Essays in equilibrium asset pricing

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Curriculum Vitae

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Contents

Introduction

This thesis is a collection of three research papers. The first paper is a collaboration with my supervisor about non-convexities in general equilibrium. Non-convexities and discrete choices have become important modeling tools in modern macro-economics. Unfortunately, existence of competitive equilibria in the presence of such non-convexities is not always ensured and most results on the existence of equilibrium that can be found in the literature consider a very general model and are not directly applicable to the macro-models used in the literature. In this paper we explain the three main problems one needs to face when proving existence and give simple sufficient conditions for the existence of competitive equilibria in stochastic OLG models with discrete choices and non-convex preferences. We also consider a version of the model without aggregate uncertainty but with bankruptcy and default and prove existence of a steady state equilibrium.

The second and third papers are about bounded rational recursive equilibrium. The second paper is a theoretical existence result and the third paper is an application to solve macro-finance puzzles.

The second paper provides a general way of modeling bounded rationality in the dynamic stochastic general equilibrium framework with infinitely lived heterogeneous agents and incomplete markets. Different from a rational agent, a bounded rational agent is associated with an extra parameter ϵ, which can be interpreted as the "level of irrationality". The bounded rational agent does not know the true probability distribution of the economy fundamentals. To make decisions, the bounded rational agent forms a belief of a stationary distribution of the fundamentals and then use the Markov transition associated with it to maximize utility. If a distribution of the fundamentals stays “closer” to its next-period transition than ϵ, the agent would consider it as ϵ-stationary. In equilibrium, each agent maximizes utility with an ϵ-stationary belief and markets clear. The main theorem of this paper shows that for any strictly positive ϵ, a recursive equilibrium exists. This result provides a potential way of measuring the “level of irrationality” for many behavioral models. Besides, there are two applications for a special case of the model, when ϵ is extremely close to zero: It lays foundation for numerically computed equilibria of models with the rational expectation assumption; and it can be viewed as an epsilon-equilibrium existence result for models with heterogeneous heuristics.

The third paper provides a simple dynamic general equilibrium model that can generate short-term momentum and long-term reversal effect of excess stock returns with incomplete markets due to collateral constraints. The model also helps to understand quantitatively some of the puzzling empirical regularities in macro-finance stated by Campbell (2003) and Gabiax (2012). We assume there are two types of bounded rational agents: the fundamentalists and the speculators. The fundamentalists believes that future asset prices are determined by the exogenous shocks and dividends. The speculator believes the excess stock return has a short-term momentum and long-term reversal regardless of exogenous shocks. These beliefs are not common knowledge. In equilibrium, both agents maximize utilities with these beliefs and markets clear. Both types of agents are bounded rational in the sense that they both partially capture the law of motion of the asset prices in equilibrium. We show that in calibrated simulations, both types of agents survive and there is a significant short-term momentum of excess stock returns. The calibrated data helps to explain eleven puzzling empirical regularities.

Chapter 1

Existence of equilibrium in stochastic overlapping generations economies with nonconvexities

Non-convexities and discrete choices have become important modeling tools in modern macro-economics. Unfortunately, existence of competitive equilibria in the presence of such non-convexities is not always ensured and most results on the existence of equilibrium that can be found in the literature consider a very general model and are not directly applicable to the macro-models used in the literature.

In this paper we explain the three main problems one needs to face when proving existence and give simple sufficient conditions for the existence of competitive equilibria in stochastic OLG models with discrete choices and non-convex preferences. We also consider a version of the model without aggregate uncertainty but with bankruptcy and default and prove existence of a steady state equilibrium.

Introduction

In this paper we derive simple sufficient conditions for the existence of equilibria in dynamic stochastic models with non-convexities in budget-sets and preferences. Households make many important discrete decisions over the life-cycle. Examples of such decisions include investment in human capital, labor-supply decisions, housing decisions, and bankruptcy- or default decisions. Consequently, there are now many applied dynamic general equilibrium models where agents make discrete choices (see e.g. Chatterjee et al. (2001), Chang and Kim (2007), Chambers et al. (2009) Kumhof et al. (2015), Sommer and Sullivan (2018), among many others). While Chatterjee et a. (2001) carefully prove existence of a competitive equilibria by assuming that all choices are discrete, most of the other literature takes existence as given. Unfortunately this is not always an innocuous assumption.

Three technical issues arise when one introduces discrete choices into a standard dynamic stochastic model with heterogeneous agents. First, individual best responses are not longer convex-valued. Following Aumann (1964) and Starr(1969) who establish the existence of Arrow Debreu equilibria in economies with non-convex preferences the assumption of a continuum of agents is crucial to ensure convexity of the best response correspondences. Second, individual budget-correspondences are no longer continuous and choices can fail to be upper-hemi-continuous. As Mas-Colell (1976) points out, in many models with discrete decisions (in his paper the indivisible commodity case) a continuum of agents alone does not guarantee existence. He presents a simple example that illustrates that non-convexity of consumption sets can lead to non-existence even with a continuum of agents and derives condition on the distribution of agents' characteristics that are sufficient for existence. Third in models with infinitely many commodities and a continuum of agents general existence proofs are often technically difficult and require very specific assumptions on preferences and the commodity space (see, e.g., Ostroy (1984)). We circumvent many of the technical problems by considering models with finitely lived agents and overlapping generations.

In this paper we prove existence of a competitive equilibrium in a general dynamic general equilibrium model with overlapping generations, non-convex preferences, and discrete decisions. Ex ante identical agents distinguish themselves ex post by their choices. Our key insight is to model discrete decisions by assuming that agents choose a sequence of intertemporal budget sets. We present a series of simple and natural assumptions that guarantee that best responses are upper-hemi-continuous and convexify these best responses through the standard approach (that is to say, using versions of Shapley Folkman and Lyapunov's theorem). We perform this analysis in a very abstract framework that incorporates production economies, incomplete financial markets and trading constraints. We show in the paper how the model can incorporate indivisible commodities, segmented financial markets and many other discrete choices.

We also consider a more concrete model with bankruptcy and default. This model is not a special case of our general approach since in this case budget sets depend on other agents' choices through default and repayment-rates. For a long time, modeling default and bankruptcy has been an important challenge for general equilibrium models. Geanakoplos (1997) introduces collateral and default into general equilibrium; Dubey et al. (2005) develop a two-period model with default decisions and incomplete financial market. In both of these models default is modeled as a continuous decision, the resulting economies are convex and can be analyzed with standard models. Chatterjee et al. (2001) consider Bewley-style overlapping generations model with a continuum of agents and discrete bankruptcy decisions. In their model the bond holding is chosen from a finite discrete set. We extend the literature by allowing for more complex bonds markets and durable goods. These durable goods might be indivisible and they can be used as collateral for short positions in the bonds. The punishment for default on certain bond will be the combination of losing certain amount of collateral and being excluded from certain bonds markets for certain periods. With different punishment terms, different borrowing limit, we have different bonds. As in Bewley (1984) a continuum of ex ante identical agents faces uninsurable individual risks which are assumed to cancel out in the aggregate. Agents then differ ex post by the realization of the idiosyncratic shocks and by the choices they made. Since agents are finitely lived and there are finitely many possible idiosyncratic shocks, at each date, there is a large, finite number of *different agents*.

Duffie et al. (1994) prove the existence of stationary equilibria in convex economies. We employ their basic proof strategy to show existence of a generalized Markov equilibrium in the presence of non-convexities. Halket (2014) considers a Bewley style OLG model without aggregate uncertainty but with non-convex budget sets that are somewhat similar to ours. He proves the existence of a steady state equilibrium and provides a nice motivation for examining models with non-convexities. In our setup, the presence of aggregate uncertainty implies that there is no steady state equilibrium (in fact the existence of Markov equilibria typically requires very strong conditions (see Brumm et al. (2017)). We also consider a special case of the model without aggregate uncertainty. For this case we prove the existence of a steady state equilibrium. The main difference between our proof and Halket's work is that he considers a model with several 'islands' (which is more general than our setup) but restricts attention to the case of indivisible commodities (that is less general than our setup). Araujo and Pascoa (2002) prove existence of a competitive equilibrium in a two-period model with bankruptcy. Their model is more general than ours but it is not obvious how any of these proofs extent to infinite dimensional models.

The paper is organized as follows. In Section 2 we discuss a series of motivating examples. In Section 3 we present an abstract model. In Section 4 we prove existence of a generalized Markov equilibrium. In Section 5 we consider the special case of no aggregate uncertainty and in Section 6 we prove existence of a steady state equilibrium for this model.

Some examples

We first illustrate the three main issues with some simple examples.

Segmented markets and the need for many agents

Consider a simple Arrow-Debreu exchange economy with two types of agents,$h = 1, 2$, and two commodities, $l = 1, 2$. Suppose endowments are $e^1 = (1, 8)$ and $e^2 = (7, 1)$ and utilities

$$U_1(x_1, x_2) = x_1 + x_2, \quad U_2(x_1, x_2) = \log x_1 + \log x_2.$$

Suppose that markets are segmented in that agent 1 can only trade in good 1 unless he pays a real cost of one unit of good 2 which allows him to enter the market for good 2. There is no competitive equilibrium. If agent 1 does not enter the market, prices are determined by agent 2 and $p_2 = 7$, $p_1 = 1$. At these prices agent 1 is willing to enter the markets and pay a real cost of one unit of good. However, if agent 1 enters the market, prices are $p_1 = p_2 = 1$ and it is not worthwhile to give up one unit of good 2 to be able to trade. Following Aumann (1964) and Starr(1969) the solution is simply to assume that there is a continuum of agents of type 1 and a fraction of agents enters a market while the rest stays in autarky. The (extended) competitive equilibrium is then as follows. Prices are $p_1 = 1, p_2 = 1 + \frac{1}{7}$, agents of type 1 are indifferent between paying 1 unit of good 2, entering the market and buying 8 units of good 1, and staying in autarky – in both cases utility is 9. The fraction of agents entering the markets, ν, is determined by market clearing. The demand of agent 2 for good 1 is $4 + \frac{1}{14}$, hence the market clearing condition becomes

$$4 + \frac{1}{14} + \nu 9 = 8 \Rightarrow \nu = \frac{3 + \frac{13}{14}}{9} \simeq 0.43651.$$

Indivisible commodities and the need for an Inada condition

Suppose now the first commodity are houses and can only be consumed in integer-amounts. Suppose agents of type 2 have endowments of one unit of this good but there is only a measure 1/2 of such agents, while there is a measure 1 of agents of type 1. Utility-functions and endowments are

$$U_1(x_1, x_2) = 3x_1 + x_2,\ U_2(x_1, x_2) = x_1 + x_2, \quad e^1 = (0, 1),\ e^2 = (1, 1)$$

There is no competitive equilibrium. If $p_2 < 1$ agent 1 cannot buy a house because his consumption in good 2 must be non-negative. Agent 2 wants to sell the house and markets cannot clear. If $p_2 \geq 1$ all agents of type 1 want to buy a house, demand jumps to 1 and there is no competitive equilibrium because total supply of housing is only 1/2.

Mas-Colell (1977) suggests to solve the problem by assuming that agents of type 1 are not identical and there is a continuous distribution of agents. This turns out to be technically challenging in our framework. Instead, we assume that agent 1's utility satisfies an Inada condition, i.e. $U_1(x_1, x_2) = 10x_1 + v(x_2)$ with $\lim_{x\to 0} v'(x) = \infty$. Normalize $p_1 = 1$. At $p_2 = 1$ the agent is not willing to sell all of good 2, because marginal utility at zero is infinite. There is some $p_2 > 1$ at which agent 1 is indifferent between buying or not buying the house and a fraction 1/2 of agents buys the house. All agents of type 2 sell the house and markets clear.

Intertemporal economies and heterogneity

Consider the previous example but assume that there are two time periods. In the first period only the divisible good and a risk-free bond are traded. Utility functions are

$$U_1(x(0), x_1(1), x_2(1)) = \log(x(0)) + 3x_1(1) + \log(x_2(1)), U_2(x_1(1), x_2(1)) = x_1(1) + x_2(1)$$

As before the mass of agents of type 1 is one and the mass of agents of type 2 is 1/2. Endowments are $e^1(0) = 1, e^1_1(1) = 0, e^1_2(1) = 1$ and $e^2(0) = 0, e^2_1(1) = 1, e^2_1(2) = 1$. Clearly in spot markets in the second period 1/2 of agents of type 1 must buy the house. In a rational expectations model, these agents already save for the house at t=0, while the agents that do not buy the house borrow at t=0. The indivisibility in the second period leads to the fact that ex ante identical agents already make different choices in the first period. The competitive equilibrium is as follows. The price of housing, p, the price of the first period bond, q, and bond demand of the agents that buy a house, θ must satisfy

$$\begin{aligned}
3+\log(1-p+\theta)+\log(1-q\theta) &= \log(1-\theta)+\log(1+q\theta) \\
q\frac{1}{1-q\theta} &= \frac{1}{1-p+\theta} \\
q\frac{1}{1+q\theta} &= \frac{1}{1-\theta}
\end{aligned}$$

Reducing this to one equation we obtain $\theta = \frac{1-\frac{1}{q}}{2}$ and

$$3+\log(\frac{1-q\theta}{q})+\log(1-q\theta)=\log(1-\theta)+\log(1+q\theta)$$

Solving the latter numerically, we obtain $q \simeq 2.66553$, $\theta \simeq 0.31242$ and $p = 1.2497$.

The example illustrates that discrete decisions in the future can lead ex ante identical agents to make very different decisions in the current period. In order to convexify an agents' best response correspondence we therefore have to consider all the decisions she makes over her life-cycle.

The model

We first derive sufficient conditions for the existence of a competitive equilibrium in an abstract framework. In Section 5 below we will consider a slight variation of this model without aggregate uncertainty that incorporates default and we prove the existence of a steady-state equilibrium for this case.

An abstract intertemporal economy

We consider a stochastic overlapping generations economy with a continuum of ex ante identical agents. Time is indexed by $t \in \mathbb{N}_0$. Exogenous shocks z_t realize in a finite set $\mathbf{Z}$. A history of shocks up to some date t is denoted by $z^t = (z_0, z_1, \ldots, z_t)$ and called a date event.

At each date event a continuum of ex ante identical agents of mass one enters the economy, each agent lives for A periods, and agents differ ex post by their choices. At each z^t agents trade in L commodities and assets. The intertemporal budget correspondence of an agent is indexed by $i \in \mathbf{I} = \{1, ..., I\}$, a finite set of indices. The intertemporal budget set of an agent depends on $i \in \mathbf{I}$, the agent's age, a, her choices in commodities and assets in the last period, $x^- \in^L$. Define the budget correspondence $\mathbf{B}_{a,i}(x^-, p, z)$,

$$\mathbf{B}_{a,i} :^L \times \boldsymbol{\Delta} \times \mathbf{Z} \rightrightarrows^L, \quad \boldsymbol{\Delta} = \{p \in_+^L : \sum_{i=1}^{L} p_i = 1\}$$

as well as the opportunity correspondence

$$\mathbf{O}_a : \mathbf{I} \times^L \times \mathbf{Z} \rightrightarrows \mathcal{I},$$

where $\mathcal{I}$ denotes the set of all subsets of $\mathbf{I}$. An agent of age a who is in budget i in the current period and who chooses x can decide to move to another budget i' in the subsequent period in shock z' if $i' \in \mathbf{O}_{a+1}(i, x, z')$.

At birth all agents at node z^t are ex ante identical and each one maximizes utility subject to the budget set, i.e. he chooses budget-sets, consumptions and asset-holdings for all histories of aggregate and idiosyncratic shocks over his life-cycle. It will be useful to define $K = \sum_{a=0}^{A-1} Z^a$ to denote the total number of aggregate date events relevant to the agent, and to use the notation $\vec{x} \in^{KL}, \vec{i} \in \mathbf{I}^K$ to denote consumption, asset holdings and budget-choices over the life cycle. If associated with a specific agent who was born at node z^t, elements of $\vec{x}_{z^t}$ (and $\vec{i}$ analogously) are denoted by $x_{z^t}(z^{t+a})$, otherwise we associate them with the nodes $z_0, \ldots, (z^{A-1})$.

An agent has a utility function over life-cycle consumption that can depend on the shock at birth and that is denoted by $U_{z_t} : \mathbf{C} \subset^{kL} \rightarrow$, where $\mathbf{C}$ is a closed and convex set. At this point no assumptions are made about time- or state-separability of utility. Special cases obviously include time-separable expected utility, recursive utility, and utility models with habit formation.

The optimization problem of individual z^t is as follows.

$$\max_{\vec{x},\vec{i}} U_{z_t}(\vec{x}) \text{ s.t.} \tag{1.1}$$

$$i(z^{t+a-1}) \in \mathbf{O}_a(i(z^{t+a-2}), x(z^{t+a-2}), z_{t+a-1})$$

$$x(z^{t+a-1}) \in \mathbf{B}_{a,i(z^{t+a-1})}(x(z^{t+a-2}), p(z^{t+a-1}), z_{t+a-1})$$

$$\text{for all } z^{t+a-1} \succeq z^t, a = 1, \ldots, A$$

where $x_{z^t}(z^{t-1}) = 0$.

Aggregate resources at each z^t are assumed to depend on the shock and on last period's aggregate choices and they denoted by $\omega(z_t, x(z^{t-1})) \in^L_+$. At $t = 0$ there are initially alive agents of ages $a = 2, ..., A$, who are in some initial budget $i_{y^{A-1}}(z_0)$ and who have some initial choices $x_{z^{-a}}(z^{-1}) \geq 0$. They have utility functions $u_{z^{-a}}$ that are defined only over remaining consumption in their lifetimes.

Their utility maximization problems are straightforward modifications of (1.1).

Equilibrium

Since budget correspondences are non-convex we need to work with the concept of extended sequential equilibrium, where ex ante identical agents possibly make different choices. We focus on equilibria where there are only finitely many different choices made by each type of agent and where choices can be indexed by some $j \in \mathbf{J} = \{1, \ldots, J\}$.

We are now in a position to define a sequential competitive equilibrium.

DEFINITION 1 *A sequential competitive equilibrium consists of prices* $p(z^t)$ *for all* z^t, $t \in \mathbb{N}_,$, *fractions of agents,* $(\nu_{z^t}(j))_{j=1,\ldots,J}$, $\sum_{j \in \mathbf{J}} \nu_{z^t}(j) = 1$ *for all* z^t, *as well consumption and asset holdings* $(\vec{x}_{z^t,j})$ *and budget choices* $(\vec{i}_{z^t,j})$ *for all* j *with* $\nu_{z^t}(j) > 0$ *and for all* z^t *such that*

1. *Each agent maximizes utility, i.e. for all* z^t *and* $j \in \mathbf{J}$, *if* $\nu_{z^t}(j) > 0$ *then* $(\vec{x}_{z^t,j}, \vec{i}_{z^t,j})$, *solves (1).*

   The maximization problem of the initially alive is analogous.

2. *Markets clear at all nodes, i.e. for all* z^t

$$\sum_{a=1}^{A} \sum_{j \in \mathbf{J}} \nu_{z^{t-a+1}}(j) x_{z^{t-a+1},j}(z^t) = \omega \left(z_t, \sum_{a=1}^{A-1} \sum_{j \in \mathbf{J}} \nu_{z^{t-a}}(j) x_{z^{t-a},j}(z^{t-1}) \right).$$

Existence of a competitive equilibrium

We follow the construction from Duffie et al. (1994) to show that there exists a sequential competitive equilibrium for which all endogenous variables, jointly with the aggregate shock, follow a Markov process.

Assumptions

Let

$$\mathbf{\Delta}_{\epsilon}^{K} = \{\vec{p} \in_{+}^{LK}\text{: For all } z^t, t = 0, \ldots, A-1, p(z^t) \in \mathbf{\Delta},\ p_l(z^t) \geq \epsilon \text{ for all } l = 1, \ldots, L\}.$$

Analogous to our definitions of $\vec{x}$ and $\vec{i}$, we define $\vec{p} \in \mathbf{\Delta}_{\epsilon}^{K}$ as a sequence of state-contingent prices over an agent's life-cycle. For sufficiently small $\epsilon > 0$ and any $\vec{p} \in \mathbf{\Delta}_{\epsilon}^{k}$ the budget set of an agent born at shock $z_0 \in \mathbf{Z}$ who chooses $\vec{i}$ can be written as

$$\vec{\mathbf{B}}_{z_0,\vec{i}}(\vec{p}) = \{\vec{x} \in^{KL}\text{: } x(z^t) \in \mathbf{B}_{a,\vec{i}(z^a)}(x(z^{t-1}), p(z^t)),\ i(z^t)$$
$$\in \mathbf{O}_a(i(z^{t-1}, x(z^{t-1}), z_t),\ \text{for all } z^t, t = 0, \ldots, A-1\}.$$

We define the set of prices for which the choice $\vec{i}$ is admissible as

$$\vec{\mathbf{P}}_{z,\vec{i}} = \{\vec{p} \in \mathbf{\Delta}_{\epsilon}^{k} : \mathbf{B}_{z\vec{i}}(\vec{p}) \text{ is non-empty}\}.$$

Throughout we think of the vector x as choices in consumption and assets. To state the assumptions we need for existence it is useful to think of the first G, $1 \leq G \leq L$ elements of x to denote choices in commodities.

The following assumptions are made on fundamentals.

Assumption 1

1. *For each z the utility function, $U_z : \mathbf{C} \to$ is continuous.*
2. *For each $a = 2, \ldots, A$, $z \in \mathbf{Z}$, the function $\omega_a(z, x)$ is continuous in x and $\omega_1(z, x)$ does not depend on x.*
3. *There is a $U = \overline{x} > 0$ such that for each $a = 1, \ldots, A,$, for all $z \in \mathbf{Z}$ and all $x \in \mathbf{C}$,*

$$\overline{x} \geq \omega_{al}(z, x) \geq 0, \ l = 1, \ldots, G$$

While Assumption 1.1 is very weak and standard, Assumption 1.2 is rather strong. However, this should be viewed as a reduced-form assumption that ensures that in equilibrium aggregate consumption is bounded above.

The following assumptions are made on budget-correspondences.

Assumption 2

1. *For each $\vec{i}$ and z, if $\vec{\mathbf{P}}_{z,\vec{i}}$ is non-empty then it is a closed set and $\vec{\mathbf{B}}_{z,\vec{i}}(.)$ is continuous on $\vec{\mathbf{P}}_{z,\vec{i}}$.*
2. *For all $\epsilon > 0$, for any z there is a $\vec{i}^*$ such that $\vec{\mathbf{P}}_{z,\vec{i}^*} = \mathbf{\Delta}^K_\epsilon$.*
3. *For all $\vec{i}, z$ and all $\vec{p} \in \mathbf{\Delta}^K_\epsilon$, if $\vec{x} \in \vec{\mathbf{B}}_{z_0,\vec{i}}(\vec{p})$ then $x_l(z^t) \geq 0$ for all $l = 1, \ldots, G$, and all z^t, $t = 0, \ldots, A-1$.*

Assumption 2.1 ensures that after the choice of a given (feasible) budget, the budget correspondence is well behaved. Assumption 2.2 ensures that an agent can always choose a budget set that is feasible under all prices. Assumption 2.3 ensures that there are consumption goods that have to be consumed in non-negative quantities.

We make the following joint assumptions on utilities and the budget correspondene

Assumption 3

1. *For each* z *there exists an* $\epsilon > 0$ *such that if* $\vec{p_i} < \epsilon$ *and* $(\vec{x}, \vec{y})$ *solves (1.1) for some* i *then* $x_l(z^t) > (A+1)\overline{x}$ *for some* $l = 1, \dots, G$ *and some* z^t, $t = 0, \dots, A-1$.
2. *For any* $\vec{i}$, z, *if* $\vec{p}$ *lies on the boundary of the set* $\vec{\mathbf{P}}_{z,\vec{i}}$ *but in the interior of* $\boldsymbol{\Delta}_\epsilon^K$ *then*

$$\max_{\vec{x}} U_z(\vec{x}) \ s.t. \ x \in \vec{\mathbf{B}}_{z\vec{i}}(\vec{p}) < \max_{\vec{x},\vec{j}\in\mathbf{I}} U_z(\vec{x}) \ s.t. \ x \in \vec{\mathbf{B}}_{z\vec{j}}(\vec{p})$$

3. *If* $(\vec{x}, \vec{i})$ *solves (1.1) then*

$$p(z^t) \cdot x(z^t) = p(z^t)\omega_{t+1}(z_t, x(z^{t-1})) \ for \ all \ z^t, t = 0, \dots, A-1$$

Existence of competitive equilibrium

Define $\vec{p}(z^t) = (p(z^{t+\tau}))_{z^{t+\tau}\succeq z^t, \tau=0,\dots,A-1}$ as the collection of current prices and all possible (state-contigent) prices over the next $A-1$ periods. We define all endogeneous and exogenous variables at that are relevant for equilibrium at some z^t as

$$\xi(z^t) = \left((z_{t-a})_{a=0}^{A-1}, (\vec{x}_{z^{t-a},j})_{a=0\ldots,A-1,\vec{j}\in\mathbf{J}}, (\nu_{z^{t-a}}(.))_{a=0}^{A-1}, \vec{p}(z^t) \right).$$

In order to prove existence of a generalized Markov equilibria it is useful to change notation a bit and denote consumption of an agent of age a whose choices are indexed by j as $x_{a,j}$ and to denote the measure of choices j within agents of age a as $\nu_a(j)$. We can then define a generic ξ by

$$\xi = \left((z_{-a})_{a=0}^{A-1}, (\vec{x}_{a,j})_{a=1,\ldots A,j\in\mathbf{J}}, (\nu_a(.))_{a=1}^{A}, \vec{p} \right).$$

We define $\mathbf{\Xi}_z$ as the set of all ξ that are consistent with market clearing at a given shock $z \in \mathbf{Z}$. We have

$$\mathbf{\Xi}_z = \{\xi : \sum_{a=1}^{A} \sum_{j\in\vec{\mathbf{J}}} \nu_a(j) x_{a,j}(z_{-a+1}, \ldots, z_0) = \omega(z, \sum_{a=2}^{A} \sum_{\vec{i}} \nu_a(\vec{i}) x_{a,\vec{i}}(z_{-a+1}, \ldots, z_{-1})$$

For each shock $z \in \mathbf{Z}$ we define the expectations correspondence,

$$\mathbf{G}_z : \mathbf{\Xi}_z \rightrightarrows \mathbf{\Xi}^Z,$$

as follows.

$$\begin{array}{rcl}\mathbf{G}_{\bar{z}}(\bar{\xi}) & = & \{(\xi_1, \ldots, \xi_z) \in \mathbf{\Xi}_1 \times \ldots \times \mathbf{\Xi}_Z : \\ \vec{p} & = & \vec{S}(p, \vec{p}_1, \ldots, \vec{p}_Z) \\ (\vec{x}_{1,j}, \vec{i}) & \text{solves} & (1.1) \text{ if } \nu_1(j) > 0 \\ x_{a+1,\vec{i},z} & = & x_{a,\vec{i}} \text{ for all } a = 1, \ldots A-1, j \in \mathbf{J}, z \in \mathbf{Z} \\ \nu_{a+1,z}(.) & = & \nu_a \text{ for all } a = 1, \ldots, A = 1, z \in \mathbf{Z}\}\end{array}$$

where

$$\vec{S}(p, \vec{p}_1, \ldots, \vec{p}_Z) = (p, (p_1(z^a))_{a=0,\ldots A-2}, \ldots, (p_Z(z^a))_{a=1,\ldots,A-2}).$$

We have the following existence result.

Proposition 1 *Under Assumptions 1.1-1.3 there exists a compact sets* $\mathbf{\Xi}^* \subset \mathbf{\Xi}$ *such that for all* $\xi(0) \in \mathbf{\Xi}^*$ *there are* $\xi(z) \in \mathbf{\Xi}^*$, $z = 1, \ldots, Z$ *such that*

$$(\xi(1), \ldots, \xi(Z)) \in \mathbf{G}(\xi(0)).$$

It is easy to see that the proposition, together with our definition of the expectations correspondence implies the existence of a competitive equilibrium.

To prove the proposition we follow Duffie et al. (1994) and construct the set $\mathbf{\Xi}^*$ by backward induction. Given a compact $\overline{\mathbf{\Xi}}$, define a backward operator , $\mathcal{E}(.)$ on

$\mathbf{K} \subset \overline{\mathbf{\Xi}}$ as follows

$$\mathcal{E}^1(\mathbf{K}) = \{\xi \in \overline{\mathbf{\Xi}} : \exists(\xi(1), \ldots, \xi(Z)) \in \mathbf{K}^Z \cap \mathbf{G}(\xi)\}.$$

For $n = 2, 3, \ldots,$ let $\mathcal{E}^n(\mathbf{K}) = \mathcal{E}\left(\mathcal{E}^{n-1}(\mathbf{K})\right)$.

We use the following lemma whose proof is standard (see e.g. Duffie et al (1996) or Kubler and Polemarchakis (2004) for similar results in slightly different environments).

Lemma 1 *Suppose there is some compact* $\overline{\mathbf{\Xi}}$ *such that for each* n, $\mathcal{E}^n(\overline{\mathbf{\Xi}})$ *is non-empty and closed. Then the statement of Proposition 1 holds.*

By the definition of $\mathbf{G}$ it is clear that each $\mathcal{E}^n$ is closed. In order to establish non-emptiness, we establish existence of equilibrium for a truncated economy. For any $T = 2, \ldots$ take a T-truncated economy to be a T period economy that is identical with respect to asset payoffs, endowments and utility to the original economy at all nodes for the first $T - A$ periods. For the last $A - 1$ periods the utility functions of have to be modified to take into account the fact that there are no longer defined over a life-cycle of length A. Details on how to do this are largely irrelevant as long as the modified utility function is well defined (and continuous). We will prove that in this economy a sequential equilibrium always exists and that we can bound all endogenous variables uniformly across T.

This gives the following lemma.

Lemma 2 *There is a compact set $\overline{\mathbf{\Xi}} \subset \mathbf{\Xi}$ such that for any T, there exists a sequential equilibrium for the T-truncated economy with $\xi(z^t) \in \overline{\mathbf{\Xi}}$ for all z^t, $t \leq T$.*

To prove this lemma we need Caratheodory's theorem. For a set A let $conv(A)$ be its convex hull, i.e. the smallest convex set containing A.

Lemma 3 *Let $A \subset^n$ and $x \in conv(A)$. Then there are $z_1, ..., z_{n+1} \in A$ such that $x = \sum_{i=1}^{n+1} \alpha_i z_i$ for some $\alpha \in \Delta^n$*

Proof of Lemma 2. Given $\epsilon > 0$, Assumptions 1-3 guarantee that each z^t's best response correspondence in upper-hemi-continous on $\mathbf{\Delta}_\epsilon^K$. Let $\Phi_{z^t}(\vec{p}; z^{t+a})$ denote the convex hull of this correspondence projected to z^{t+a}. At each z^t a price player takes as given $x_a(z^t) \in \Phi_{z^{t-a+1}}(\vec{p}; z^t)$, $a = 1, \ldots, A$ and solves

$$\max_{p \in \mathbf{\Delta}_\epsilon} p \cdot \left(\sum_{a=1}^{a} x_a(z^t) - \omega(z_t, \sum_{a=1}^{A} x_a(z^{t-1})) \right)$$

A standard argument gives the existence of a fixed point. For sufficiently small ϵ, Assumption 2 implies that the price player in fact maximizes over all $\mathbf{\Delta}$. Assumption 3 then implies that market clearing must hold at the fixed point. Lemma 3 implies that for each agent z^t there exist $KL + 1$ different choices $j = 1, \ldots, KL + 1$ and weights $\nu_{z^{t+a-1}}(j)$ such that

$$x_{a,z^{t-a+1}}(z^t) = \sum_j \nu_{z^t}(j) x_{a,z^{t+a-1}}(z^t).$$

Finally Assumption 1-3 imply that all endogenous variables are bounded uniformly in T.

Stationary equilibrium

It is a natural question to ask if there exist Markov equilibria if one makes stronger assumptions on preferences (in particular if one assumes time-separable expected utility) and budget-sets. As Kubler and Polemarchakis (2004) explain, the assumption of rational expectations implies that in the presence of multiple equilibria in spot markets, the past plays a crucial rule in selecting the equilibrium that is consistent with agents' expectations. Brumm et al. (2017) prove the existence of a Markov equilibria in a model with finitely many agents – it seems impossible to extend their proof to this framework. It would seem that our assumption that agents are ex ante identical which guarantees that ex post there are finitely many agents in the economy might make it easier to prove existence of a Markov equilibrium. However, as our example in Section 2 above illustrates, the fractions of agents making different choices is determined by all prices over the agent's life-cycle and not just by current prices. If one assumes that utility is concave one can index an agent by his life-cycle budget choice $\vec{i}$. This potentially decreases the number of possible choices substantially. However, it is not clear how it helps to establish existence of a simple Markov equilibrium.

As we will see now, it is possible to establish the existence of a steady state equilibrium in a Bewley-style model without aggregate uncertainty.

No aggregate uncertainty and steady states

We provide a Bewley-style model with heterogeneous agents and only idiosyncratic endowment shocks. The financial market consists of several bonds that differ on the punishments of default. This special case can be seen as an application on the agents' housing decisions and default decisions. We briefly rewrite the notations to avoid confusion.

The Model

We set up an exchange economy from a Bewley-style model with heterogeneous agents and idiosyncratic endowment shocks. Time is discrete and denoted by $t \in \mathbb{N}_+$. There are H types of heterogeneous agents with an overlapping-generation structure. For each type of agent, new agents are born with age 1 and will live for A periods. Agents face idiosyncratic endowment shocks and choose consumption among M perishable commodities, N durable commoditie and holdings of J bonds. $H, M, N, J, A \in \mathbb{N}_{++}$ are finite. There is a continuum of agents distributed on the interval $[0, AH]$ according to the Lebesgue measure λ. Denote $\mathbf{H} = \{1, 2, ..., H\}, \mathbf{A} = \{1, 2, ..., A\}$, and denote the measure of age a type h agents as λ_{ah} . At every period of time, a measure 1 of each type $h \in \mathbf{H}$ is born with age 0, which makes $\lambda_{ah} = 1, \forall a \in \mathbf{A}, h \in \mathbf{H}$. An agent can be identified by (a, h, k), where $a \in \mathbf{A}, h \in \mathbf{H}, k \in [0, 1]$.

For each agent (a, h, k), the exogenous shock $\{y_t^{(a,h,k)}\}_{t \geq 0}$ follows an ergodic Markov process with finite states $y_t^{(a,h,k)} \in \mathbf{Y}, \mathbf{Y} = \{1, 2, ..., Y\}$, with transition function $Q : \mathbf{Y} \times 2^{\mathbf{Y}} \to [0, 1]$. Since it is an ergodic process, it converges to a stationary distribution $P : 2^{\mathbf{Y}} \to [0, 1]$. We take the Feldman and Gills (1985) construction among agents, so that the empirical distribution equals the limit distribution. That is, for $\mathbf{S} = \{k : y^{(a,h,k)} = s\}$ we have $\lambda_{ah}(\mathbf{S}) = P(s)$ for all a, h.

We assume the perishable commodities(consumption goods) are continuously valued and the durable commodities are discrete valued. It is trivial to also include discrete valued perishable commodities and continuous valued durable commodities into the model, yet our set up catches the essence of the underling problem and simplifies notations. The durable commodities' supplies are exogenous given as $\bar{\phi}$. Agents are born with no durable commodities. And agents have endowments of consumption goods each period depending on the endowment shocks $e : \mathbf{A} \times \mathbf{H} \times \mathbf{Y} \to \mathbb{R}_{++}$. And we denote $e(a, h, y)$ as $e_{ah}(y)$ for short.

There are J one period bonds in the market traded every period. Without lose of generality, we assume one share of bond promises to pay one unit of consumption good m_1 on the next period. The bonds are traded at price $q \in \mathbb{R}_+^J$ and the returns are collected the next period. The return vector on the bonds is denoted as $r \in \mathbb{R}_+^J$. The net supplies of bonds are 0.

We incorporate two types of discrete decisions an agent can make that induce non-convexities. The first is that the durable commodities take integer values only; the second is the decision to default on certain bonds. For the first kind of discrete decisions, perishable commodities M are continuous valued and durable commodities N are discrete valued. We denote an agent's consumption or holding of commodities as $x \in \mathbb{R}_+^M, \phi \in \mathbb{N}_+^N$. All the bonds take continuous values, $\theta \in \mathbb{R}^J$. Correspondingly, we have the price vectors: $p \in \mathbb{R}_+^M$, $\pi \in \mathbb{R}_+^N$, $q \in \mathbb{R}_+^J$. For the other type of discrete decisions, we need to address the punishment of default. We assume that the punishments are combinations of two forms: to take away durable goods, and (or) to restrict the agent's participation in the financial market. We use the opportunity correspondence to describe this.

We define the opportunity correspondence as $\mathbf{O} : \mathbf{I} \rightrightarrows \mathcal{I}$, $\mathbf{I} = \{1, 2, ..., i, ...\}$ is a countable set and $\mathcal{I}$ denotes the set of all $\mathbf{I}$'s subsets. For each $i \in \mathbf{I}$ there are several sets and functions to define: $\phi_i \in \mathbb{N}_+$ is the decision on holdings of the durable commodities; $\underline{\theta}_i \in \mathbb{R}$ which is the borrowing constraint on the bonds; for each bond j, $\delta_{ij} : \mathbb{N}_+^N \to \mathbb{N}_+^N$ is the change function of the durable goods holding ϕ^-, and we have $\varrho_{ji} : \mathbb{R} \to \mathbb{R}$ is the portfolio holding after change of θ_j^-.

Assumption 4 *For all i, $\phi_i \geq 0$. $\underline{\theta}_i$ are bounded below as $\underline{\theta}_i \geq \underline{\theta}$. For each discrete choice ϕ, the subset $\mathcal{I}_\phi = \{i \in \mathbf{I} : \phi_i = \phi\}$ is non-empty and finite.*

Assumption 5 *ϱ is continuous, $\varrho_j(0) = 0$, $\varrho_j(\theta_j) = \theta_j$ if $\theta_j > 0$, $\varrho(\theta_j) \geq \theta_j$ if $\theta_j < 0$ for all $j \in \mathbf{J}$; $\sum_j \delta_j(\phi) \leq \phi$.*

The assumption on ϱ states that the shares of the lenders will not change, and the shares of the borrowers may change due to default. The assumption on ϕ states that the punishment of taking away one's durable goods cannot take more durable goods than the amount one owns.

Then we define the budgets sets $\mathbf{B}(a, h, \theta^-, i^-, i, y, p, \pi, q, r)$ as follows:

$$
\begin{aligned}
&\mathbf{B}(a,h,\theta^-,i^-,i,y,p,\pi,q,r):\\
=&\{(x,\phi,\theta):\phi=\phi_i,\\
&p\cdot x+\pi\cdot\phi_i+q\cdot\theta\le\pi\cdot\phi_{i^-}\\
&-\sum_j \pi\cdot\delta_{ij}(\phi_{i^-})+\sum_j p_1\varrho_{ji}([1-\mathbf{1}_{\theta_j^->0}(1-r_j)]\theta_j^-)+p\cdot e_{ah}(y)\}
\end{aligned}
$$

For the new born agents $a=0$, of type h, we set $\theta_1^-=0,\phi_1^-=0,i^-=h$.

Assumption 6 *For all $i^-\in\mathbf{I}$, there exists $i_0\in\mathbf{O}(i^-)$ such that $B(a,h,i^-,i_0,y)=\{(x,\phi,\theta):\phi=0,\theta=0,px\le pe_{ah}(y)\}$*

This assumption allows any agent to default on every bond and consume the endowments only.

Utility functions

The utility function for agents with (a,h) is defined as $u_{ah}:\mathbb{R}_+^M\times\mathbb{N}_+^N\to\mathbb{R}$.

Assumption 7 *For all $a\in\mathbf{A},h\in\mathbf{H}$,*

i). Given $\phi\in\mathbb{N}_+^N$, the utility function $u_{ah}(\cdot,\phi)$ is strictly increasing and concave, continuous, and twice differentiable. There is a consumption commodity m_1 such that $lim_{m_1\to 0}u_{m_1}(\cdot,\phi)=+\infty, u_{ah}(m_1=0,\cdot,\phi)=-\infty$, and $u_{ah}(m_1,\cdot,\phi)>-\infty$ for $m_1>0$.

ii). Given $x\in\mathbb{R}_+^M$, the utility function $u_{ah}(x,\cdot)$ is strictly increasing.

This Inada condition on one consumption good plays an important role on getting an upper-hemi continuous optimal policy. The agent would not "jump" to a new budget set once she could afford it because that implies the zero consumption on m_1 and thus infinite negative utility.

Besides the punishments on default, we introduce a cost of default $l : \mathbf{I} \to \mathbb{R}_+$ in the form of a negative term on the utility: $u_i(\cdot) = u(\cdot) - l(i)$.

Assumption 8

$$l(i) = \begin{cases} 0 & \textit{if } \varrho_i(\theta) = \theta \\ l_0 > 0 & \textit{else} \end{cases}$$

This cost term represents the aggregate effect of time and effort spent on the default procedure, and can be arbitrarily small. It is crucial to have this term when the punishments on default is too "mild" that agents would always default regardless of the bond price. This will be further illustrated in the next section.

We define the value function in the recursive form as follows

$$V(a, h, \theta^-, i^-, p, \pi, q, r, y) = \sup_{i \in \mathcal{I}(i^-), (x, \phi, \theta) \in B(i^-, i)} \left\{ u_{ah}(x, \phi_{i^-}) - l(i) + \beta \sum_{y'} V(a+1, h, \theta, i) Q(y, y') \right\}$$

with the terminal value defined as $V(A-1,\cdot)=0$. The policy correspondence is

$$G(a,h,\theta^-,i^-,p,\pi,q,r,y)$$
$$=\arg\max_{i\in\mathcal{I}(i^-),(x,\phi,\theta)\in B(i^-,i)}\left\{u_{ah}(x,\phi_{i^-})-l(i)+\beta\sum_{y'}V(a+1,h,\theta,i)Q(y,y')\right\}$$

The equilibrium

For each agent (a,h,k), there is a vector $(a,h,\theta^-,i^-,y,x,\theta,i)$ as defined before. Define a distribution μ as

$$\mu(X)=\lambda(\{(a,h,k):(a,h,\theta^-,i^-,y,x,\theta,i)_{a,h,k}\in X\})$$

The equilibrium $\mathbf{E}$ consists of stationary prices p^*,π^*,q^*, returns of bonds r^* decision rules $(x^*,\phi^*,\theta^*,i^*)_{a,h,k}, a\in\mathbf{A}, h\in\mathbf{H}, k\in[0,1]$ and a distribution μ^* such that:

(i) $(x^*,\phi^*,\theta^*,i^*)_{a,h,k}\in G(a,h,\theta^{-*},i^{-*},p^*,\pi^*,q^*,r^*,y)$;

(ii) $\int x^*d\mu^*=\int ed\mu^*, \int\phi^*\mu^*=\bar{\phi}$;

(iii) $\int\theta^*d\mu^*=0$;

(iv) $\int\pi\cdot\delta_{ji^*}(\phi^-)d\mu^*=\int p_1\varrho_{ji}([1-\mathbf{1}_{\theta_j^->0}(1-r_j^*)]\theta_j^-)d\mu^*$ for all j;

(v) μ^* is invariant.

We can redefine the equilibrium with the structure of a single period exchange economy setting.

Define $G^{y^{A-1}}(h)\equiv\{(\tau_1,\tau_2):\tau_1\in G^{y^{a-1}}(h), y^{A-1}\succ y^{a-1};\tau_2\in G(\theta_{a-1},i_{a-1},y_a),$ $(\theta_{a-1},\phi_{a-1})=(\theta_{a-1}(\tau_1),\phi_{a-1}(\tau_1))\}, G^{y^0}(h)=G(a=1,h,\theta^-=0,\phi^-=0,i^-=h,y_0)$.

So for one type h agent, the total demand over her life cycle with shocks y^{A-1} can be defined as

$$
\begin{aligned}
d(h, y^{A-1}) =& \{(\bar{x_h}, \bar{\phi_h}, \bar{\theta_h}, \delta_h, \varrho_h(-), \varrho_h(+)) : \\
& \bar{x_h} = \sum_a x_a(\tau), \bar{\phi_h} = \sum_a \phi_a(\tau), \bar{\theta_h} = \sum_a \theta_a(\tau), \\
& \delta_h = \sum_a \delta_{i_a(\tau)}(\phi_{a-1}), \\
& \varrho_h(-) = -\sum_a \varrho_{i_a(\tau)}(\mathbf{1}_{\theta_{a-1}(\tau)\leq 0}\theta_{a-1}(\tau)), \varrho_h(+) \\
& \quad = \sum_a \varrho_{i_a(\tau)}(\mathbf{1}_{\theta_{a-1}(\tau)\geq 0}\theta_{a-1}(\tau)), , \tau \in G^{y^{A-1}}(h)\}
\end{aligned}
$$

Since we have continuum of agents, the total demand of agent h with shocks y^{A-1} is the convex hull of $d(h, y^{A-1})$.

Given shocks y converges to a stationary distribution, the total demand of type h agent(with measure A) in equilibrium is equivalent to the expected total demand over her life cycle.

Then the total demand of type h agents in the economy can be defined as

$$
D(h) = \{\tau(h) : \tau(h) = \sum_{y^{A-1}} \tau(h, y^{A-1})\mathbf{P}(y^{A-1}), \tau(h, y^{A-1}) \in Cov(d(h, y^{A-1}))\}
$$

So the original equilibrium is equivalent to the following:

The equilibrium consists of stationary prices p^*, π^*, q^*, returns r^* , demads $\bar{x_h^*}, \bar{\phi_h^*}, \bar{\theta_h^*}, \delta_{jh}^*, \varrho_{jh}^*(-), \varrho_{jh}^*(+)$ such that:

(i) For all h, $(\bar{x_h^*}, \bar{\phi_h^*}, \bar{\theta_h^*}, \delta_{jh}^*, \varrho_{jh}^*(-), \varrho_{jh}^*(+)) \in D(h)$;

(ii) $\sum_h \bar{x_h^*} = \sum_h e_h, \sum_h \bar{\phi_h^*} = \bar{\phi}$;

(iii)$\sum_h \bar{\theta_h^*} = 0$;

(iv) $r_j^* = \frac{\sum_h \pi^* \delta_{jh}^* + \sum_h p_1^* \varrho_{jh}^*(-)}{\sum_h p_1^* \varrho_{jh}^*(+)}$ for all j.

Existence of a steady state equilibrium

We prove the existence of a refined equilibrium in the sense of Dubey, Geanakopolos and Shubik (2005).

Trembling refined equilibrium

First we define a trembling equilibrium. We introduce a measure of A external ε-agents who sell and buy $\varepsilon = (\varepsilon_j)_{j \in J} \gg 0$ every period and pay back debt fully every period. The transactions are financed by fixed endowment of $\frac{(1-r)\varepsilon}{p_1}$ each period in continuous consumption good m_1.

So we define the ε-boosted equilibrium $\mathbf{E}(\varepsilon)$ as follows:

The equilibrium $\mathbf{E}(\varepsilon)$ consists of stationary prices $p^*(\varepsilon), q^*(\varepsilon)$, returns $r^*(\varepsilon)$, demads $x^*(\varepsilon)$, $\phi^*(\varepsilon)$, $\theta^*(\varepsilon)$, $\delta_j^*(\varepsilon)$,$\varrho_j^*(-)(\varepsilon)$, $\varrho_j^*(+)(\varepsilon)$ such that:

(i) For all h, $(x^*(\varepsilon), \phi^*(\varepsilon), \theta^*(\varepsilon), \delta_j^*(\varepsilon), \varrho_j^*(-)(\varepsilon), \varrho_j^*(+)(\varepsilon)) \in D(h)$;

(ii) $\sum_h x_{-1}^*(\varepsilon) = \sum_h e_{-1}, \sum_h x_1^*(\varepsilon) = \sum_h e_1 + \frac{(1-r)\varepsilon}{p_1}, \sum_h \phi^*(\varepsilon) = \bar{\phi}$;

(iii)$\sum_h \theta^*(\varepsilon) = 0$;

(iv) $r_j^*(\varepsilon) = \frac{\sum_h \pi^*(\varepsilon)\delta_j^*(\varepsilon) + \sum_h p_1 \varrho_j^*(-)(\varepsilon) + p_1^* \varepsilon_j}{\sum_h p_1 \varrho_j^*(+)(\varepsilon) + p_1^* \varepsilon_j}$ for all j;

Existence of ε-boosted Fixed Point

First, for any small lower bound $\epsilon > 0$, define

$$\Delta_\epsilon = \{(p,\pi,q) : \sum p + \sum \pi + \sum q = 1, \text{and} \min\{p,\pi,q\} \geq \epsilon\}$$

The lower bound ϵ on prices, along with lower bound of asset holding, put upper bounds of consumptions, durable goods holdings and asset holdings, and indirectly on returns r. Denote the maximum of those upper bounds as M_ϵ. Denote $\mathbf{I}_\epsilon$ as the finite subset of $\mathbf{I}$ bounded by M_ϵ.

We detour from the procedure of proving ε-boosted equilibrium. This is due to the fact that the limit point when $\epsilon \to 0$ of the fixed point is not necessarily a ε-boosted equilibrium. Nonetheless, when $\epsilon \to 0, \varepsilon \to 0$, we can show that the limit point is actually the original equilibrium. So instead, we set $\varepsilon = \epsilon$, and show that the limit fixed point is the original equilibrium.

The modified budget set

We define $\hat{\mathbf{B}}$ as modified (non-empty, compact, and continuous) budget set.

Let the modified budget set be

$$\hat{\mathbf{B}}(a,h,\theta^-,i^-,i,y,p,\pi,q,r) = \mathbf{B}(a,h,\theta^-,i^-,i,y,p,\pi,q,r) \cup \{(0,0,\underline{\theta}_i)\}$$

LEMMA 4 *$\hat{\mathbf{B}}(a,h,\theta^-,i^-,i,y,p,\pi,q,r)$ is non-empty, compact, and continuous in θ^-,p,π,q,r.*

Proof: First, $\hat{\mathbf{B}}(i^-,i)$ is non-empty and compact by construction.

Second, for $(p,\pi,q)\in\Delta_\epsilon$, $\mathbf{B}(a,h,\theta^-,i^-,i,y,p,\pi,q,r)$ and $\{(0,0,\underline{\theta}_i)\}$ are both continuous. So the union is also continuous.□

The total demand of type h agents

The value function

$$\begin{aligned}&\hat{V}(a,h,\theta^-,i^-,p,\pi,q,r,y)\\=&\sup_{i\in\mathcal{I}(i^-),(x,\phi,\theta)\in\hat{B}(i^-,i)}\left\{u(x,\phi^-)+\beta\sum_{y'}\hat{V}(a+1,h,\theta,i,p,\pi,q,r,y')Q(y,y')\right\}\end{aligned}$$

The policy correspondence is

$$\begin{aligned}&\hat{G}(a,h,\theta^-,i^-,p,\pi,q,r,y)\\=&\arg\max_{i\in\mathcal{I}(i^-),(x,\phi,\theta)\in\hat{B}(i^-,i)}\left\{u(x,\phi^-)+\beta\sum_{y'}\hat{V}(a+1,h,\theta,i,p,\pi,q,r,y')Q(y,y')\right\}\end{aligned}$$

Lemma 5 *$\hat{G}$ is u.h.c. in (θ^-,i^-,p,π,q,r).*

Proof: Using Theorem 2 in the Appendix, we have the policy correspondence $\hat{G}$ is u.h.c. Theorem 2 assumes infinitely-living agent, but the results easily apply to the finite-living agent case.□

Analogiously, define $\hat{G}^{y^{A-1}}(h),\hat{D}(h)$.

Lemma 6 *Given Δ_ϵ, $\hat{D}(h)$ is compact, convex-valued, and u.h.c. in (p,π,q,r).*

Proof: First, we show that $\hat{G}^{y^{A-1}}(h)$ is u.h.c. in (p,π,q,r).

By Lemma, we have $\hat{G}^{y^0}(h)$ is u.h.c. in (p, π, q, r). So for $(p_n, \pi_n, q_n, r_n) \to (p, \pi, q, r)$, and $(x_n, \phi_n, \theta_n, i_n) \in \hat{G}^{y^0}(h, p_n, \pi_n, q_n, r_n)$, if $(x_n, \phi_n, \theta_n, i_n) \to (x, \phi, \theta, i)$, then $(x, \phi, \theta, i) \in \hat{G}^{y^0}(h, p, \pi, q, r)$.

For $\hat{G}^{y^1}(h)$, $(p_n, \pi_n, q_n, r_n) \to (p, \pi, q, r)$, and $((x_{1n}, \phi_{1n}, \theta_{1n}, i_{1n}), (x_{2n}, \phi_{2n}, \theta_{2n}, i_{2n})) \in \hat{G}^{y^1}(h, p_n, \pi_n, \pi_n, q_n, r_n)$, if $((x_{1n}, \phi_{1n}, \theta_{1n}, i_{1n}), (x_{2n}, \phi_{2n}, \theta_{2n}, i_{2n})) \to ((x_1, \phi_1, \theta_1, i_1), (x_2, \phi_2, \theta_2, i_2))$, we know from above that $(x_1, \phi_1, \theta_1, i_1) \in \hat{G}^{y^0}(h, p, \pi, q, r)$. We just need to show that $(x_2, \phi_2, \theta_2, i_2) \in G(\phi_1, \theta_1, i_1, p, \pi, q, r, y_1)$. Use the formal Lemma again, we have $\hat{G}^{y^1}(h)$ is u.h.c. in (p, π, q, r). By induction, we can get $\hat{G}^{y^{A-1}}(h)$ is u.h.c. in (p, π, q, r).

By construction, we get $d(h, y^{A-1})$ is u.h.c. in (p, π, q, r). And further, $\hat{D}(h)$ is compact, convex-valued, and u.h.c. in (p, π, q, r). $\square$

LEMMA 7 *Given assumption 5, 6, for $(p, \pi, q) \in \Delta_\epsilon$, $\hat{V} = V, \hat{G} = G$.*

Proof: Given assumption 5, 6, $(0, 0, \underline{\theta}_i)$ is never an optimal choice. So the agents choose from the true budget set. Thus, $\hat{V} = V, \hat{G} = G$. $\square$

So we have for any $(\bar{x_h}, \bar{\theta_h}, \delta_h \varrho_h(+), \varrho_h(-)) \in \hat{D}_b(h)$, the following inequality holds.

$$p\bar{x_h} + q\bar{\theta_h} \leq -\pi\delta_h + p_1 r \varrho_h(+) - p_1 \varrho_h(-) + pe_h \tag{1.2}$$

The fixed point

Given ϵ, first define the return on bonds

$$r_j = \frac{\sum_h \pi\delta_j + \sum_h p_1\varrho_j(-) + p_1\epsilon}{\sum_h p_1\varrho_j(+) + p_1\epsilon} \tag{1.3}$$

$0 \leq r_j \leq M_\epsilon$ for all j.

The demand correspondences are

$$\Phi_\epsilon^h = \hat{D}(h, p, \pi, q, r)$$

Define the price player as

$$\Phi_\epsilon^0 = \arg \max_{(p,\pi,q)\in\Delta_\epsilon} p\sum_h(x-e) + \pi(\sum_h \phi - \bar{\phi}) + q\sum_h \theta$$

By Kakutani's Theorem the correspondence $r_\epsilon \times_{h=0}^{H} \Phi_\epsilon^h$ has a fixed point η_ϵ.

Sum across agents with (2) and plug in (3), we get

$$p\sum_h(x-e) + q\sum_h \theta - p_1\sum_j(1-r_j)\epsilon \leq 0 \tag{1.4}$$

With increasing utility, the equality holds at the fixed point.

LEMMA 8 *When $\epsilon \to 0$, $p(\eta_\epsilon), \pi(\eta_\epsilon), q(\eta_\epsilon), r(\eta_\epsilon)$ are bounded away from 0, $r(\eta_\epsilon)$ is bounded above.*

Proof: First, we show that p, π are bounded away from 0. Denote $\underline{p} = \min\{p, \pi\}$, $\hat{p} = \max\{p\}$, and $\bar{p} = \max\{p, \pi, q\}$. We prove by contradiction: assume when $\epsilon \to 0$,

$\underline{p} \to 0$. Then if $\hat{p} \not\to 0$, the consumption of the good with price $\underline{p}$ will go to infinity. This contradicts with the price player's best response. If $\hat{p} \to 0$, then $p \to 0$. It is easy to check that this cannot be at a fixed point. So p, π are bounded away from 0.

Next, we show that q, r are bounded away from 0 and r is bounded above. We prove by contradiction. If $r_j \to +\infty$, then $\varrho_{h,j}(+) \geq 0$ for all h, which in turn implies by (3) that $r_j \leq 1$. Contradiction.

If $q_j \to 0$, then due to the positive cost of default. There exists an $\underline{q}_j$ that if $q_j \leq \underline{q}_j$, $\varrho_{h,j}(-) = 0$ for all h. Also $\varrho_{h,j}(+) > 0$, otherwise $r_j \to 1$ and it contradicts with agent's utility maximization. So $\sum_h \theta_j > 0$ and it contradicts the price player's maximization problem.

If $r_j \to 0$, There exists an $\underline{r}_j$ that if $r_j \leq \underline{r}_j$, $\varrho_{h,j}(+) = 0$ for all h. This implies by (3) that $r_j \geq 0$. Contradiction. $\square$

It is crucial to have Assumption 8 for Lemma 8. If there is no cost of default, it is possible to have a zero-price, zero-return bond in the limit point. This may happen when the punishments are too "mild" and every one that borrows would default.

Existence of equilibrium

From Lemma 8, there exists a subsequence of $\eta(\epsilon)$ that converges to η^* when $\epsilon \to 0$. Next we show η^* is an equilibrium.

THEOREM 1 *Given Assumptions 4-8,there exists an equilibrium defined in section 5.3.*

Proof: When $\epsilon = 0$, (3) becomes

$$p \sum_h (x-e) + q \sum_h \theta \leq 0 \tag{1.5}$$

The equality holds at the fixed point.

$$p* \sum_h (x-e) + q* \sum_h \theta = 0 \tag{1.6}$$

$$\frac{p*}{\sum p* + \sum q*} \sum_h (x-e) + \frac{q*}{\sum p* + \sum q*} \sum_h \theta = 0 \tag{1.7}$$

So we get

$$\begin{aligned} &\max_{(p,q)\in\Delta} p \sum_h (x-e) + \pi(\sum_h \phi - C) + q \sum_h \theta \\ \geq &\frac{p*}{\sum p* + \sum q*} \sum_h (x-e) + \frac{q*}{\sum p* + \sum q*} \sum_h \theta + 0(\sum_h \phi - \bar{\phi}) \\ = &0 \end{aligned}$$

This implies $\sum(x-e) \geq 0, (\sum \phi - C) \geq 0, \sum \theta \geq 0$. Combined with (5), we can get $\sum(x-e) = 0, \sum \theta = 0$. This further implies $(\sum \phi - C) = 0$. Because if $(\sum \phi - C) > 0$, by setting price to $\pi + \Delta, p - \Delta$, the price player is better off, which contradicts that the price player optimizes at the fixed point.

Also easy to see, when $\epsilon = 0$, the agents optimize on the true budget sets. So there exists an equilibrium. $\square$

Appendix

Similarly to Stockey & Lucas (1989),

Assumption 9.4 X is a convex Borel set in $\mathbf{R}^l$, with its Borel subsets $\mathcal{X}$.

Assumption 9.5 One of the following conditions holds:

a. Z is a countable set and $\mathcal{Z}$ is the σ-algebra containing all subsets of Z;

b. Z is a compact (Borel) set in $\mathbf{R}^k$, with its Borel subsets $\mathcal{Z}$, and the transition function Q on $(Z, \mathcal{Z})$ has the Feller property.

Assumption 9.6 The correspondence $\Gamma : X \times Z \to X$ is nonempty, compact-valued, and continuous.

Assumption 9.7 The function $F : A \to \mathbf{R}$ is bounded and continuous, and $\beta \in (0, 1)$.

Assumption 9.6' For all $i \in I$, $i' \in \mathcal{I}(i)$, the correspondence $\Gamma_{i,i'} : X \times Z \to X$ is nonempty, compact-valued, and continuous.

THEOREM 2 *Let $(X, \mathcal{X}), (Z, \mathcal{Z}), Q, \Gamma, F, \beta,$ and I satisfy Assumptions 9.4,9.5,9.6',9.7, and define the operator T on $C(S)$ by*

$$(Tf)(x, z, i) = \sup_{i' \in \mathcal{I}(i), y \in \Gamma_{i,i'}(x.z)} \left\{ F(x, y, z) + \beta \int f(y, z', i') Q(z, dz') \right\}$$

We use discrete metric on I, all functions are continuous on I.

Then $T : C(S) \to C(S)$*; T has a unique fixed point v in* $C(S)$ *and for any* $v_0 \in C(S)$,

$$\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|, n = 1, 2, ...$$

Moreover, for all $i \in I$ *the correspondence* $G_i : S \to X$ *defined by*

$$G(x, z, i)$$
$$= \left\{ (i', y)) : i' \in \mathcal{I}(i), y \in \Gamma_{i,i'}(x, z), v(x, z, i) = F(x, y, z) + \beta \int v(y, z', i')Q(y, dz') \right\},$$

is nonempty, compact-valued, and u.h.c.

Proof: It is easy to see that

$$\begin{aligned}
(Tf)(x, z, i) &= \sup_{i' \in \mathcal{I}(i), y \in \Gamma_{i,i'}(x.z)} \left\{ F(x, y, z) + \beta \int f(y, z', i')Q(z, dz') \right\} \\
&= \max_{i' \in \mathcal{I}(i)} \sup_{y \in \Gamma_{i,i'}(x.z)} \left\{ F(x, y, z) + \beta \int f(y, z', i')Q(z, dz') \right\} \\
&= \max_{i' \in \mathcal{I}(i)} (T_{i'} f)(x, z, i)
\end{aligned}$$

From Lemma 9.5, we have $(T_{i'} f)(x, z, i)$ are bounded and continuous. Since I is a finite set, $(Tf)(x, z, i) \in C(S)$. So we have $T : C(S) \to C(S)$.

Immediately we can see that T satisfies Blackwell's sufficient conditions. So T has a unique fixed point and the equation $\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|$ holds.

Then $G(x, z, i)$ is u.h.c based on similar argument as the theorem of Maximum. Define $f(x, z, i, y, i') := F(x, y, z) + \beta \int v(y, z', i')Q(y, dz')$. Let $(x_n, z_n, i_n) \to (x, z, i)$, for large enough n,$i_n = i$. Let $\Gamma_i = \cup_{i' \in \mathcal{I}(i)} \Gamma_{i,i'}$, with discrete metric on i', Γ_i is continuous.

Choose $(i'_n, y_n) \in G(x_n, z_n, i)$. Since Γ_i is u.h.c., there exists a subsequence $\{i'_{n_k}, y_{n_k}\}$ converging to $(i', y) \in \Gamma_i(x, z)$. Also since Γ_i is l.h.c., for all $(i'_l, y_l) \in \Gamma_i(x, z)$, there exists $\{i'_{ln_k}, y_{ln_k}\} \to (i'_l, y_l)$, with $\{i'_{ln_k}, y_{ln_k}\} \in \Gamma_i(x_{n_k}, z_{n_k})$. Since $f(x_{n_k}, z_{n_k}, i, i'_{n_k}, y_{n_k}) \geq f(x_{n_k}, z_{n_k}, i, i'_{ln_k}, y_{ln_k})$ and f is continuous. So $f(x, z, i, i', y) \geq f(x, z, i, i'_l, y_l)$, with leads to $(i', y) \in G(x, y, i)$.

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Chapter 2

Recursive equilibria in dynamic economies with bounded rationality

This paper provides a general way of modeling bounded rationality in the dynamic stochastic general equilibrium framework with infinitely lived heterogeneous agents and incomplete markets. Different from a rational agent, a bounded rational agent is associated with an extra parameter ϵ, which can be interpreted as the "level of irrationality". The bounded rational agent does not know the true probability distribution of the economy fundamentals. To make decisions, the bounded rational agent forms a belief of a stationary distribution of the fundamentals and then use the Markov transition associated with it to maximize utility. If a distribution of the fundamentals stays "closer" to its next-period transition than ϵ, the agent would consider it as ϵ-stationary. In equilibrium, each agent maximizes utility with an ϵ-stationary belief and markets clear. The main theorem of this paper shows that for any strictly positive ϵ, a recursive equilibrium exists. This result provides a potential way of measuring the "level of irrationality" for many behavioral models. Besides, there are two applications for a special case of the model, when ϵ is extremely close to zero: It lays foundation for numerically computed equilibria of models with the rational expectation assumption; and it can be viewed as an epsilon-equilibrium existence result for models with heterogeneous heuristics.

Introduction

Since it is first proposed by Simon (1955), bounded rationality has been extensively studied in the literature of economics and finance, especially behavioral and experimental economics. The idea that agents are limited by the tractability of the decision problem and the cognitive limitations of their minds to make optimal decisions should not be seen as a compromise for tractable models with the rational expectation assumption, but a more realistic way of modeling human behavior. Indeed, if the collective knowledge of human kind cannot solve a model, it is rather harsh to assume that each and every agent in the model can solve it perfectly. Experimental economists has shown that agents behavior deviates from the theoretical optimal systematically in so many aspects, that dynamic stochastic models with bounded rationality are getting more and more popular than their alternatives with the rational expectation assumption. Despite that there are many convincing findings in the literature of behavioral economics and experimental economics showing that agents are bounded rational, there is not yet a consistent theory to explain these findings. This paper tries to provide a general way to model bounded rationality in the framework of dynamic stochastic general equilibrium models with infinitely lived heterogeneous agents and incomplete financial markets.

Different kinds of heuristics the agents use not only have been found in experiments but also have been used in macro models, such as Krusell and Smith (1998), Gabaix (2014). The fact that many of these heuristics are seemingly independent of each other makes it challenging to find a general model. Dixon (2001) made an attempt to unify different kinds of bounded rationality: If the bounded rational agents' choices get them "close" to the optimum, then we can use the notion of ϵ-optimization, which means agents make choices so that the payoff is within ϵ of the optimum. Define the optimum (best possible) payoff as U^*, then the set of ϵ-optimizing choices $C(\epsilon)$ can be defined as all those choices c such that: $U(c) \geq U^* - \epsilon$. The notion of strict rationality is then a special case ($\epsilon = 0$). Difficulties emerge if we try to incorporate this idea into dynamic general equilibrium models. On the one hand, we do not know the "true" optimum in a general equilibrium model since the utility level achieved with strict rationality is not necessarily the optimum any more. This is consistent with the idea of Gigerenzer and Reinhard (2002) that simple heuristics may lead to better decisions than theoretically optimal under strict rationality. On the other hand, if the agents know the optimum choices, it makes less sense that they deliberately choose those actions that are sub-optimal. And hence there is no satisfactory way to interpret the meaning of ϵ.

When it comes to stochastic models and agents make choices under uncertainty, there can be an alternative way to incorporate Dixon's (2001) idea. Instead of ϵ-optimization, we argue that the rational expectation assumption should be relaxed and agents optimize with an "ϵ-belief" of the equilibrium distribution of the economic fundamentals. Given the "ϵ-belief", the agent would forecast future fundamentals using the Bayes' rule and then make optimal choices consistent with the forecasts. This way of modeling bounded rationality embodies three philosophical considerations besides its practical virtues. First, *the "modelees" behave the same way as the "modeler"*. Since economic modeling itself is usually forming a belief about how the economy evolves and make (sometimes terrible) predictions on future economic variables, it is natural to assume the agents in the model behave with this procedure as well. Second, it captures the notion that *a small mistake may cause a large loss.* Compared to Dixon (2001), an "ϵ-belief" does not necessarily mean an ϵ loss of the total utility. A small misperception may cause a large loss. This is analogous to Akerlof and Yellen (1985), where small deviations from rational choices make significant differences to utility functions. Third, it conveys the idea that *rational agents are all alike; every bounded rational agent is bounded rational in its own way.* The error term ϵ may differ among agents. And even with the same ϵ, there are different kinds of incorrect beliefs. So the agents choices (policy functions) may still be different. This is consistent with the fact that many different kinds of heuristics are found in experiments.

Although we now have a more natural and general idea to model bounded rationality, how to define the "ϵ-belief" remains a challenge. In a static partial equilibrium model, if we assume there is a true distribution μ^* exogenously given, it is easy to find the "ϵ-belief" once we have a metric $\|\cdot\|$ on the space of probability measures. An "ϵ-belief" then can be defined as a probability measure μ such that $\|\mu - \mu^*\| \leq \epsilon$. But in a dynamic stochastic general equilibrium model instead, the task is way more difficult in two aspects. First, with standard assumptions, a recursive equilibrium does not exist generically.[1] So we may not be able to find such μ^*. Even if we assume recursive equilibrium always exists, the equilibrium distribution of the fundamentals

1. See Hellwig (1983), Kubler and Schmedders (2002), Santos (2002), Kubler and Polemarchakis (2004).

changes if any agent's belief changed—i.e. the equilibrium distribution μ^* depends on μ and has to be determined simultaneously with μ. Second, in dynamic stochastic models, agents need not only the information of the current period but also forecasts of economic variables in future periods to make decisions. So we need μ to be a distribution over an adequate state space so that the agent can compute the distribution over the same state space of the next time period μ' conditional on μ.

The second difficulty can be dealt with by using the state space introduced by Duffie et al. (1994) when considering an exchange economy with incomplete financial market. The state space consists of the exogenous shock, the initial asset holdings of agents, the prices, and the end-of-period asset holdings. This is an extension of the natural state space in the sense of Maskin and Tirole (2001). With this state space, given a belief μ, we can get the next time period μ' with the Bayes' rule. To circumvent the first difficulty, instead of finding an "ϵ-belief", we define an ϵ-stationary belief. We assume that all agents believe the economy is in stationary equilibrium and form beliefs of the stationary distribution over the state space. If the agent's belief μ satisfies $\|\mu - \mu'\|_0 \leq \epsilon$ ($\|\cdot\|_0$ is the Kantorovich-Rubinstein norm), we can then say that it is ϵ- stationary. The error term ϵ can be interpreted as the "level of irrationality", or the limit of the agent's learning ability. The rational expectation equilibrium is then a special case of the model ($\epsilon = 0$). In equilibrium with heterogeneous agents, each agent is associated with a "level of irrationality" ϵ and maximizes her utility with an ϵ-stationary belief.

The main result of this paper is that for any positive ϵ, there exists a recursive equilibrium (or stationary Markov equilibria in the terminology of stochastic games).

Recursive equilibria are characterized by a pair of functions (characteristic functions): a transition function mapping this period's "state" into probability distributions over next period's state, and a "policy function" mapping the current state into current prices and choices. The state space is set to be the current exogenous shock and the beginning-of-period distribution of assets across individuals. This way of restricting the state space is often considered "minimal" or "natural" and widely used in the literature for both philosophical and practical reasons as follows. The philosophical reasons are the same as Maskin and Tirole's (2001) reasons to consider Markov perfect equilibria (MPE). On the practical side, recursive methods can be used to approximate stationary Markov equilibria numerically. Heaton and Lucas (1996), Krusell and Smith (1998), and Kubler and Schmedders (2003) are early examples of papers that approximate stationary Markov equilibria in models with infinitely lived, heterogeneous agents.

The recursive equilibrium of our model is compatible with the literature of behavioral and experimental economics. Bounded rational choices are often made not because the agent does not "try hard enough" to optimize, but because of an incorrect belief. Psychological effects found in the literature of behavioral and experimental economics can be seen as biases from the correct belief. Intuitively, the larger the ϵ, the more beliefs will the bounded rational agent accept and hence the more kinds of heuristics may be used by the agent. On the theoretical side, focusing on recursive equilibrium carries on our philosophy of modeling: since the economic models are to compute the stationary Markov equilibria and make forecasts recursively, the assumption that the agents believe the equilibrium is stationary and behave recursively comes natural.

Besides the general theoretical framework we show for modeling bounded rationality, we contribute to the literature of general equilibrium theory by showing existence of a recursive equilibrium. With rational expectations, recursive equilibrium does not exist generically. This problem was first illustrated by Hellwig (1983) and since then has been demonstrated in different contexts. Kubler and Schmedders (2002) give an example showing the nonexistence of stationary Markov equilibria in models with incomplete asset markets and infinitely lived agents. Santos (2002) provides examples of nonexistence for economies with externalities. Kubler and Polemarchakis (2004) present such examples for overlapping generations (OLG) models. Brumm et al. (2017) further provide one simple example demonstrating the possibility of nonexistence.

The nonexistence problem has been explained (dealt with) in the literature from different angles. Hellwig (1983) ascribed the reason of nonexistence to the "the simultaneous determination of prices for different periods". He stated that "The concept of a rational expectations equilibrium involves the simultaneous determination of prices for different periods. This simultaneity conflicts with the sequential nature of the temporary equilibrium process, so that rational expectations equilibria can only be implemented if the auctioneer in each period is informed of past expectations and uses them to select his temporary equilibria." Duffie et al. (1994) showed existence of competitive equilibria for general Markovian exchange economies with a different state space. The authors also proved that the equilibrium process is a stationary Markov process. However, we follow the well-established terminology in dynamic games and do not refer to these equilibria as stationary Markov equilibria, because the state space also contains consumption choices and prices from the previous period. Citanna and Siconolfi (2010 and 2012) provided sufficient conditions for the generic existence of stationary Markov equilibria in OLG models. However, their arguments cannot be extended to models with infinitely lived agents or to models with occasionally binding constraints on agentsâĂŹ choices, and for their argument to work in their OLG framework they need to assume a very large number of heterogeneous agents within each generation. Brumm et al. (2017) provided existence for stochastic economies with stochastic productions by assuming that there are two atomless shocks that are stochastically independent (conditional on a possible third shock that can be arbitrary). This construction is provided first by Dugan (2012), who proved the existence of Markov Perfect Equilibrium in noisy stochastic games.

The literature showing the existence of recursive equilibria usually take the way of specifying the model set up such that although the agents do not make decisions recursively, the optimal strategies take a recursive form in equilibrium. We differ from the literature by reexamining the rational expectation assumption. Looking back at why we are interested in recursive equilibria in the first place, rational expectation assumption is actually inconsistent with the reasons: Philosophically, a rational agent would not put any restrictions on the form of the strategy and would consider the entire history to maximize utility; practically, the rational agent does not need an efficient algorithm to compute the equilibria or to approximate equilibria since she knows the analytical solution to the maximization problem. Simply put, when considering large complex economic systems, it is unrealistic to solve or compute rational expectation strategies. So it is natural and necessary to consider bounded rational behavior with recursive strategies that result in recursive equilibria.

There are two immediate applications of our model with the special case when ϵ is very small. First, it lays foundation for numerically computed rational expectation equilibria. With this set up, usual convergence criteria used in the computational economics literature can be incorporated and thus many computed equilibria in the literature using recursive methods can be categorized as bounded rational recursive equilibria in the sense of this paper. Second, it can be viewed as an epsilon-equilibrium existence result for models with heterogeneous heuristics. As we will show in more detail later, this paper brings up one more dimension of heterogeneity. The epsilon-equilibrium exists even if agents use different heuristics, e.g., some agents may use the first moment of the distribution of asset holdings to optimize as in Krusell and Smith (1998), while some agents may use sparse max in the sense of Gabiax (2014).

The rest of the paper is organized as follows: Section 2 describes the model set up. Section 3 defines the bounded rational recursive equilibrium. Section 4 proves existence of bounded rational equilibrium. Section 5 concludes the paper. Detailed proofs can be found in the appendix.

The Economic Model

In this section we describe the model set up. To illustrate how we model bounded rationality, the markets structure is adapted from the Lucas (1978) asset pricing model. We can get the same result for more complex market structure with bounded asset holdings, but we use the Lucas (1978) model to simplify notations. The first part of this section is the physical economy setup. The second part describes how a bounded rational agent maximizes utility given a belief of the distribution of the state space. The third part are some standard or weaker assumptions on the utility function.

The Physical Economy

Time is discrete and denoted by $t \in \mathbb{N}_0$. The exogenous shocks denoted by z realize from an Euclidean space $\mathbf{Z}$,[2] and follow a first-order Markov process with transition probability $\mathbb{P}(\cdot \mid z)$ defined on the Borel σ-algebra $\mathcal{Z}$ on $\mathbf{Z}$, $\mathbb{P} : \mathbf{Z} \times \mathcal{Z} \to [0,1]$. Let $(z_t)_{t=0}^{\infty}$, or in short (z_t), denote the stochastic process and let $(\mathcal{F}_t)$ denote its natural filtration. A history of shocks up to some date t is denoted by $z^t = (z_0, z_1, ..., z_t)$.

There are H types of agents, $h \in \mathbf{H} = \{1, ..., H\}$. There is one perishable commodity and J Lucas trees, $j \in \mathbf{J} = \{1, ..., J\}$, denote the prices normalized to a simplex by $p = (p_c, p_J) \in \mathbf{\Delta}^J$. The Lucas trees are long-lived assets in unit net supply that pay exogenous positive dividends in terms of a single consumption good $d : \mathbf{Z} \to \mathbb{R}_+^J$. The agent h's endowment is denoted by $e_h \colon \mathbf{Z} \to \mathbb{R}_+$. The dividends and endowments are time-invariant and measurable functions of the current shock. $\mathbf{X}, \mathbf{\Xi}$ are Euclidean spaces. The agent h's consumption for period t is denoted by $x_{h,t} \in \mathbf{X}_h$; her holding of financial assets is denoted by $\alpha_{h,t} \in \mathbf{\Xi}_h$. The budget set is

2. In this paper, we use the Euclidean space because it is sufficient for the later proofs and general enough to represent economic variables. It is not a necessary condition for our results.

denoted by $\mathbf{\Gamma}_h \colon \mathbf{Z} \times \mathbf{\Xi}_h \times \mathbf{\Delta}^J \rightrightarrows \mathbf{X}_h \times \mathbf{\Xi}_h$. The consumption vector of the economy is $x \in \mathbf{X}$, the asset holdings vector is $\alpha \in \mathbf{\Xi}$. There is no short selling on trees, and we restrict the asset holdings vector with financial markets clearing conditions, $\mathbf{\Xi} = (\mathbf{\Delta}^{H-1})^J$.

Denote the minimal state space by $\mathbf{S} = \mathbf{Z} \times \mathbf{\Xi}^-$ and associated Borel algebra by $\mathcal{S}$. Also, we specify an augmented state space that contains the exogenous shock, the beginning-of-period asset holdings, the end-of-period asset holdings, and the prices similar as Duffie et al. (1994). Denote the augmented state space by $\tilde{\mathbf{S}} = \mathbf{Z} \times (\mathbf{\Xi}^- \times \mathbf{\Xi} \times \mathbf{\Delta}^J)$ and associated Borel algebra by $\tilde{\mathcal{S}}$.

The characteristic transition of the economy is

$$F \colon \mathbf{S} \to \mathcal{P}(\mathbf{\Xi} \times \mathbf{\Delta}^J).$$

In the sense of Aliprantis and Border (2006) Chapter 19.2, we denote the associated transition kernel k_F.

Then the Markov transition of the augmented state space $\tilde{F} \colon \tilde{\mathbf{S}} \to \mathcal{P}(\tilde{\mathbf{S}})$ generated by $F, \mathbb{P}$ is defined as follows, for any $\tilde{s}_0 = [z_0, (\alpha_0^-, \alpha_0, p_0)]$ and letting $\mu = \tilde{F}(s_0)$, we have:

(i) the marginal on $\mathbf{Z}$ is denoted by $\mu_{\mathbf{Z}} = \mathbb{P}(\cdot \mid z_0)$, and the marginal on $\mathbf{\Xi}^-$ is $\mu_{\mathbf{\Xi}^-} = \delta_{\alpha_0}$ almost surely;

(ii) for any $z_0' \in \mathrm{supp}(\mathbb{P}(\cdot \mid z_0))$, the conditional probability $\mu(\cdot \mid (z_0', \alpha_0)) = F(z_0', \alpha_0)$.

Similarly, we denote the associated transition kernel by $k_{\tilde{F}}$.

Utility Maximization of Agents

Given the economy described above, the agent h believes that the economy is in stationary equilibrium and the stationary distribution is $\mu_h \in \mathcal{P}(\tilde{\mathbf{S}})$. Then she would maximize utility using the Bayes' rule with an information set $I_h \in \mathbf{I}_h$, where $\mathbf{I}_h$ is a finite dimensional Euclidean space. We first specify the structure of the information set, and then introduce the utility maximization problem.

Information structure

The information set of agent $I_h \in \mathbf{I}_h$ consists of two parts, the price $p \in \mathbf{\Delta}^J$ and the information of asset holdings of other agents $f_I^h \colon \mathbf{\Xi}_{-h} \times \mathbf{\Xi}_{-h} \to \mathbf{I}_{h,\alpha}$.

ASSUMPTION 9 *1.* $\mathbf{I}_{h,\alpha}$ *is compact and convex for all* $h \in \mathbf{H}$*;*

2. f_I^h *is continuous for all* $h \in \mathbf{H}$*.*

This set up is general and can incorporate many forms of information structures commonly used in the literature. We give three special cases that satisfy Assumption 9 as examples.

Special Case 1. $f_I^h(\alpha_{-h}^-, \alpha_{-h})$ is a subset of $(\alpha_{-h}^-, \alpha_{-h})$ for all $h \in \mathbf{H}$. The agents would have partial information of asset holdings of the economy. This setting is the same as the "sparse max" agents used by Gabaix (2014), when agents do not put any weight on the omitted information.

Special Case 2. $f_I^h(\alpha_{-h}^-, \alpha_{-h})$ is the first several moments of α_{-h}^-. The agents would use the match of moments to describe the distribution of asset holdings among agents. This setting is the same as the bounded rational agents used by Krusell and

Smith (1998). Although Krusell and Smith (1998) assumed a continuum of agents in the model, the computation part had to use a large number of agents.

Special Case 3. $f_I^h(\alpha_{-h}^-, \alpha_{-h}) = (\alpha_{-h}^-, \alpha_{-h})$ for all $h \in \mathbf{H}$. The agents would have full information of asset holdings of the economy. This setting also shows that Assumption 9 is harmless in the sense that it incorporate the full information setting. It will be shown that with this special case, many computed equilibira with recursive method in the literature with rational expectation assumption are in fact bounded rational recursive equilibria as described in the next section. One example is the computed equilibira by Kubler and Schmedders (2003).

Utility function given a belief of an equilibrium distribution

Given probability space $(\tilde{\mathbf{S}}, \tilde{\mathcal{S}}, \mu_h)$, each agent h chooses an arbitrary random variable $T_h \colon \tilde{\mathbf{S}} \to \tilde{\mathbf{S}}$.[3] Denote the regular conditional probability measure of μ_h on A associated with T_h as $\mu_h(\cdot \mid A)$, and denote the marginal of $\mu_h(\cdot \mid A)$ on b as $\mu_{h,b}(\cdot \mid A)$. Then the prediction of the distribution of the next period's information set $\mathbb{Q}_h[I_h' \mid (z', I_h)]$ can be written as follows:

$$\mathbb{Q}_h[I_h' \mid (z', I_h)] = \int_{\alpha \in \mathbf{\Xi}} \mu_I[I_h' \mid (\alpha, z')] \mu_\alpha[d\alpha \mid I_h].$$

With $\mathbb{P}$ and $\mathbb{Q}_h$, we get the overall predicted transition probability as

$$\mathbb{R}_h[(z', I_h') \mid (z, I_h)] = \mathbb{Q}_h[I_h' \mid (z', I_h)] \mathbb{P}(z' \mid z).$$

3. T_h is defined so that conditional probability can be well defined. In the construction of a recursive equilibrium later, T_h does not affect the equilibrium.

Easy to check that $\mathbb{R}_h[(z', I'_h) \mid (z, I_h)]$ is a transition probability. Denote $\mathbb{R}_h[(z_{t+1}, I_{h,t+1}) \mid (z_t, I_{h,t})]$ by $\mathbb{R}_h(t)$ for short, and define $\mu_h^t, t \geq 0$ in the same way as in 8.2 of Stokey (1989):

$$\mu_h^t(A_1 \times \cdots \times A_t \mid (z_0, I_{h,0})) = \int_{A_1} \cdots \int_{A_{t-1}} \int_{A_t} \mathbb{R}_h(t-1)\mathbb{R}_h(t-1) \cdots \mathbb{R}_h(0),$$

where $A_1, ..., A_t \subset \mathbf{Z} \times \mathbf{I}_h$.

With the single period utility function $u_h \colon \mathbf{Z} \times \mathbf{X}_h \to \mathbb{R}$, each agent maximizes a time-separable expected utility function. For a given sequence $(x_{h,t}(z_t, I_{h,t}), \alpha_{h,t}(z_t, I_{h,t}))_{t=0}^{\infty}$,

$$\begin{aligned} &\mathbb{E}_{\mu_h}\left[\sum_{t=0}^{\infty} \delta^t u_h(z_t, x_{h,t}(z_t, I_{h,t}))\right] \\ &\quad = u_h(z_0, x_{h,0}(I_{h,0})) + \sum_{t=1}^{\infty} \int_{(z,I_h)^t} \delta^t u(z_t, x_{h,t}(z_t, I_{h,t}))\mu^t(d(z, I_h)^t \mid (z_0, I_{h,0})) \end{aligned}$$

Each agent maximizes the total utility subject to the budget constraint.

$$\begin{aligned} U_h^{\mu_h}(z_0, \alpha_{h,0}^-, I_{h,0}) &= \max_{x_{h,t}, \alpha_{h,t}} \mathbb{E}_{\mu_h}\left[\sum_{t=0}^{\infty} \delta^t u_h(z_t, x_{h,t}(z_t, I_{h,t}))\right] \\ \text{s.t.} \quad & (x_{h,t}(z_t, I_{h,t}), \alpha_{h,t}(z_t, I_{h,t})) \in \mathbf{\Gamma}(z_t, \alpha_{h,t}^-, p_t) \end{aligned}$$

To compare this utility to the rational expectation utility, consider the special case 3 mentioned above, where the agents have full information about asset holdings in the economy. If the belief distribution μ_h is the "true" stationary equilibrium (if it exists), then the utility function coincides with rational expectation utility.

Assumptions

A few more standard assumptions are needed for later proofs.

Assumption 10 *$\mathbf{Z}$ is compact and convex. For all $z \in \mathbf{Z}$, $\mathbb{P}(\cdot \mid z)$ is absolutely continuous with respect to Lebesgue measure with density $p(\cdot \mid z)$.*

Assumption 11 *1. e_h, d are bounded measurable functions. Further, there are $\underline{e}, \bar{e}, \bar{d} \in \mathbb{R}_{++}$ such that for all z, h, j*

$$e_h(z) \geq \underline{e}, \sum_{h \in \mathbf{H}} e_h(z) \leq \bar{e}, 0 < d_j(z) < \bar{d}.$$

2. The agents' discount factor satisfies $\delta \in (0,1)$.

3. The utility functions $u_h \colon \mathbf{Z} \times \mathbf{X} \to \mathbb{R}$ are measurable in z, continuous, strictly increasing and concave in x.

The two assumptions above are standard in the literature, and we relaxed the assumption that the utility functions are differentiable. Another assumption is needed to keep the consumption and asset holding choices bounded.

Assumption 12 *The agent cannot consume more than the aggregate amount of the commodity and cannot hold more than one unit of any tree–that is, $(x_h, \alpha_h) \in \mathbf{B}$. Where*

$$\mathbf{B} := \left\{(x, \alpha) : 0 \leq x \leq \bar{e} + J\bar{d}; 0 \leq \alpha_j \leq 1, \forall j \in \mathbf{J}\right\}.$$

This assumption is needed because the bounded rational agent perceives the future differently from the true distribution. Combined with Assumption 11, the budget set is

$$\mathbf{\Gamma}(z, \alpha_h^-, p_t) = \left\{ p_c(x_h + d(z)\alpha_h^- - e_h(z)) + p_J(\alpha_h^- - \alpha_h) \leq 0 \right\} \cap \mathbf{B}.$$

Bounded Rational Recursive Equilibrium

With the model set up above, we go on to define the bounded rational recursive equilibrium in this section. First, we introduce the concept of an ϵ-stationary measure. And then we define a bounded rational competitive equilibrium given belief distributions. Last we define the bounded rational recursive equilibrium.

ϵ-Stationary Measure

Consider a state space $(\mathbf{A}, \mathcal{A})$, where $\mathbf{A}$ is a compact convex subset of a finite dimensional Euclidean space and $\mathcal{A}$ is the Borel algebra. Denote a Markov transition as a Borel measurable function by $P\colon \mathbf{A} \to \mathcal{P}(\mathbf{A})$. Denote the associated Markov transition kernel by $k_P\colon \mathbf{A} \times \mathcal{A} \to [0, 1]$.

For a measure $\mu \in \mathcal{P}(\mathbf{A})$, denote its transition under P by μ'_P. We have

$$\mu'_P(S) = \int_{\mathbf{A}} k_P(a, S) d\mu(a).$$

To characterize distance between measures, we equip the space $\mathcal{P}(\mathbf{A})$ with the following Kantorovich-Rubinshtein norm:

$$\|\mu\|_0 = \sup\left\{\int_{\mathbf{A}} f d\mu \middle| f \in \mathrm{Lip}_1(\mathbf{A}), \|f\|_\infty \leq 1\right\},$$

$$\text{where} \quad \mathrm{Lip}_1(\mathbf{A}) := \{f \colon \mathbf{S} \to \mathbb{R} | \mid f(x) - f(y) \mid \leq \mid x - y \mid, \forall x, y \in \mathbf{A}\}.$$

Denote the space of those test functions as $\mathbf{L}(\mathbf{A}) = \{f \in \mathrm{Lip}_1(\mathbf{A}) : \|f\|_\infty \leq 1\}$. By theorem 8.3.2 of Billingsley(2013), the topology generated by $\|\cdot\|_0$ coincides with the weak topology on $\mathcal{P}(\mathbf{A})$.

The agent is associated with an arbitrarily small parameter $\epsilon > 0$. The agent's rationality is bounded in the sense that she would consider two measures "close" enough if the Kantorovich-Rubinshtein distance is within ϵ. We call a measure ϵ-stationary with respect to P if it is "close" enough to its transition under P.

Definition 2 (ϵ-stationary measure) *Measure μ is said to be ϵ-stationary with respect to P if $\|\mu - \mu'_P\|_0 \leq \epsilon$.*

Definition of Recursive Equilibrium

μ_h-Competitive Equilibrium

Different from rational expectation models, the agents perceive the future differently. So the maximization problems need to be restated for each time period. To put it more formally, we define the bounded rational competitive equilibrium as follows:

a μ_h-*competitive equilibrium*, given initial conditions $z \in \mathbf{Z}, \alpha_h^- \in \mathbf{\Xi}$, consists of prices and choices,

$$(p, (x_h, \alpha_h)_{h \in \mathbf{H}})$$

such that markets clear and each agent h optimizes utility with respect to μ_h–that is to say, (A), (B), and (C) hold.

(A) Commodity market clearing:

$$\sum_{h \in \mathbf{H}} x_h = \sum_{h \in \mathbf{H}} e_h(z) + \sum_{j \in \mathbf{J}} d_j(z).$$

(B) Financial markets clearing:

$$\sum_{h \in \mathbf{H}} \alpha_h = \mathbf{1}.$$

(C) Utility maximization:

$$(x_{h,t}, \alpha_{h,t})_{t=0}^{\infty} \in \arg\max \mathbb{E}_{\mu_h} \left[\sum_{t=0}^{\infty} \delta^s u_h(z_t, x_{h,t}(z_t, I_{h,t})) \right], \forall h \in \mathbf{H},$$

where the budget set is

$$\mathbf{\Gamma}(z_t, \alpha_{h,t}^-, p_t) = \{p_{c,t}(x_{h,t} + d(z_t)\alpha_{h,t}^- - e_h(z_t)) + p_{J,t}(\alpha_{h,t}^- - \alpha_{h,t}) \leq 0\} \cap \mathbf{B},$$

and

$$(x_{h,0}, \alpha_{h,0}) = (x_h, \alpha_h), \qquad I_{h,0} = (p, f_h^I(\alpha_{-h}^-, \alpha_{-h})).$$

With strictly increasing utility functions, this equilibrium can be represented by $((\alpha_h)_{h\in\mathbf{H}}, p)$ using the budget constraint. Since there is only one commodity, the consumption is fixed once the asset holdings are known.

Bounded Rational Recursive Equilibrium

With the μ_h-competitive equilibrium defined, the bounded rational recursive equilibrium is essentially a μ_h-competitive equilibrium with ϵ_h-stationary measures μ_h as beliefs.

An *bounded rational recursive equilibrium* $(F, (\mu_h)_{h\in\mathbf{H}})$ consists of a characteristic transition

$$F\colon \mathbf{S} \to \mathcal{P}(\mathbf{\Xi} \times \mathbf{\Delta}^J)$$

and ϵ_h-stationary measures μ_h for all $h \in \mathbf{H}$,

$$\|\mu_h - \mu'_{h,\tilde{F}}\|_0 \leq \epsilon_h.$$

Such that for all initial conditions $s_0 \in \mathbf{S}$, $((\alpha_h)_{h\in\mathbf{H}}, p)$ is a μ_h-competitive equilibrium if $((\alpha_h)_{h\in\mathbf{H}}, p) \in \mathrm{supp}F(s_0)$.

Again, recall the special case 3, if $\epsilon_h = 0$, we will have a recursive equilibrium with rational agents.

Existence of Bounded Rational Recursive Equilibrium

In this section, we show that for arbitrary $\epsilon_h > 0, h \in \mathbf{H}$, a bounded rational recursive equilibrium exists. Since we are most concerned with the smallest ϵ_h among agents, we write ϵ in this section to simplify notations. One crucial part of the proof is to find continuous (or upper hemi-continuous and convex valued) policy functions (or correspondences) of the agents. With standard assumptions and rational expectation, continuous policy functions may not exist. But intuitively we can find continuous functions that are very "close" to the "true" policy functions. Although bounded rational agents are less "smart" than rational agents, the policy functions of bounded rational agents are more complex. This is due to the fact that the belief is a distribution over the state space. The policy of a bounded rational agent is a function of not only the natural state variables and the prices, but also the characteristic transition of the whole economy from which the ϵ-stationary belief is formed. This also distinguish our model from traditional epsilon-equilibrium models.

The essence of the proof is finding a fixed point of a characteristic transition. We construct a recursive equilibrium with continuous policy functions. To do so, the bounded rational agent would not only form an ϵ-stationary belief on the equilibrium path, but also knows how to form beliefs given any characteristic transitions. We call the function that maps a characteristic transition to an ϵ-stationary belief the perceivable model constructed by the bounded rational agent. With the beliefs, the agents optimize with the Bayes' rule, and the agents' strategies forms a new characteristic transition. Then the fixed point of the characteristic transition gives us a recursive equilibrium. This section has three parts: First, we construct a continuous perceivable model for any $\epsilon > 0$. Second, we construct a continuous policy function. Third, we show the main theorem of this paper.

Perceivable Model Constructed by Bounded Rational Agents

We first show that for any characteristic transition, there exists an ϵ-stationary distribution with a continuous density function. Then we define a characteristic perceivable model as a function that maps a characteristic transition to a ϵ-stationary continuous density function. Further, we show the existence of a continuous perceivable model: a continuous function that maps a characteristic transition with its weak topology to a continuous density function with the sup norm.

Continuous Density Functions

The first challenge of the proof is to find an ϵ-stationary belief with a continuous density function. Given an arbitrary characteristic transition, it is not trivial that there exists an ϵ-stationary belief. Further, given an ϵ-stationary belief, it does not necessarily have a density function. So we show in the first theorem that for any strictly positive ϵ, there exists an ϵ-stationary belief with a continuous density function.

Let $(\mathbf{N}(\mathbf{A}), \|\cdot\|_\infty)$ be the subset of $(\mathbf{C}_b(\mathbf{A}), \|\cdot\|_\infty)$ such that for any $f \in \mathbf{N}(\mathbf{A})$, we have $f \geq 0, \int_{\mathbf{A}} f d\lambda = 1$ so that the elements are all density functions. Denote the measure with density function f as μ_f. Then we define an ϵ-stationary density function for agent h as follows.

Definition 3 (ϵ-stationary continuous density) *A function f is an ϵ-stationary continuous density function with respect to P if $f \in \mathbf{N}(\mathbf{A})$ and $\|\mu_f - \mu'_{f,P}\|_0 \leq \epsilon$.*

THEOREM 3 *For any $\epsilon > 0$, the set of ϵ-stationary continuous density functions $\boldsymbol{\Upsilon}(\epsilon, P)$ is non-empty, convex, and closed.*

Continuous Perceivable Model

Now that we know there exists a continuous density function given a characteristic transition, we go on to find a way the bounded rational agent construct a continuous perceivable model. Intuitively, the agents would form similar beliefs given similar characteristic transitions.

Denote the space of the λ-equivalent class of Markov kernels $k_P\colon \mathbf{A}\times\mathcal{A} \to [0,1]$ as $\mathbf{K}_A$, and equip it with the weak topology $\tau(\lambda)$ defined as in 2.2 of Häusler, Erich, and Harald Luschgy (2015):

$$K_\alpha \to K \iff \int_\lambda \int_a f(\lambda)h(a)K_\alpha(\lambda, da)d\lambda \to \int_\lambda \int_a f(\lambda)h(a)K(\lambda, da)d\lambda$$

for every $f \in L^1$ and $h \in \mathbf{C}_b$.

Then by Theorem 2.7 of Häusler, Erich, and Harald Luschgy (2015), we have $\mathbf{K}_A$ with $\tau(\lambda)$ is a non-empty, weakly compact and convex subset of a locally convex Hausdorff space. We call a function mapping $\mathbf{K}_A$ into $\mathcal{P}(\mathbf{A})$ a perceivable model. The agent gets a computed distribution given any Markov transition. Then we define a continuous perceivable model as follows.

Definition 4 (continuous perceivable model) *A continuous perceivable model is a function* $M\colon \mathbf{K}_A \to \mathbf{N}(\mathbf{A})$ *that is continuous with respect to the* $\tau(\lambda)$ *weak topology and the* $\|\cdot\|_\infty$ *uniform norm.*

Theorem 4 *For any* $\epsilon > 0$*, there exists a continuous perceivable model* M *such that for all* $k_p \in \mathbf{K}_A$*,* $\|\mu_{M(k_p)} - \mu'_{M(k_p),P}\|_0 \le \epsilon$*.*

Corollary 1 *For any* $\epsilon > 0$*, there exists a continuous perceivable model* M *with a positive lower bound* $\underline{b} > 0$ *such that for all* $k_p \in \mathbf{K}_A$*,* $\|\mu_{M(k_p)} - \mu'_{M(k_p),P}\|_0 \le \epsilon$*, and* $\inf_{k_p} M(k_p) \ge \underline{b}$*.*

Denote the set of all continuous perceivable models satisfying conditions in Corollary 1 as $\mathbf{M}_\epsilon(\mathbf{A})$, and denote the space of continuous density functions bounded below by $\underline{b}$ as $\mathbf{N}_{\underline{b}}(\mathbf{A})$.

Existence of u.h.c. Policy Functions

In this section, we construct continuous policy functions for bounded rational agents. We first provide sufficient conditions under which the following functional equation is equivalent as the utility function,

$$V_h^{\mu_h}(z, \alpha_h^-, I_h) = \max_{\alpha_h, x_h} u_h(z, x_h) + \delta \int V_h^{\mu_h}(z', \alpha_h, I_h')\mathbb{Q}(d(I_h') \mid (z', I_h))\mathbb{P}(z' \mid z)$$

With the budget set

$$\alpha_h, x_h \in \mathbf{\Gamma}(z, \alpha_h^-, p)$$

And the corresponding policy correspondence attains the maximum utility level.

ASSUMPTION 13 $\mathbf{Z}, \mathbf{X}, \mathbf{\Xi}$ *are compact and convex.*

ASSUMPTION 14 $\mathbf{\Gamma}_h \colon \mathbf{Z} \times \mathbf{\Xi}_h \times \mathbf{\Delta}^J \rightrightarrows \mathbf{X}_h \times \mathbf{\Xi}_h$ *is weakly measurable, non empty, compact valued, and continuous in* $\mathbf{\Xi}_h \times \mathbf{\Delta}^J$ *for all* h.

ASSUMPTION 15 $u_h \colon \mathbf{Z} \times \mathbf{X} \to \mathbb{R}$ *is a bounded Caratheodry function for all* h*, and is strictly increasing, and concave in* x *.*

Define a continuous pricing density function as follows, and denote the Banach space of all those functions as $(\mathbf{Q}(\mathbf{I}_h \times (\mathbf{Z} \times \mathbf{I}_h)), \| \cdot \|_\infty)$.

DEFINITION 5 (CONTINUOUS PRICING DENSITY FUNCTION) $q \colon (\mathbf{I}_h \times (\mathbf{Z} \times \mathbf{I}_h)) \to \mathbb{R}$ *is a continuous pricing density function if*

1. q *is jointly continuous in all arguments;*

2. *For all* $s \in (\mathbf{Z} \times \mathbf{I}_h)$*,* $q(\cdot \mid s)$ *is a continuous probability density function (that is* $\int_{\mathbf{I}_h} q(i \mid s) di = 1, q(\cdot \mid s) \geq 0$*).*

ASSUMPTION 16 $\mathbb{Q}_h$ *is absolutely continuous with respect to Lebesgue measure with density function* q_{μ_h}*. And* $q_{\mu_h} \in \mathbf{Q}$*.*

THEOREM 5 *Under Assumption 13- 16, for all* $q_{\mu_h} \in \mathbf{Q}$*, there exists a unique bounded function* $V_h^{q_{\mu_h}} \colon \mathbf{Z} \times \mathbf{\Xi}_h \times \mathbf{I}_h \to \mathbb{R}$*, which satisfies* $V_h^{q_{\mu_h}}(z, \alpha_h^-, I_h) = U_h^{\mu_h}(z, \alpha_h^-, I_h)$*, is jointly continuous in* $(\alpha_h^-, I_h, q_{\mu_h})$*, strictly increasing, and concave in* α_h^-*, and is measurable in* z*. The policy correspondence* $G_h^{q_{\mu_h}}(z, \alpha_h^-, I_h)$ *is u.h.c., and convex valued in* (z, α_h^-, I_h)*. Furthermore,* $G_h^{q_{\mu_h}}(z, \alpha_h^-, I_h)$ *is measurable in* z *and admits a measurable selector.*

The proof has a similar structure as Theorem 9.2 in Stokey(1989) and is given in detail in the appendix.

The Main Theorem

In this section, we state the main steps of the fixed point argument and our main theorem.

Recall the compact set $\mathbf{A}$. Let $\mathbf{A} = \tilde{\mathbf{S}}$ for Corollary 1 and denote the corresponding lower bound as $\underline{b}'$. Denote $\mathbf{N}_{\underline{b}'}(\tilde{\mathbf{S}})$ as the space of continuous density functions bounded below by $\underline{b}'$. Then we can find a continuous mapping from a continuous density function to a continuous pricing density function.

LEMMA 9 *Under Assumption 10, for any $f \in \mathbf{N}_{\underline{b}'}(\tilde{\mathbf{S}})$ with associated μ_f, q_{μ_f} is continuous in all its arguments and uniformly continuous in f.*

With this Lemma and Theorem 5, we can prove the existence of a bounded rational competitive equilibrium.

LEMMA 10 *Under Assumption 9- 12, for any function $f_h \in \mathbf{N}_{\underline{b}'}(\tilde{\mathbf{S}})$ and initial conditions $z_t \in \mathbf{Z}, \alpha^-_{h,t} \in \Xi$, there exists a μ_{f_h}-competitive equilibrium, where μ_{f_h} is the measure with density function f_h.*

Denote the space of the λ-equivalent class of Markov kernels $k\colon \mathbf{S} \times \mathcal{S} \to [0,1]$ as $\mathbf{K}$, and the space of the λ-equivalent class of Markov kernels $k\colon \tilde{\mathbf{S}} \times \tilde{\mathcal{S}} \to [0,1]$ as $\mathbf{K}_{\tilde{F}}$. Equip both of them with the $\tau(\lambda)$ weak topology, and recall the definition of $k_{\tilde{F}}$, we have the following lemma.

LEMMA 11 *Under Assumption 10, $k_{\tilde{F}}$ is continuous in k_F under weak topologies.*

Theorem 6 *Under Assumption 9- 12, for arbitrarily small $\epsilon > 0$, bounded rational recursive equilibrium exists.*

By Corollary 1, we can choose a continuous perceivable model $M_h \in \mathbf{M}_\epsilon(\tilde{\mathbf{S}})$ for each agent h. Then by Lemma 11, $k_{\tilde{F}}$ is continuous in k_F. So we can find for each agent a continuous mapping $N_h \colon \mathbf{K} \to \mathbf{N}_{\underline{b}'}(\tilde{\mathbf{S}})$, where $\mathbf{K}$ is equipped with $\tau(\lambda)$ and $\mathbf{N}_{\underline{b}'}(\tilde{\mathbf{S}})$ with $\|\cdot\|_\infty$.

For any $k_F \in \mathbf{K}$, given $s \in \tilde{\mathbf{S}}$, choose $f_h = N_h(k_F)$. Denote the set of μ_{f_h}-competitive equilibria as $\mathbf{R}_F(s)$. Let $\mathbf{R}(k_F)$ be the set of (equivalence classes of) measurable selections of $\mathbf{R}_F$. For all $Y \in \mathbf{R}(k_F)$, we embed $Y(\tilde{\mathbf{S}})$ into $\mathcal{P}(\tilde{\mathbf{S}})$ via $Y(s) \mapsto \delta_{Y(s)}$, and denote δ_Y as the transition kernel generated by the embedding. Since Y is measurable by definition, we get $\delta_Y \in \mathbf{K}$. Denote the set $\mathbf{K}'(k_F) := \{\delta_Y : Y \in \mathbf{R}(k_F)\}$, and the closure of the convex hull of $\mathbf{K}'(k_F)$ as $\bar{\text{co}}\mathbf{K}'(k_F)$. Clearly, $\bar{\text{co}}\mathbf{K}'(k_F) \subset \mathbf{K}$. We show in the appendix that the map $\mathbf{K} \to \bar{\text{co}}\mathbf{K}'(k_F)$ has a fixed point and generate the bounded rational recursive equilibrium from the fixed point.

Conclusion

This paper provides a general way of modeling bounded rationality. Yet, many aspects of behavior economics literature cannot be incorporated, especially the findings about preferences such as loss aversion and hyperbolic discounting. we prove the existence of the recursive equilibrium in a dynamic stochastic model with bounded rational agents. This set up is realistic when the economy is large and complex, which is a fact for a lot of markets. A large and complex economy would affect the agents in two ways: first, each agent would rely on recursive methods to optimize utility;

second, each agent cannot solve for the stationary equilibrium analytically and has to approximate an equilibrium distribution.

The entire part of constructing a continuous value function does not put extra assumptions on the model, it provides one additional interpretation of what the rationality is bounded by – that is, the agent's choices are continuous in the information she observes. This is consistent with the philosophical principles stated in Maskin and Tirole (2001). Also, the continuous value function is analogous to the continuity assumption of the utility function in the static exchange economy models.

Although relatively abstract, this model has the potential to explore dynamic economic models further in multiple ways. First, it may provide foundation for many algorithms computing general equilibrium models. As we stated before, examples like the computationally part of Krusell and Smith (1998) and Kubler and Schmedders (2003) can be incorporated into my set up. Second, it may generate interesting economic processes from standard set up. As we can see from the proof, although the equilibrium is recursive, the Markov process of the whole economy may not be ergodic. This may provide a new angle to study business cycles and financial crisis from a standard model set up. Third, it provides a new dimension of heterogeneity among agents, i.e. they may use different heuristics. And last, it contributes to finding a general theory of bounded rationality.

Proofs of Section 4.1

Proof of Theorem 3

By the way it is defined, $(\mathbf{N}(\mathbf{A}), \|\cdot\|_\infty)$ is a convex, Hausdorff locally convex topological vector space. The very first task is to find a compact subset of $\mathbf{N}(\mathbf{A})$. Consider the space of Hölder continuous functions $\mathbf{C}^\alpha(\mathbf{A})$, $\alpha \in (0,1)$ with the norm

$$\|f\|_\alpha = \|f\|_\infty + \mid f \mid_\alpha, \text{ where } \mid f \mid_\alpha = \sup_{x \neq y} \frac{\mid f(x) - f(y) \mid}{\mid x - y \mid^\alpha}.$$

Denote the subset of $\mathbf{C}^\alpha(\mathbf{A})$ uniformly bounded in $\|\cdot\|_\alpha$ by M as $\mathbf{C}^\alpha_M(\mathbf{A})$, it is compact. Further we have $\mathbf{N}_M(\mathbf{A}) := \mathbf{N}(\mathbf{A}) \cap \mathbf{C}^\alpha_M(\mathbf{A})$ is compact for any $M \in (0, +\infty)$.

For any Markov transition P, denote the associated transition operator by $\mathbf{P}$ and its adjoint operator by $\mathbf{P}'$, in the sense of Aliprantis and Border (2006) Chapter 19.2. We have

$$\mu'_P = \mathbf{P}'\mu.$$

Define a correspondence G_M as:

$$G_M(f) \rightrightarrows \{f' \in \mathbf{N}_M(\mathbf{A}) : \|\mu_{f'} - \mathbf{P}'\mu_f\|_0 \leq \epsilon\}, \forall f \in \mathbf{N}(\mathbf{A}).$$

To prove the theorem, it is equivalent to show that for a large enough M, there exists a fixed point of G_M. The proof is structured as follows.

First we show that for large enough M, G_M is non-empty, compact, convex valued. Equip $\mathcal{P}(\mathbf{A})$ with the weak* topology $\sigma(\mathcal{P}, \mathbf{C}_b)$. Since $\mathbf{A}$ is compact, this coincide with the vague topology $\sigma(\mathcal{P}, \mathbf{C}_c)$. And let $L^\infty(\mathbf{A}, \mathcal{A}, \lambda)$ be the space of essentially bounded measurable (equivalent classes of) functions equipped with the weak* topology $\sigma(L^\infty, L^1)$. By Theorem 19.9(2) of Aliprantis and Border (2006), for all $f \in \mathbf{N}(\mathbf{A})$, $\mathbf{P}'\mu_f \in \mathcal{P}(\mathbf{A})$. The following lemma provides a function $\hat{f} \in \mathbf{N}_M(\mathbf{A})$ as an approximate density function for $\mathbf{P}'\mu_f$.

LEMMA 12 *Given $\epsilon > 0$, there exists an $N < +\infty$ such that $\forall M \geq N, \forall f \in \mathbf{N}(\mathbf{A})$, $G_M(f)$ is non-empty valued.*

Proof: Consider $s \in \mathbf{A}$ with the embedding $\delta_s \in \mathcal{P}(\mathbf{A})$, we can construct a sequence of nascent delta functions $\eta_N(\cdot \mid s) \in \mathbf{N}_N(\mathbf{A})$ such that $\lim_{N \to +\infty} \eta_N(\cdot \mid s) \to \delta_s$ in the sense of vague topology (which coincides with the weak*-topology on $\mathbf{A}$). As an example, consider the following mollifier:

$$\eta_N(x \mid s) = N^{\dim(\mathbf{A})} \varphi(Nx \mid s),$$

where

$$\varphi(x \mid s) = \begin{cases} e^{-1/(1-|x-s|^2)}/Const & \text{if } \mid x - s \mid < 1 \\ 0 & \text{if } \mid x - s \mid \geq 1 \end{cases}.$$

Given the norm $\|\cdot\|_0$ generates weak*-topology, there exists $N < +\infty$, such that

$$\sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} \eta_N(u \mid s) f_L(u) d\lambda(u) - f_L(s) \right| \leq \epsilon.$$

Then by the symmetry of set $\mathbf{A}$, we can easily construct a set of functions $\hat{\eta}_s \in \mathbf{N}_N(\mathbf{A})$ such that for all $s \in \mathbf{A}$,

$$\sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(u)\hat{\eta}_s(u)d\lambda(u) - f_L(s) \right| \leq \epsilon.$$

This means we found an approximate density functions for all $\delta_s, s \in \mathbf{A}$, namely all extreme points in $\mathcal{P}(\mathbf{A})$. By theorem 15.10 of Aliprantis and Border (2006), $\mathcal{P}(\mathbf{A}) \subset \bar{\text{co}}\mathbf{A}$. Take the closure of convex hull of $\hat{\eta}_s$ accordingly. Since the convex hull of $\hat{\eta}_s$ is uniformly bounded by N in the $\|\cdot\|_\alpha$, the limit points also lie in $\mathbf{N}_N(\mathbf{A})$. Then we can find $\hat{f} \in \mathbf{N}_N(\mathbf{A})$ for all $\mu \in \mathcal{P}(\mathbf{A})$ such that

$$\sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s)\hat{f}(s)d\lambda(s) - \int_{\mathbf{A}} f_L(s)d\mu(s) \right| \leq \epsilon.$$

Combined with the fact that $\mathbf{P}'\mu_f \in \mathcal{P}(\mathbf{A})$, we have $G_M(f)$ is not empty. □

LEMMA 13 *For any $f \in \mathbf{N}(\mathbf{A})$, $G_M(f)$ is convex and close valued.*

Proof: For any $f_1', f_2' \in G_M(f)$, consider the convex combination $f' = \beta f_1' + (1-\beta)f_2', \beta \in (0,1)$,

$$\begin{aligned}
\|\mu_{f'} - \mathbf{P}'\mu_f\|_0 &= \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'(s) d\lambda(s) - \int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s) \right| \\
&= \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s)[\beta f_1'(s) + (1-\beta) f_2'(s)] d\lambda(s) - \int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s) \right| \\
&= \sup \left| \beta[\int_{\mathbf{A}} f_L(s) f_1'(s) - \int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s)] + (1-\beta)[f_2'(s) d\lambda(s) - \int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s)] \right| \\
&\leq \beta \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f_1'(s) - \int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s) \right| \\
&\quad + (1-\beta) \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| f_2'(s) d\lambda(s) - \int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s) \right| \\
&\leq \beta\epsilon + (1-\beta)\epsilon \\
&= \epsilon.
\end{aligned}$$

So $f' \in G_M(f)$, it is convex. Next, consider $f_n' \xrightarrow{unif.} f', f_n' \in G_M(f)$. Notice first $f' \in \mathbf{N}(\mathbf{A})$ under uniform convergence. Using prove by contradiction, assume that $\|\mu_{f'} - \mathbf{P}'\mu_f\|_0 = \epsilon' > \epsilon$,

$$\begin{aligned}
\|\mu_{f'} - \mathbf{P}'\mu_f\|_0 &= \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'(s) d\lambda(s) - \int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s) \right| \\
&\leq \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'(s) d\lambda(s) - \int_{\mathbf{A}} f_L(s) f_n'(s) d\lambda(s) \right| \\
&\quad + \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f_n'(s) d\lambda(s) - \int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s) \right| \\
&\leq \int_{\mathbf{A}} |f'(s) - f_n'(s)| \, d\lambda(s) + \epsilon.
\end{aligned}$$

This means $\int_{\mathbf{A}} |f'(s) - f_n'(s)|\, d\lambda(s) \geq \epsilon' - \epsilon > 0$ for all n, which contradicts uniform convergence of f_n'. So $G_M(\cdot)$ is close valued. $\Box$

Now we have G_M is non-empty, compact, convex valued. Then we show that it is also *u.h.c.*, and then the Kakutani-Glicksberg-Fan theorem applies.

Lemma 14 *$G_M(\cdot)$ is u.h.c..*

Proof: Now we characterize the integral with $\mathbf{P}$.

$\int_{\mathbf{A}} f_L(s) d\mathbf{P}'\mu_f(s) = \int_{\mathbf{A}} (\mathbf{P}f_L)(s) d\mu_f(s)$ by the way $\mathbf{P}, \mathbf{P}'$ are defined. By Theorem 19.7(4) of Aliprantis and Border (2006), $\mathbf{P}f_L \in \mathbf{B}_b(\mathbf{A})$. It is also easy to check that for any $f_L \in \mathbf{L}(\mathbf{A})$, $\mathbf{P}f_L \in \mathbf{B}_b(\mathbf{A})$ is bounded by 1. So for all $f \in \mathbf{N}(\mathbf{A}), f' \in \mathbf{N}_M(\mathbf{A})$,

$$\begin{aligned}
\|\mu_{f'} - \mathbf{P}'\mu_f\|_0 &= \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'(s) d\lambda(s) - \int_{\mathbf{A}} (\mathbf{P}f_L)(s) d\mu_f(s) \right| \\
&= \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'(s) d\lambda(s) - \int_{\mathbf{A}} (\mathbf{P}f_L)(s) f(s) d\lambda(s) \right|.
\end{aligned}$$

Now consider a sequence of functions $f_n \xrightarrow{unif.} f$, and let $f_n' \xrightarrow{unif.} f'$ where $f_n' \in G_M(f_n)$, we go on to show that $f' \in G_M(f)$.

$$
\begin{aligned}
&\sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'(s) d\lambda(s) - \int_{\mathbf{A}} (\mathbf{P} f_L)(s) f(s) d\lambda(s) \right| \\
&\quad \leq \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'_n(s) d\lambda(s) - \int_{\mathbf{A}} f_L(s) f'(s) d\lambda(s) \right| \\
&\qquad + \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'_n(s) d\lambda(s) - \int_{\mathbf{A}} (\mathbf{P} f_L)(s) f_n(s) d\lambda(s) \right| \\
&\qquad + \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} (\mathbf{P} f_L)(s) f_n(s) d\lambda(s) - \int_{\mathbf{A}} (\mathbf{P} f_L)(s) f(s) d\lambda(s) \right| \\
&\quad \leq \sup_{f_L \in \mathbf{L}(\mathbf{A})} \left| \int_{\mathbf{A}} f_L(s) f'_n(s) d\lambda(s) - \int_{\mathbf{A}} (\mathbf{P} f_L)(s) f_n(s) d\lambda(s) \right| \\
&\qquad + \int_{\mathbf{A}} |f_n - f| \, d\lambda + \int_{\mathbf{A}} |f'_n - f'| \, d\lambda \\
&\quad \leq \epsilon + \int_{\mathbf{A}} |f_n - f| \, d\lambda + \int_{\mathbf{A}} |f'_n - f'| \, d\lambda \\
&\quad \longrightarrow \epsilon.
\end{aligned}
$$

So we have $f' \in G_M(f)$. $\square$

Now choose $M > N$ as in Lemma A.1. By the Kakutani-Glicksberg-Fan theorem, the set of fixed points of $G_M \colon \mathbf{N}_M(\mathbf{A}) \rightrightarrows \mathbf{N}_M(\mathbf{A})$ is non-empty. So the set of computationally invariant density functions $\boldsymbol{\Upsilon}(\epsilon, P)$ is non-empty. The final part of the proof is to show that it is also convex and close valued. It is analogous to Lemma 5 and concludes the proof of the theorem.

Proof of Theorem 4

To simplify notations, redefine $\boldsymbol{\Upsilon}(\epsilon, P)$ in a slightly different way: Define a correspondence H as:

$$H(k_P) \rightrightarrows \{f \in \mathbf{N}(\mathbf{A})\colon \|\mu_f - \mathbf{P}'\mu_f\|_0 \le \epsilon\}, \forall k_P \in \mathbf{K}_A.$$

LEMMA 15 *H is l.h.c..*

Proof: By definition, we need to show $\forall k_{P_m} \rightharpoonup k_P, \forall f \in H(k_P)$, there exists a subsequence of $k_{P_{m_k}}$ such that $\exists f_k \in H(k_{P_{m_k}}), f_k \to f$.

First, $k_{P_m} \rightharpoonup k_P$, for all $f_c \in \mathbf{C}_b(\mathbf{A})$,

$$\int_{\mathbf{A}}\int_{\mathbf{A}} f_c(u)k_{P_m}(s,du)f(s)ds \to \int_{\mathbf{A}}\int_{\mathbf{A}} f_c(u)k_P(s,du)f(s)ds.$$

By the definitions of $k_P, \mathbf{P}'$, this is equivalent to

$$\int_{\mathbf{A}} f_c(u)d\mathbf{P}'_m\mu_f(u) \to \int_{\mathbf{A}} f_c(u)d\mathbf{P}'\mu_f(u).$$

Then further we have,

$$\|\mu_f - \mathbf{P}'_m\mu_f\|_0 \to \|\mu_f - \mathbf{P}'\mu_f\|_0.$$

So if $f \in H(k_P)$ and $\|\mu_f - \mathbf{P}'\mu_f\|_0 < \epsilon$, there exists m_1 such that for $m \ge m_1$, $\|\mu_f - \mathbf{P}'_m\mu_f\|_0 < \epsilon$. Choose $m_k = m_1 + k$ and $f_k = f$, we get $f_k \in H(k_{P_{m_k}}), f_k \xrightarrow{unif.} f$.

Otherwise, if $f \in H(k_P)$ and $\|\mu_f - \mathbf{P}'\mu_f\|_0 = \epsilon$. By theorem 3, there is also $f' \in H(k_P)$ such that $\|v_3 - v_3\Pi\| \le \epsilon/2 < \epsilon$ (By choosing $\epsilon/2$ as the lower bound). By the convexity of $\boldsymbol{\Upsilon}(\epsilon, P)$, there exists a sequence $f_k \xrightarrow{unif.} f$ such that $\|\mu_{f_k} - \mathbf{P}'\mu_{f_k}\|_0 < \epsilon$. So by similar argument as the first scenario, there exists $m_{(k)}$ such that for $m \ge m_{(k)}$,

$\|\mu_{f_k} - \mathbf{P'}_m \mu_{f_k}\|_0 < \epsilon$. Then let $m_1 = m_{(1)}$ and $m_k = \max\{m_{(k)}, m_{k-1}+1\}, k \geq 2$. And we have $f_k \in H(k_{P_{m_k}}), f_k \xrightarrow{unif.} f$. $\square$

Given in Theorem 3 that the correspondence is convex, close valued, by Michael selection theorem, there exists a continuous selection.

Proof of Corollary 1

The proof is analogous to Lemma 6 and can be provided upon request.

Proofs of Section 4.2 (Theorem 5)

First, we define an operator $M_{q_\mu} f$ that projects bounded, Caratheodry functions to bounded, Caratheodry functions for all bounded, Caratheodry pricing density $q_\mu \in \mathbf{Q}$. And $M_{q_\mu} f$ is jointly continuous in $((\alpha_h, I_h), q_\mu)$. $M_{q_\mu} f$ is defined by

$$(M_{q_\mu} f)(z, \alpha_h, I_h) = \int_{z' \in \mathbf{Z}} \int_{I'_h} f(z', \alpha_h, I'_h) q_\mu(I'_h \mid (z', I_h)) d(I'_h) p(z' \mid z) dz.$$

Given $z' \in \mathbf{Z}$ and q_μ is continuous with compact support. So q_μ is bounded. Then by Theorem 19.7 of Aliprantis and Border (2006), $M_{q_\mu} f$ is measurable in z. Now fix z

and consider a sequence $\{(\alpha_h^m, I_h^m, q_\mu^m)\}$ converging to (α_h, I_h, q_μ). Then we have

$$
\begin{aligned}
&\left|M_{q_\mu^m} f(z, \alpha_h^m, I_h^m) - M_{q_\mu} f(z, \alpha_h, I_h)\right| \\
&\quad \leq \left|M_{q_\mu^m} f(z, \alpha_h^m, I_h^m) - M_{q_\mu^m} f(z, \alpha_h, I_h)\right| + \left|M_{q_\mu^m} f(z, \alpha_h, I_h) - M_{q_\mu} f(z, \alpha_h, I_h)\right| \\
&\quad = \left|\int\int [f(z', \alpha_h^m, I_h')q_\mu(I_h' \mid (z', I_h^m)) - f(z', \alpha_h, I_h')q_\mu(I_h' \mid (z', I_h))]d(I_h')p(z' \mid z)dz\right| \\
&\qquad + \left|\int\int [q_\mu^m(I_h' \mid (z', I_h)) - q_\mu(I_h' \mid (z', I_h))]f(z', \alpha_h I_h')d(I_h')p(z' \mid z)dz\right| \\
&\quad \xrightarrow{m\to\infty} 0.
\end{aligned}
$$

The first term converges to 0 because f and q_μ are continuous in (α_h, I_h); the second term converges because of the uniform convergence of q_μ^m. Then we define another operator $T_{q_\mu} f$ as

$$
\begin{aligned}
(T_{q_\mu} f)(z, \alpha_h^-, I_h)) &= \max_{\alpha_h, x_h} u_h(z, x_h) + \delta(M_{q_\mu} f)(z, \alpha_h, I_h) \\
\alpha_h, x_h &\in \mathbf{\Gamma}(z, \alpha_h^-, p)
\end{aligned}
$$

For given (α_h^-, I_h, q_μ), by Theorem 18.19 in Aliprantis and Border (2006), $T_{q_\mu} f$ is measurable in z. Fix z and consider a sequence $\{(\alpha_h^{-m}, I_h^m, q_\mu^m)\}$ converging to (α_h^-, I_h, q_μ). Then we have

$$
\begin{aligned}
&\left|T_{q_\mu^m} f(z, \alpha_h^{-m}, I_h^m) - T_{q_\mu} f(z, \alpha_h^-, I_h)\right| \\
&\quad \leq \left|T_{q_\mu^m} f(z, \alpha_h^{-m}, I_h^m) - T_{q_\mu^m} f(z, \alpha_h^-, I_h)\right| + \left|T_{q_\mu^m} f(z, \alpha_h^-, I_h) - T_{q_\mu} f(z, \alpha_h^-, I_h)\right| \\
&\quad \longrightarrow 0.
\end{aligned}
$$

The first term converges to 0 because $T_{q_\mu^m} f(z,\cdot)$ is continuous according to the maximum theorem. For the second term, since f is bounded, for a given z, $u_h(z,x_h) + \delta(M_{q_\mu} f)(z,\alpha_h,I_h)$ is uniformly continuous in q_μ. So the maximum over the same compact budget set also converges to 0.

Then by induction with the same argument as above, we have $T^n_{q_\mu} f$ is measurable in z and jointly continuous in (α_h^-, I_h, q_μ) for all $n \in \mathbb{N}_{++}$

Easy to check that Blackwell's sufficient conditions are satisfied for T_{q_μ} for all $q_\mu \in \mathbf{Q}$. Then take any Caratheodry function f_0, and we have the value function $V^{q_\mu} = T^\infty_{q_\mu} f_0$. Obviously, V^{q_μ} is measurable in z and jointly continuous in α_h^-, I_h, q_μ.

More specifically, choose a Caratheodry function $\hat{f}_0$ that is strictly increasing, and concave in α_h^-. Analogously, we get $V^{q_\mu} = T^\infty_{q_\mu} \hat{f}_0$ is measurable in z and jointly continuous in α_h^-, I_h, q_μ and strictly increasing, and concave in α_h^-. The equivalence of the functional equation and the utility function is directly implied. To describe the policy correspondence, notice

$$V^{q_\mu}(z,\alpha_h^-,I_h) = \max_{\alpha_h,x_h} u_h(z,x_h) + \delta \int_{z'\in\mathbf{Z}} \int_{I_h'} V^{q_\mu}(z',\alpha_h,I_h') q_\mu(I_h' \mid (z',I_h)) d(I_h') p(z' \mid z) dz$$

$$\alpha_h, x_h \in \mathbf{\Gamma}(z,\alpha_h^-,I_h).$$

Fix z, by maximum theorem, we have $G_h^{q_{\mu_h}}(z,\alpha_h^-,I_h)$ is u.h.c., and convex valued in $(\alpha_h^-, I_h, q_{\mu_h})$. And fix $(\alpha_h^-, I_h, q_{\mu_h})$, by Theorem 18.19 of Aliprantis and Border (2006), $G_h^{q_{\mu_h}}(z,\alpha_h^-,I_h)$ is measurable in z and admits a measurable selector. □

Proofs in Section 4.3

Proof of Lemma 9

As described in Section 3, we can write q_{μ_f} as follows, when f is a continuous density function:

$$q_{\mu_f}(I_h' \mid (z', I_h)) = \int_\alpha f_I(I_h' \mid (z', \alpha)) f_\alpha(\alpha \mid I_h) d\lambda(\alpha), \tag{2.1}$$

f_I and f_α are defined as follows:

$$f_\alpha[\alpha \mid I_h] = \frac{\int_{s \in S_{\alpha, I_h}} f(s) ds}{\int_{s \in S_{I_h}} f(s) ds}$$

where $S_{\alpha,I_h} = \{s : z(s) = z, I_h(s) = I_h, \alpha(s) = \alpha\}$ and $S_{I_h} = \{s : z(s) = z, I_h(s) = I_h\}$; Similarly, we have

$$f_I[I_h' \mid (z', \alpha)] = \frac{\int_{s \in S_{I_h', (z', \alpha)}} f(s) ds}{\int_{s \in S_{(z', \alpha)}} f(s) ds}$$

where $S_{I_h',(z',\alpha)} = \{s : z(s) = z', \alpha^-(s) = \alpha, I_h(s) = I_h'\}$ and $S_{(z',\alpha)} = \{s : z(s) = z', \alpha^-(s) = \alpha\}$.

Easy to check, the four integrals in the definitions of f_I, f_α are bounded linear transformations of f. Hence, they are all uniformly continuous in f. Further, if $f \in \mathbf{N}_{J\underline{b}'}(\mathbf{S})$, all these four integrals are bounded away from 0. Then we have f_I, f_α are uniformly continuous in f and hence $f_I \cdot f_\alpha$ is uniformly continuous in f. Next, notice the integration in (1) is also a bounded linear operator. So we have q_{μ_f} is uniformly continuous in f. And it is trivial to check that q_{μ_f} is jointly continuous in all its arguments. □

Proof of Lemma 11

To show that $k_{\tilde{F}}$ is continuous in k_F, consider a sequence $k_{F_n} \rightharpoonup k_F$ in $\tau(\lambda)$, so we have for all $f \in L^1(\mathbf{S}), g \in \mathbf{C}_b(\boldsymbol{\Xi} \times \boldsymbol{\Delta}^J)$,

$$\int_{\mathbf{S}} \int_{\boldsymbol{\Xi} \times \boldsymbol{\Delta}^J} f(s)g(a)dF_{n,s}(a)d\lambda(s) \to \int_{\mathbf{S}} \int_{\boldsymbol{\Xi} \times \boldsymbol{\Delta}^J} f(s)g(a)dF_s(a)d\lambda(s). \tag{2.2}$$

By Theorem 2.6 of Häusler, Erich, and Harald Luschgy (2015), this is equivalent to for all measurable, bounded function $h\colon (\mathbf{S}, \boldsymbol{\Xi} \times \boldsymbol{\Delta}^J) \to \mathbb{R}$ such that $h(s, \cdot) \in \mathbf{C}_b$,

$$\int_{\mathbf{S}} \int_{\boldsymbol{\Xi} \times \boldsymbol{\Delta}^J} h(s,a)dF_{n,s}(a)d\lambda(s) \to \int_{\mathbf{S}} \int_{\boldsymbol{\Xi} \times \boldsymbol{\Delta}^J} h(s,a)dF_s(a)d\lambda(s). \tag{2.3}$$

We go on to show that for all $f' \in L^1(\tilde{\mathbf{S}}), g' \in \mathbf{C}_b(\tilde{\mathbf{S}})$

$$\int_{\tilde{\mathbf{S}}} \int_{\tilde{\mathbf{S}}} f'(s)g'(a)d\tilde{F}_{n,s}(a)d\lambda(s) \to \int_{\tilde{\mathbf{S}}} \int_{\tilde{\mathbf{S}}} f'(s)g'(a)d\tilde{F}_s(a)d\lambda(s)$$

where

$$\int_{\tilde{\mathbf{S}}} \int_{\tilde{\mathbf{S}}} f'(s)g'(a)d\tilde{F}_{n,s}(a)d\lambda(s) = \int_{\tilde{\mathbf{S}}} \int_{\mathbf{Z}} \int_{\boldsymbol{\Xi} \times \boldsymbol{\Delta}^J} f'(s)g'(a)d\mathbb{P}(z' \mid z)dF_{n,(z',\alpha)}(a)d\lambda(s).$$

First, for all z, α, letting $h := p(z' \mid z)g'(z', \alpha, \alpha', p')$, from (2) we have

$$\int_{\mathbf{Z}} \int_{\mathbf{\Xi} \times \mathbf{\Delta}^J} p(z' \mid z) g'(z', \alpha, \alpha', p') dF_{n,(z',\alpha)}(\alpha', p') d\lambda(z')$$
$$\longrightarrow \int_{\mathbf{Z}} \int_{\mathbf{\Xi} \times \mathbf{\Delta}^J} p(z' \mid z) g'(z', \alpha, \alpha', p') dF_{(z',\alpha)}(\alpha', p') d\lambda(z').$$

Since $g' \in \mathbf{C}_b$, g' is bounded. Denote the bound as B, we have

$$\int_{\mathbf{Z}} \int_{\mathbf{\Xi} \times \mathbf{\Delta}^J} p(z' \mid z) g'(z', \alpha, \alpha', p') dF_{n,(z',\alpha)}(\alpha', p') d\lambda(z') \leq \int_{\mathbf{Z}} p(z' \mid z) B d\lambda(z') = B.$$

So by dominated convergence theorem, we have

$$\int_{\tilde{\mathbf{S}}} \int_{\tilde{\mathbf{S}}} f'(s) g'(a) d\tilde{F}_{n,s}(a) d\lambda(s) \to \int_{\tilde{\mathbf{S}}} \int_{\tilde{\mathbf{S}}} f'(s) g'(a) d\tilde{F}_s(a) d\lambda(s).$$

□

Proof of Lemma 10

Given $f \in \mathbf{N}_{\underline{b}'}(\tilde{\mathbf{S}})$, by Lemma 3.2, we have $q_{\mu_f} \in \mathbf{Q}$. Combined with Assumptions 4.1-4.3, all the assumptions in section 3 are satisfied. So the equilibrium conditions are equivalent to the following conditions:

(A) Commodity market clearing:

$$\sum_{h \in \mathbf{H}} x_{h,t} = \sum_{h \in \mathbf{H}} e_h(z_t) + \sum_{j \in \mathbf{J}} d_j(z_t)$$

(B) Financial market clearing:

$$\sum_{h \in \mathbf{H}} \alpha_{h,t} = \mathbf{1}$$

(C') Utility maximization:

$$x_{h,t}, \alpha_{h,t} \in G_h^{q_{\mu_h}}(z_t, \alpha^-_{h,t}, I_{h,t})$$

Define a price player's best response $\psi\colon \mathbf{X} \times \mathbf{\Xi} \rightrightarrows \mathbf{\Delta}^J$ by

$$\psi((x_{h,t}, \alpha_{h,t})_{h \in \mathbf{H}}) = \arg\max_{p \in \mathbf{\Delta}^J} \{p_{c,s} \sum_h (x_{h,s} + d(z_s)\alpha^-_{h,s} - e_h(z_s)) + p_{J,s}(\sum_h \alpha_{h,s} - 1)\}$$

Then the standard existence proof of a static exchange economy applies. See, for example, Lemma 3 of Brumm et al. (2017). □

Proof of Theorem 6

First, notice that $s \rightrightarrows \mathbf{R}_F(s)$ is measurable. This is shown in the proof of Lemma 4 of Brumm et al.(2017). Then by theorem 18.13 of Aliprantis and Border(2006), $\mathbf{R}_F$ has a measurable selector. So the map $k_F \rightrightarrows \mathbf{R}(k_F)$ is non-empty valued. Then in the same way as the second part of the proof of Lemma 4 of Brumm et al.(2017), we can get $k_F \rightrightarrows \mathbf{R}(k_F)$ has a closed graph. Then by the definition of $\mathbf{K}'$ as a subset of $\mathbf{K}$ equipped with $\tau(\lambda)$, we have $k_F \rightrightarrows \mathbf{K}'(k_F)$ has a closed graph. Then theorem 17.35(1) in Aliprantis and Border(2006) implies that $k_F \rightrightarrows \bar{\text{co}}\mathbf{K}'(k_F)$

has a closed graph. Apply Kakutani-Glicksberg-Fan theorem, we have a fixed point $k_F^* \in \bar{\text{co}}\mathbf{K}'(k_F^*)$.

If $k_F^* \in \text{co}\mathbf{K}'(k_F^*)$, choose $f_h^* = N_h(k_F^*)$. Then $(F^*, (\mu_{f_h^*})_{h\in\mathbf{H}})$ is a ϵ-bounded rational equilibrium.

If $k_F^* \notin \text{co}\mathbf{K}'(k_F^*)$, we make use of the fact that ϵ is arbitrarily small, and repeat the procedure above for $\epsilon' = \epsilon/2$. And we find another fixed point $k_F' \in \bar{\text{co}}\mathbf{K}'(k_F')$. If $k_F' \notin \text{co}\mathbf{K}'(k_F')$, then we can choose $k_F'' \in \text{co}\mathbf{K}'(k_F')$ within an arbitrary ε-neighbourhood of k_F', and $f_h' = N_h(k_F')$ such that $(F'', (\mu_{f_h'})_{h\in\mathbf{H}})$ is a ϵ-bounded rational equilibrium. □

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Chapter 3

Self-justified momentum and eleven puzzles in macro-finance

This article provides a simple dynamic general equilibrium model that can generate short-term momentum and long-term reversal effect of excess stock returns with incomplete markets due to collateral constraints. The model also helps to understand quantitatively some of the puzzling empirical regularities in macro-finance stated by Campbell (2003) and Gabiax (2012). We assume there are two types of bounded rational agents: the fundamentalists and the speculators. The fundamentalists believes that future asset prices are determined by the exogenous shocks and dividends. The speculator believes the excess stock return has a short-term momentum and long-term reversal regardless of exogenous shocks. These beliefs are not common knowledge. In equilibrium, both agents maximize utilities with these beliefs and markets clear. Both types of agents are bounded rational in the sense that they both partially capture the law of motion of the asset prices in equilibrium. We show that in calibrated simulations, both types of agents survive and there is a significant short-term momentum of excess stock returns. The calibrated data helps to explain eleven puzzling empirical regularities.

Introduction

We use Mehra and Prescott (1985) as the starting point of our introduction. Since the famous equity premium puzzle was presented, numerous attempts have been made to explain the effect on both theoretical and empirical sides. Meanwhile, more and more puzzles in the field of macro-finance have been found in recent empirical studies. Campbell (2000) provides a thorough overview of outstanding papers in the field. With all these advances in the literature, it is almost common sense that concepts such as representative agent, complete market, rational expectations are too-strong-to-be-realistic assumptions. With my limited understanding of the literature, I categorize the approaches to solve macro-finance puzzles on the spectrum from "determinismists” to "free-willists". The pure "determinismists" believes that the economy evolves "deterministically" once the exogenous shocks are given and those puzzling findings exists because we did not characterize the exogenous shocks perfectly. Barro (2006), Gabiax (2012) showed that rare disasters help to explain asset pricing puzzles. Bansal and Yoran (2004) shows that there exists a long-run persistent risk. With more complex exogenous shocks, there are inevitably more parameters to help fitting empirical data. The game of the pure "determinismists" is then to use as few parameters as possible to fit more moments of the data. The pure "free-willists" take the tool of behavioral finance. They believe the agents are free to make economic decisions even if they are irrational and does not maximize utility functions. Psychology and neuroscience are brought into the field of economics and finance. Besides, they are also the provider of many of the existing puzzles. Findings of Shiller (2015) showed many empirical facts that cannot be explained by traditional models with the rational expectation assumption. However, there is still no unified theory to explain the consistent existence of the irrational behaviors. In between these two approaches, there are also near "determinismists" who believes that instead of the researcher herself, the agents in the model does not characterize the exogenous shocks perfectly. They focus on asymmetric information, common knowledge, and partially observable shocks. Huo and Takayama (2015) explains business cycles with higher-order beliefs. And then more to the side of free-will, there are models with bounded rational agents. Krusell and Smith (1998) method is wildly used in macro economic studies, where the agents in the model use simple policy functions to maximize utility and those functions are in return justified by the equilibrium outcome. Gabiax (2014) claims the agents use the sparse-max tool to make decision.

This paper lies between the near "free-willists" and the pure "free-willists". We argue that the heterogeneity of agents are larger than proposed by the near "free-willists". One example is the observation of Hong and Stein (2007). The large trading volume can not be generated by the models mentioned above, only with a larger disagreement among agents can this effect be explained. This paper can be seem as an application of Geng (2018), where existence result of a more general model in provided. In Geng (2018), the author proposed the idea that there can be a measure of how irrational the agents are. And with a certain degree of irrationality, different agents may have different strategies and make different decisions under the same circumstance. One paper with a similar philosophy is Cao (2017), where the author showed that with incomplete market, an overly optimistic agents not only survive but also prosper by speculation. In this paper, we push further to assume both types of agents in the economy are bounded rational. In this way, the disagreement between agents may generate large trading volumes along with other properties found in empirical studies. We focus on the following eleven puzzling empirical regularities: (i) equity premium puzzle; (ii) excess volatility puzzle; (iii) fat tail of excess stock returns; (iv) predictability of aggregate stock returns with price-dividend ratios; (v) short-term momentum of excess stock returns; (vi) long-term reversal of excess stock returns; (vii) momentum effect of real interest rate; (viii) high trading volume puzzle; (ix) asset price bubbles and market crashes; (x) often greater explanatory power of characteristics than covariances for asset returns; (xi) common use of technical analysis in the finance industry.

Back to Mehra and Prescott (1985), we use the same simple market structure and two types of agents with heterogeneous beliefs to explain macro-finance puzzles. We assume the two types of agents are fundamentalists and speculators. The fundamentalists believes the asset prices are determined by exogenous shocks and future dividend payoffs. The speculators believes there exists a short-term momentum effect of the excess stock return and speculates on it. In equilibrium, both types of agents are justified in the sense that they both captures the law of motion of the asset prices partially. There is another way to justify the existence of the two types of agents. The fundamentalists believes there should not be speculators since if there is a consistent momentum effect, everyone would speculate on it and the effect would disappear in equilibrium. If there were no speculators, the fundamentalists will have the rational belief and the model degress to Mehra and Prescott (1985)'s model. The speculators exist because there is indeed a momentum effect in equilibrium. The simulation shows that both types of agents survive in the long run and the simulated equilibrium exhibits properties that helps to explain the listed asset pricing puzzles.

The rest of the paper is organized as follows. Section 2 set up the model; section 3 shows the calibration and the simulation results; section 4 concluses the paper.

The Economic Model

In this section we describe the model set up in three subsections. The first part is the physical economy setup.The markets structure is adapted from the Mehra and Prescott (1985). There is a one-period bond and a long-lived stock in the financial market and one consumption good. The exogenous shock affects the dividend payoffs of the stock. The financial market is incomplete due to a borrowing constraint on the bond and a short-selling ban on the stock. The second part describes how both types of bounded rational agents maximizes utility. The third part defines the equilibrium and shows some properties about the equilibrium.

The Physical Economy

Time is discrete and denoted by $t \in \mathbb{N}_0$. The exogenous shocks denoted by z realize from a two-state space $\mathbf{Z} = \{1, 2\}$, and follow a first-order Markov process with transition probability $\mathbb{P}(\cdot \mid z)$ defined on the Borel σ-algebra $\mathcal{Z}$ on $\mathbf{Z}$, $\mathbb{P} : \mathbf{Z} \times \mathcal{Z} \to [0, 1]$. Let $(z_t)_{t=0}^{\infty}$, or in short (z_t), denote the stochastic process and let $(\mathcal{F}_t)$ denote its natural filtration. A history of shocks up to some date t is denoted by $z^t = (z_0, z_1, ..., z_t)$.

There are two types of agents, $h \in \mathbf{H} = \{F, S\}$, i.e. fundamentalists and speculators. There is one perishable commodity , a one-period bond and a long-lived stock. The bond pays one unit of commodity on the next period, the stock pays dividends each period in terms of consumption good $d : \mathbf{Z} \to \mathbb{R}_+$. The agent h's endowment is zero consistent with Mehra and Prescott (1985). The agent h's consumption for period t is denoted by $c_{h,t} \in \mathbb{R}_+$; her holding of financial assets is denoted by $(\theta_{h,t}, \phi_{h,t}) \in \mathbf{\Xi}_h$. The prices of the stock and the bond are denoted by $(p, q) \in \mathbb{R}^2_{++}$. The budget set is denoted by $\mathbf{\Gamma}_h \colon \mathbf{Z} \times \mathbf{\Xi}_h \times \mathbb{R}^2_{++} \rightrightarrows \mathbb{R}_+ \times \mathbf{\Xi}_h$. The consumption vector of the economy is $c \in \mathbb{R}^2_+$, the asset holdings vector is $(\theta, \phi) \in \mathbf{\Xi}$.

Utility Maximization of Agents

Given the economy described above, the fundamentalist believes that the future asset prices are determined by exogenous shocks (z_t) and the dividends (d_t). So in our model with two different exogenous shocks, the fundamentalist would assume the future asset prices takes two values $p_F(z), q_F(z)$. The fundamentalist would then maximize the expected utility given the current asset prices and the belief of future prices. Then speculator believes that there is a one period momentum of the excess return of the stock and a reversal afterwards. Since the speculator ignores the exogenous shock, the future excess stock return is then a deterministic process in her belief. The speculator would then strictly prefer the stock or the bond given the current excess stock return. Given these subjective beliefs, we state their utility maximization in more detail.

We assume that both agents have time separable utility functions with the same CRRA Bernoulli functions:

$$u(c,\gamma) = \frac{c^{1-\gamma}}{1-\gamma}, 0 < \gamma < +\infty.$$

The value functions with subjective expectations are then

$$U_h = \mathbb{E}_h \left\{ \sum_t^{\infty} \beta^t u(c_t) \right\},$$

where β is the discount factor. We set a collateral constraint so that there will not be default with any prices. With a collateral constraint on the bond and the short-selling ban on the stock, the budget set of both agents can be described with

$$\mathbf{\Gamma}(z, \theta_h^-, \phi_h^-, p, q) = \left\{ p\theta_h + q\phi_h + x_h \leq (p + d(z))\theta_h^- + \phi_h^-, \theta_h \geq 0, -\theta_h \underline{d} \leq \phi_h \leq (1-\theta_h)\underline{d} \right\}.$$

This borrowing constraint is relatively tight, it ensures that the borrower can always repay the debt with the dividend next period even if the price of the stock drops to zero. With this borrowing constraint, we make sure that there is no default in equilibrium.

The fundamentalist

For the fundamentalist, the future prices are determined by the exogenous shock, we denote the future prices predicted by the fundamentalist by $p_F(z), q_F(z)$.

The maximization problem of the fundamentalist is then

$$V_F(z_0, \theta^-, \phi^-, p, q) = \max_{\{c_t\}_0^\infty} U_F,$$

$$\text{subject to } (c_t, \theta_t, \phi_t) \in \mathbf{\Gamma}(z_t, \theta_{t-1}, \phi_{t-1}, p_F(z_t), q_F(z_t)), \forall z^t, t \geq 0.$$

We assume the agent makes decisions recursively, and we write the value function and the policy correspondence of the fundamentalist as

$$V_F(z_0, \theta^-, \phi^-, p, q) = \max_{c_0, \theta_0, \phi_0} u(c_0) + \mathbb{E}_F \left\{ V_F(z', \theta_0, \phi_0, p_F(z'), q_F(z')) \right\},$$

$$\text{subject to } (c_0, \theta_0, \phi_0) \in \mathbf{\Gamma}(z_0, \theta^-, \phi^-, p, q);$$

$$G_F(z_0, \theta^-, \phi^-, p, q) = \arg\max_{c_0, \theta_0, \phi_0} u(c_0) + \mathbb{E}_F \left\{ V_F(z', \theta_0, \phi_0, p_F(z'), q_F(z')) \right\},$$

$$\text{subject to } (c_0, \theta_0, \phi_0) \in \mathbf{\Gamma}(z_0, \theta^-, \phi^-, p, q).$$

We make a simplifying assumption that the fundamentalist takes the complete-market asset prices ans forecasts. In this way, the belief of the fundamentalist can be simply summarized as: the fundamentalist does not know there are speculators in the economy. Indeed, if there are only fundamentalists in the economy, the borrowing constraint would never bind and the market evolves in the same way as a complete market. The asset prices would be the same as the forecasted prices by the fundamentalists. The future prices forecasted by the fundamentalist is then given by:

$$p_F(1) = \frac{\beta}{1-\beta} d^\gamma(1) \left(\frac{\mathbf{P}(1 \mid 1)}{d^{\gamma-1}(1)} + \frac{\mathbf{P}(2 \mid 1)}{d^{\gamma-1}(2)} \right), \quad q_F(1) = \beta d^\gamma(1) \left(\frac{\mathbf{P}(1 \mid 1)}{d^\gamma(1)} + \frac{\mathbf{P}(2 \mid 1)}{d^\gamma(2)} \right);$$
$$p_F(2) = \frac{\beta}{1-\beta} d^\gamma(2) \left(\frac{\mathbf{P}(1 \mid 2)}{d^{\gamma-1}(2)} + \frac{\mathbf{P}(2 \mid 2)}{d^{\gamma-1}(1)} \right), \quad q_F(2) = \beta d^\gamma(2) \left(\frac{\mathbf{P}(1 \mid 2)}{d^\gamma(2)} + \frac{\mathbf{P}(2 \mid 2)}{d^\gamma(1)} \right).$$

These prices also implicate that the fundamentalist would predict to buy a fixed number of shares of the stock and no bond from the next period and always consume the stock dividend from then on.

The speculator

For the speculator, she has beliefs about the future excess stock returns over the bond. First, the speculator will observe the excess stock return of the current period $r_0^{\text{excess}} = \frac{p+d(z_0)}{p^-} - \frac{1}{q^-}$. Denote the future risk free rate by r_f. She believes that the future excess stock return follows the function $r_t = f_t(r_0^{\text{excess}})$. Given the excess stock returns, the speculator would decide to hold stock or bond, the optimal choices always lie on the boundary. The realized return on the portfolio will be r_0. Notice that r_0 is different from $\max\{r_0^{\text{excess}} + r_f, r_f\}$. This is because the speculator is constrained to short arbitrary amount on either asset. We make one simplified assumption that the future (after the next period) portfolio return is $\max\{r_f, r_t + r_f\}$ (i.e. The speculator can hold only stock or only bond and will not be constrained). We will further justify this assumption when we specify how to characterize r_t. The future value function can be defined recursively as

$$\tilde{V}_S(z_t, \omega_{t-1}, r_{t-1}) = \max_{c_t, \omega_t} u(c_t) + \mathbb{E}_S \left\{ \tilde{V}_S(z_{t+1}, \omega_t, r_t) \right\},$$

$$\text{subject to } c_t + \omega_t \leq r_{t-1}\omega_{t-1}, t \geq 1.$$

We wrote the expectation sign because of the subjective belief of the speculator. But the exogenous shock does not matter for the speculator since she predicts the future excess returns deterministically and ignores the exogenous shocks. The value function and the policy correspondence of the speculator are

$$V_S(z_0, \theta^-, \phi^-, p^-, q^-, p, q) = \max_{c_0, \theta_0, \phi_0} u(c_0) + \mathbb{E}_S \left\{ \tilde{V}_S(z', p\theta_0 + q\phi_0, r_0) \right\},$$

$$\text{subject to } c_0 \in \mathbf{\Gamma}(z_0, \theta^-, \phi^-, p, q);$$

$$G_S(z_0, \theta^-, \phi^-, p^-, q^-, p, q) = \arg \max_{c_0, \theta_0, \phi_0} u(c_0) + \mathbb{E}_S \left\{ \tilde{V}_S(z', p\theta_0 + q\phi_0, r_0) \right\},$$

$$\text{subject to } c_0 \in \mathbf{\Gamma}(z_0, \theta^-, \phi^-, p, q).$$

With the setup above, the decision problem of the speculator is simplified to deciding how much to consume today. And then the speculator will use the rest of her wealth buy more higher return asset and short more lower return asset until she is constrained. Next we define the recursive equilibrium with the two types of agents.

Bounded Rational Recursive Equilibrium

Because of the speculator, the state space of the economy is enlarged to include the past asset prices p^-, q^-. A bounded rational recursive equilibrium, given initial conditions $z \in \mathbf{Z}, (\theta_h^-, \phi_h^-) \in \boldsymbol{\Xi}, (p^-, q^-) \in \mathbb{R}^2_{++}$, consists of prices and choices,

$$((p, q), (c_h, \theta_h, \phi_h)_{h \in \{F,S\}})$$

such that markets clear and each agent h optimizes utility–that is to say, (A), (B), and (C) hold.

(A) Commodity market clearing:

$$\sum_{h \in \{F,S\}} c_h = d(z).$$

(B) Financial markets clearing:

$$\sum_{h \in \{F,S\}} \theta_h = 1; \sum_{h \in \{F,S\}} \phi_h = 0$$

(C) Utility maximization:

$$(c_h, \theta_h, \phi_h) \in G_h(z, \theta_h^-, \phi_h^-, p^-, q^-, p, q), \forall h \in \{F, S\}.$$

An existence result of this kind of equilibrium can be found in Geng (2018).One characteristic of the bounded rational recursive equilibrium is that there are usually multiple equilibria. To circumvent this complication, in later sections, we use calibrated parameters and show that the simulated equilibrium path exhibits properties that help explain asset pricing puzzles in the literature.

Results

This section shows the simulation results and discussions about the eleven puzzles. The first part explains the calibrated parameter values; the second part explains the simulation process in detail; the last part shows properties of the generated equilibrium and discussions.

Parameter calibrations

We use $\beta = 0.98$ and $\gamma = 4$. The value of β is widely used in the literature and the risk aversion coefficient is within the reasonable range of acceptable values (not bigger than 5). The dividends are consistent with Mehra and Prescott (1985) with $\mu = 0.018, \delta = 0.036$, and $d(1) = 1 + \mu + \delta, d(2) = 1 + \mu - \delta$. The exogenous shock follows an iid process with $P(1) = P(2) = 0.5$. This is slightly different from Mehra and Prescott (1985) to simplify computations. The predicting function of excess returns $f_1(r_0) = 0.8r_0$,and $f_t(r_0) \leq 0, t \geq 2$. The prediction of future risk free rate is $\frac{1}{\beta}$. The speculator believes the excess stock return will have a one-period momentum, and then disappear afterwards.

Solution and simulation parameters

Given the same state variables, there are multiple equilibria. This is due to the fact that the speculator would behave differently give today's equity premium, yet the behavior of the speculator in turn affect today's asset prices and hence today's equity premium. First, more obviously, there may be two equilibria when the fundamentalist is not constrained by the collateral requirement: One start with the speculator behaves as if there is a positive equity premium and the equilibrium equity premium is positive; the other with the speculator behaves as if there is a non-positive equity premium and the equilibrium equity premium is non-positive. These two equilibria may not exist simultaneously. Also, there may exist other equilibria when both agents are constrained. Because of the disagreement between agents, both of them would want to borrow(sell) more if not constrained. In this case, we also need to fix a risk-free rate. For the simulation, we take 1.048 and 0.85 with respect to positive and non-positive equity premiums. We start with an initial wealth of 0.5 share of the total stock and no debt for both agents. We simulate 20,000 periods and discard the first 5,000 periods.

Equilibrium properties

The properties of the simulated equilibrium helps to understand the following asset pricing puzzles quantitatively. (i) equity premium puzzle; (ii) excess volatility puzzle; (iii) fat tail of excess stock returns; (iv) predictability of aggregate stock returns with price-dividend ratios; (v) short-term momentum of excess stock returns; (vi) long-term reversal of excess stock returns; (vii) momentum effect of real interest rate; (viii) high trading volume puzzle; (ix) asset price bubbles and market crashes; (x) often greater explanatory power of characteristics than covariances for asset returns; (xi) common use of technical analysis in the finance industry.

Autocorrelations of asset returns

The simulation shows that there is a first-order autocorrelation of excess stock returns, consistent with the speculator's belief. Also, the two to fourth-order autocorrelation are significantly negative, also consistent with the speculator's belief. These results help to explain the short-term momentum and long-term reversal effect of excess stock returns.(puzzle (v), (vi)) The simulation also shows significant first-order autocorrelation of bond prices.(puzzle (vii)) Figure 1 shows significant short-term momentum and long-term reversal effect of excess stock returns. Figure 2 shows a significant momentum effect of bond prices.

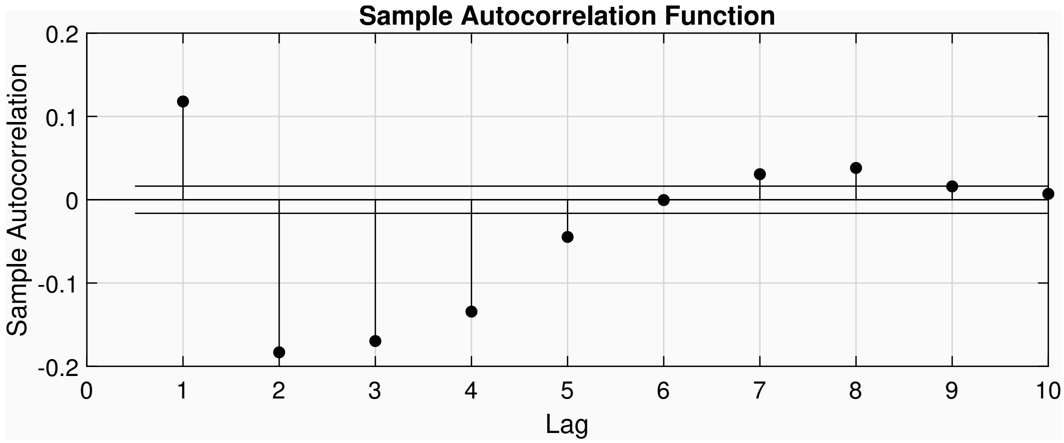

Figure 3.1. Auto-correlation of excess stock returns

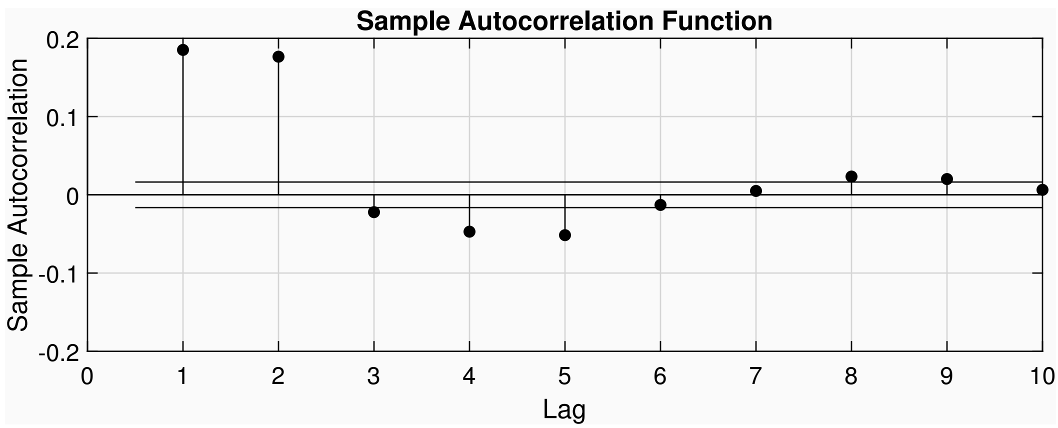

Figure 3.2. Auto-correlation of one period interest rate

Equity premium puzzle

We compare the first four moments of excess stock returns of the data from Shiller (2015), the Mehra and Prescott (1985) model, and our model. Our simulation exhibits a much higher equity premium than the baseline model of Mehra and Prescott (1985), (puzzle (i)) It also has a higher second (puzzle (ii)) and fourth (puzzle (iii)) moment of excess stock returns than the baseline model and the data. The results are displayed in Table 1.

Table 3.1. Excess stock return moments

	Mean	Volatility	Skewness	Kurtosis
Data	5.36%	20.21%	-0.76	2.90
Model	4.45%	37.72%	1.12	4.85
Baseline	2.01%	14.58%	0.03	1.08

Notes: The data are from Shiller (2015)'s calculation for the United States 1871-2015.

Aggregate return predictability

We compare the predictability of excess stock returns with dividend price rations of the data from Gabiax (2012), the Mehra and Prescott (1985) model, and our model. Our simulation shows similar slop and $\mathbf{R}^2$ with the data, significantly better than the baseline model of Mehra and Prescott (1985).(puzzle (iv)) The interest rate has a much lower predictability of the excess stock return.(puzzle (x)) The results are displayed in Table 2.

High trading volume

Our model exhibits significantly higher trading volumes than many other heterogeneous agents models, the bond position shits hand between agents every time

Table 3.2. Prediction excess stock returns with the dividend/price ratio

	Slope	s.e.	$\mathbf{R}^2$
Data	0.11	(0.053)	0.04
Model	0.15	(0.012)	0.03
Baseline	0.06	(0.017)	0.00

Notes: The data are from Gabiax (2012, Table IV)'s calculation for the United States 1891-1997.

the excess return switches signs. This contributes to explain regularities stated in Hong and Stein (2007). Also, the high trading volume co-moves with extreme returns. This is consistent with Hong and Stein (2007). Although the co-movement exists, our model shows no forecasting power of the trading volume since the high trading volume and extreme returns always appear on the same period.

Asset price bubble and market crashes

Our simulations shows the price dividend ratios can get very high, which can be seen as a sign of asset price bubbles. With different parameter values, the model can provide the effect of asset price bubbles and market crashes, the speculator keeps on shorting on bond and speculate on the positive excess returns, this will drive the price up. Yet with the price getting higher and higher, the consumption of the speculator will eventually go up. When the high consumption of the speculator causes the price to decrease, the asset price crashes because of the sign change of the excess return.

Technical analysis

Our model shows that technical analysis can be effective with incomplete financial markets. The traditional theory usually predicts no effectiveness of technical analysis. In our simulation, we showed the momentum strategy can be self-justified in equilibrium. This opens the gate to studies of other kinds of technical indexes to see if other technical trading strategies can generate the effects they predict in equilibrium.

Conclusions

This paper provides a simple model that can generate short-term momentum and long-term reversal of excess stock returns. The calibrated simulation also contributes to solving many asset pricing puzzles. The paper can be seen an application of Geng (2018). It provides a new view on the heterogeneity of agents. Former papers of heterogeneous agents usually focus on idiosyncratic shocks, asymmetric information, or different beliefs about the exogenous shocks. This paper shows that the agents may differ in the decision policy when all types of agents have subjective beliefs. The paper also opens the gate to study what kind of policies can be self-justified in equilibrium. This can be seen as a theoretical foundation for agent-based models.

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Conclusion

My contribution to the first paper is the model without aggregate uncertainty but with bankruptcy and default and prove existence of a steady state equilibrium.

The second paper provides a general way of modeling bounded rationality. Yet, many aspects of behavior economics literature cannot be incorporated, especially the findings about preferences such as loss aversion and hyperbolic discounting. we prove the existence of the recursive equilibrium in a dynamic stochastic model with bounded rational agents. This set up is realistic when the economy is large and complex, which is a fact for a lot of markets. A large and complex economy would affect the agents in two ways: first, each agent would rely on recursive methods to optimize utility; second, each agent cannot solve for the stationary equilibrium analytically and has to approximate an equilibrium distribution.

The entire part of constructing a continuous value function does not put extra assumptions on the model, it provides one additional interpretation of what the rationality is bounded by – that is, the agent's choices are continuous in the information she observes. This is consistent with the philosophical principles stated in Maskin and Tirole (2001). Also, the continuous value function is analogous to the continuity assumption of the utility function in the static exchange economy models.

Although relatively abstract, this model has the potential to explore dynamic economic models further in multiple ways. First, it may provide foundation for many algorithms computing general equilibrium models. As we stated before, examples like the computationally part of Krusell and Smith (1998) and Kubler and Schmedders

(2003) can be incorporated into my set up. Second, it may generate interesting economic processes from standard set up. As we can see from the proof, although the equilibrium is recursive, the Markov process of the whole economy may not be ergodic. This may provide a new angle to study business cycles and financial crisis from a standard model set up. Third, it provides a new dimension of heterogeneity among agents, i.e. they may use different heuristics. And last, it contributes to finding a general theory of bounded rationality.

I further justifies the above remarks using calibrated simulations in the third paper. This paper provides a simple model that can generate short-term momentum and long-term reversal of excess stock returns. The calibrated simulation also contributes to solving many asset pricing puzzles. The paper can be seen an application of the second paper. It provides a new view on the heterogeneity of agents. Former papers of heterogeneous agents usually focus on idiosyncratic shocks, asymmetric information, or different beliefs about the exogenous shocks. This paper shows that the agents may differ in the decision policy when all types of agents have subjective beliefs. The paper also opens the gate to study what kind of policies can be self-justified in equilibrium. This can be seen as a theoretical foundation for agent-based models.