



University of
Zurich^{UZH}

University of Zurich
Department of Economics

Working Paper Series

ISSN 1664-7041 (print)
ISSN 1664-705X (online)

Working Paper No. 336

A Class of N -Player Colonel Blotto Games with Multidimensional Private Information

Christian Ewerhart and Dan Kovenock

November 2019

A Class of N -Player Colonel Blotto Games with Multidimensional Private Information

Christian Ewerhart*

Dan Kovenock†

November 14, 2019

Abstract. We consider a class of incomplete-information Colonel Blotto games in which $N \geq 2$ agents are engaged in $(N + 1)$ battlefields. An agent's vector of battlefield valuations is drawn from a generalized sphere in L_p -space. We identify a Bayes-Nash equilibrium in which any agent's resource allocation to a given battlefield is strictly monotone in the agent's valuation of that battlefield. In contrast to the single-unit case, however, agents never enjoy any information rent. We also outline an extension to networks of Blotto games.

Keywords. Colonel Blotto games · Private information · Bayes-Nash equilibrium · Information rents · Networks

JEL-Codes. C72 – Noncooperative Games; D72 – Political Processes: Rent-Seeking, Lobbying, Elections, Legislatures, and Voting Behavior; D82 – Asymmetric and Private Information, Mechanism Design

*) (corresponding) Department of Economics, University of Zurich, Schönberggasse 1, CH-8001 Zurich, Switzerland; christian.ewerhart@econ.uzh.ch.

†) Economic Science Institute, Chapman University; kovenock@chapman.edu.

1. Introduction

In a Colonel Blotto game, finitely many agents (parties in a military conflict, say) simultaneously allocate their limited resources to a given set of battlefields. The agent that deploys the largest amount of the resource to a given battlefield wins in that battlefield. Moreover, each agent's payoff depends on the set of battlefields won. This type of game has been among the first games seriously studied in the literature (Borel, 1921; Borel and Ville, 1938), and has since found widespread application in military theory, political economy, and security analysis, for instance (cf. Roberson, 2010).

In this paper, we introduce a new class of N -player Colonel Blotto games with $(N + 1)$ battlefields and multidimensional incomplete information regarding battlefield valuations. We assume that valuation vectors are private information and independently distributed across agents. Only the joint distribution of valuation vectors is common knowledge. Each agent maximizes the expected sum of valuations of battlefields won, where resource budgets are fixed and homogeneous across agents, and where unused resources do not create any positive value. In this framework, we identify a Bayes-Nash equilibrium in which any agent's resource allocation to a battlefield is strictly monotone increasing in her valuation of that battlefield. The construction of equilibria for more than two agents relies on a new method. Specifically, we exploit the particular properties of uniform distributions on generalized spheres in finite-dimensional vector spaces equipped with an L_p -norm. The necessary mathematical background will be reviewed in Section 2.

Past years have seen a rise in interest in Colonel Blotto games with incomplete information. In a model with $N \geq 2$ agents and private information about budgets, Adamo and Matros (2009) identified a symmetric monotone Bayes-Nash equilibrium. A higher budget allows an agent to scale up her resource allocation, while the share of the resource allocated to individual battlefields remains constant. This reduces the analysis to a one-dimensional problem which is tractable.¹ No general results are available, however, when agents possess private information about battlefield valuations.² Kovenock and Roberson (2011) presented an example with two agents and three battlefields.³ Private valuations of battlefields are drawn independently from a uniform distribution over a two-dimensional surface element that corresponds to the intersection of the nonnegative orthant with the Euclidean unit sphere. Since, in this case, marginal distributions are uniform, the budget constraint may be kept by bidding the squared valuation on each battlefield. It turns out that this strategy constitutes a symmetric Bayes-Nash equilibrium in the Colonel Blotto game.

Extending this example in a substantial way, Akyol (2014) considered a setting with two agents and $N \geq 2$ battlefields, where agents' valuation vectors are drawn from absolutely continuous, possibly heterogeneous distributions.⁴

¹Kim and Kim (2017) considered a similar set-up for the lottery contest.

²This lack of general results contrasts with the rich theory on single-unit all-pay auctions with incomplete information. See Weber (1985), Hillman and Riley (1989), Amann and Leininger (1995, 1996), and Krishna and Morgan (1997), for example.

³Kovenock and Roberson (2011) note that their approach easily generalizes to any number of battlefields N that is an integer multiple of three.

⁴In the exceptional case of $N = 2$ battlefields, Akyol (2014) even allows for arbitrary distributions and identifies a Bayes-Nash equilibrium in weakly dominant strategies.

He constructed explicit examples using the Generalized Gamma Distribution, the uniform distribution on a three-dimensional Euclidean ball, and the uniform distribution on a three-dimensional volume bounded by two Euclidean spheres of different radius. The present paper goes beyond Akyol's (2014) important contribution by presenting a class of examples of N -agent Colonel Blotto games with incomplete information about valuations.⁵

The remainder of the paper is structured as follows. Section 2 reviews the necessary background on uniform spherical distributions. The set-up is introduced in Section 3. Section 4 contains the equilibrium analysis. The issue of information rents is discussed in Section 5. An extension to network games is considered in Section 6. Section 7 concludes.

2. L_p -norm uniform distributions

The results of this section are essentially well-known in the mathematical literature. We will follow the exposition in Song and Gupta (1997).⁶

For an integer $n \geq 3$, and a real parameter $p > 0$, we consider Euclidean

⁵There are a number of less closely related papers. Powell (2007) studied a signaling game with private information about vulnerability. Paarporn et al. (2019) assumed one-sided incomplete information in a Colonel Blotto game with a finite state space. Finally, in a model of price setting with menu costs for multiproduct firms, Alvarez and Lippi (2014) made use of the marginals of a uniform distribution on a higher-dimensional Euclidean sphere that represents a vector of price changes. They, however, studied the problem of a monopolist, i.e., there is no Colonel Blotto game.

⁶A helpful overview is given in the recent paper by Ahmadi-Javid and Moeini (2019), even though we caution the reader that their definition of the uniform distribution on the generalized sphere as a conditional probability measure on a set of measure zero might be subject to the Borel-Kolmogorov paradox. Another recent contribution in this literature is Richter (2019).

n -space equipped with the L_p -norm

$$\|y\|_p = (|y_1|^p + \dots + |y_n|^p)^{1/p} \quad (y \in \mathbb{R}^n). \quad (1)$$

The *generalized sphere* \mathcal{S}_p^{n-1} is defined as the subset

$$\mathcal{S}_p^{n-1} = \{(y_1, y_2, \dots, y_n) \in \mathbb{R}^n : \|y\|_p = 1\} \quad (2)$$

of vectors of L_p -norm one in \mathbb{R}^n . Provided that $p > 1$, which will be assumed below, the set \mathcal{S}_p^{n-1} is a smooth $(n-1)$ -dimensional submanifold embedded in \mathbb{R}^n .⁷ Figure 1 illustrates this property for the case where $n = p = 3$.

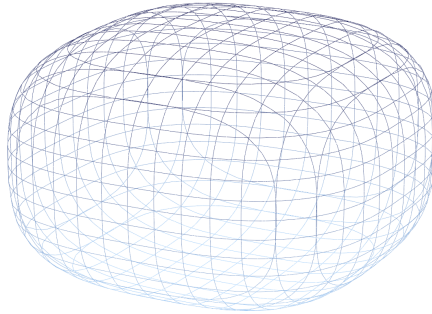


Figure 1. The generalized sphere \mathcal{S}_p^{n-1} .

To construct a uniform distribution on \mathcal{S}_p^{n-1} , we define a particular L_p -norm *spherical distribution* on \mathbb{R}^n and consider its projection to the generalized sphere. Let

$$\mathbf{X} = (X_1, \dots, X_n), \quad (3)$$

where the components X_j are i.i.d. random variables with p.d.f.

$$f_p(x) = \frac{p^{1-1/p}}{2\Gamma(\frac{1}{p})} \exp\left(-\frac{|x|^p}{p}\right) \quad (x \in \mathbb{R}), \quad (4)$$

⁷For $p \leq 1$, there may be kinks, and the generalized sphere corresponds to a union of bordered manifolds.

with $\Gamma(\cdot)$ denoting the Gamma function. Let $U_j = X_j / \|\mathbf{X}\|_p$, where $j = 1, \dots, n$. Then, $\sum_{j=1}^n |U_j|^p = 1$, and the joint distribution of the vector

$$\mathbf{U} = (U_1, \dots, U_n) \quad (5)$$

is known as the L_p -norm uniform distribution. For $p = 2$, this probability distribution corresponds to the uniform distribution on the Euclidean unit sphere. We have the following result.

Lemma 1 (Song and Gupta, 1997). *The joint p.d.f. of $\mathbf{U} = (U_1, \dots, U_n)$ is given by*

$$g(u_1, \dots, u_{n-1}) = \frac{p^{n-1} \Gamma(\frac{n}{p})}{2^{n-1} \left(\Gamma(\frac{1}{p})\right)^n} \left(1 - \sum_{j=1}^{n-1} |u_j|^p\right)^{\frac{1-p}{p}}, \quad (6)$$

$$\left(-1 < u_j < 1; j = 1, 2, \dots, n-1; \sum_{j=1}^{n-1} |u_j|^p < 1\right).$$

Proof. See Song and Gupta (1997, Theorem 1.1). \square

Since the spherical distribution is multidimensional, it is of interest to know if a stochastic representation in terms of simpler distributions exists. In addition to the definition in terms of a collection of independent one-dimensional exponential distributions, stochastic representations may be derived from a collection of independent one-dimensional uniform distributions, and from a Dirichlet distribution (which may be decomposed using Beta distributions). For further details, we refer the reader to Song and Gupta (1997, Theorem 2.1, part (2) and (3)).

The marginal distribution of the j -th component U_j may be determined using the Jacobi integral formula. The corresponding calculation leads to the following result.

Lemma 2 (Song and Gupta, 1997). *The univariate marginal density of g with respect to any of the components u_1, \dots, u_n is given as*

$$m_j(u_j) = \frac{p\Gamma(\frac{n}{p})}{2\Gamma(\frac{1}{p})\Gamma(\frac{n-1}{p})} (1 - |u_j|^p)^{\frac{n-1}{p}-1} \quad (j = 1, \dots, n; -1 < u_j < 1). \quad (7)$$

In particular, if $p = n - 1$, then the univariate marginal is uniform.

Proof. See Song and Gupta (1997, Theorem 2.1, part (1)) for $k = 1$. \square

For example, in the special case where $n = 3$ and $p = 2$, the marginal density of the j 's component U_j is given as

$$m_j(u_j) = \frac{2\Gamma(\frac{3}{2})}{2\Gamma(\frac{1}{2})\Gamma(1)} = \frac{1}{2}, \quad (8)$$

hence constant, so that the marginal distribution is uniform on $[-1, 1]$. This special case of Lemma 2 has been used by Kovenock and Roberson (2011). Below, we will exploit Lemma 2 more generally.

3. Set-up

There are $N \geq 2$ risk-neutral agents, denoted by $i \in \{1, \dots, N\}$, and $(N + 1)$ battlefields, denoted by $j \in \{1, \dots, N + 1\}$.⁸ Each agent is endowed with an

⁸Thus, the number of battlefields exceeds the number of agents by precisely one. This specific assumption will be relaxed in Section 6.

identical budget of a perfectly divisible resource. For convenience, we normalize budgets to one. An agent's *resource allocation* is a vector

$$b = (b_1, \dots, b_{N+1}) \in \mathbb{R}_+^{N+1}, \quad (9)$$

where $b_j \in \mathbb{R}_+ = [0, \infty)$ denotes the amount of the resource allocated to battlefield j . We call a resource allocation $b = (b_1, \dots, b_{N+1})$ *feasible* if

$$\sum_{j=1}^{N+1} b_j \leq 1. \quad (10)$$

Denote by \mathcal{B}^{N+1} the set of feasible resource allocations over $N + 1$ battlefields. The N agents simultaneously and independently choose feasible resource allocations. In each battlefield, the agent that allocates the largest amount of the resource wins. In the case of a tie, each of the agents that allocated the largest amount of the resource to a battlefield wins in that battlefield with equal probability. Each agent's payoff equals the sum of the valuations of the battlefields won.

Agents are ex-ante identical but privately learn, before deciding about the resource allocation, their respective vector of battlefield valuations, $v = (v_1, v_2, \dots, v_{N+1})$. The vector v is commonly known to be drawn, independently across agents, from a given $(N + 1)$ -variate probability distribution. Since, as discussed in the Introduction, a general solution of Colonel Blotto games with incomplete information about valuations is presently out of reach, we will work under parametric assumptions. The type space for an agent is given as the intersection of the nonnegative orthant \mathbb{R}_+^{N+1} with a generalized sphere \mathcal{S}_p^{n-1} , where the parameters are given by $n = N + 1$ and $p = N$ (cf.

Section 2). Thus,

$$\mathcal{V}^N = \mathbb{R}_+^{N+1} \cap \mathcal{S}_p^{n-1} \quad (11)$$

$$= \{(v_1, v_2, \dots, v_{N+1}) \in \mathbb{R}_+^{N+1} : v_1^N + \dots + v_{N+1}^N = 1\}. \quad (12)$$

For example, in the special case $N = 2$, the type space \mathcal{V}^2 corresponds to the intersection of \mathbb{R}_+^3 with the two-dimensional Euclidean unit sphere. For general $N \geq 2$, the dimension of the bordered type manifold \mathcal{V}^N is N , and the dimension of the embedding space is $N + 1$. It will be assumed that each agent's valuation vector v is drawn from the L_p -norm uniform distribution on \mathcal{S}_p^{n-1} , conditional on $v \in \mathbb{R}_+^{N+1}$. The corresponding probability measure on the Borel subsets of \mathcal{V}^N will be denoted by μ^N .

4. Equilibrium analysis

A *pure strategy* is a measurable mapping $\mathbf{b} : \mathcal{V}^N \rightarrow \mathcal{B}^{N+1}$. Suppose that an agent's type is

$$v = (v_1, v_2, \dots, v_{N+1}) \in \mathcal{V}^N. \quad (13)$$

When adhering to strategy \mathbf{b} , type v 's resource allocation is

$$\mathbf{b}(v) = (b_1(v), \dots, b_{N+1}(v)) \in \mathbb{R}_+^{N+1}. \quad (14)$$

Clearly, any strategy of an opponent induces a probability measure over feasible resource allocations. Therefore, given strategies for the $(N - 1)$ opponents, type v 's resource allocation translates into a vector of winning probabilities, and hence, into an expected payoff for type v . A strategy \mathbf{b}^* is called a

symmetric Bayes-Nash equilibrium strategy in the Colonel Blotto game with incomplete information if, for any type realization $v \in \mathcal{V}^N$, the resource allocation $\mathbf{b}^*(v)$ maximizes the expected payoff of type v under the assumption that the other $(N - 1)$ agents individually adhere to strategy \mathbf{b}^* . We will say that an equilibrium strategy \mathbf{b}^* is *strict* if each type $v \in \mathcal{V}^N$ even has a strict incentive to choose the resource allocation $\mathbf{b}^*(v)$.

Proposition 1. *Suppose that each agent's $(N + 1)$ -vector of battlefield valuations is independently drawn according to the probability measure μ^N on \mathcal{V}^N . Then, the pure strategy \mathbf{b}^* defined through*

$$\mathbf{b}^*(v) = ((v_1)^N, \dots, (v_{N+1})^N) \tag{15}$$

is a strict symmetric Bayes-Nash equilibrium strategy in the Colonel Blotto game with incomplete information.

The strictness property contrasts with the case of Colonel Blotto games with complete information where Nash equilibria are typically non-strict.

Proposition 1 extends known equilibrium characterizations for Colonel Blotto games with incomplete information about valuations. The analysis of Kovenock and Roberson (2011) is contained as a special case where $N = 2$. Akyol (2014) allowed for any finite number of battlefields, but restricted attention to the case of two agents. In addition, to account for his assumption that the distributions of valuation vectors are absolutely continuous, his characterization of the equilibrium strategy entails a normalization factor, which would be one in Proposition 1.

In comparison with the analysis of Adamo and Matros (2009), who considered incomplete information about budgets, we find that their Corollaries 1 and 2 hold likewise in the present setting, i.e., all agents compete for all prizes, and each agent spends more on prizes she values higher. However, their Corollaries 3 and 4 do not transfer to the present setting. Indeed, there is no highest type in our setting and, as will be shown in the next section, the expected payoff is constant across types in our model.

Proof of Proposition 1. For each battlefield $j \in \{1, \dots, N + 1\}$, we denote by $F_j(\cdot)$ the cumulative distribution function describing the marginal distribution of the component v_j . Thus, $F_j(z) = \mu^N \{ \mathcal{V}_j^N(z) \}$, where $\mathcal{V}_j^N(z) = \{v \in \mathcal{V}^N : v_j \leq z\}$ denotes the set of valuation vectors whose j -th component does not exceed a given $z \in [0, 1]$. Suppose that all agents $i \in \{2, \dots, N\}$ adhere to strategy \mathbf{b}^* . Then, any agent $i \in \{2, \dots, N\}$ of type

$$v^{(i)} = (v_1^{(i)}, \dots, v_{N+1}^{(i)}) \in \mathcal{V}^N \quad (16)$$

allocates an amount $b_j^*(v_j^{(i)})$ of the resource to battlefield $j \in \{1, \dots, N + 1\}$. Suppose further that agent 1 is of type

$$v^{(1)} = (v_1^{(1)}, \dots, v_{N+1}^{(1)}) \in \mathcal{V}^N.$$

Then, since there are no mass points, agent 1's optimization problem may be written as

$$\max_{(b_1^{(1)}, \dots, b_{N+1}^{(1)}) \in \mathcal{B}^{N+1}} \sum_{j=1}^{N+1} \Pr \left\{ b_j^{(1)} \geq \max_{i \in \{2, \dots, N\}} \left(b_j^*(v_j^{(i)}) \right) \right\} v_j^{(1)}. \quad (17)$$

Now, for any resource allocation $(b_1^{(1)}, \dots, b_{N+1}^{(1)}) \in \mathcal{B}^{N+1}$, and for any battle-field $j \in \{1, \dots, N+1\}$, the independence of type distributions across agents yields

$$\Pr \left\{ b_j^{(1)} \geq \max_{i \in \{2, \dots, N\}} \left(b_j^*(v_j^{(i)}) \right) \right\} = \prod_{i=2}^N \Pr \left\{ b_j^{(1)} \geq b_j^*(v_j^{(i)}) \right\}. \quad (18)$$

Moreover, for any $i \in \{2, \dots, N\}$,

$$\Pr \left\{ b_j^{(1)} \geq b_j^*(v_j^{(i)}) \right\} = \Pr \left\{ b_j^{(1)} \geq \left(v_j^{(i)} \right)^N \right\} \quad (19)$$

$$= \Pr \left\{ \left(b_j^{(1)} \right)^{1/N} \geq v_j^{(i)} \right\} \quad (20)$$

$$= F_j \left\{ \left(b_j^{(1)} \right)^{1/N} \right\}. \quad (21)$$

Applying Lemma 2 with $n = N + 1$ and $p = N$, we see that

$$F_j(z) = 2 \int_0^z m_j(u_j) du_j \quad (22)$$

$$= \frac{p \Gamma\left(\frac{n}{p}\right)}{\Gamma\left(\frac{1}{p}\right) \Gamma\left(\frac{n-1}{p}\right)} \int_0^z (1 - |u_j|^p)^{\frac{n-1}{p}-1} du_j \quad (23)$$

$$= \frac{N \Gamma\left(\frac{1}{N} + 1\right)}{\Gamma\left(\frac{1}{N}\right) \Gamma(1)} \int_0^z du_j \quad (24)$$

$$= z, \quad (25)$$

for any $z \in [0, 1]$. Hence, $F_j(\cdot)$ is uniform. Putting the pieces together, we find that

$$\Pr \left\{ b_j^{(1)} \geq \max_{i \in \{2, \dots, N\}} \left(b_j^*(v_j^{(i)}) \right) \right\} = \prod_{i=2}^N \left(b_j^{(1)} \right)^{1/N} = \left(b_j^{(1)} \right)^{(N-1)/N}. \quad (26)$$

Let $\lambda \equiv \lambda(v^{(1)})$ denote the shadow cost of the budget constraint in agent 1's problem. Using the calculation above, agent 1's problem in battlefield

$j \in \{1, \dots, N + 1\}$ may be written as

$$\max_{b_j^{(1)} \in [0,1]} \left(b_j^{(1)} \right)^{(N-1)/N} v_j^{(1)} - \lambda b_j^{(1)}. \quad (27)$$

Note that agent 1's objective function in (17) is strictly concave since $N \geq 2$.

Solving the first-order condition,

$$\frac{N-1}{N} \left(b_j^{(1)} \right)^{-(1/N)} v_j^{(1)} - \lambda = 0, \quad (28)$$

delivers

$$b_j^{(1)} = \left(\frac{N-1}{N\lambda} \right)^N \cdot \left(v_j^{(1)} \right)^N. \quad (29)$$

Clearly, in an optimal allocation, no resources remain unused, i.e.,

$$b_1^{(1)} + \dots + b_{N+1}^{(1)} = 1. \quad (30)$$

Moreover, since $v^{(1)}$ is drawn from \mathcal{V}^N , we have

$$\left(v_1^{(1)} \right)^N + \dots + \left(v_{N+1}^{(1)} \right)^N = 1. \quad (31)$$

Therefore, summing Eq. (29) over all battlefields $j \in \{1, \dots, N + 1\}$, we find that $\lambda = (N - 1)/N$ and, hence,

$$b_j^{(1)} = \left(v_j^{(1)} \right)^N. \quad (32)$$

Thus, it is indeed optimal for type $v^{(1)}$ of agent 1 to allocate the resource as prescribed by the symmetric equilibrium strategy \mathbf{b}^* . Obviously, the same is true for agents $i \in \{2, \dots, N\}$. This concludes the proof of Proposition 1. \square

5. Information rents

The symmetric equilibrium strategy identified above has the property that the amount of the resource deployed in any given battlefield increases strictly in the agent's valuation of that battlefield. Thus, the identified Bayes-Nash equilibrium leads to an efficient selection of battlefield winners, just as in the symmetric single-unit all-pay auction with independent types.⁹

It is noteworthy, however, that the expected payoff of an agent does not depend on the type realization. Indeed, using our intermediary result (26), the expected payoff of type $v^{(1)} \in \mathcal{V}^N$ of agent 1, say, is given by

$$R = \sum_{j=1}^{N+1} \Pr \left\{ b_j^*(v_j^{(1)}) \geq \max_{i \in \{2, \dots, N\}} \left(b_j^*(v_j^{(i)}) \right) \right\} \cdot v_j^{(1)} \quad (33)$$

$$= \sum_{j=1}^{N+1} \left(b_j^*(v_j^{(1)}) \right)^{(N-1)/N} \cdot v_j^{(1)} \quad (34)$$

$$= \sum_{j=1}^{N+1} \left(v_j^{(1)} \right)^{N-1} \cdot v_j^{(1)} \quad (35)$$

$$= \sum_{j=1}^{N+1} \left(v_j^{(1)} \right)^N \quad (36)$$

$$= 1. \quad (37)$$

We have shown the following.

Proposition 2. *In the considered class of Colonel Blotto games with incomplete information, all types realize the same expected payoff.*

Thus, in contrast to the single-unit auction, where higher types realize positive information rents as a consequence of incentive compatibility, any such benefits net out over battlefields in the considered class of Colonel Blotto games.¹⁰

⁹This conclusion is not robust, however, with respect to a rescaling of the valuation vector for a single player.

¹⁰Clearly, this is so because the set of possible realizations of valuation vectors forms an

6. Networks of Blotto games

As an extension, we consider networks of Colonel Blotto games with $N \cdot K$ agents and $(N + 1) \cdot K$ battlefields, where $K \geq 1$ is an integer. Each agent is restricted to be active in $(N + 1)$ given battlefields, and draws a type from \mathcal{V}^N . For example, any triangulation of a globe, say, may be understood as a network of Blotto games, where each triangle represents an agent, and each edge shared with a neighboring triangle represents a battlefield. In this case, $N = 2$ and $K \geq 2$. Figure 2(a) illustrates this for $K = 5$. Another example is a cube where each side represents an agent, and each adjacent node represents a battlefield. In this case, $N = 3$ and $K = 2$. See Figure 2(b) for illustration. It is immediate to see that examples exist for any combination of $N \geq 2$ and $K \geq 2$. The equilibrium analysis extends in a straightforward way. Intuitively, an agent does not care whether she is facing, in any two distinct battlefields, the same opponent or two different opponents. This is so because the marginal distribution of resource bids in each battlefield does not depend on this.

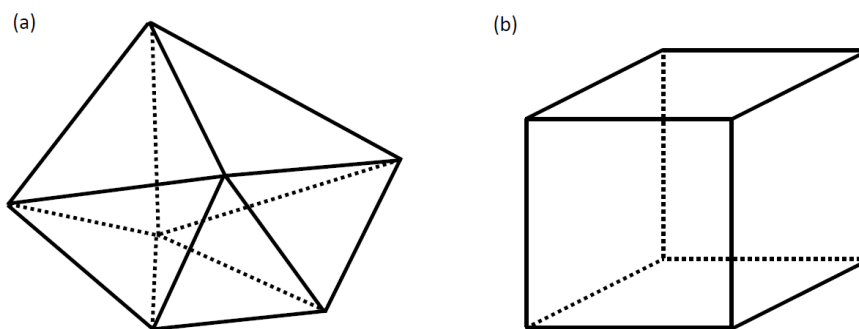


Figure 2. Networks of Blotto games

antichain with respect to the product order on \mathbb{R}^{N+1} .

7. Concluding remark

The methods of this paper may be used to construct also new classes of mixed-strategy equilibria in two-player Colonel Blotto games with complete information and $2(N + 1)$ homogeneous battlefields. These may be understood as further generalizations of the disc solution (Borel and Ville, 1938; Gross and Wagner, 1950; Laslier and Picard, 2002; Thomas, 2018). However, since the number of battlefields is even, such new solutions in the case of complete information would ultimately be of limited interest, for the same reasons discussed by Laslier and Picard (2002), viz. that agents partition battlefields into pairs, and perfectly negatively correlate within each pair of battlefields.¹¹

By combining the insights of Akyol (2014) and the result of the present paper, it seems feasible to construct additional examples involving any finite number of agents and any finite number of battlefields. We leave that extension for future work.

References

- Adamo, A., Matros A. (2009), Blotto game with incomplete information, *Economic Letters* **105**, 100-102.
- Ahmadi-Javid, A., Moeini, A. (2019), Uniform distributions and random variate generation over generalized l_p balls and spheres, *Journal of Statistical Planning and Inference* **201**, 1-19.

¹¹However, the construction of network games can be accomplished in an analogous fashion for Colonel Blotto games with complete information.

- Akyol, E. (2014), "Allocation with points: Colonel Blotto game under incomplete information," Ch. 3 in: *Essays in Mechanism Design without Transfers*, Ph.D. Thesis, Penn State University.
- Alvarez, F., Lippi, F. (2014), Price setting with menu cost for multiproduct firms, *Econometrica* **82**, 89-135.
- Amann, E., Leininger, W. (1995), Expected revenue of all-pay and first-price sealed-bid auctions with affiliated signals, *Journal of Economics* **3**, 273-279.
- Amann, E., Leininger, W. (1996), Asymmetric all-pay auctions with incomplete information: The two-player case, *Games and Economic Behavior* **14**, 1-18.
- Borel, E. (1921), La théorie du jeu les équations intégrales à noyau symétrique, *Comptes Rendus de l'Académie* **173**, 1304-1308, English translation by Savage, L. (1953), The theory of play and integral equations with skew symmetric kernels, *Econometrica* **21**, 97-100.
- Borel, E., Ville, J. (1938), *Applications aux Jeux de Hasard: Cours Professe a la faculte des Sciences de Paris: Traite du Calcul des Probabilites et de ses Applications: Tome IV: Applications Diverses et Conclusion: Fascicule II*, Gauthier-Villars.
- Gross, O., Wagner, R. (1950), A continuous Colonel Blotto game, RAND Corporation, Research Memorandum 408.
- Hillman, A.L., Riley, J. G. (1989), Politically contestable rents and transfers, *Economics & Politics* **1**, 17-39.

- Kim, J., Kim, B. (2017), An asymmetric lottery Blotto game with a possible budget surplus and incomplete information, *Economics Letters* **152**, 31-35.
- Krishna, V., Morgan, J. (1997), An analysis of the war of attrition and the all-pay auction, *Journal of Economic Theory* **72**, 343-362.
- Kovenock, D., Roberson, B. (2011), A Blotto game with multi-dimensional incomplete information, *Economics Letters* **113**, 273-275.
- Laslier, J.-F., Picard, N. (2002), Distributive politics and electoral competition, *Journal of Economic Theory* **103**, 106-130.
- Paarporn, K., Chandan, R., Alizadeh, M., Marden, J.R. (2019), Characterizing the interplay between information and strength in Blotto games, *IEEE Conference on Decision and Control (CDC)*, March 15, 2019.
- Powell, R. (2007), Allocating defensive resources with private information about vulnerability, *American Political Science Review* **101**, 799-809.
- Richter, W.D. (2019), On (p_1, \dots, p_k) -spherical distributions, *Journal of Statistical Distributions and Applications* **6**, 1-18.
- Roberson, B. (2010), Allocation games, *Wiley Encyclopedia of Operations Research and Management Science*, Hoboken.
- Song, D., Gupta, A. (1997), L_p -norm uniform distribution, *Proceedings of the American Mathematical Society* **125**, 595-601.
- Thomas, C. (2018), N -dimensional Blotto game with heterogeneous battle values, *Economic Theory* **65**, 509-544.
- Weber, R. (1985), "Auctions and Competitive Bidding," in *Fair Allocation* (H.P. Young, Ed.), Am. Math. Soc., Providence, RI.