2k} \times \{0\}$ and $\left\{ (r,s) \,\middle|\, \frac{r}{2k} + \frac{s}{\ell} \in \mathbb{Z} \right\}$ of $\mathbb{Z}_{2k} \times \mathbb{Z}_\ell$ are invariant under the action of $(\mathbb{Z}_\Pi)^*$, $\mathfrak{R}_r$ is invariant under the action of $(\mathbb{Z}_\Pi)^*$ iff the subset $\{\pm r\}$ of $\mathbb{Z}_{2k}$ is invariant under the action of $(\mathbb{Z}_\Pi)^*$, which has the same orbits in $\mathbb{Z}_{2k}$ that the action of $(\mathbb{Z}_{2k})^*$. But the only integers $n > 1$ such that $(\mathbb{Z}_n)^* = \{\pm 1\}$ are $2, 3, 4, 6$. This proves the proposition. □

REMARK 2.4. The subspace H_k is identified with the $H_1(\mathcal{M}_{1,\ell},\mathbb{R})$ under the natural projection $\mathcal{M}_{k,\ell} \to \mathcal{M}_{1,\ell}$. In general, the subspaces H_r are given by the eigenspaces of the generator of the group of deck transformations of $\mathcal{M}_{k,\ell} \to \mathcal{M}_{1,\ell}$. In particular, the decomposition $H_1(\mathcal{M}_{k,\ell},\mathbb{R}) = \bigoplus_{0<r\leqslant k} H_r$ can be extended to all translation surfaces of the Veech–McMullen family $\mathcal{F}_{k,\ell}$.

2.7. **Relation with the symplectic intersection form.** Let χ_σ be the 1-dimensional character on G with kernel $\mathbb{Z}_{2k} \times \mathbb{Z}_\ell$. The intersection form ω on $H_1(\mathcal{M}_{k,\ell}, \mathbb{Q})$ satisfies, for any $g \in G$, $v, w \in H_1(\mathcal{M}_{k,\ell}, \mathbb{Q})$

$$\omega(g.v, g.w) = \chi_\sigma(g)\omega(v, w).$$

PROPOSITION 2.5. *The* 2*-dimensional summands in the decomposition of the* G*-module* $H_1(\mathcal{M}_{k,\ell}, \mathbb{Q})$ *in the last subsection are mutually* ω*-orthogonal.*

Proof. Let E, F be two such distinct summands.The intersection form defines an homomorphism u from E to the dual F^* of F. We have to prove that $u = 0$. If $v \in E$ belongs to the kernel of u, then the same is true for $g.v$, for any $g \in G$. As E is irreducible, this proves that u is either invertible or equal to 0. If u is invertible, the formula above shows that it is an isomorphism of G-modules from E to the tensor product of the contragredient representation of F by χ_σ. But this tensor product is isomorphic as G-module to F itself. As E, F are distinct, we conclude that u cannot be invertible, hence $u = 0$. □

For $0 < r \leqslant k$, let H_r be the sum of the summands in the decomposition of $H_1(\mathcal{M}_{k,\ell}, \mathbb{Q})$ with characters $\chi_{r,s}$, s varying according to the prescription above. Assume now that $0 < r < k$. We equip H_r with the complex structure[1] such that

$$1_{2k}.v = \exp\left(i\pi\frac{r}{k}\right)v.$$

The formula

$$\langle v, v\rangle := \left(\sin\left(\pi\frac{r}{k}\right)\right)^{-1}\omega(1_{2k}.v, v)$$

defines an hermitian[2] form on H_r whose imaginary part is ω.

The signature of this hermitian form is calculated in Subsection 3.6.

3. HYPERELLIPTIC RAUZY DIAGRAMS AND RAUZY–VEECH GROUPS $RV(k,\ell)$

3.1. **Review of the description of hyperelliptic Rauzy diagrams.** In this subsection, we recall the content of Subsection 2.1 of our previous paper [1].

Given an integer $d \geqslant 2$, denote by $\mathcal{A}_d$ the arithmetic progression $d-1, d-3, \ldots, 1-d$. Note that $\mathcal{A}_d$ has a natural involution $\iota(k) = -k$.

The hyperelliptic Rauzy class $\mathcal{R}_d$ over $\mathcal{A}_d$ and the associated Rauzy diagram $\mathcal{D}_d$ are inductively defined as follows.

The Rauzy class $\mathcal{R}_d$ contains the central vertex $\pi^* = \pi^*(d)$:

$$\pi_t^*(k) = \frac{1}{2}(d+1+k), \quad \pi_b^*(k) = \frac{1}{2}(d+1-k).$$

For $d = 2$, this is the sole vertex. For $d \geqslant 2$, $\mathcal{R}_{d+1}$ is the disjoint union of $\pi^*(d+1)$, $j_t(\mathcal{R}_d)$ and $j_b(\mathcal{R}_d)$, where j_t, j_b are the following injective maps: for

[1] It comes from the fact that H_r is associated to eigenspaces of the eigenvalues $\exp(\pm\frac{2\pi i r}{2k})$ for the action on complex cohomology of the generator 1_{2k} of the deck group of $\mathcal{M}_{k,\ell} \to \mathcal{M}_{1,\ell}$ (cf. Remark 2.4).

[2] In the literature, this hermitian form is called *Hodge form*.

$\pi \in \mathcal{R}_d$, if we denote $j_t(\pi) = t\pi$, $j_b(\pi) = b\pi$, then

$$\begin{aligned} t\pi_t(-d) &= 1, \\ t\pi_b(-d) &= \pi_b(d-3), \\ t\pi_t(k) &= 1 + \pi_t(k-1), \\ t\pi_b(k) &= \begin{cases} \pi_b(k-1) & \text{if } \pi_b(k-1) < \pi_b(d-3), \\ \pi_b(k-1)+1 & \text{if } \pi_b(k-1) \geqslant \pi_b(d-3), \end{cases} \end{aligned}$$

for $2-d \leqslant k \leqslant d$, and

$$\begin{aligned} b\pi_b(d) &= 1, \\ b\pi_t(d) &= \pi_t(3-d), \\ b\pi_b(k) &= 1 + \pi_b(k+1), \\ b\pi_t(k) &= \begin{cases} \pi_t(k+1) & \text{if } \pi_t(k+1) < \pi_t(3-d), \\ \pi_t(k+1)+1 & \text{if } \pi_b(k+1) \geqslant \pi_t(3-d), \end{cases} \end{aligned}$$

for $-d \leqslant k \leqslant d-2$.

The arrows of $\mathcal{D}_d$ are given by the following one-to-one maps R_t, R_b from $\mathcal{R}_d$ to itself:

$$\begin{cases} R_t(\pi^*(d+1)) = j_t(\pi^*(d)), \\ R_b(\pi^*(d+1)) = j_b(\pi^*(d)), \end{cases}$$

$$\begin{cases} R_t \circ j_b \circ R_t^{-1} = j_b, \\ R_b \circ j_t \circ R_b^{-1} = j_t, \end{cases}$$

$$\begin{cases} R_t \circ j_t \circ R_t^{-1}(\pi) = j_t(\pi), & \pi \neq \pi^*(d), \\ R_b \circ j_b \circ R_b^{-1}(\pi) = j_b(\pi), & \pi \neq \pi^*(d), \end{cases}$$

$$R_t \circ j_t \circ R_t^{-1}(\pi^*(d)) = \pi^*(d+1) = R_b \circ j_b \circ R_b^{-1}(\pi^*(d)).$$

The elements of $\mathcal{R}_d$ correspond bijectively to the words in $\{t, b\}$ of length $< d-1$ via the following map W_d: let $W_d(\pi^*(d))$ be the empty word, $W_d(j_t(\pi))$ is the word $tW_{d-1}(\pi)$ and $W_d(j_b(\pi))$ is the word $bW_{d-1}(\pi)$.

One recovers from $W_d(\pi)$ the winners of the arrows starting from π as follows: the winner of the arrow of top type starting from π is the letter $d-1-2w_b(\pi)$ of $\mathcal{A}_d$, where $w_b(\pi)$ is the number of occurrences of b in $W_d(\pi)$; similarly, the winner of the arrow of bottom type starting from π is the letter $1-d+2w_t(\pi)$ of $\mathcal{A}_d$. Observe that we have always

$$d-1-2w_b(\pi) > 1-d+2w_t(\pi).$$

A vertex $\pi \in \mathcal{R}_d$ is connected to the central vertex $\pi^*(d)$ by a *unique* oriented *simple* path $\gamma^*(\pi)$ in $\mathcal{D}_d$ from $\pi^*(d)$ to π.

As it turns out, all non-trivial *simple* loops in $\mathcal{R}_d$ are *elementary*: they consist of arrows of the same type. Any such loop γ contains a unique vertex π such that γ passes through π but $\gamma^*(\pi)$ does not contain any arrow of γ, and, furthermore, π is the vertex of γ such that $|W_d(\pi)|$ is minimal, and its value is $d-1-|\gamma|$. In the sequel, we denote by γ' the non-oriented loop at $\pi^*(d)$ defined by

$$\gamma' := \gamma^*(\pi) * \gamma * (\gamma^*(\pi))^{-1}.$$

3.2. **Mapping classes attached to the arrows of $\mathcal{D}_d$.** In this subsection, we essentially review the content of Section 4 of our previous paper [1].

Given $\pi \in \mathcal{R}_d$, denote by M_π the *canonical* translation surface with combinatorial data π whose length data λ^{can} and suspension data τ^{can} are:

$$\lambda_\alpha^{can} = 1, \quad \tau_\alpha^{can} = \pi_b(\alpha) - \pi_t(\alpha), \quad \forall \alpha \in \mathcal{A}.$$

We obtain M_π by identifying parallel sides of an appropriate polygon P_π. The set of marked points of M_π is denoted by Σ_π, and the middle points of the sides of P_π together with a base point O_π in the interior of P_π form a subset Σ_π^* of M_π.

The Rauzy–Veech operation associated to each arrow $\gamma : \pi \to \pi'$ of $\mathcal{D}_d$ is encoded by the isotopy class of a homeomorphism

$$H_\gamma : (M_\pi, \Sigma_\pi \cup \Sigma_\pi^*) \to (M_{\pi'}, \Sigma_{\pi'} \cup \Sigma_{\pi'}^*)$$

constructed in Subsection 4.1 of [1]. The map $\gamma \mapsto [H_\gamma]$ induces (by functoriality[3]) a morphism $\pi_1(\widetilde{\mathcal{D}_d}, \pi^*) \to \mathrm{Mod}(\pi^*)$ from the fundamental group $\pi_1(\widetilde{\mathcal{D}_d}, \pi^*)$ of the non-oriented Rauzy diagram $\widetilde{\mathcal{D}_d}$ associated to $\mathcal{D}_d$ based at the central vertex π^* to the mapping class group $\mathrm{Mod}(\pi^*)$ of $(M_{\pi^*}, \Sigma_{\pi^*})$.

In this context, given γ a simple loop in $\mathcal{D}_d$, the action of γ' as a isotopy class on M_{π^*} was computed in Proposition 4.5 of [1]: it is a Dehn twist about the straight line joining the midpoints of the sides of P_{π^*} indexed by the letter of $\mathcal{A}_d$ winning in the loop γ.

Note that the elements of $\mathrm{Mod}(\pi^*)$ can be viewed also as mapping classes on the translation surface $\mathcal{M}_{1,\ell}$ where $\ell = d+1$, and, *a fortiori*, they can be lifted to $\mathcal{M}_{k,\ell}$ via the natural projection $\mathcal{M}_{k,\ell} \to \mathcal{M}_{1,\ell}$.

DEFINITION 3.1. The Rauzy–Veech group $RV(k,\ell)$ is the group generated by the actions on $H_1(\mathcal{M}_{k,\ell}, \mathbb{R})$ of all γ' associated to all elementary loops γ in $\mathcal{R}_d$ (where $\ell = d+1$).

The Rauzy–Veech group $RV(1,\ell)$ was computed in our previous paper [1]: it is isomorphic to an explicit finite-index subgroup of $Sp(2g, \mathbb{Z})$ (where g is the genus of $\mathcal{M}_{1,\ell}$).

REMARK 3.2. In general, natural projection $\mathcal{M}_{k,\ell} \to \mathcal{M}_{1,\ell}$ takes H_k to $H_1(\mathcal{M}_{1,\ell}, \mathbb{R})$ in such a way that $RV(k,\ell)|_{H_k}$ is isomorphic to $RV(1,\ell)$, so that $RV(k,\ell)|_{H_k}$ is the explicit finite-index subgroup described in our previous paper [1, Theorem 2.9].

3.3. **Lifting the action of the loops in $\mathcal{D}_d$: top case.** Let γ, π, γ' be as in the previous subsection. Let k be an integer $\geqslant 2$. Let $\mathcal{M}_{k,d+1}$ be the surface considered in the first section. We have a canonical projection $\mathcal{M}_{k,d+1} \to \mathcal{M}_{1,d+1}$. The action of γ', as an isotopy class of the translation surface $\mathcal{M}_{1,d+1}$ with marked points at the $M(j)$ and the $A(\delta)$, was already discussed in the previous subsection. We describe now the lift of this action to $\mathcal{M}_{k,d+1}$.

[3] I.e., the image of a loop under the morphism is the composition of the $[H_\gamma]$ attached to its arrows.

Assume first that γ is of top type. Let $w := w_b(\pi) \in \{0, \dots, d-2\}$. The winner[4] of π is then $p := d-1-2w$. We write L_p^t for the action of γ' on the homology of $\mathcal{M}_{k,d+1}$.

For $i \in \mathbb{Z}_{2k}$, $j \in \mathbb{Z}_{d+1}$, the image of the relative homology class $\aleph((i,i+1),(j,j+1))$ by L_p^t is equal to

- $\aleph((i,i+1),(j,j+1))$ if j is neither 0 nor $w+1$;
- $\aleph((i-1,i),(0,1)) + \sum_1^w (\aleph((i-1,i),(m,m+1)) - \aleph((i,i+1),(m,m+1)))$ if $j = w+1$;
- $\aleph((i-1,i),(w+1,w+2)) + \sum_{w+2}^d (\aleph((i-1,i),(m,m+1)) - \aleph((i,i+1),(m,m+1)))$ if $j = 0$.

We define

$$V_i(p) := \sum_0^w \aleph((i-1,i),(m,m+1)) - \sum_1^{w+1} \aleph((i,i+1),(m,m+1)),$$

so that we have

$$L_p^t(\aleph((i,i+1),(j,j+1))) = \aleph((i,i+1),(j,j+1)) + (\delta_{j,w+1} - \delta_{j,0}) V_i(p).$$

This gives

$$L_p^t(V_i(p)) = -V_{i-1}(p).$$

Let $0 < r < k, 0 < s < \ell := d+1$ with $\frac{r}{2k} + \frac{s}{\ell} \neq 1$; write

$$x_{i,j} = \cos 2\pi \left(\frac{ri}{2k} + \frac{sj}{\ell} \right), \quad y_{i,j} = \sin 2\pi \left(\frac{ri}{2k} + \frac{sj}{\ell} \right).$$

Define

$$X_{r,s}^t := \sum_{i \in \mathbb{Z}_{2k}, j \in \mathbb{Z}_\ell} x_{i,j} \aleph((i,i+1),(j,j+1)), \quad Y_{r,s}^t := \sum_{i \in \mathbb{Z}_{2k}, j \in \mathbb{Z}_\ell} y_{i,j} \aleph((i,i+1),(j,j+1)).$$

We have

$$L_p^t(X_{r,s}^t) = X_{r,s}^t + \sum_{i \in \mathbb{Z}_{2k}} (x_{i,w+1} - x_{i,0}) V_i(p),$$

$$L_p^t(Y_{r,s}^t) = Y_{r,s}^t + \sum_{i \in \mathbb{Z}_{2k}} (y_{i,w+1} - y_{i,0}) V_i(p).$$

Set

$$V_{\cos}(p,r) := \sum_{i \in \mathbb{Z}_{2k}} x_{i,0} V_i(p), \quad V_{\sin}(p,r) := \sum_{i \in \mathbb{Z}_{2k}} y_{i,0} V_i(p).$$

Then we have

$$\begin{aligned} L_p^t(V_{\cos}(p,r)) &= \sum_{i \in \mathbb{Z}_{2k}} x_{i,0} L_p^t(V_i(p)) = -\sum_{i \in \mathbb{Z}_{2k}} x_{i,0} V_{i-1}(p) = -\sum_{i \in \mathbb{Z}_{2k}} x_{i+1,0} V_i(p) \\ &= \cos\left(\pi\left(1+\frac{r}{k}\right)\right) V_{\cos}(p,r) - \sin\left(\pi\left(1+\frac{r}{k}\right)\right) V_{\sin}(p,r) \end{aligned}$$

and

$$L_p^t(V_{\sin}(p,r)) = \sin\left(\pi\left(1+\frac{r}{k}\right)\right) V_{\cos}(p,r) + \cos\left(\pi\left(1+\frac{r}{k}\right)\right) V_{\sin}(p,r).$$

[4]In the sequel, it is useful to recall that an elementary loop of given (top or bottom) type is uniquely determined the parameter w, or equivalently, the parameter p.

We also compute the image of $V_{\cos}(p',r)$ and $V_{\sin}(p',r)$ for $p' \neq p$. We write $p' = d-1-2w'$. Assume first that $p' < p$ (i.e., $w' > w$). One has

$$
\begin{aligned}
&L_p^t(V_{\cos}(p',r)) \\
&= \sum_{i\in\mathbb{Z}_{2k}} x_{i,0}\Big(\sum_0^{w'} L_p^t(\aleph((i-1,i),(m,m+1))) - \sum_1^{w'+1} L_p^t(\aleph((i,i+1),(m,m+1)))\Big) \\
&= V_{\cos}(p',r) + \sum_{i\in\mathbb{Z}_{2k}} x_{i,0}\Big(\sum_0^{w'}(\delta_{m,w+1}-\delta_{m,0})V_{i-1}(p) - \sum_1^{w'+1}(\delta_{m,w+1}-\delta_{m,0})V_i(p)\Big) \\
&= V_{\cos}(p',r) - \sum_{i\in\mathbb{Z}_{2k}} x_{i,0}V_i(p)) \\
&= V_{\cos}(p',r) - V_{\cos}(p,r).
\end{aligned}
$$

Similarly

$$L_p^t(V_{\sin}(p',r)) = V_{\sin}(p',r) - V_{\sin}(p,r).$$

When $p' > p$ (i.e., $w' < w$), one obtains

$$
\begin{aligned}
&L_p^t(V_{\cos}(p',r)) \\
&= V_{\cos}(p',r) + \sum_{i\in\mathbb{Z}_{2k}} x_{i,0}\Big(\sum_0^{w'}(\delta_{m,w+1}-\delta_{m,0})V_{i-1}(p) - \sum_1^{w'+1}(\delta_{m,w+1}-\delta_{m,0})V_i(p)\Big) \\
&= V_{\cos}(p',r) - \sum_{i\in\mathbb{Z}_{2k}} x_{i,0}V_{i-1}(p) \\
&= V_{\cos}(p',r) + L_p^t(V_{\cos}(p,r)),
\end{aligned}
$$

and similarly

$$L_p^t(V_{\sin}(p',r)) = V_{\sin}(p',r) + L_p^t(V_{\sin}(p,r)).$$

In summary, for each $0 < r < k$, we deduce that

- The subspace H_r is fixed by the lift of the action of γ' (which commutes with the action of $\mathbb{Z}_{2k}$); therefore, the Rauzy–Veech group $RV(k,\ell)$ gives rise to well-defined groups $RV(k,\ell)|_{H_r}$ (obtained by restriction to H_r);
- H_r is the direct sum of the 2-dimensional subspace generated by $V_{\cos}(p,r)$ and $V_{\sin}(p,r)$ on which γ' acts by a rotation of $-\pi(1+\frac{r}{k})$, and a subspace of codimension 2 on which γ' acts by the identity.

3.4. **Lifting the action of the loops in $\mathcal{D}_d$: bottom case.** In the same setting that in the last subsection, we now assume that γ is of bottom type.

Let $w := w_t(\pi) \in \{0,\dots,d-2\}$. The winner of π is then $p := -d+1+2w$. We write L_p^b for the action of γ' on the homology of $\mathcal{M}_{k,d+1}$.

For $i \in \mathbb{Z}_{2k}$, $j \in \mathbb{Z}_{d+1}$, the image of $\aleph((i-1,i),(-j,-j+1))$ by L_p^b is equal to

- $\aleph((i-1,i),(-j,-j+1))$, if j is neither 0 nor $w+1$;
- $\aleph((i,i+1),(0,1)) + \sum_1^w(\aleph((i,i+1),(-m,-m+1)) - \aleph((i-1,i),(-m,-m+1)))$ if $j = w+1$;
- $\aleph((i,i+1),(-w-1,-w)) + \sum_{w+2}^d(\aleph((i,i+1),(-m,-m+1)) - \aleph((i-1,i),(-m,-m+1)))$ if $j = 0$.

We now have

$$\sum_0^w \aleph((i,i+1),(-m,-m+1)) - \sum_1^{w+1} \aleph((i-1,i),(-m,-m+1)) = V_i(p)$$

hence

$$L_p^b(\aleph((i-1,i),(-j,-j+1))) = \aleph((i-1,i),(-j,-j+1)) + (\delta_{j,w+1} - \delta_{j,0})V_i(p).$$

This gives

$$L_p^b(V_i(p)) = -V_{i+1}(p).$$

Let $0 < r < k, 0 < s < \ell$ with $\frac{r}{2k} + \frac{s}{\ell} \neq 1$; we have

$$\begin{aligned}
L_p^b(V_{\cos}(p,r)) &= \sum_{i\in\mathbb{Z}_{2k}} x_{i,0} L_p^b(V_i(p)) \\
&= -\sum_{i\in\mathbb{Z}_{2k}} x_{i,0} V_{i+1}(p) \\
&= -\sum_{i\in\mathbb{Z}_{2k}} x_{i-1,0} V_i(p) \\
&= \cos\left(\pi\left(1+\frac{r}{k}\right)\right) V_{\cos}(p,r) + \sin\left(\pi\left(1+\frac{r}{k}\right)\right) V_{\sin}(p,r)
\end{aligned}$$

and

$$L_p^b(V_{\sin}(p,r)) = -\sin\left(\pi\left(1+\frac{r}{k}\right)\right) V_{\cos}(p,r) + \cos\left(\pi\left(1+\frac{r}{k}\right)\right) V_{\sin}(p,r).$$

We also compute the image of $V_{\cos}(p',r)$ and $V_{\sin}(p',r)$ for $p' \neq p$. We write $p' = -d+1+2w'$. Assume first that $p' > p$ (i.e., $w' > w$). One has

$$\begin{aligned}
&L_p^b(V_{\cos}(p',r)) \\
&= \sum_{i\in\mathbb{Z}_{2k}} x_{i,0}\Big(\sum_0^{w'} L_p^b(\aleph((i,i+1),(-m,-m+1))) - \sum_1^{w'+1} L_p^b(\aleph((i-1,i),(m,m+1)))\Big) \\
&= V_{\cos}(p',r) + \sum_{i\in\mathbb{Z}_{2k}} x_{i,0}\Big(\sum_0^{w'} (\delta_{m,w+1} - \delta_{m,0}) V_{i+1}(p) - \sum_1^{w'+1} (\delta_{m,w+1} - \delta_{m,0}) V_i(p)\Big) \\
&= V_{\cos}(p',r) - \sum_{i\in\mathbb{Z}_{2k}} x_{i,0} V_i(p) \\
&= V_{\cos}(p',r) - V_{\cos}(p,r).
\end{aligned}$$

Similarly

$$L_p^b(V_{\sin}(p',r)) = V_{\sin}(p',r) - V_{\sin}(p,r).$$

When $p' < p$ (i.e., $w' < w$), one obtains

$$\begin{aligned}
&L_p^b(V_{\cos}(p',r)) \\
&= V_{\cos}(p',r) + \sum_{i\in\mathbb{Z}_{2k}} x_{i,0}\Big(\sum_0^{w'} (\delta_{m,w+1} - \delta_{m,0}) V_{i+1}(p) - \sum_1^{w'+1} (\delta_{m,w+1} - \delta_{m,0}) V_i(p)\Big) \\
&= V_{\cos}(p',r) - \sum_{i\in\mathbb{Z}_{2k}} x_{i,0} V_{i+1}(p) \\
&= V_{\cos}(p',r) + L_p^b(V_{\cos}(p,r)),
\end{aligned}$$

and similarly

$$L_p^b(V_{\sin}(p',r)) = V_{\sin}(p',r) + L_p^b(V_{\sin}(p,r)).$$

Thus we see that L_p^b is the inverse of L_p^t.

In summary, the group $RV(k,\ell)|_{H_r}$ is generated by the operators $L_p^t|_{H_r}$.

3.5. **Formulas for L_p^t as a complex operator.** Let $\rho := \exp(i\pi\frac{r}{k})$. The complex structure on H_r is given by $1_{2k}.v = \rho v$. We have thus

$$\begin{aligned} L_p^t(V_{\cos}(p,r)) &= \sum_{i\in\mathbb{Z}_{2k}} x_{i,0} L_p^t(V_i(p)) \\ &= -\sum_{i\in\mathbb{Z}_{2k}} x_{i,0} V_{i-1}(p) \\ &= -\rho^{-1} V_{\cos}(p,r). \end{aligned}$$

For $p' > p$,

$$L_p^t(V_{\cos}(p',r)) = V_{\cos}(p',r) - \rho^{-1} V_{\cos}(p,r),$$

and for $p' < p$

$$L_p^t(V_{\cos}(p',r)) = V_{\cos}(p',r) - V_{\cos}(p,r).$$

3.6. **Computation of the hermitian form on H_r.** We fix $0 < r < k$ and abbreviate $Z(p) := V_{\cos}(p,r)$, $x_i := x_{i,0}$. We first compute the hermitian product

$$\langle Z(p), Z(p)\rangle := \left(\sin\left(\pi\frac{r}{k}\right)\right)^{-1} \omega(1_{2k}.Z(p), Z(p)).$$

LEMMA 3.3. *Let $(a_i)_{i\in\mathbb{Z}_{2k}}$ and $(b_i)_{i\in\mathbb{Z}_{2k}}$ be real numbers with $\sum_i a_i = \sum_i b_i = 0$. Then, we have*

$$\omega\Big(\sum_i a_i V_i(p), \sum_i b_i V_i(p)\Big) = \sum_{1\leqslant i\leqslant i'<2k} (a_i b_{i'} - a_{i'} b_i).$$

For $p > p'$, we have

$$\omega\Big(\sum_i a_i V_i(p), \sum_i b_i V_i(p')\Big) = \sum_{1\leqslant i\leqslant i'<2k} a_i b_{i'}.$$

Proof. Observe first that, although $V_i(p)$ is only a relative homology class with nonzero boundary equal to $M(0,1) - M(w+1,w+2)$ (where $p = d-1-2w$), the condition $\sum_i a_i = 0$ insures that $\sum_i a_i V_i(p)$ is an absolute homology class so the intersection form ω is well-defined in the formulas of the lemma.

In the second case, it is easy to represent the cycles $V_i(p)$ (resp. $V_i(p')$) by paths from $M(w+1,w+2)$ (resp. $M(w'+1,w'+2)$) to $M(0,1)$ in Q_i so that the intersection takes place only at $M(0,1)$. A direct inspection at this point gives the formula of the lemma.

In the first case, we choose two distinct representations for each $V_i(p)$ as paths from $M(w+1,w+2)$ to $M(0,1)$ so that the intersection takes place only at $M(w+1,w+2)$ and $M(0,1)$. Again a direct inspection at these points gives the formula of the lemma. □

Using the first part of the lemma, we get (as $\sum_i x_i = 0$)

$$\begin{aligned}
\omega\big(1_{2k}.Z(p), Z(p)\big) &= \omega\big(\sum x_i 1_{2k}.V_i(p), \sum x_i V_i(p)\big)\\
&= \omega\big(\sum x_i V_{i+1}(p), \sum x_i V_i(p)\big)\\
&= \omega\big(\sum x_{i-1} V_i(p), \sum x_i V_i(p)\big)\\
&= \sum_{1\leqslant i\leqslant i'<2k} (x_{i-1}x_{i'} - x_i x_{i'-1})\\
&= \sum_1^{2k-1} x_0 x_{i'} + \sum_1^{2k-2} x_i x_{2k-1} - \sum_1^{2k-1} x_i x_{i-1} - \sum_1^{2k-2} x_i^2\\
&= -x_0^2 - x_{-1}(x_0 + x_{-1}) - \sum_1^{2k-1} x_i x_{i-1} - \sum_1^{2k-2} x_i^2\\
&= -\sum_{i\in\mathbb{Z}_{2k}} (x_i x_{i-1} + x_i^2)\\
&= -\frac{1}{4}\sum_{i\in\mathbb{Z}_{2k}} (\rho^i + \rho^{-i})(\rho^i + \rho^{-i} + \rho^{i-1} + \rho^{-i+1})\\
&= -\frac{k}{2}\big(2 + \rho + \rho^{-1}\big)\\
&= -k\left(1 + \cos\pi\frac{r}{k}\right).
\end{aligned}$$

We have thus

$$\langle Z(p), Z(p)\rangle := -k\frac{1+\cos\pi\frac{r}{k}}{\sin(\pi\frac{r}{k})}.$$

We now compute

$$2\Re(\langle Z(p), Z(p')\rangle) = \langle Z(p)+Z(p'), Z(p)+Z(p')\rangle - \langle Z(p), Z(p)\rangle - \langle Z(p'), Z(p')\rangle$$

for $p > p'$. We have (using the second part of Lemma 3.3)

$$\begin{aligned}
&2\sin\left(\pi\frac{r}{k}\right)\Re(\langle Z(p), Z(p')\rangle)\\
&\quad= \omega\big(1_{2k}.Z(p), Z(p')\big) + \omega\big(1_{2k}.Z(p'), Z(p)\big)\\
&\quad= \omega\Big(\sum x_{i-1}V_i(p), \sum x_i V_i(p')\Big) - \omega\Big(\sum x_i V_i(p), \sum x_{i-1} V_i(p')\Big)\\
&\quad= \sum_{1\leqslant i\leqslant i'<2k} (x_{i-1}x_{i'} - x_i x_{i'-1})\\
&\quad= -k\left(1 + \cos\pi\frac{r}{k}\right).
\end{aligned}$$

This gives

$$\Re(\langle Z(p), Z(p')\rangle) = -\frac{k}{2}\frac{1+\cos\pi\frac{r}{k}}{\sin(\pi\frac{r}{k})}.$$

We finally compute

$$2\Re(\rho\langle Z(p), Z(p')\rangle) = \langle \rho Z(p) + Z(p')), \rho Z(p) + Z(p')\rangle - \langle Z(p), Z(p)\rangle - \langle Z(p'), Z(p')\rangle$$

for $p > p'$. We have

$$\begin{aligned}
2\sin\left(\pi\frac{r}{k}\right)&\Re(\rho\langle Z(p),Z(p')\rangle)\\
&= \omega\left(1_{2k}^2.Z(p),Z(p')\right)+\omega\left(1_{2k}.Z(p'),1_{2k}Z(p)\right)\\
&= \omega\left(1_{2k}.Z(p),1_{2k}^{-1}.Z(p')\right)+\omega\left(Z(p'),Z(p)\right)\\
&= \omega\Big(\sum x_{i-1}V_i(p),\sum x_{i+1}V_i(p')\Big)-\omega\Big(\sum x_iV_i(p),\sum x_iV_i(p')\Big)\\
&= \sum_{1\leqslant i\leqslant i'<2k}(x_{i-1}x_{i'+1}-x_ix_{i'})\\
&= \sum_1^{2k-1}x_0x_{i'+1}+\sum_2^{2k-1}x_{i-1}x_{2k}-\sum_1^{2k-2}x_ix_{i+1}-\sum_1^{2k-1}x_i^2\\
&= -\sum_{i\in\mathbb{Z}_{2k}}(x_ix_{i+1}+x_i^2)\\
&= -k\left(1+\cos\pi\frac{r}{k}\right).
\end{aligned}$$

This gives

$$\Re(\rho\langle Z(p),Z(p')\rangle)=-\frac{k}{2}\frac{1+\cos\pi\frac{r}{k}}{\sin(\pi\frac{r}{k})}.$$

We finally get, for $p > p'$,

$$\begin{aligned}
\Im(\langle Z(p),Z(p')\rangle) &= (\Im(\rho))^{-1}(\Re(\rho)\Re(\langle Z(p),Z(p')\rangle)-\Re(\rho\langle Z(p),Z(p')\rangle))\\
&= -\frac{k}{2}\frac{1+\cos\pi\frac{r}{k}}{\sin(\pi\frac{r}{k})}\frac{-1+\cos\pi\frac{r}{k}}{\sin(\pi\frac{r}{k})}\\
&= \frac{k}{2}.
\end{aligned}$$

It is probably nicer to scale the hermitian form by the factor $-k\frac{1+\cos\pi\frac{r}{k}}{\sin(\pi\frac{r}{k})}$ in order to have $\langle Z(p),Z(p)\rangle = 1$ for all p. One has then, for $p > p'$

$$\langle Z(p),Z(p')\rangle=\frac{1}{2}\left(1-i\tan\frac{\pi r}{2k}\right).$$

Let $u := \tan\frac{\pi r}{2k}$. Consider the hermitian form on $\mathbb{C}^{\mathcal{A}_d}$ defined by

$$A((w(p))_{p\in\mathcal{A}_d}) := \left\langle\sum w(p)Z(p),\sum w(p)Z(p)\right\rangle = A_1-uA_2,$$

with

$$A_1=\frac{1}{2}\left(\sum_p|w(p)|^2+\Big|\sum_p w(p)\Big|^2\right)$$

and

$$A_2=\sum_{p>p'}\Im(w(p')\overline{w(p)}).$$

Observe that A_1 is positive, so we can diagonalize simultaneously A_1 and A_2. Let indeed $\xi := \exp\frac{2i\pi}{\ell}$ (recall that $\ell = d+1$). Define, for $0 < s < \ell$, $p = d-1-2w$

$$w_s(p)=\xi^{sw}.$$

We have then, for $z_1,\dots,z_d \in \mathbb{C}$

$$A_1(\sum_s z_s w_s) = \frac{1}{2}\left(\sum_p \left|\sum_s z_s \xi^{sw}\right|^2 + \left|\sum_p\sum_s z_s\xi^{sw}\right|^2\right)$$
$$= \frac{1}{2}\left(\sum_p\sum_s\sum_{s'} z_s\overline{z_{s'}}\xi^{(s-s')w} + \left|\sum z_s\xi^{-s}\right|^2\right)$$
$$= \frac{\ell}{2}\sum_s |z_s|^2,$$

and

$$A_2(\sum_s z_s w_s) = \sum_{p>p'} \Im\left(\sum_{s'}\sum_s z_{s'}\overline{z_s}w_{s'}(p')\overline{w_s(p)}\right)$$
$$= \sum_{s'}\sum_s \Im\left(z_{s'}\overline{z_s}\sum_{0\leqslant w<w'<\ell-1}\xi^{s'w'-sw}\right).$$

For $s\neq s'$, one has

$$\sum_{0\leqslant w<w'<\ell-1}\xi^{s'w'-sw} = \sum_{0<w'<\ell-1}\xi^{s'w'}\frac{\xi^{-sw'}-1}{\xi^{-s}-1}$$
$$= \frac{-1-\xi^{s-s'}+1+\xi^{-s'}}{\xi^{-s}-1}$$
$$= \xi^{s-s'},$$

and

$$\Im\left(z_{s'}\overline{z_s}\xi^{s-s'} + z_s\overline{z'_s}\xi^{s'-s}\right) = 0.$$

On the other hand, for $0<s<\ell$, we have

$$\sum_{0\leqslant w<w'<\ell-1}\xi^{sw'-sw} = \sum_{0<w'<\ell-1}\xi^{sw'}\frac{\xi^{-sw'}-1}{\xi^{-s}-1} = \frac{\ell+\xi^{-s}-1}{\xi^{-s}-1}$$

We have therefore

$$A_2\left(\sum_s z_s w_s\right) = \sum_s \Im\left(|z_s|^2\sum_{0\leqslant w<w'<\ell-1}\xi^{sw'-sw}\right)$$
$$= \ell\sum_s |z_s|^2\Im\frac{1}{\xi^{-s}-1}$$
$$= \frac{\ell}{2}\sum_s\left(\tan\frac{\pi s}{\ell}\right)^{-1}|z_s|^2.$$

In particular, this shows that the hermitian form on H_r has the signature described in Theorem 1.5 (as expected from McMullen's paper [7]).

3.7. **Matricial description of $RV(k,\ell)|_{H_r}$.** Our discussion so far is summarized as follows. The group $RV(k,\ell)|_{H_r}$ is generated by the operators L_p^t, $p\in\mathcal{A}_d$, given by

$$L_p^t(V_{\cos}(p',r)) = \begin{cases} V_{\cos}(p',r)-\rho^{-1}V_{\cos}(p,r) & \text{if } p'>p \\ V_{\cos}(p',r)-V_{\cos}(p,r) & \text{if } p'<p \\ -\rho^{-1}V_{\cos}(p,r) & \text{if } p'=p \end{cases}$$

These operators preserve the hermitian form

$$\langle V_{\cos}(p,r), V_{\cos}(p,r)\rangle = 1, \qquad \langle V_{\cos}(p,r), V_{\cos}(p',r)\rangle = \frac{1}{2}\left(1 - i\tan\frac{\pi r}{2k}\right) \quad \forall\, p > p'$$

At this point, we reduced the proof of Theorem 1.5 to the analysis of the group generated by the matrices above.

4. MATRICES

In this section, we complete the proof of Theorem 1.5 by studying the groups of matrices. Since we do not need anymore to make reference to the homology groups translation surfaces $\mathcal{M}_{k,\ell}$, we are going to rewrite below the formulas from Subsection 3.7 using a slightly more abstract notation.

4.1. **Setting.**

- d is an integer ≥ 2, $\ell = d+1$.
- ρ is a complex number of modulus 1 with positive imaginary part, frequently a root of unity; $\zeta := \rho^{-1}$. We write $\rho = \exp 2\pi i\alpha$, $\alpha \in (0, \frac{1}{2})$.
- p is an integer running from $1-d$ to $d-1$ with step 2, and thus taking d values. We denote by $\mathcal{A}_d$ the set of values of p.
- (e_p) is the canonical basis of $\mathbb{C}^{\mathcal{A}_d}$.

4.2. **The operators.**

For $p \in \mathcal{A}_d$, we define an operator $L_p = L_p^t$ on $\mathbb{C}^{\mathcal{A}_d}$ by

$$L_p(e_q) = \begin{cases} e_q - e_p & \text{if } q > p \\ e_q - \zeta e_p & \text{if } q < p \\ -\zeta e_p & \text{if } q = p. \end{cases}$$

Observe that the inverse $L_p^{-1} = L_p^b$ is given by

$$L_p^b(e_q) = \begin{cases} e_q - e_p & \text{if } q < p \\ e_q - \rho e_p & \text{if } q > p \\ -\rho e_p & \text{if } q = p, \end{cases}$$

i.e., the same formula, changing ζ to ρ and inverting the order on $\mathcal{A}_d$.

4.3. **The invariant hermitian form.**

Let Q_α be the hermitian form on $\mathbb{C}^{\mathcal{A}_d}$ such that

$$Q_\alpha(e_p) = 1, \quad \forall\, p \in \mathcal{A}_d, \quad Q_\alpha(e_p, e_{p'}) = (1+\zeta)^{-1} = \frac{1}{2}(1 + i\tan\pi\alpha), \forall\, p > p'.$$

4.4. The case $d = 2$.

4.4.1. *A special element.* We compute $L^b_{-1} \circ L^t_1$ as

$$\begin{cases} L^b_{-1} \circ L^t_1(e_{-1}) & = L^b_{-1}(e_{-1} - \zeta e_1) & = -\rho e_{-1} - \zeta(e_1 - \rho e_{-1}) & = (1-\rho)e_{-1} - \zeta e_1, \\ L^b_{-1} \circ L^t_1(e_1) & = L^b_{-1}(-\zeta e_1) & = -\zeta(e_1 - \rho e_{-1}) & = e_{-1} - \zeta e_1. \end{cases}$$

We thus have $\det(L^b_{-1} \circ L^t_1) = 1$ and $\mathrm{tr}(L^b_{-1} \circ L^t_1) = 1 - (\rho + \rho^{-1})$.

LEMMA 4.1. *The operator $L^b_{-1} \circ L^t_1$ has infinite order if ρ is a root of unity but $\rho^4 \neq 1$, $\rho^6 \neq 1$. It is hyperbolic if $\rho + \rho^{-1} < -1$.*

Proof. The second assertion is clear. For the first, if ρ is a root of unity but $\rho^4 \neq 1$, $\rho^6 \neq 1$, there exists a Galois-conjugate ρ' of ρ such that $\rho' + \rho'^{-1} < -1$. Then the corresponding Galois-conjugate of the matrix of $L^b_{-1} \circ L^t_1$ has infinite order. The same is true of the matrix of $L^b_{-1} \circ L^t_1$. □

4.4.2. *The cases $\alpha = \frac{1}{6}, \frac{1}{4}$.* In these cases, the hermitian form has signature $(2,0)$. The coefficients of the matrices of the group generated by L_1 and L_{-1} belong to the ring of integers of the quadratic field $\mathbb{Q}(\rho)$ and are bounded, hence the group generated by L_1 and L_{-1} is finite.

4.4.3. *The case $\alpha = \frac{1}{3}$.* Let $\rho = j := \exp \frac{2\pi i}{3}$, $e := e_{-1} + j e_1$, $f := e_{-1} + j^2 e_1$. We have $L_{-1}(e) = L_1(e) = e$ and $L_{-1}(f) = -j^2 e - j^2 f$, $L_1(f) = -je - j^2 f$. The subgroup generated by L_{-1} and L_1 is therefore contained in

$$\Gamma := \left\{ L \in GL(2,\mathbb{C}) \;\middle|\; L(e) = e, L(f) = \mu f + \omega e, \mu^6 = 1, \omega \in \mathbb{Z} \oplus \mathbb{Z} j \right\}.$$

Observe that $L^b_{-1} \circ L^t_1$ is parabolic, i.e satisfies $\mu = 1$, $\omega \neq 0$.

4.4.4. *The case $0 < \alpha < \frac{1}{3}$, $\alpha \neq \frac{1}{4}, \frac{1}{6}$.* In this case, the hermitian form has signature $(2,0)$. Denote by $U(Q_\alpha)$ and $SU(Q_\alpha)$ the associated unitary and special unitary groups.

The operator $L^b_{-1} \circ L^t_1$ belongs to $SU(Q_\alpha)$ and has infinite order hence the closed (for the usual topology) subgroup generated by $L^b_{-1} \circ L^t_1$ is a one-parameter group isomorphic to a circle, consisting of those transformations of $SU(Q_\alpha)$ having the same eigenvectors than $L^b_{-1} \circ L^t_1$. An infinitesimal generator of this one-parameter group is the element X of the Lie algebra $su(Q_\alpha)$ which satisfies $X.v_\pm = \pm i v_\pm$, where $v_\pm$ are the eigenvectors of $L^b_{-1} \circ L^t_1$.

The eigenvalues $\lambda_\pm$ of $L^b_{-1} \circ L^t_1$ are solutions of $\lambda^2 - (1 - \zeta - \rho)\lambda + 1 = 0$, and the corresponding eigenvectors are

$$v_\pm = a_\pm e_{-1} + e_1, \qquad a_\pm = -1 - \rho\lambda_\pm.$$

The coefficients of X are given by

$$X_{-1\,-1} = \frac{i(a_+ + a_-)}{a_+ - a_-} = -X_{11}, \quad X_{1\,-1} = \frac{2i}{a_+ - a_-}, \quad X_{-1\,1} = \frac{2i a_+ a_-}{a_+ - a_-}.$$

For $n \in \mathbb{Z}$, write $\mathrm{ad}(L_{-1}^n)X =: X(n)$, which is the infinitesimal generator of the previous one-parameter group conjugated by L_{-1}^n. Setting also $L_{-1}^n v_\pm =: v_\pm(n) =: a_\pm(n)e_{-1} + e_1$, we have

$$X_{-1\,-1}(n) = \frac{i(a_+(n) + a_-(n))}{a_+(n) - a_-(n)} = -X_{1\,1}(n), \quad X_{1\,-1}(n) = \frac{2i}{a_+(n) - a_-(n)},$$

$$X_{-1\,1}(n) = \frac{2i\,a_+(n)a_-(n)}{a_+(n) - a_-(n)}$$

with the recurrence relation $a(n+1) = -\zeta a(n) - 1$.

We now show that the vector fields $X = X(0), X(1), X(2)$ are linearly independent (over $\mathbb{C}$). The sequences $s(n) := a_+(n) + a_-(n)$ and $p(n) := a_+(n)a_-(n)$ satisfy the recurrence relations $s(n+1) = -\zeta s(n) - 2$, $p(n+1) = \zeta^2 p(n) + \zeta s(n) + 1$ with initial conditions $s(0) = -1 - \rho + \rho^2$, $p(0) = \rho$. We have thus

$$s(1) = \zeta - 1 - \rho, \quad s(2) = -\zeta^2 + \zeta - 1, \quad p(1) = \rho, \quad p(2) = \zeta^2.$$

As we have

$$\det \begin{pmatrix} 1 & p(0) & s(0) \\ 1 & p(1) & s(1) \\ 1 & p(2) & s(2) \end{pmatrix} = (1 - \zeta^3)(1 - \rho^3) \neq 0,$$

the vector fields $X = X(0), X(1), X(2)$ are indeed linearly independent. The Lie algebra $su(Q_\alpha)$ has dimension 3, hence is generated by $X(0), X(1), X(2)$.

We conclude that the intersection of $SU(Q_\alpha)$ with the group generated by L_1 and L_{-1} is dense (for the usual topology) in $SU(Q_\alpha)$.

4.4.5. *The case* $\frac{1}{3} < \alpha < \frac{1}{2}$. In this case, the hermitian form has signature $(1,1)$. Denote by $U(Q_\alpha)$ and $SU(Q_\alpha)$ the associated unitary and special unitary groups.

The operator $L_{-1}^b \circ L_1^t$ belongs to $SU(Q_\alpha)$ and has eigenvalues $\lambda_+ > 1$ and $\lambda_- = \lambda_+^{-1}$. Let $v_\pm$ be the associated eigenvectors. The Zariski closure of the group generated by $L_{-1}^b \circ L_1^t$ is the one-parameter group having for infinitesimal generator a vectorfield X satisfying $X.v_\pm = \pm v_\pm$. The same calculation as in the previous case shows that X, $\mathrm{ad}(L_{-1})X$ and $\mathrm{ad}(L_{-1}^2)X$ span $su(Q_\alpha)$. We conclude that the intersection of $SU(Q_\alpha)$ with the group generated by L_1 and L_{-1} is dense (for the Zariski topology) in $SU(Q_\alpha)$.

4.5. **The induction step in the generic case.**

4.5.1. *Restrictions.* For $d \geq 3$, $p \in \mathcal{A}_d$, let ι_p be the embedding of $\mathcal{A}_{d-1}$ into $\mathcal{A}_d$ sending q to $q-1$ if $q < p$ and to $q+1$ if $q > p$ (observe that $p \notin \mathcal{A}_d$). The image of ι_p is $\mathcal{A}_d - \{p\}$. We also denote by ι_p the embedding of $\mathbb{C}^{\mathcal{A}_{d-1}}$ into $\mathbb{C}^{\mathcal{A}_d}$ such that $\iota_p(e_q) = e_{\iota_p(q)}$, and by H_p the hyperplane of $\mathbb{C}^{\mathcal{A}_d}$ which is the image of this embedding.

From the defining formulas, for all $p \in \mathcal{A}_d, q \in \mathcal{A}_{d-1}$, the hyperplane H_p is invariant under $L_{\iota_p(q)}$ and we have

$$\iota_p \circ L_q = L_{\iota_p(q)} \circ \iota_p.$$

In order to avoid confusion, we denote by Q'_α the hermitian form on $\mathbb{C}^{\mathcal{A}_{d-1}}$ denoted by Q_α previously, and keep the notation Q_α for the hermitian form on $\mathbb{C}^{\mathcal{A}_d}$.

LEMMA 4.2. *For any $p \in \mathcal{A}_d$, the restriction to H_p of the form Q_α on $\mathbb{C}^{\mathcal{A}_d}$ is equal to the image of Q'_α under ι_p.*

Proof. This is clear from Subsection 4.2. □

When $\rho^{d+1} \neq 1$, we denote by H'_p the 1-dimensional subspace which is the Q_α-orthogonal of H_p. As Q_α is non-degenerate and $\bigcap_{p\in\mathcal{A}_d} H_p = \{0\}$, $\mathbb{C}^{\mathcal{A}_d}$ is the direct sum of the H'_p.

When moreover $\rho^d \neq 1$, the restriction of Q_α to each H_p is non-degenerate by the lemma. Therefore $\mathbb{C}^{\mathcal{A}_d}$ is the direct sum of H_p and H'_p. From the formulas for L_q, $q \neq p$, we see that the line H'_p is point wise fixed under all L_q, $q \neq p$.

4.5.2. *A result on stabilizers.*

PROPOSITION 4.3. *Let $d \geq 3$, Q a non-degenerate hermitian form on $\mathbb{C}^d$, $u_1, u_2 \in \mathbb{C}^d$ linearly independent vectors such that $Q(u_1) = Q(u_2) \neq 0$. Let $SU(Q)$ be the special unitary group of Q, and , for $j = 1,2$, let G_j be the stabilizer of u_j in $SU(Q)$. Then the smallest closed subgroup containing $G_1 \cup G_2$ is $SU(Q)$.*

Proof. Let G be the smallest closed subgroup of $SU(Q)$ containing $G_1 \cup G_2$. Let c be the common value of the $Q(u_j)$. It is sufficient to prove that G acts transitively on $\{Q(u) = c\}$. Indeed, assume this is true, and let h be an element of $SU(Q)$; there exists $g \in G$ such that $g(u_1) = h(u_1)$; then $g^{-1}h \in G_1$ and $h \in G$. To prove that G acts transitively on $\{Q(u) = c\}$, we observe that, as $\{Q(u) = c\}$ is connected[5], it is sufficient to show that the orbits of G have non-empty interior (and so are open) in $\{Q(u) = c\}$.

Let u_0 be a vector such that $Q(u_0) = c$.

LEMMA 4.4. *If either $Q(Gu_0, u_1)$ or $Q(Gu_0, u_2)$ have non-empty interior in $\mathbb{C}$, then Gu_0 has non-empty interior in $\{Q(u) = c\}$.*

Proof. Recall the following fact: for $p + q \geq 2$ and $b \neq 0$, $SU(p, q)$ acts transitively on $\left\{\sum_1^p |z_i|^2 - \sum_{p+1}^{q+1} |z_i|^2 = b\right\} \subset \mathbb{C}^{p+q}$.

Assume for instance that $Q(Gu_0, u_1)$ has non-empty interior in $\mathbb{C}$. Let W be a non-empty open set which is contained in $Q(Gu_0, u_1)$ and is disjoint from the circle $\{|z| = Q(u_1)\}$. For any $w \in W$, the intersection $\{Q(u) = Q(u_1)\} \cap \{Q(u, u_1) = w\}$ consists of vectors of the form $u = \alpha u_1 + v$, with $\alpha = \frac{w}{Q(u_1)}$, $Q(v, u_1) = 0$, and $Q(v) = (1 - |\alpha|^2)Q(u_1)$. As $|\alpha| \neq 1$, it follows from the fact recalled above that the intersection of $\{Q(u) = Q(u_1)\}$ with $\{Q(u, u_1) = w\}$ is contained in Gu_0. □

LEMMA 4.5. *Let $i \in \{1,2\}$. If $Q(Gu_0, u_i)$ is not contained in the circle $\{|z| = c\}$, then $Q(Gu_0, u_{3-i})$ has non-empty interior in $\mathbb{C}$.*

[5]Since we are assuming that $c = Q(u_1) = Q(u_2) \neq 0$, the desired connectedness follows from Witt's theorem (see also the claim in the proof of Lemma 4.4 below).

Proof. We may assume for instance that $u_0 = \alpha u_1 + v_0$ with $|\alpha| \neq 1$, $Q(v_0, u_1) = 0$, $Q(v_0) = (1-|\alpha|^2)Q(u_1) \neq 0$. Write $u_2 = \beta u_1 + v_2$ with $Q(v_2, u_1) = 0$, $v_2 \neq 0$. For $g \in G_1$, we have

$$Q(g.u_0, u_2) = \alpha\bar{\beta}Q(u_1) + Q(g.v_0, v_2).$$

From the fact recalled above, the set $G_1 v_0$ consists of the vectors v orthogonal to u_1 satisfying $Q(v) = (1-|\alpha|^2)Q(u_1)$. Any linear projection of this set on $\mathbb{C}$ has nonempty interior, which proves the assertion of the lemma. □

We can now end the proof of the proposition. In view of the two lemmas above, we know that Gu_0 has non-empty interior in $\{Q(u) = c\}$ except perhaps if both $Q(Gu_0, u_1)$ and $Q(Gu_0, u_2)$ are contained in the circle $\{|z| = c\}$. We will now prove that this exceptional case is impossible. Write as before $u_0 = \alpha u_1 + v_0$, $u_2 = \beta u_1 + v_2$, with $Q(v_0, u_1) = Q(v_2, u_1) = 0$, $v_2 \neq 0$. Exchanging u_1, u_2 if necessary, we may assume that $v_0 \neq 0$. We may also assume that $|\alpha| = 1$, $Q(v_0) = 0$. Choose a 2-dimensional subspace E of the hyperplane H orthogonal to u_1 with the following properties:

- The subspace E contains v_0.
- The restriction of Q to E is non-degenerate.
- The orthogonal projection of v_2 on E is $\neq 0$.

Such a choice is possible because the last two conditions are open and dense amongst 2-dimensional subspaces of H containing v_0. Choose a basis e, f of E such that $v_0 = e+f$ and $Q(xe+yf) = |x|^2-|y|^2$. For any $a, b \in \mathbb{C}$ with $|a|^2-|b|^2 = 1$, we can find $g \in G_1$ such that

$$g.e = ae + bf, \quad g.f = \bar{b}e + \bar{a}f.$$

Therefore any vector of the form $ze + \bar{z}f$, with $z \in \mathbb{C}^*$ belongs to $G_1 v_0$.

Let $se + tf \neq 0$ be the orthogonal projection of v_2 on E. Then $z\bar{s} - \bar{z}\bar{t}$ belongs to $Q(G_1 v_0, v_2)$ for any $z \in \mathbb{C}^*$. As the set $\left\{z\bar{s} - \bar{z}\bar{t} \,\middle|\, z \in \mathbb{C}^*\right\}$ contains a straight segment in $\mathbb{C}$, the set $Q(G_1 u_0, u_2) = \alpha\bar{\beta}c + Q(G_1 v_0, v_2)$ is not contained in the circle $\{|z| = c\}$.

This concludes the proof of the proposition. □

4.5.3. *Application.*

PROPOSITION 4.6. *Assume that $d \geq 3$, $\rho^d, \rho^{d+1} \neq 1$. Assume also that the intersection of the group generated by the operators L_q, $q \in \mathcal{A}_{d-1}$, on $\mathbb{C}^{\mathcal{A}_{d-1}}$ with the special unitary group $SU(Q'_\alpha)$ is dense (resp. Zariski dense) in $SU(Q'_\alpha)$. Then the intersection of the group generated by the operators L_p, $p \in \mathcal{A}_d$, on $\mathbb{C}^{\mathcal{A}_d}$ with the special unitary group $SU(Q_\alpha)$ is dense (resp. Zariski dense) in $SU(Q_\alpha)$.*

Proof. Let p_1, p_2 be two distinct elements of $\mathcal{A}_d$. Denote by u_1, u_2 generators of H'_{p_1}, H'_{p_2} respectively, satisfying $Q_\alpha(u_1) = Q_\alpha(u_2) \neq 0$. Such a choice is possible because the restrictions of Q_α to H_{p_1}, H_{p_2} have the same signature by lemma 4.2. By the assumption of the proposition, for $i \in \{1,2\}$, the intersection of the group generated by the operators L_p, $p \in \mathcal{A}_d$, $p \neq p_i$, with the special unitary group $SU(Q_\alpha)$ is dense (resp. Zariski dense) in the stabilizer G_i of u_i in $SU(Q_\alpha)$. By

Proposition 4.3, the smallest closed group containing $G_1 \cup G_2$ is $SU(Q_\alpha)$. We thus obtain the conclusion of the proposition . □

4.6. **The induction step in the non-exceptional degenerate case.**

4.6.1. *The setting.* We assume in this subsection that $d \geqslant 4$ and that $(d+1)\alpha$ is an integer. Therefore the hermitian form Q_α on $\mathbb{C}^{\mathcal{A}_d}$ is degenerate. As $0 < \alpha < \frac{1}{2}$, $d\alpha$ is not an integer and the hermitian form Q'_α on $\mathbb{C}^{\mathcal{A}_{d-1}}$ is non-degenerate. As the restriction of Q_α to each hyperplane H_p is isomorphic to Q'_α (Lemma 4.2), the kernel of Q_α has dimension 1.

REMARK 4.7. In the case $d = 3$, we must have $\alpha = \frac{1}{4}$; then the induction hypothesis (see below) is not satisfied.

As Q_α is invariant under each L_p the kernel of Q_α is invariant under the L_p. But the eigenvalues of L_p are 1 with multiplicity $(d-1)$ and $-\zeta$ with multiplicity 1, and the eigenvector associated to the eigenvalue $-\zeta$ is e_p, which is not an eigenvector of L_q for $q \neq p$. We conclude that the kernel of Q_α is pointwise fixed by each L_p.

We make the following **induction hypothesis**: on $\mathbb{C}^{\mathcal{A}_{d-1}}$, the intersection of the subgroup generated by the L_p, $p \in \mathcal{A}_{d-1}$ with the special unitary group $SU(Q'_\alpha)$ is dense for the ordinary topology (resp. Zariski dense) in $SU(Q'_\alpha)$.

4.6.2. *The induction step.* As the restriction of Q_α to each H_p is non-degenerate, the kernel $\mathbb{C}e$ of Q_α is not contained in any H_p.

Let us denote by $SU^*(Q_\alpha)$ the subgroup of $GL(\mathbb{C}^{\mathcal{A}_d})$ formed by linear automorphisms which preserve Q_α, fix e (and not simply the line $\mathbb{C}e$) and have determinant 1. If one writes these automorphisms in the basis $(e, e_{3-d}, \dots, e_{d-1})$ (using that $\mathbb{C}^{\mathcal{A}_d} = \mathbb{C}e \oplus H_{1-d}$), the matrix takes a block triangular form

$$M = \begin{pmatrix} 1 & v \\ 0 & g \end{pmatrix},$$

with $g \in SU(Q'_\alpha)$. Let D be the subgroup of $SU^*(Q_\alpha)$ formed of automorphisms whose matrix in the selected basis satisfies $v = 0$.

PROPOSITION 4.8. *A subgroup of $SU^*(Q_\alpha)$ which contains D is equal to D or to $SU^*(Q_\alpha)$.*

Proof. Let D' be such a subgroup . We identify H_{1-d} to $\mathbb{C}^{\mathcal{A}_{d-1}}$ through ι_{1-d}. Let

$$V(D') := \left\{ v \in (\mathbb{C}^{\mathcal{A}_{d-1}})^* \;\middle|\; \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \in D' \right\}.$$

Clearly $V(D')$ is an additive subgroup of $(\mathbb{C}^{\mathcal{A}_{d-1}})^*$. Moreover, as

$$\begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix}\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} 1 & v.g \\ 0 & 1 \end{pmatrix},$$

$V(D')$ is invariant under the natural action of $SU(Q'_\alpha)$ on $(\mathbb{C}^{\mathcal{A}_{d-1}})^*$. In view of the lemma below, $V(D')$ must be equal to $\{0\}$ or $(\mathbb{C}^{\mathcal{A}_{d-1}})^*$, which corresponds to $D' = D$ and $D' = SU^*(Q_\alpha)*$. □

LEMMA 4.9. *Let Q be a non-degenerate hermitian form on $\mathbb{C}^N$, $N \geqslant 3$. The only additive subgroups of $\mathbb{C}^N$ which are invariant under $SU(Q)$ are $\{0\}$ and $\mathbb{C}^N$.*

Proof. We first observe that , as $N \geqslant 3$, the orbit $SU(Q).v_0$ of a vector v_0 is equal to

- $\{0\}$ if $v_0 = 0$;
- $\{v \neq 0 \mid Q(v) = 0\}$ if $v_0 \neq 0$, $Q(v_0) = 0$;
- $\{Q(v) = c\}$ if $Q(v_0) = c \neq 0$.

Indeed, Witt's theorem implies that the orbit $U(Q).v_0$ is as stated. As $N \geqslant 3$, there exists a vector v_1 in this orbit with at least one coordinate vanishing (in an orthogonal basis for Q) . But then the image of the stabilizer of v_1 (in $U(Q)$) by the determinant map is the full unit circle. This means that the orbits $U(Q).v_1$ and $SU(Q).v_1$ are equal.

Let V be an additive subgroup of $\mathbb{C}^N$ which is also $SU(Q)$-invariant. It is sufficient to show that , if V contains a non-zero vector, then V has non-empty interior. Let $v_0 \in V$, $v_0 \neq 0$. Then V contains $V_0 := \{v \neq 0 \mid Q(v) = Q(v_0)\}$. The set $V_0 - v_0$ is also contained in V. The map

$$v \to Q(v - v_0) = 2Q(v_0) - 2\Re Q(v, v_0)$$

is not constant in a neighborhood of v_0 in V_0, hence its image contains a non-trivial interval. This implies that V has non-empty interior. □

COROLLARY 4.10. *Under the induction hypothesis stated above, the intersection of the subgroup generated by the L_p with $SU(Q_\alpha)$ is dense (resp. Zariski dense) in $SU^*(Q_\alpha)$.*

Proof. Let G be the closure (resp. the Zariski closure) of the intersection of the subgroup generated by the L_p, $1 - d \leqslant p \leqslant d - 1$ with $SL(n, \mathbb{C})$. We have seen earlier that G is contained in $SU^*(Q_\alpha)$.

Let G' be the closure (resp. the Zariski closure) of the intersection of the subgroup generated by the L_p, $3 - d \leqslant p \leqslant d - 1$ with $SL(n, \mathbb{C})$. We have seen earlier that $G' \subset D$. It follows from the induction hypothesis that $G' = D$. As $L_{1-d} \circ L_{d-1}^{-1}$ has determinant 1 but does not preserve H_{1-d}, G is not equal to D. It follows from the proposition that $G = SU^*(Q_\alpha)$. □

4.7. **From the degenerate case to the non-degenerate case.**

4.7.1. *The setting.* We assume in this subsection that $d \geqslant 5$ and that $d\alpha$ is an integer. Therefore the hermitian form Q_α is non-degenerate , but the restrictions of Q_α to each hyperplane H_p, which are isomorphic to Q'_α, are degenerate. It means that the Q_α-orthogonal of H_p is a line $\mathbb{C}w_p$ contained in H_p.

We make the following **induction hypothesis**: The intersection of the subgroup generated by the L_p, $p \in \mathcal{A}_{d-1}$ with $SL(\mathbb{C}^{\mathcal{A}_{d-1}})$ is dense for the ordinary topology (resp. Zariski dense) in the subgroup $SU^*(Q'_\alpha)$ defined in the previous subsection.

4.7.2. *Stabilizers.* Let Q be a non-degenerate non-definite hermitian form on $\mathbb{C}^N$. Let w be a nonzero vector such that $Q(w) = 0$. Let H be the hyperplane Q-orthogonal to $\mathbb{C}w$. It contains $\mathbb{C}w$. Denote by Q' the restriction of Q to H, which is degenerate. Choose a vector w' such that $Q(w, w') = 1$. Denote by H' the orthogonal of the plane $\mathbb{C}w \oplus \mathbb{C}w'$, and by Q'' the restriction of Q to H', which is non-degenerate. We have direct sums

$$\mathbb{C}^N = H \oplus \mathbb{C}w', \qquad H = \mathbb{C}w \oplus H'.$$

Write $\mathrm{Stab}(w)$ for the stabilizer of w in $SU(Q)$. The matrix of an element of $\mathrm{Stab}(w)$ in the decomposition $\mathbb{C}w \oplus H' \oplus \mathbb{C}w'$ of $\mathbb{C}^N$ takes a block-triangular form

$$M = \begin{pmatrix} 1 & v & t \\ 0 & g & h \\ 0 & 0 & \omega \end{pmatrix} \tag{4}$$

However, not all such matrices M are associated to elements of $SU(Q)$. Define a one-dimensional subgroup

$$K := \{\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \Re t = 0\}.$$

In the next proposition, the group $SU^*(Q')$ was defined in the last subsection.

PROPOSITION 4.11. *We have an exact sequence*

$$1 \longrightarrow K \longrightarrow \mathrm{Stab}(w) \longrightarrow SU^*(Q') \longrightarrow 1$$

The homomorphism θ from $\mathrm{Stab}(w)$ *to $SU^*(Q')$ is induced by restriction to H. The exact sequence is* ***not*** *split.*

Proof. If a matrix of the form (4) is unitary, we must have $\omega = 1$ because the scalar product $Q(w, w')$ is preserved.

As $\omega \equiv 1$, the homomorphism θ takes values in $SU^*(Q')$. It is onto by Witt's theorem.

An elementary computation shows that the kernel of θ is equal to K (see also below).

It remains to show that θ has no section. Assume by contradiction that such a section σ exists. Consider

$$u(v) := \sigma\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & v & t(v) \\ 0 & 1 & h(v) \\ 0 & 0 & 1 \end{pmatrix}.$$

As $u(v)$ is unitary, we have, for all $v \in (H')^*, x \in H'$

$$Q(u(v)x, u(v)w') = Q(v(x)w + x,\ w' + h(v) + t(v)w) = v(x) + Q(x, h(v)) = 0.$$

This determines h as a semi-linear isomorphism from $(H')^*$ to H'. On the other hand, as σ is a homomorphism, we have

$$t(v + v') = t(v) + t(v') + v(h(v')),$$

for all $v, v' \in (H')^*$. Therefore $v(h(v'))$ is a symmetric function of v, v'. As h is $\mathbb{C}$-antilinear, we should have $v(h(v')) \equiv 0$. This is not true because h is an isomorphism. □

We now obtain in our particular setting:

COROLLARY 4.12. *Let $p \in \mathcal{A}_d$. The intersection with $SU(Q_\alpha)$ of the subgroup generated by the L_q, $q \neq p$, is Zariski dense in the stabilizer* $\mathrm{Stab}(w_p)$.

REMARK 4.13. Probably one doesn't get the density in the usual topology, even if we started from this form of the induction hypothesis.

Proof. We already know that $L_q(w_p) = w_p$ for $q \neq p$. Therefore the Zariski closure G_p of the intersection with $SU(Q_\alpha)$ of the subgroup generated by the L_q, $q \neq p$ is a Zariski closed subgroup contained in $\mathrm{Stab}(w_p)$. On the other hand, the induction hypothesis (applied to the restriction of Q_α to H_p, which is isomorphic to Q'_α) implies that restriction to H_p induces an homomorphism of G_p onto $SU^*(Q'_\alpha)$.

The kernel of the homomorphism from G_p onto $SU^*(Q'_\alpha)$ is a Zariski closed subgroup of K hence it is either equal to K (in which case $G_p = \mathrm{Stab}(w_p)$) or to $\{1\}$. (If this subgroup is only closed for the usual topology, it could be an infinite discrete subgroup of K). But the second case is impossible since the exact sequence of the proposition is not split. □

4.7.3. *More stabilizers.* Let Q be a non-degenerate non-definite hermitian form on $\mathbb{C}^N$. Let w be a nonzero vector such that $Q(w) = 0$. Let $\mathrm{Stab}(w)$ be the stabilizer of w in $SU(Q)$. From Witt's theorem, one gets

LEMMA 4.14. *($N \geqslant 3$) For any $c \in \mathbb{C}$, $c \neq 0$, the group* $\mathrm{Stab}(w)$ *acts transitively on*

$$N(w,c) := \left\{ u \in \mathbb{C}^N \,\middle|\, Q(u) = 0, Q(u,w) = c \right\}.$$

Proof. Let $u \in N(w,c)$. It is sufficient to see that the determinant of an element $g \in SU(Q)$ which stabilizes w and u can have any determinant of modulus one. This is clear since the restriction of Q to the orthogonal of $\langle u, w\rangle$ is non-degenerate and the restriction of g to this subspace is any unitary matrix. □

LEMMA 4.15. *Let F be a nontrivial linear subspace of $\mathbb{C}^N$. Assume that the restriction of Q to F is non-degenerate.*

1. *If the restriction of Q to F is indefinite, any translate $u + F$ intersects $\{Q = 0\}$.*
2. *Assume that the restriction of Q to F is positive definite (resp. negative definite). Then a translate $u + F$, $u \in F^\perp$, intersects $\{Q = 0\}$ iff $Q(u) \leqslant 0$ (resp. $Q(u) \geqslant 0$).*

Proof.

1. The real-valued function $f \mapsto Q(u + f) = Q(u) + Q(f) + 2\Re Q(u, f)$ takes on F arbitrarily large positive and negative values, hence must vanish somewhere.
2. One has now $Q(u + f) = Q(u) + Q(f) \geqslant Q(u)$; the conclusion follows. □

LEMMA 4.16. *Assume $N = 2$. Let w, w' be a basis of $\mathbb{C}^2$, such that $Q(w) = 0$. Then $z \to Q(w + zw')$ takes positive and negative values in any neighborhood of* 0.

Proof. Indeed, one has $Q(w+zw') = |z|^2 Q(w') + 2\Re(zQ(w', w))$ with $Q(w', w) \neq 0$ as Q is non-degenerate and $Q(w) = 0$. □

PROPOSITION 4.17. *Let $(w_1, \dots, w_N)$ be a basis of $\mathbb{C}^N$ with $Q(w_i) = 0$ for $1 \leqslant i \leqslant N$, $\langle w_i, w_j \rangle \neq 0$ for $1 \leqslant i \neq j \leqslant N$. Let G_i be the stabilizer of w_i in $SU(Q)$. Then the smallest closed subgroup containing $G_1, \dots, G_N$ is $SU(Q)$.*

Proof. Let G be the smallest subgroup containing $G_1, \dots, G_N$. It is sufficient to show that G acts transitively on $\mathfrak{Q} := \{Q(u) = 0 \mid u \neq 0\}$. As this last set is connected, it is sufficient to show that any orbit of G in $\mathfrak{Q}$ has non-empty interior.

LEMMA 4.18. *Let $u_0 \in \mathfrak{Q}$. There exists an index $1 \leqslant i \leqslant N$ such that the image of $G.u_0$ by the map $u \mapsto Q(u, w_i)$ has non-empty interior in $\mathbb{C}$.*

From this lemma, we may assume $c_i := Q(u_0, w_i)$ is different from 0 and that a small neighborhood of c_i in $\mathbb{C}$ is contained in the image of $G.u_0$ by the map $u \mapsto Q(u, w_i)$. Let $u \in \mathfrak{Q}$ be close to u_0. There exists $u_1 \in G.u_0$ such that $Q(u_1, w_i) = Q(u, w_i) \neq 0$. By Lemma 4.14, one has $u \in G_i.u_1 \subset G.u_0$. □

Proof of Lemma 4.18. We first claim that there exist distinct indices i, j and $u_1 \in G.u_0$ such that $Q(u_1, w_i) \neq 0$, $Q(u_1, w_j) \neq 0$.

Let i be an index such that $c_i := Q(u_0, w_i) \neq 0$. Let $j \neq i$. As $Q(w_i, w_j) \neq 0$, there exists $\lambda \in \mathbb{C}$ such that $\Re(\bar{\lambda} c_i) = 0$ and $Q(u_0 + \lambda w_i, w_j) \neq 0$. Take $u_1 := u_0 + \lambda w_i$. One has $Q(u_1) = 0$, $Q(u_1, w_j) =: c_j \neq 0$, $Q(u_1, w_i) = c_i$ and $u_1 \in G_i.u_0$ by Lemma 4.14. This proves the claim.

Let F be the codimension 2 subspace of $\mathbb{C}^N$ orthogonal to w_i, w_j. Since $Q(w_i, w_j) \neq 0$, the restriction of Q to F is nondegenerate.

- If the restriction of Q to F is indefinite, Lemma 4.15 implies that for any c_i' close to c_i there exist $u_2 \in \mathfrak{Q}$ such that $Q(u_2, w_i) = c_i'$, $Q(u_2, w_j) = c_j$. By Lemma 4.14, one has $u_2 \in G_j.u_1$. This proves the assertion of the lemma in this case.
- Assume that the restriction of Q to F is positive definite (the negative case is symmetric). Then the restriction of Q to $F^\perp$ is indefinite. Identify $F^\perp$ to $\mathbb{C}^2$ through $u \mapsto (Q(u, w_i), Q(u, w_j))$. We have $Q(c_i, c_j) \leqslant 0$ by Lemma 4.15. If $Q(c_i, c_j) < 0$, we proceed as in the first case to get the conclusion of the lemma (using again Lemma 4.15). If $Q(c_i, c_j) = 0$, by Lemma 4.16 there exists c_j' close to c_j such that $Q(c_i, c_j') < 0$. Then there exists $u_1' \in \mathfrak{Q}$ such that $Q(u_1', w_i) = c_i$, $Q(u_1', w_j) = c_j'$. One has $u_1' \in G_i.u_1$ from Lemma 4.14. Then the end of the argument is the same than for $Q(c_i, c_j) < 0$. □

Putting together Proposition 4.17 and Corollary 4.12, we obtain

COROLLARY 4.19. *The intersection with $SU(Q_\alpha)$ of the subgroup generated by the L_p is Zariski dense in $SU(Q_\alpha)$.*

Proof. We apply Proposition 4.17, taking for w_p ($p \in \mathcal{A}_d$) a generator of the orthogonal H'_p of H_p. We have $Q_\alpha(w_p) = 0$. If we had $Q_\alpha(w_p, w_q) = 0$ for some distint $p, q \in \mathcal{A}_d$, the vector w_p would belong to $H_p \cap H_q$, and two of its coordinates in the canonical basis would vanish. But we know from the diagonalization formulas of subsection 4.3 that it is not so. From Corollary 4.12, the Zariski closure of the intersection with $SU(Q_\alpha)$ of the subgroup generated by the L_p contains the stabilizer of each w_p. Therefore it is equal to $SU(Q_\alpha)$. □

4.8. **Exceptional case I: $\boldsymbol{\alpha} = \mathbf{1/3}$.** At this stage, we can conclude that the Zariski closure of the intersection with $SU(Q_\alpha)$ of the subgroup generated by the L_p is equal to $SU(Q_\alpha)$ when $\alpha \neq \frac{1}{6}, \frac{1}{4}, \frac{1}{3}$ and $(d+1)\alpha$ is not an integer (so that Q_α is non-degenerate). We may even replace Zariski closure by closure for the usual topology when the form is definite, i.e $(d+1)\alpha < 1$. Indeed it is sufficient to proceed by induction on the dimension d, starting with the results of Subsection 4.4, and applying successively either Proposition 4.6, Corollary 4.10, or Corollary 4.19.

In the two exceptional cases $\alpha = \frac{1}{6}, \frac{1}{4}$, the group generated by L_1, L_{-1} in dimension 2 is a finite group so it is not a basis for a successful induction. These two cases will be dealt with in later subsections. However, the case $\alpha = \frac{1}{3}$ does not need any supplemental work.

Remember that $Q_{\frac{1}{3}}$ is degenerate for $d = 2$. The kernel of Q_α is generated by $e := e_{-1} + j e_1$ and fixed by L_{-1} and L_1. It was observed in Subsection 4.4 that the intersection of the group generated by L_{-1} and L_1 with $SL(2,\mathbb{C})$ contains a parabolic matrix. The determinant of L_{-1} and L_1 is $-\zeta$, a sixth root of unity. This is sufficient to show that the Zariski closure G of the intersection of the group generated by L_{-1} and L_1 with $SL(2,\mathbb{C})$ is the stabilizer of e in $SL(2,\mathbb{C})$. Indeed, G is contained in this stabilizer. As $\mathbb{R}$-Lie groups, the stabilizer of e in $SL(2,\mathbb{C})$ has only three types of Zariski closed subgroups: the two trivial subgroups[6], and (given any vector f independent of e) the subgroup G_f of the stabilizer formed of elements g such that $g.f - f$ is a **real** multiple of e (there is a one-parameter family, parametrized by the 1-dimensional real projective space, of such subgroups). Here, the existence of a parabolic element guarantees that G is not reduced to the identity. It cannot be of the intermediate form, because conjugating by powers of L_1 an element g such that $g.f - f = e$, we get elements g' such that $g'.f - f = \omega e$ for any sixth root of unity. Therefore G is equal to the full stabilizer.

Thus the argument from Subsection 4.7 can be used to conclude the desired result (even though the dimension d is not as high as assumed there).

4.9. **Exceptional case II: $\boldsymbol{\alpha} = \mathbf{1/4}$.** We assume in this subsection that $\alpha = \frac{1}{4}$. Then, we have seen in Subsection 4.4 that the group generated by L_{-1} and L_1 is finite. For $d = 3$, the form $Q_{\frac{1}{4}}$ is degenerate, but the Zariski closure G of the intersection with $SL(3,\mathbb{C})$ of the group generated by L_{-2}, L_0 and L_2 is strictly smaller

[6] I.e., {Id} and the full stabilizer itself.

than the group $SU^*(Q_{\frac{1}{4}})$ described in Subsection 4.6 (see below). It is only from dimension 4 that we get a "big" group generated by the L_p.

Consider first the case $d = 2$. It can be checked that the group Γ generated by L_{-1} and L_1 has order 96. The property of Γ that will be useful in the sequel is

LEMMA 4.20. *The representation of Γ on $\mathbb{C}^2 \simeq \mathbb{R}^4$ induced by the inclusion $\Gamma \subset U(Q_{\frac{1}{4}})$ is irreducible over $\mathbb{R}$.*

Proof. It is clear that L_1 and L_{-1} do not have a common eigenvector, therefore the representation is irreducible over $\mathbb{C}$. But Γ contains the scalar multiplications by the fourth roots of unity, hence any $\mathbb{R}$-subspace invariant under Γ has to be complex. □

We can now determine what is the Zariski closure Γ_3 of the group generated by the L_p for $d = 3$. As in Subsection 4.6, let us denote by e a generator of the kernel of $Q_{\frac{1}{4}}$, and write the operators in the basis (e, e_0, e_2) (using that all coordinates of e are non-zero). All L_p fix e so the matrices take the block-form

$$\begin{pmatrix} 1 & v \\ 0 & \gamma \end{pmatrix}.$$

Here, the 2×2 block γ will vary exactly in the finite group Γ mentioned above (hence G will not contain $SU^*(Q_{\frac{1}{4}})$). Therefore we have an exact sequence

$$1 \longrightarrow K \longrightarrow \Gamma_3 \longrightarrow \Gamma \longrightarrow 1,$$

where the kernel K is the Zariski closed subgroup of $(H_{-2})^*$ formed of those v such that $\left(\begin{smallmatrix} 1 & v \\ 0 & \mathbf{1} \end{smallmatrix}\right)$ belongs to Γ_3. Note that K is not only a Zariski closed subgroup of $(H_{-2})^*$, it is also invariant under the representation of Γ dual to that induced by the inclusion of Γ in $U(Q'_{\frac{1}{4}})$: indeed, conjugating an element of the kernel by an element of Γ_3 with diagonal block γ changes v into $v.\gamma^{-1}$. By Lemma 4.20 this representation is irreducible over $\mathbb{R}$. We conclude that K must be equal to either $\{0\}$ or $(H_{-2})^*$.

LEMMA 4.21. *The kernel K is equal to $(H_{-2})^*$.*

Proof. Otherwise, K would be trivial and Γ_3 would be finite. As any representation of a finite group is semi-simple, there would exist a Γ_3-invariant complex 2-plane supplemented by $\mathbb{C}e$. But then the transposed matrices ${}^tL_{-2}$, ${}^tL_0$, ${}^tL_2$ would have a common eigenvector. This is clearly not the case. □

Having described Γ_3, we now go to the case $d = 4$. To represent the operators L_p, we chose a basis w_{-3}, f_{-1}, f_1, w_3 with the following properties:

- w_{-3} is a generator of the orthogonal of the hyperplane H_{-3};
- w_3 is a generator of the orthogonal of the hyperplane H_3;
- $Q_{\frac{1}{4}}(w_{-3}, w_3) = 1$
- the subspace F generated by f_{-1}, f_1 is the orthogonal of the subspace W generated by w_{-3}, w_3; note that the restriction of $Q_{\frac{1}{4}}$ to W has signature

$(1,1)$, hence W and F are indeed transverse and the restriction of $Q_{\frac{1}{4}}$ to F has signature $(2,0)$.

- f_{-1}, f_1 form an orthonormal basis of F.

Observe that the first three vectors form a basis of H_{-3}, and the last three form a basis of H_3.

Consider the Zariski closure of the subgroup generated by L_{-1}, L_1, L_3. An element in this group preserves the hyperplane H_{-3} and its restriction to H_{-3} is constrained exactly by the case $d = 3$: we can write it in $1+2+1$ block form as

$$\begin{pmatrix} 1 & v & s \\ 0 & \gamma & v' \\ 0 & 0 & \omega \end{pmatrix}$$

with $\gamma \in \Gamma$. We restrict to the subgroup G_{-3} of finite index such that $\gamma = \mathbf{1}_\Gamma$. As in Subsection 4.7, writing that the form $Q_{\frac{1}{4}}$ is preserved gives (when $\gamma = \mathbf{1}_\Gamma$)

$$\omega = 1, \qquad v' = -{}^t\bar{v}, \qquad \Re s = -\frac{1}{2}||v||^2.$$

LEMMA 4.22. *Conversely, any matrix of the prescribed form satisfying these relations belongs to G_{-3}.*

Proof. Essentially the same as in Proposition 4.11 and Corollary 4.12. □

The 5-dimensional Lie algebra $\mathfrak{g}_{-3}$ of G_{-3} is therefore the set of matrices of the form

$$A = \begin{pmatrix} 0 & v_{-1} & v_1 & i\,s \\ 0 & 0 & 0 & -\bar{v}_{-1} \\ 0 & 0 & 0 & -\bar{v}_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with v_{-1}, $v_1 \in \mathbb{C}$ and $s \in \mathbb{R}$.

Similarly, from the action of L_{-3}, L_{-1}, L_1, one obtains a Zariski closed group G_3 whose Lie algebra $\mathfrak{g}_3$ is the set of matrices of the form

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\bar{u}_{-1} & 0 & 0 & 0 \\ -\bar{u}_1 & 0 & 0 & 0 \\ i\,r & u_{-1} & u_1 & 0 \end{pmatrix}$$

with u_{-1}, $u_1 \in \mathbb{C}$ and $r \in \mathbb{R}$.

LEMMA 4.23. *The smallest Lie algebra containing $\mathfrak{g}_{-3}$ and $\mathfrak{g}_3$ is the Lie algebra $\mathfrak{su}(Q_{14})$.*

Proof. With A, B as above, we have

$$AB - BA = \begin{pmatrix} -rs - v_{-1}\bar{u}_{-1} - v_1\bar{u}_1 & isu_{-1} & isu_1 & 0 \\ -ir\bar{v}_{-1} & v_{-1}\bar{u}_{-1} - u_{-1}\bar{v}_{-1} & v_1\bar{u}_{-1} - u_1\bar{v}_{-1} & is\bar{u}_{-1} \\ -ir\bar{v}_1 & v_{-1}\bar{u}_1 - u_{-1}\bar{v}_1 & v_1\bar{u}_1 - u_1\bar{v}_1 & is\bar{u}_1 \\ 0 & -irv_{-1} & -irv_1 & rs + u_{-1}\bar{v}_{-1} + u_1\bar{v}_1 \end{pmatrix}.$$

After adding appropriate elements A', B' of $\mathfrak{g}_{-3}$, $\mathfrak{g}_3$ respectively, the matrix $AB - BA + A' + B'$ is equal to

$$C = \begin{pmatrix} -rs - v_{-1}\bar{u}_{-1} - v_1\bar{u}_1 & 0 & 0 & 0 \\ 0 & v_{-1}\bar{u}_{-1} - u_{-1}\bar{v}_{-1} & v_1\bar{u}_{-1} - u_1\bar{v}_{-1} & 0 \\ 0 & v_{-1}\bar{u}_1 - u_{-1}\bar{v}_1 & v_1\bar{u}_1 - u_1\bar{v}_1 & 0 \\ 0 & 0 & 0 & rs + u_{-1}\bar{v}_{-1} + u_1\bar{v}_1 \end{pmatrix}.$$

For $u_{-1} = u_1 = v_{-1} = v_1 = 0$, $r = s = 1$, we get

$$C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $r = s = v_{-1} = u_1 = 0$, $v_1 = u_{-1} = 1$, we get

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $r = s = v_{-1} = u_1 = 0$, $v_1 = i$, $u_{-1} = 1$, we get

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $r = s = v_{-1} = u_{-1} = 0$, $v_1 = 1$, $u_1 = i$, we get

$$C = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}.$$

For $r = s = v_1 = u_1 = 0$, $v_{-1} = 1$, $u_{-1} = i$, we get

$$C = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -2i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}.$$

These five matrices, together with $\mathfrak{g}_{-3}$ and $\mathfrak{g}_3$, span a 15-dimensional vector space. As $\mathfrak{s}u(Q_{\frac{1}{4}})$, being isomorphic to $\mathfrak{s}u(3,1)$, has also dimension 15, the lemma is proved. □

COROLLARY 4.24. *The intersection of the subgroup generated by L_{-3}, L_{-1}, L_1 and L_3 with $SL(\mathbb{C}^{\mathcal{A}_4})$ is Zariski dense in $SU(Q_{\frac{1}{4}})$.*

This provides an appropriate starting point for the induction for $\alpha = 1/4$. The results in higher dimension follow.

4.10. **Exceptional case III: $\alpha = 1/6$.** The proof is identical to the previous case $\alpha = 1/4$ due to the following fact, an improvement (in generality) on Lemma 4.20.

PROPOSITION 4.25. *For any $\alpha \in (0, 1/2)$ and any $d \geqslant 2$ such that $(d+1)\alpha$ is not an integer, there is no non-trivial $\mathbb{R}$-subspace of $\mathbb{C}^{\mathcal{A}_d}$ which is invariant under every L_p, $p \in \mathcal{A}_d$.*

Proof. The kernel of $L_p - \mathrm{id}$ is the hyperplane

$$\mathcal{H}_p = \left\{ \zeta \sum_{q \leqslant p} x_q + \sum_{q \geqslant p} x_q = 0 \right\}.$$

The other eigenvalue of L_p is equal to $-\zeta$ and is simple. The associated eigenspace is $\mathbb{C}e_p$. We claim that the intersection $\bigcap_{p \in \mathcal{A}_d} \mathcal{H}_p$ is trivial (if $\zeta^{d+1} \neq 1$). Indeed, the intersection $\mathcal{H}_p \cap \mathcal{H}_{p+2}$ is contained in $\{x_p = \zeta x_{p+2}\}$ for $p < d-1$, and the intersection $\mathcal{H}_{d-1} \cap \mathcal{H}_{1-d}$ is contained in $\{\zeta x_{1-d} = \zeta^{-1} x_{d-1}\}$.

Let $p \in \mathcal{A}_d$, and let W be an $\mathbb{R}$-subspace of $\mathbb{C}^{\mathcal{A}_d}$ that is invariant under L_p. If W is not contained in $\mathcal{H}_p$, it contains $\mathbb{C}e_p$: indeed, if $w \in W$ has the form $h + te_p$ with $h \in \mathcal{H}_p$ and $t \neq 0$, then $w - L_p(w) = (1+\zeta)te_p$ belongs to W and $\mathbb{C}e_p \subset W$ as ζ is not real.

Assume now that W is an $\mathbb{R}$-subspace of $\mathbb{C}^{\mathcal{A}_d}$ that is invariant under all L_p, $p \in \mathcal{A}_d$. If W is contained in $\mathcal{H}_p$ for every $p \in \mathcal{A}_d$, it is equal to $\{0\}$. Otherwise, there exists $p \in \mathcal{A}_d$ such that $\mathbb{C}e_p \subset W$. For $q \neq p$, $e_p - L_q(e_p)$ is equal to e_q or ζe_q, therefore we have also $\mathbb{C}e_q \subset W$. Then $W = \mathbb{C}^{\mathcal{A}_d}$. □

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