

## Topological many-body scar states in dimensions one, two, and three

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We propose an exact construction for atypical excited states of a class of nonintegrable quantum many-body Hamiltonians in one, two, and three dimensions that display area-law entanglement entropy. These examples of many-body “scar” states have, by design, other properties, such as topological degeneracies, usually associated with the gapped ground states of symmetry-protected topological phases or topologically ordered phases of matter.

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### I. INTRODUCTION

Until recently, the study of many-body quantum systems has largely focused on ground-state properties and low-energy excitations, implicitly assuming the eigenstate thermalization hypothesis (ETH) dictating that highly excited states of generic nonintegrable models are void of interesting structures [1–4]. With the discovery of quantum systems that violate the ETH, a broader interest in the physics of many-body excited states emerged [5]. This modern development is complemented by the growing potential of quantum simulators, predominantly using ultracold atomic gases, to prepare and study quantum many-body systems that are well isolated from the environment [6–8].

Theoretical indicators for the violation of the ETH by a conserved quantum many-body Hamiltonian include (i) a subvolume law scaling for the entanglement entropy of eigenstates [9], (ii) emergent local constants of motion in a nonintegrable system [10,11], and (iii) oscillations in the expectation value of suitably chosen local observables under the unitary time evolution [12].

Two examples of ETH-violating conserved quantum Hamiltonians are those that either support (1) quantum many-body localized phase [13–20], in which nearly all eigenstates at finite-energy density share properties (i) and (ii), and (2) a phase with many-body quantum scar states, in which only a small set of states embedded in a continuum of thermalizing states show such exotic behavior [9,12,21–26]. Here, we will be concerned with examples of type (2).

Theoretical studies of such ETH-violating systems are challenging for two reasons. Progress is hard because the

models in question are, by definition, nonintegrable and ETH violation is a nonperturbative effect. Analytical approaches for many-body localization have been used in Refs. [15,20]. A countably infinite [9,24] and finite [26] series of low-entanglement excited states in thermalizing nonintegrable systems have been constructed exactly. Numerical techniques to obtain highly excited states rely on exact diagonalization [27] and, in some cases, matrix-product-state calculations [28]. These numerical techniques are limited in that the range of available system sizes is often too small to allow an extrapolation to the thermodynamic limit. For these reasons, the majority of studies on ETH violation have been focused on one-dimensional (1D) models.

In this work we present a generic construction that places a scar state in the spectrum of nonintegrable many-body quantum systems in one, two, and three dimensions (1D, 2D, and 3D). While the construction of such states applies to many systems, our primary focus is on topological scar states. In 1D, we construct symmetry-protected topological (SPT) states [29]. In 2D, we present a nonintegrable deformation of the toric code, with fourfold-degenerate scar states on the torus. Finally, in 3D we present a deformation of the  $X$ -cube model [30,31] as an example of a system with scars that display fracton topological order [30–35].

### II. CONDITIONS AND GENERAL FORMALISM

Our construction is inspired by families of Hamiltonians that have been studied in the context of quantum dimer models and spin liquids [36–42]. In those studies, the emphasis was on the construction of parent Hamiltonians for a given ground state.

Consider any Hamiltonian of the form

$$H_0 := \sum_s A_s, \quad (2.1)$$

where the set  $\{A_s\}$  is made of mutually commuting and Hermitian operators that square to the identity. We also assume

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TABLE I. Definition of the operators introduced in Sec. II for each of the models studied in this paper.

| Sec. II: General construction | $A_s$                 | $\alpha_s$                 | $M_s$                                   | $Q_s$                                                        | $M$                                |
|-------------------------------|-----------------------|----------------------------|-----------------------------------------|--------------------------------------------------------------|------------------------------------|
| Sec. III: Warmup example      | $X_i$                 | $\alpha_i$                 | $Z_{i-1} Z_i + Z_i Z_{i+1}$             | $e^{-\beta(Z_{i-1} Z_i + Z_i Z_{i+1})} - X_i$                | $\sum_j Z_j Z_{j+1}$               |
| Sec. IV: 1D ( $a = 1, 2$ )    | $Z_{j-1} X_j Z_{j+1}$ | $\alpha_{1,j}^{\text{1D}}$ | $X_{j-1} + X_{j+1}$                     | $e^{-\beta_a(X_{j-1} + X_{j+1})} - Z_{j-1} X_j Z_{j+1}$      | $\sum_{j \in \text{SL}_a} X_{j-1}$ |
| Sec. V: 2D                    | $A_s$                 | $\alpha_s$                 | $\sum_{i \in s \cap \mathcal{P}_1} Z_i$ | $\exp(-\beta_1 \sum_{i \in s \cap \mathcal{P}_1} Z_i) - A_s$ | $\sum_{i \in \mathcal{P}_1} Z_i$   |
|                               | $B_p$                 | $\alpha_p$                 | $\sum_{i \in p \cap \mathcal{P}_1} X_i$ | $\exp(-\beta_2 \sum_{i \in p \cap \mathcal{P}_2} X_i) - B_p$ | $\sum_{i \in \mathcal{P}_2} X_i$   |
| Sec. VI: 3D                   | $A_s$                 | $\alpha_s$                 | $\sum_{i \in s \cap \mathcal{P}_1} Z_i$ | $\exp(-\beta_1 \sum_{i \in s \cap \mathcal{P}_1} Z_i) - A_s$ | $\sum_{i \in \mathcal{P}_1} Z_i$   |
|                               | $B_c$                 | $\alpha_c$                 | $\sum_{i \in c \cap \mathcal{P}_1} X_i$ | $\exp(-\beta_2 \sum_{i \in c \cap \mathcal{P}_2} X_i) - B_c$ | $\sum_{i \in \mathcal{P}_2} X_i$   |

that  $H_0$  does not commute with an extensive set  $\{W_n\}$  of operators  $W_n$  other than products of  $A_s$ . We are interested in the deformation

$$H(\beta) := \sum_s \alpha_s Q_s(\beta),$$

$$Q_s(\beta) := e^{-\beta M_s} - A_s, \tag{2.2}$$

where  $\alpha_s$  and  $\beta$  are real-valued parameters. Additionally, we have introduced the operators

$$M_s := \sum_{u: \{A_s, B_u\} = 0} B_u, \tag{2.3}$$

where  $B_u$  are local operators labeled by a set of sites  $u$  chosen such that  $\{M_s\}$  anticommutes with sufficiently many of the  $\{A_s\}$  so that, by design,  $A_s$  are no longer constants of the motion for  $H(\beta \neq 0)$ . This is a necessary condition for  $H(\beta \neq 0)$  not to be integrable. We will use numerical techniques (the computation of level statistics [16,43]) to verify that  $H(\beta \neq 0)$  is indeed not integrable.

The operators  $Q_s(\beta)$  were built by design so as to share a common null state  $|\Psi(\beta)\rangle$ ,

$$Q_s(\beta) |\Psi(\beta)\rangle = 0, \quad \forall s. \tag{2.4a}$$

This common null state  $|\Psi(\beta)\rangle$  is obtained from the common eigenstate  $|\Psi_0\rangle$  of all  $A_s$ ,

$$A_s |\Psi_0\rangle = a_0 |\Psi_0\rangle, \quad a_0 \in \mathbb{R}, \quad \forall s, \tag{2.4b}$$

by the similarity transformation

$$|\Psi(\beta)\rangle := e^{\frac{\beta}{2} M} |\Psi_0\rangle, \quad M := \sum_u B_u. \tag{2.4c}$$

[For instance, at the Rokhsar-Kivelson point of the quantum dimer model on the square lattice,  $s$  would be a plaquette and the operators  $Q_s(\beta)$  are projectors that encode both the potential and kinetic (plaquette flip) terms [36,39,41].]

If all the couplings  $\alpha_s$  are positive, the state  $|\Psi(\beta)\rangle$  is the ground state of  $H(\beta)$ , as the  $Q_s(\beta)$  are positive-semidefinite in view of the identity

$$(Q_s(\beta))^2 = 2 \cosh(\beta M_s) Q_s(\beta). \tag{2.5}$$

If the  $\alpha_s$  take positive or negative values depending on  $s$ , then one cannot guarantee anymore that  $|\Psi(\beta)\rangle$  is the ground state. It is, nonetheless, still an eigenstate with energy  $E = 0$ . Even though the state is in the middle of the spectrum of  $H(\beta)$ , it is an atypical state in that it displays area-law entanglement entropy since it is also a ground state of a *different* local Hamiltonian  $\hat{H}(\beta) := \sum_s |\alpha_s| Q_s(\beta)$ . Hence,  $|\Psi(\beta)\rangle$  is a scar

state if, furthermore,  $H(\beta)$  is nonintegrable. Reference [22] presents an alternative analytical construction of scar states; we explain the connection to ours in Appendix A.

An interesting possibility arises when there exists a nonextensive set  $\{W_n\}$  of operators that are (i) nonlocal, (ii) commute with  $H_0$  and  $H(\beta)$  for all  $\beta$ , and (iii) endows the scar states with topological attributes (through their  $W_n$  eigenvalues). This is possible in 2D and higher dimensions. If so, a topological degeneracy of the zero-energy eigenspace of  $H_0$  is retained for  $H(\beta)$ . There follows a set of degenerate scar states that are topologically distinct (through their  $W_n$  eigenvalues).

By deforming exactly solvable models (the toric code, for instance), one can break integrability while retaining the  $E = 0$  scar state. [In Appendix B we show how to construct noncommuting  $Q_s(\beta)$  operators with the desired properties starting from solvable models with commuting projectors.] In what follows, we construct topological scar states in 1D, 2D, and 3D.

For convenience, we present, for each model in this paper, the precise definition of the quantities corresponding to  $A_s, \alpha_s, M_s, Q_s$ , and  $M$  in Table I.

### III. A WARMUP EXAMPLE

We start with a simple example in 1D, which is topologically trivial, but illustrates the general ideas in a straightforward way. Consider a quantum spin- $\frac{1}{2}$  1D chain with periodic boundary conditions, i.e., a ring, with  $L$  sites. On each site  $i = 1, \dots, L$ , we denote the three Pauli operators by  $X_i, Y_i$ , and  $Z_i$ . For any  $\beta \geq 0$ , we define the local Hamiltonian

$$H(\beta) := \sum_i \alpha_i Q_i(\beta), \tag{3.1a}$$

$$\alpha_i := \alpha + (-1)^i,$$

$$Q_i(\beta) := e^{-\beta(Z_{i-1} Z_i + Z_i Z_{i+1})} - X_i, \tag{3.1b}$$

with  $0 < |\alpha| < 1$ . The condition  $|\alpha| < 1$  is required to place the scar state in the middle of the spectrum; the condition  $\alpha \neq 0$  is needed so as not to break the system into two independent (and integrable) transverse-field Ising chains.

At  $\beta = 0$ , the system is equivalent to a paramagnetic spin chain in a Zeeman field, which is integrable. With  $\beta \neq 0$ , all the nearest-neighbor terms no longer commute, i.e.,  $[Q_i(\beta), Q_{i\pm 1}(\beta)] \neq 0$ . In this case,  $H(\beta)$  should no longer be integrable, a fact confirmed by analysis of the energy-level statistics obtained numerically as we now explain. We study the statistics of the spacings between consecutive energy levels,  $s_n := E_{n+1} - E_n$ , as well as the  $r$  value defined as the

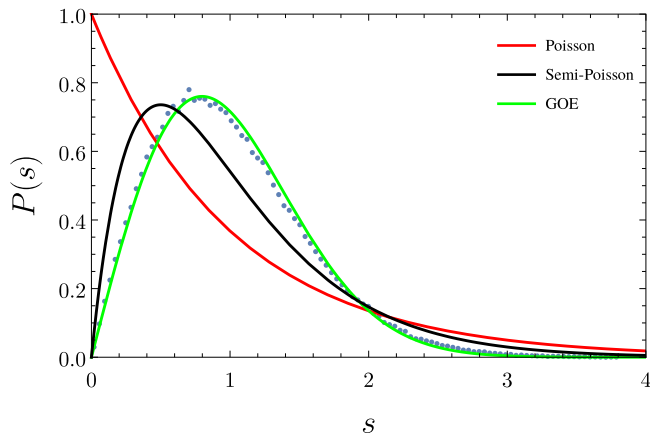


FIG. 1. Distribution of consecutive energy-level spacings  $s_n$  for the 1D Hamiltonian  $H$  defined in Eq. (3.1) with  $L = 20$ ,  $\alpha = 0.3$ , and  $\beta = 0.5$ . The distributions for the  $s_n$  from all momentum sectors, except for  $k = 0, \pi$ , have been joined. The middle 60% of the spectrum in each sector is taken. The distribution obtained can be seen to be well approximated by the Gaussian orthogonal ensemble (GOE) of random matrix theory.

average  $\langle r_n \rangle$  of the ratios  $r_n := \min(s_n, s_{n-1})/\max(s_n, s_{n-1})$  [16,43]. We analyze the spectrum in common eigenspaces of a maximal set of commuting symmetries of the system, namely translation, parity under inversion, and an additional  $\mathbb{Z}_2$ -valued parity defined by  $\prod_i X_i = \pm 1$ . Figure 1 contains the result of this analysis for  $\alpha = 0.3$ ,  $\beta = 0.5$ , and  $L = 20$ . The distribution matches the distribution of eigenvalue spacings for the Gaussian orthogonal ensemble (GOE) of random matrices, thus supporting the claim that Hamiltonian (3.1) is nonintegrable. The corresponding mean  $r$  value for our distribution (averaged over the different momentum sectors) is  $\langle r \rangle = 0.531$ , close to that of the GOE,  $r_{\text{GOE}} = 0.5359$ , and clearly distinct from the value of the Poisson distribution  $r_{\text{Poisson}} = 0.3863$ .

One can verify that the state

$$|\text{scar}(\beta)\rangle := G(\beta) \bigotimes_i |+\rangle_i^x, \quad (3.2a)$$

where  $|+\rangle_i^x$  is the eigenstate of  $X_i$  with the eigenvalue +1 and

$$G(\beta) := \exp\left(\frac{\beta}{2} \sum_j Z_j Z_{j+1}\right) \quad (3.2b)$$

is annihilated by the operators  $Q_i(\beta)$  for all  $i$ . Therefore,  $|\text{scar}(\beta)\rangle$  is an eigenstate of  $H(\beta)$  with eigenvalue 0. (In Appendix C, we generalize this model to have multiple scars at different energies.)

That this eigenstate obeys area-law entanglement entropy can be seen as follows. The operators  $Q_i(\beta)$  are positive-semidefinite definite, owing to the identity  $Q_i^2(\beta) = 2 \cosh(\beta(Z_{i-1}Z_i + Z_iZ_{i+1}))Q_i(\beta)$ . Therefore,  $|\text{scar}\rangle$  is the ground state of another (local) Hamiltonian  $\hat{H}(\beta) := \sum_i |\alpha_i|Q_i(\beta)$ . The spectrum of  $\hat{H}(0)$  has a gap between its ground state and all excited states, a gap that remains for a finite range of values of  $\beta$ . Therefore,  $|\text{scar}(\beta)\rangle$  obeys area-law entanglement entropy for a range of  $\beta$  [44]. Alternatively,

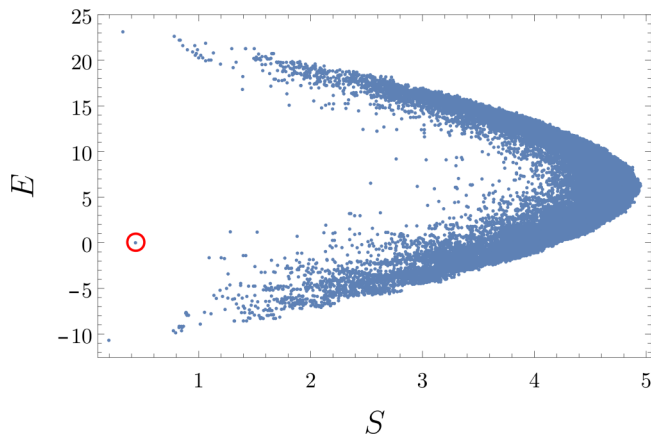


FIG. 2. Entanglement entropy of the eigenstates of Hamiltonian (3.1) for a real-space bipartition of the system into two equal halves. The parameters are set at  $L = 16$ ,  $\beta = 0.5$ , and  $\alpha = 0.3$ . The analytically obtained scar state has  $E = 0$  (red circle) and is well separated from the highly entangled states.

the area-law property of  $|\text{scar}(\beta)\rangle$  can be argued from the form of Eq. (3.2) for any  $\beta$ , by noting that it can be represented by a quantum circuit of constant depth (independent of both  $\beta$  and system size), applied to a product state [45,46].

In Fig. 2, we present the entanglement entropy for the different eigenstates of  $H(\beta)$  for  $\alpha = 0.3$ ,  $\beta = 0.5$ , and  $L = 16$ . Notice that the  $E = 0$  scar state is embedded within highly entangled states.

#### IV. 1D: SPT CLUSTER MODEL

Consider a quantum spin- $\frac{1}{2}$  ring with  $2L$  sites. Odd and even sites are denoted by  $\text{SL}_1 := \{1, 3, \dots, 2L-1\}$  and  $\text{SL}_2 := \{2, 4, \dots, 2L\}$ , respectively. For any  $\beta_a \geq 0$  with  $a = 1, 2$ , we define the Hamiltonians

$$H^{\text{1D}} := H_1^{\text{1D}} + H_2^{\text{1D}},$$

$$H_a^{\text{1D}} := \sum_{j \in \text{SL}_a} \alpha_{a,j}^{\text{1D}} Q_{a,j}^{\text{1D}}, \quad (4.1a)$$

$$\alpha_{a,j}^{\text{1D}} := \alpha + (-1)^{\frac{j-a}{2}},$$

$$Q_{a,j}^{\text{1D}} := e^{-\beta_a(X_{j-1} + X_{j+1})} - Z_{j-1} X_j Z_{j+1}. \quad (4.1b)$$

Note that  $[H_1^{\text{1D}}, H_2^{\text{1D}}] = 0$  for any  $\beta_1$  and  $\beta_2$ . For  $\beta_1 = \beta_2 = 0$ ,  $H^{\text{1D}}$  is exactly solvable and its ground state is a gapped SPT state [47,48]. Its topological attributes originate from symmetry-protected zero modes that are localized at the two ends of an open chain when open boundary conditions are imposed instead of periodic ones. The symmetry protecting the boundary states is  $\mathbb{Z}_2 \times \mathbb{Z}_2$  as shown in the Appendix D [29]. Being gapped at  $\beta_1 = \beta_2 = 0$ , the SPT phase extends to nonvanishing but sufficiently small  $\beta_1 > 0$  and  $\beta_2 > 0$ . (See Ref. [49] for another deformation of 1D SPT Hamiltonians.)

The null state for  $\beta_1 = \beta_2 = 0$  is an eigenstate of the  $Z_{i-1} X_i Z_{i+1}$  operators,  $i = 1, \dots, 2L$ , with eigenvalue +1. We

denote this state by  $|+, \dots, +\rangle$ . For  $\beta_1 > 0$  and  $\beta_2 > 0$ , the null state of Eq. (4.1a) is

$$|\text{scar}^{\text{1D}}\rangle := G_1^{\text{1D}} G_2^{\text{1D}} |+, \dots, +\rangle, \quad (4.2a)$$

obtained via a similarity transformation with

$$G_a^{\text{1D}} := \exp\left(\frac{\beta_a}{2} \sum_{j \in \text{SL}_a} X_{j-1}\right). \quad (4.2b)$$

It remains to be shown that the Hamiltonian is nonintegrable. Since the Hamiltonian is made up of two commuting pieces  $H_1^{\text{1D}}$  and  $H_2^{\text{1D}}$ , one must show that each component alone is nonintegrable. We shall reduce the calculation of the energy-level statistics to the problem already solved for the topologically trivial warmup example of the Hamiltonian  $H(\beta)$  in Eq. (3.1), presented previously. The mapping is via a nonlocal unitary transformation

$$W := \exp\left(i\frac{\pi}{4} \sum_{j \in \text{SL}_1} Z_j Z_{j+1} - i\frac{\pi}{4} \sum_{j \in \text{SL}_2} Z_j Z_{j+1}\right), \quad (4.3)$$

which maps  $Q_{a,j}^{\text{1D}}$  into  $\tilde{Q}_{a,j}^{\text{1D}} := W Q_{a,j}^{\text{1D}} W^\dagger$  where

$$\tilde{Q}_{a,j}^{\text{1D}} = e^{-\beta_a (Z_{j-2} X_{j-1} Z_j + Z_j X_{j+1} Z_{j+2})} - X_j. \quad (4.4)$$

The spectrum of  $H_a^{\text{1D}}$  can be related to that of  $H$  by noticing that the operators  $X_i$  with  $i \in \text{SL}_2$  that appear in the exponentials in Eq. (4.4) have no dynamics within  $H_1^{\text{1D}}$ , and *vice versa*, the  $X_i$  with  $i \in \text{SL}_1$  have no dynamics within  $H_2^{\text{1D}}$ . For the purpose of obtaining the eigenvalues of  $H_1^{\text{1D}}$ , one can freeze the  $X_i$ ,  $i \in \text{SL}_2$ ; there are only two gauge-inequivalent choices depending on the  $\mathbb{Z}_2$  sector selected, i.e., the choice of  $\prod_{i \in \text{SL}_2} X_i = \pm 1$ . (This symmetry is one of the two  $\mathbb{Z}_2$ 's in the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  in  $\beta_1 = \beta_2 = 0$  case.) The spectrum of  $H_1^{\text{1D}}$  in the  $+$  sector (equivalent to fixing  $X_i = +1$ ,  $i \in \text{SL}_2$ ) reduces to that of  $H$  that we studied previously. We thus conclude that the 1D SPT scar from Eq. (4.2a) is an exceptional state in the spectrum of a nonintegrable Hamiltonian  $H_1^{\text{1D}} + H_2^{\text{1D}}$ .

## V. EXAMPLE IN 2D: TORIC CODE

In 2D, we study a lattice model derived from the toric code [50]. The Hamiltonian

$$H^{2\text{D}} := H_1^{2\text{D}} + H_2^{2\text{D}} \quad (5.1a)$$

is defined by the pair of commuting operators

$$H_1^{2\text{D}} := \sum_s \alpha_s \left[ \exp\left(-\beta_1 \sum_{i \in s \cap \mathcal{P}_1} Z_i\right) - A_s \right], \quad (5.1b)$$

$$H_2^{2\text{D}} := \sum_p \alpha_p \left[ \exp\left(-\beta_2 \sum_{i \in p \cap \mathcal{P}_2} X_i\right) - B_p \right], \quad (5.1c)$$

where  $s$  labels a star and  $p$  a plaquette (see Fig. 3):

$$A_s := \prod_{i \in s} X_i \quad (5.1d)$$

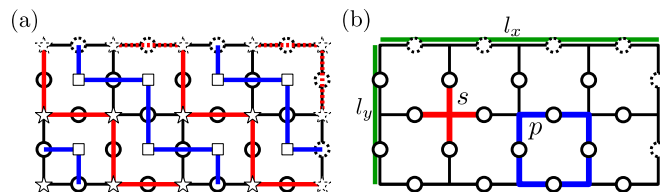


FIG. 3. Example of a lattice structure of the 2D model. Dashed sites and lines are used to represent periodic boundary conditions. (a) Starting from a  $(N_x \times N_y = 2 \times 4)$  square lattice  $\Lambda_\star$ , we define the median and dual lattices  $\Lambda_\circ$  and  $\Lambda_\square$  in such a way that sites of  $\Lambda_\star$ ,  $\Lambda_\circ$ , and  $\Lambda_\square$  are represented by the symbols  $\star$ ,  $\circ$ , and  $\square$ , respectively. The red (blue) path  $\mathcal{P}_1$  ( $\mathcal{P}_2$ ) along the bonds of  $\Lambda_\star$  ( $\Lambda_\square$ ) goes through all sites  $\star \in \Lambda_\star$  ( $\square \in \Lambda_\square$ ) without intersecting itself. (b) The toric code assigns a local spin- $\frac{1}{2}$  degree of freedom to each site  $\circ$  of the median lattice  $\Lambda_\circ$ . To each site  $\star$  ( $\square$ ) of the lattice  $\Lambda_\star$  ( $\Lambda_\square$ ), we assign the subset  $s$  ( $p$ ) consisting of the four sites of  $\Lambda_\circ$  on the red cross (blue square) at the site  $\star$  ( $\square$ ) and define the star (plaquette) operator  $A_s := \prod_{i \in s} X_i$  ( $B_p := \prod_{i \in p} Z_i$ ). The two orthogonal green lines are the “electric” paths  $l_x$  and  $l_y$  needed to define two Wilson loops  $W_\mu := \prod_{i \in l_\mu \cap \Lambda_\circ} Z_i$  with  $\mu = x, y$ , respectively.

and

$$B_p := \prod_{i \in p} Z_i. \quad (5.1e)$$

When  $\beta_{1,2} = 0$ ,  $H^{2\text{D}}$  reduces to the usual toric code, up to an additive constant. This means that the spectrum of  $H^{2\text{D}}$  is then fully gapped. In addition, there exist two pairs of topological operators, each of which generates an independent  $\mathbb{Z}_2$  symmetry group. As was the case for the SPT phase from Sec. IV, this gap persists for small  $\beta_1 > 0$  and  $\beta_2 > 0$ , while it vanishes at and for values larger than some nonvanishing threshold values of  $\beta_1 > 0$  and  $\beta_2 > 0$ .

We define

$$\alpha_s := \alpha + (-1)^{\rho_s}, \quad \alpha_p := \alpha + (-1)^{\rho_p}, \quad (5.2)$$

such that  $\rho_s$  ( $\rho_p$ ) is equal to 0 on one sublattice and 1 on the other sublattice of the lattice  $\Lambda_\star$  ( $\Lambda_\square$ ). Here,  $\Lambda_\star$  is the lattice formed by the centers of all the stars, and  $\Lambda_\square$  is the lattice formed by the centers of all the plaquettes. Our deformation of the toric code for  $\beta_{1,2} \neq 0$  uses the paths  $\mathcal{P}_1$  and  $\mathcal{P}_2$  on  $\Lambda_\star$  and  $\Lambda_\square$ , respectively. These paths are connected, nonintersecting, and chosen such that all the spins are on either of the two paths. (An example of such paths  $\mathcal{P}_{1,2}$  is presented in Fig. 3, and we give further examples in Appendix G.) These conditions on  $\mathcal{P}_{1,2}$  guarantee that (a)  $H_1^{2\text{D}}$  commutes with  $H_2^{2\text{D}}$ , (b)  $H_1^{2\text{D}}$  and  $H_2^{2\text{D}}$  commute with all the generators of the space group of the lattice, (c) there are no further constants of motion, and (d) the spectrum of  $H_1^{2\text{D}}$  alone is equal to that of  $H_1^{\text{1D}}$  for a path  $\mathcal{P}_1$  of length  $L$  (up to exact degeneracies due to a different number of constants of motion in 1D and 2D). To obtain (d), one notes that  $Z_i$  for spins not in  $\mathcal{P}_2$  are constants of motion of  $H_2^{2\text{D}}$ . Replacing them by their eigenvalue  $\pm 1$  reduces  $H_2^{2\text{D}}$  to the form of  $H_2^{\text{1D}}$  for an appropriate choice of its constants of motion  $X_j$  for  $j \in \text{SL}_2$  in Eq. (4.1b), upon labeling the spins along  $\mathcal{P}_2$  in the order of the 1D chain. We conclude that the level statistics of  $H_1^{2\text{D}}$  and  $H_1^{\text{1D}}$  are identical



up to exact degeneracies. Hence, the numerical evidence for the nonintegrability of  $H_1^{1D}$  directly carries over to  $H_1^{2D}$ . In our model, the extensive symmetries at  $\beta_1 = \beta_2 = 0$  arising from  $[A_s, B_p] = 0$  are lifted when  $\beta_{1,2} \neq 0$  (in which case  $H_{1,2}^{2D}$  are no longer sums of commuting projectors).

The scar states are built as follows. Because  $A_s$  and  $B_p$  square to unity and satisfy

$$\prod_s A_s = \prod_p B_p = \mathbb{1}, \quad (5.3a)$$

we can build a vector

$$\lambda \in \{-, +\}^{2N_x N_y - 2} \quad (5.3b)$$

out of the distinct eigenvalues of  $(N_x N_y - 1)$  independent  $A_s$ 's and  $(N_x N_y - 1)$  independent  $B_p$ 's to label an orthogonal basis  $|\lambda\rangle$  of a  $2^{2N_x N_y - 2}$ -dimensional subspace of the  $2^{2N_x N_y}$ -dimensional Hilbert space on which  $H^{2D}$  acts. To complete the basis of the Hilbert space, we use the eigenstates  $|\omega\rangle$  with the eigenvalues

$$\omega \equiv (\omega_x = \pm, \omega_y = \pm) \quad (5.3c)$$

of the pair of Wilson-loop operators  $W_\mu$  with  $\mu = x, y$  defined in Fig. 3. The four scar states

$$|\text{scar}^{2D}; \omega\rangle := G_1^{2D} G_2^{2D} |+, \dots, +; \omega\rangle, \quad (5.4a)$$

$$G_1^{2D} := \exp\left(\frac{\beta_1}{2} \sum_{i \in \mathcal{P}_1} Z_i\right), \quad (5.4b)$$

$$G_2^{2D} := \exp\left(\frac{\beta_2}{2} \sum_{i \in \mathcal{P}_2} X_i\right)$$

(one in each of the four topological sectors) are then eigenstates of  $H^{2D}$  with the eigenvalues  $E = 0$ .

## VI. 3D EXAMPLE: X-CUBE MODEL

The 2D construction from Sec. V can be extended in a straightforward way to 3D toric code-type Hamiltonians [51]. Here, we derive scar states for the slightly more exotic fracton topological order, which only arises in three or more dimensions [30–32,34,35]. Fracton phases carry excitations whose mobilities are limited to certain submanifolds of space and support topological ground-state degeneracies that scale exponentially with the system size. Here, we introduce a Hamiltonian based on the X-cube model [31], which supports fracton topological order in its ground state, to construct a set of 3D scar states with the same exponential degeneracy. The Hamiltonian

$$H^{3D} := H_1^{3D} + H_2^{3D} \quad (6.1a)$$

is defined by the pair of commuting operators

$$H_1^{3D} := \sum_s \alpha_s \left[ \exp\left(-\beta_1 \sum_{i \in \mathcal{P}_1} Z_i\right) - A_s \right], \quad (6.1b)$$

$$H_2^{3D} := \sum_c \alpha_c \left[ \exp\left(-\beta_2 \sum_{i \in \mathcal{P}_2} X_i\right) - B_c \right], \quad (6.1c)$$

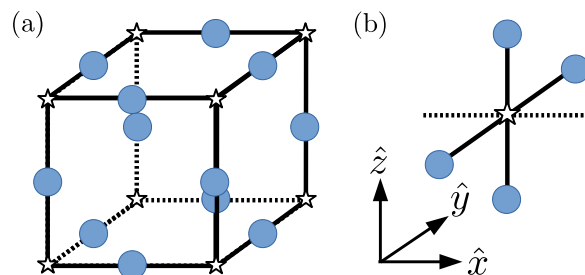


FIG. 4. The notation  $\Lambda_\star$ ,  $\Lambda_\circ$ , and  $\Lambda_\square$  of Fig. 3 becomes  $\Lambda_\star$ ,  $\Lambda_\circ$ , and  $\Lambda_{\boxplus}$ , where  $\Lambda_\star$  denotes the cubic lattice,  $\Lambda_\circ$  its median lattice, and  $\Lambda_{\boxplus}$  its dual lattice. (a) The elementary unit cell  $\boxplus$  of  $\Lambda_\star$  is cubic. Spin- $\frac{1}{2}$  degrees of freedom represented by  $\circ$  are located on its midbonds. The 12  $\circ$ 's on the bonds of a  $\boxplus$  define a subset  $c \subset \Lambda_\circ$ . The corners of  $\boxplus$  define sites  $\star$  of  $\Lambda_\star$ . The center of  $\boxplus$  defines a site from  $\Lambda_{\boxplus}$ . For any such  $\boxplus$ , we define  $B_c$  by taking the product of all 12 Pauli matrices  $Z_i$  from the neighboring bonds with  $i \in c \cap \Lambda_\circ$ . (b) The center of a cross  $\boxplus$  joining its four nearest-neighbor sites from  $\Lambda_\circ$  defines a site from  $\Lambda_{\boxplus}$  and the subset  $s \subset \Lambda_\circ$ . There are three oriented crosses for any site from  $\Lambda_\star$ . They are in one-to-one correspondence with the three oriented planes in the Cartesian coordinates of  $\mathbb{R}^3$ . For any such oriented cross, we define  $A_s$  by taking the product of all four Pauli matrices  $X_i$  with  $i \in s$ .

where  $s$  labels a star and  $c$  a cube (see Fig. 4),

$$A_s := \prod_{i \in s} X_i \quad (6.1d)$$

and

$$B_c := \prod_{i \in c} Z_i. \quad (6.1e)$$

When  $\beta_1 = \beta_2 = 0$ ,  $H^{3D}$  reduces to the usual X-cube model, up to a constant. This means that the spectrum of  $H^{3D}$  is then fully gapped. This gap persists for small  $\beta_1 > 0$  and  $\beta_2 > 0$ , while it vanishes at and for values larger than some nonvanishing threshold values of  $\beta_1 > 0$  and  $\beta_2 > 0$ . The algebraic structure of the nonlocal topological operators result in an extensive number of independent  $\mathbb{Z}_2$  symmetries,

We define

$$\alpha_s := \alpha + (-1)^{\rho_s}, \quad \alpha_c := \alpha + (-1)^{\rho_c} \quad (6.2)$$

such that  $\rho_s$  ( $\rho_c$ ) is equal to 0 on one sublattice and 1 on the other sublattice of the lattice  $\Lambda_\star$  ( $\Lambda_{\boxplus}$ ). The paths  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are defined on  $\Lambda_\star$  and  $\Lambda_{\boxplus}$ , respectively, and they obey the same conditions as their counterparts from Sec. V do. These conditions guarantee that

$$[H_1^{3D}, H_2^{3D}] = 0 \quad (6.3)$$

for any  $\beta_1, \beta_2 \geq 0$ , while lifting the extensive symmetries present when  $\beta_1 = \beta_2 = 0$  arising from

$$[A_s, B_c] = 0 \quad (6.4)$$

because neither  $H_1^{3D}$  nor  $H_2^{3D}$  are no longer sums of commuting projectors when  $\beta_1, \beta_2 > 0$ .

The Hilbert space for a cubic lattice of linear size  $L$  is  $2^{3L^3}$ -dimensional (there are  $L^3$  sites in  $\Lambda_\star$  and  $3L^3$  in  $\Lambda_\circ$ ).

By counting the number of independent stars and cubes, we obtain the extensive number of vectors of quantum numbers

$$\lambda \in \{-, +\}^{3L^3-6L+3}, \quad (6.5a)$$

which is to be complemented by the subextensive number of vectors of topological quantum numbers

$$\zeta \in \{-, +\}^{6L-3}. \quad (6.5b)$$

The number of scar states that are eigenstates of  $H^{3D}$  with the eigenenergy  $E = 0$  thus grows subextensively with the linear size  $L$  of  $\Lambda_*$ . These scar states are given by

$$|\text{scar}^{3D}; \zeta\rangle := G_1^{3D} G_2^{3D} |+, \dots, +; \zeta\rangle, \quad (6.6a)$$

$$G_1^{3D} := \exp\left(\frac{\beta_1}{2} \sum_{i \in \mathcal{P}_1} Z_i\right),$$

$$G_2^{3D} := \exp\left(\frac{\beta_2}{2} \sum_{i \in \mathcal{P}_2} X_i\right). \quad (6.6b)$$

## VII. CONCLUSIONS

We propose a construction to obtain scar states (excited states with sub-volume-law entanglement entropy scaling that are embedded in a dense spectrum of volume-law scaling states) based on stochastic matrix form Hamiltonians [39,41,42]. Starting from an integrable parent Hamiltonian, we deform it into a nonintegrable one such that the ground states of the parent model develop into quantum many-body scars while preserving their analytical constructability. When the parent models are topological, the topological degeneracy is inherited directly, resulting in a finite (toric code) or an exponential (fracton models) number of degenerate scars. Whether these degeneracies are topological in that they retain a sense of robustness against small generic local perturbations is left as a problem for future work.

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## APPENDIX A: RELATION TO THE CONSTRUCTION FOR SCAR STATES FROM PREVIOUS STUDIES

In this Appendix, we show that there exists a unitary transformation that brings Hamiltonian (2.2) with the property (2.4) into the form of the family of Hamiltonians defined

in Eqs. (1) and (2) from Ref. [22]. However, we emphasize that Hamiltonian (2.2) stems from the stochastic matrix form Hamiltonians introduced in Ref. [41], wherein the property (2.4) was proven.

We present the local Hermitian operator  $Q_s$  in Eq. (2.2) (the  $\beta$  dependence is implicit) as

$$Q_s = \sum_{a(s)} \lambda_{a(s)} |\psi_{a(s)}\rangle\langle\psi_{a(s)}|, \quad (A1a)$$

where  $a(s)$  labels the orthogonal eigenstates  $|\psi_{a(s)}\rangle$  with the real-valued eigenvalues  $\lambda_{a(s)}$  of  $Q_s$ . The consequence of the locality of  $Q_s$ , in this paper, is that its spectrum is bounded and discrete. Moreover, by construction,  $Q_s$  has zero eigenvalues. We denote by  $\mathcal{T}(s)$  the kernel of  $Q_s$ , i.e., the subspace spanned by the eigenvectors with vanishing eigenvalues  $\lambda_{a'(s)} = 0$ . [From here, we use primed label  $a'(s)$  for  $a'(s) \in \mathcal{T}(s)$  and unprimed label  $a(s)$  for  $a(s) \notin \mathcal{T}(s)$ .] We shall define the local projector

$$P_s := \sum_{a(s) \notin \mathcal{T}(s)} |\psi_{a(s)}\rangle\langle\psi_{a(s)}| \quad (A1b)$$

that assigns to all eigenspaces of  $Q_s$  with nonzero eigenvalue the eigenvalue 1. The eigenvalue of the null state  $|\Psi(\beta)\rangle$  with respect to both  $P_s$  and  $Q_s$  is 0 for all  $s$ . We define the local Hermitian operator

$$\tilde{Q}_s := \sum_{a(s) \notin \mathcal{T}(s)} \lambda_{a(s)} |\psi_{a(s)}\rangle\langle\psi_{a(s)}| + U \sum_{a'(s) \in \mathcal{T}(s)} |\psi_{a'(s)}\rangle\langle\psi_{a'(s)}| \quad (A2a)$$

together with the counterpart to Eq. (2.2) defined by

$$\begin{aligned} \tilde{H} &:= \sum_s \alpha_s \tilde{Q}_s \\ &= \sum_s \sum_{a(s) \notin \mathcal{T}(s)} \alpha_s \lambda_{a(s)} |\psi_{a(s)}\rangle\langle\psi_{a(s)}| \\ &\quad + U \sum_{s'} \sum_{a'(s') \in \mathcal{T}(s')} \alpha_{s'} |\psi_{a'(s')}\rangle\langle\psi_{a'(s')}| \\ &\equiv \sum_s P_s \tilde{h}_s P_s + \tilde{H}', \end{aligned} \quad (A2b)$$

where

$$\begin{aligned} \tilde{h}_s &:= \alpha_s \sum_{a(s)} \lambda_{a(s)} |\psi_{a(s)}\rangle\langle\psi_{a(s)}|, \\ \tilde{H}' &:= U \sum_{s'} \sum_{a'(s') \in \mathcal{T}(s')} \alpha_{s'} |\psi_{a'(s')}\rangle\langle\psi_{a'(s')}|. \end{aligned} \quad (A2c)$$

The projector defined by Eq. (A1b) and  $\tilde{H}'$  fulfill all the conditions of their counterparts in Eqs. (1) and (2) from Ref. [22], respectively. Since  $U \in \mathbb{R}$  is allowed to take the value 0, in which case  $\tilde{Q}_s = Q_s$ ,  $\tilde{H} = H$ , and  $[\tilde{H}', P_s] = 0$ , our Hamiltonian  $H$  in Eq. (2.2) belongs to the family of Hamiltonians defined by Ref. [22].

## APPENDIX B: CONSTRUCTION OF HAMILTONIANS CONTAINING NULL STATES

We are going to construct Hamiltonians hosting null eigenstates starting from a solvable model. This construction will

hinge on a similarity transformation. Consider first operators  $A_s$  satisfying

$$A_s^2 = \mathbb{1}, \quad [A_s, A_{s'}] = 0, \quad \forall s, s' \quad (\text{B1a})$$

where  $s$  labels bounded regions in space, for instance any finite subset of sites from a lattice. The notion of locality is tied to the fact that the region on which these operators act nontrivially is bounded. More precisely, for two sites  $i, j \in s$ , the distance between the sites is bounded,  $|i - j| < d_s$ , where  $d_s$  is the finite ‘‘diameter’’ of the region  $s$ . Notice that the set of operators  $\mathbb{1} - A_s$  labeled by  $s$  is a set of commuting projectors. Second, we define

$$M := \sum_i O_i, \quad M_s := \sum_{i \in s} O_i, \quad \bar{M}_s := \sum_{i \notin s} O_i, \quad (\text{B1b})$$

where the operators  $O_i$  need not just act at one site  $i$ , but on a bounded subset of sites centered around  $i$ . The operators  $O_i$  are chosen to be Hermitian and to commute

$$[O_i, O_j] = 0, \quad \forall i, j. \quad (\text{B1c})$$

Moreover, we demand that

$$[A_s, M_s] = 0, \quad [A_s, \bar{M}_s] = 0, \quad \forall s. \quad (\text{B1d})$$

(Notice that if  $O_i$  contains exclusively operators at site  $i$ , then  $[A_s, \bar{M}_s] = 0$  follows trivially from the fact that no common site belongs to  $s$  and its complement.) Third, we define

$$\begin{aligned} F_s &:= e^{+\frac{1}{2}\beta M} (\mathbb{1} - A_s) e^{-\frac{1}{2}\beta M} \\ &= \mathbb{1} - e^{+\beta M_s} A_s \\ &= e^{+\beta M_s} (e^{-\beta M_s} - A_s) \end{aligned} \quad (\text{B2a})$$

and

$$Q_s := e^{-\beta M_s} - A_s. \quad (\text{B2b})$$

Notice that  $Q_s$  is Hermitian, while  $F_s$  is not. They are related by

$$Q_s = e^{-\beta M_s} F_s. \quad (\text{B3})$$

In addition to being Hermitian,  $Q_s$  is local because  $A_s$  is local and the exponential of the local operator  $M_s$  is also local. It is positive-semidefinite, as can be inferred by squaring it,

$$Q_s^2 = 2 \cosh(\beta M_s) Q_s, \quad (\text{B4})$$

and observing that  $\cosh(\beta M_s)$  is positive-definite.

We shall now construct a common null state to all the  $Q_s$  operators. First, notice that the state

$$|\Psi_0\rangle := \prod_{s'} (\mathbb{1} + A_{s'}) |\Omega\rangle \quad (\text{B5})$$

is annihilated by  $(\mathbb{1} - A_s)$ , for all  $s$ , for

$$\begin{aligned} (\mathbb{1} - A_s) |\Psi_0\rangle &= (\mathbb{1} - A_s) \prod_{s'} (\mathbb{1} + A_{s'}) |\Omega\rangle \\ &= (\mathbb{1} - A_s) (\mathbb{1} + A_s) \prod_{s' \neq s} (\mathbb{1} + A_{s'}) |\Omega\rangle \\ &= (\mathbb{1} - A_s^2) \prod_{s' \neq s} (\mathbb{1} + A_{s'}) |\Omega\rangle \\ &= 0, \end{aligned} \quad (\text{B6})$$

where we used the fact that  $A_s^2 = \mathbb{1}$  to reach the last equality. The state  $|\Omega\rangle$  is arbitrary, as long as it is not annihilated by the projectors  $(\mathbb{1} + A_{s'})$ . Second, let

$$|\Psi_\beta\rangle := e^{+\frac{1}{2}\beta M} |\Psi_0\rangle. \quad (\text{B7})$$

It follows that, for any  $s$ ,

$$\begin{aligned} F_s |\Psi_\beta\rangle &= e^{+\frac{1}{2}\beta M} (\mathbb{1} - A_s) e^{-\frac{1}{2}\beta M} e^{+\frac{1}{2}\beta M} |\Psi_0\rangle \\ &= e^{+\frac{1}{2}\beta M} (\mathbb{1} - A_s) |\Psi_0\rangle \\ &= 0, \end{aligned} \quad (\text{B8})$$

and consequently

$$Q_s |\Psi_\beta\rangle = e^{-\beta M_s} F_s |\Psi_\beta\rangle = 0. \quad (\text{B9})$$

Therefore, the state  $|\Psi_\beta\rangle$  is a common null state of all the local operators  $Q_s$ , and also of any local Hamiltonian written as a weighted sum of the  $Q_s$ , say

$$H(\beta) := \sum_s \alpha_s Q_s, \quad (\text{B10})$$

for any weights  $\alpha_s \in \mathbb{R}$ . In Eq. (3.1), we chose, in place of  $A_s$  and  $M_s, X_i$  and  $-\beta(Z_{i-1}Z_i + Z_iZ_{i+1})$ , respectively.

### APPENDIX C: TWO SCARS AT DIFFERENT ENERGIES ACQUIRED BY DEFORMING THE HAMILTONIAN DEFINED IN EQ. (3.1)

We present a construction of a Hamiltonian, which carries two scar states with exact analytical expressions at different energies. The system is a one-dimensional spin- $\frac{1}{2}$  periodic ( $i + L \equiv i$ ) chain defined by a local Hamiltonian of the form

$$\underline{H}(\beta) := \sum_i \alpha_i Q_i^-(\beta) Q_{i+2}^+(\beta) + \sum_i P_i(\beta) \quad (\text{C1a})$$

with local terms

$$\begin{aligned} Q_i^+(\beta) &:= e^{+\beta(Z_{i-1}Z_i + Z_iZ_{i+1})} + X_i, \\ Q_i^-(\beta) &:= e^{-\beta(Z_{i-1}Z_i + Z_iZ_{i+1})} - X_i, \end{aligned} \quad (\text{C1b})$$

and

$$\begin{aligned} P_i(\beta) &:= \frac{1}{2} (\mathbb{1} - Z_{i-1}Z_{i+1}) X_i \\ &+ \frac{1}{4 \cosh 2\beta} (\mathbb{1} + Z_{i-1})(\mathbb{1} + Z_{i+1})(\sinh 2\beta Z_i + X_i) \\ &+ \frac{1}{4 \cosh 2\beta} (\mathbb{1} - Z_{i-1})(\mathbb{1} - Z_{i+1})(-\sinh 2\beta Z_i + X_i). \end{aligned} \quad (\text{C1c})$$

As shown in Fig. 5, Hamiltonian (C1a) is nonintegrable and has two scar states. The analytical expressions for these two scar states are

$$\begin{aligned} |\text{scar}^+(\beta)\rangle &:= G(\beta) \bigotimes_i |+\rangle_i^x, \\ |\text{scar}^-(\beta)\rangle &:= G(-\beta) \bigotimes_i |-\rangle_i^x, \end{aligned} \quad (\text{C2})$$

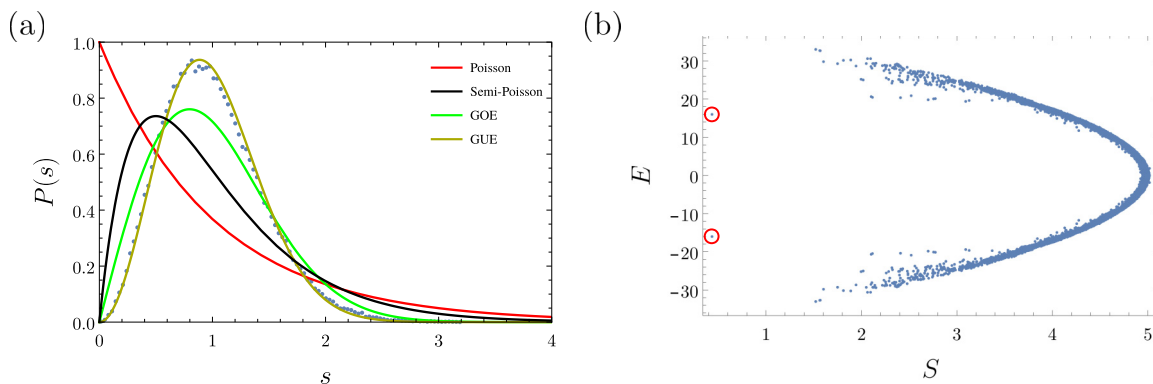


FIG. 5. (a) Level statistics for the Hamiltonian given by Eq. (C1a) with system size  $L = 20$ . As with Fig. 1, the middle 60% of the eigenvalues from all the symmetry sectors except for  $k = 0, \pi$  are used. The distribution follows the Gaussian unitary ensemble (GUE), which indicates nonintegrability. The absence of inversion symmetry in Eq. (C1a) removes the possible connection between two momentum sectors with opposite  $k$ , and makes the matrix elements nonreducible complex numbers. Hence, the trend of GUE, which is different from Fig. 1, is justified. (b) Entanglement entropy–energy plot for every eigenstate of Eq. (C1a) with  $L = 16$ . Two scar states at  $E = -L$  and  $+L$  are marked by red circles. Parameters are set at  $\beta = 0.5$ ,  $\alpha_i = (-1)^i$ .

where  $G(\beta)$  is defined in Eq. (3.2b) and repeated below for convenience:

$$G(\beta) := \exp\left(\frac{\beta}{2} \sum_j Z_j Z_{j+1}\right). \quad (\text{C3})$$

The two scars defined as such are orthogonal and satisfy

$$\begin{aligned} \underline{H}(\beta) |\text{scar}^+(\beta)\rangle &= L |\text{scar}^+(\beta)\rangle, \\ \underline{H}(\beta) |\text{scar}^-(\beta)\rangle &= -L |\text{scar}^-(\beta)\rangle. \end{aligned} \quad (\text{C4})$$

In the following paragraphs, we prove Eq. (C4).

We start by considering the first term in Eq. (C1a). The states  $|\text{scar}^+(\beta)\rangle$  and  $|\text{scar}^-(\beta)\rangle$  are simultaneous zero-energy eigenstates of  $Q_i^-(\beta)$  and  $Q_i^+(\beta)$  for all  $i$ , respectively. In addition,  $Q_i^-(\beta)$  and  $Q_i^+(\beta)$  fulfill

$$[Q_i^-(\beta), Q_{i+2}^+(\beta)] = 0, \quad \forall i. \quad (\text{C5})$$

Therefore, the two states satisfy

$$\begin{aligned} \sum_i \alpha_i Q_i^-(\beta) Q_{i+2}^+(\beta) |\text{scar}^+(\beta)\rangle &= 0, \\ \sum_i \alpha_i Q_i^-(\beta) Q_{i+2}^+(\beta) |\text{scar}^-(\beta)\rangle &= 0. \end{aligned} \quad (\text{C6})$$

The pseudoprojector  $P_i(\beta)$  in the second term in Eq. (C1a) is chosen such that it gives the eigenvalues  $+1$  for  $|\text{scar}^+(\beta)\rangle$  and  $-1$  for  $|\text{scar}^-(\beta)\rangle$ . The form (C1c) is obtained by the following procedure. Define two subspaces of the Hilbert space that are null spaces of  $Q_i^\mp(\beta)$ :

$$\begin{aligned} \mathcal{V}_0^-(\beta; i) &:= \{|\psi\rangle \mid Q_i^-(\beta) |\psi\rangle = 0\}, \\ \mathcal{V}_0^+(\beta; i) &:= \{|\psi\rangle \mid Q_i^+(\beta) |\psi\rangle = 0\}. \end{aligned} \quad (\text{C7})$$

Also define  $\mathcal{V}_\perp^\mp(\beta; i)$  as the orthogonal subspaces to  $\mathcal{V}_0^\mp(\beta; i)$ , which therefore satisfy

$$\mathcal{V}_0^\mp(\beta; i) \oplus \mathcal{V}_\perp^\mp(\beta; i) = \mathcal{H}, \quad (\text{C8})$$

with  $\mathcal{H}$  the full Hilbert space of  $L$  spin- $\frac{1}{2}$  degrees of freedom. Now, notice that (i)  $Q_i^-(\beta) Q_i^+(\beta) = Q_i^+(\beta) Q_i^-(\beta) = 0$  and

that (ii)  $Q_i^+(\beta) + Q_i^-(\beta)$  is a positive-definite operator. It follows from (i) and (ii) that

$$\mathcal{V}_\perp^\mp(\beta; i) = \mathcal{V}_0^\pm(\beta; i), \quad (\text{C9})$$

i.e., the  $\mathcal{V}_0^\pm(\beta; i)$  partition the Hilbert space  $\mathcal{H}$  into two orthogonal subspaces:

$$\mathcal{V}_0^-(\beta; i) \perp \mathcal{V}_0^+(\beta; i) \quad \text{and} \quad \mathcal{V}_0^-(\beta; i) \oplus \mathcal{V}_0^+(\beta; i) = \mathcal{H}. \quad (\text{C10})$$

Notice that

$$|\text{scar}^+(\beta)\rangle \in \mathcal{V}_0^-(\beta; i) \quad \text{and} \quad |\text{scar}^-(\beta)\rangle \in \mathcal{V}_0^+(\beta; i), \quad \forall i. \quad (\text{C11})$$

Therefore, the two scar states are orthogonal in every local subspace of three neighboring spins, which is the support of  $Q_i^-(\beta)$  and  $Q_i^+(\beta)$ . Hence, one can define  $P_i(\beta)$  as

$$P_i(\beta) := \sum_{|\psi\rangle \in \mathcal{V}_0^-(\beta; i)} |\psi\rangle \langle \psi| - \sum_{|\psi\rangle \in \mathcal{V}_0^+(\beta; i)} |\psi\rangle \langle \psi|, \quad (\text{C12})$$

where the states  $|\psi\rangle$  entering each sum are chosen as an orthogonal basis of the respective subspace. This operator has  $+1$  eigenvalue for  $\mathcal{V}_0^-(\beta; i)$ , and  $-1$  for  $\mathcal{V}_0^+(\beta; i)$ . Since  $P_i(\beta)$  is a sum of projectors onto the (orthogonal) null spaces of  $Q_i^-(\beta)$  and  $Q_i^+(\beta)$ , its support follows that of  $Q_i^-(\beta)$  and  $Q_i^+(\beta)$ , which is finite. Thus, the locality of  $P_i(\beta)$  is guaranteed. One can express

$$\begin{aligned} P_i(\beta) &:= [Q_i^+(\beta) + Q_i^-(\beta)]^{-1} [Q_i^+(\beta) - Q_i^-(\beta)] \\ &= \{\cosh[\beta(Z_{i-1} Z_i + Z_i Z_{i+1})]\}^{-1} \\ &\quad \times \{\sinh[\beta(Z_{i-1} Z_i + Z_i Z_{i+1})] + X_i\}. \end{aligned} \quad (\text{C13})$$

With some algebra, one can derive the form of Eq. (C1c) from Eq. (C13).



The model Eq. (C1a) is again a specific example of Ref. [22] with

$$\begin{aligned} \hat{P}_i &= \mathbb{1} - \sum_{|\psi\rangle \in \mathcal{V}_0^-(\beta, i) \cup \mathcal{V}_0^+(\beta, i+2)} |\psi\rangle\langle\psi|, \\ \hat{H}' &= \sum_i P_i(\beta) \end{aligned} \quad (\text{C14})$$

using the notation of Eq. (2) in Ref. [22], where again the states  $|\psi\rangle$  entering each sum are chosen as an orthogonal basis of the respective subspace.

#### APPENDIX D: SYMMETRIES IN 1D

One finds the commutation relations

$$[H_1^{1D}, H_2^{1D}] = [H^{1D}, H_a^{1D}] = 0, \quad a = 1, 2. \quad (\text{D1})$$

Therefore,  $H_1^{1D}$ ,  $H_2^{1D}$ , and  $H^{1D}$  can be diagonalized simultaneously.

*Translation symmetry.*  $H_1^{1D}$ ,  $H_2^{1D}$ , and  $H^{1D}$  are each invariant under the translations

$$i \mapsto i + 2n, \quad i = 1, \dots, 2L, \quad n \in \mathbb{Z}. \quad (\text{D2})$$

Hence,  $H_1^{1D}$ ,  $H_2^{1D}$ , and  $H^{1D}$  can be simultaneously diagonalized with the Hermitian generator of the unitary operators representing the transformations (D2), i.e., the momentum operator associated to the sublattice  $\text{SL}_1$ , say.

*Inversion symmetry.* For any site  $j \in \text{SL}_1$ ,  $H_1^{1D}$  is invariant under the inversion

$$i \mapsto i - 2(i - j), \quad i = 1, \dots, 2L. \quad (\text{D3})$$

For any site  $j \in \text{SL}_2$ ,  $H_2^{1D}$  is invariant under the inversion

$$i \mapsto i - 2(i - j), \quad i = 1, \dots, 2L. \quad (\text{D4})$$

Hence,  $H^{1D}$  has the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry that is generated by the two independent involutive unitary transformations (D3) and (D4). This is to say that  $H_1^{1D}$ ,  $H_2^{1D}$ , and  $H^{1D}$  are invariant under any inversion of the ring that leaves one site of the ring unchanged.

*Two independent involutive symmetries.* Hamiltonian  $H_1^{1D}$  is invariant under the involutive unitary transformation

$$Z_j \mapsto U_2 Z_j U_2 = -Z_j, \quad j \in \text{SL}_2, \quad U_2 := \prod_{k \in \text{SL}_2} X_k = U_2^\dagger, \quad (\text{D5a})$$

that acts trivially on the sites of the ring. Hamiltonian  $H_2^{1D}$  is invariant under the involutive unitary transformation

$$Z_j \mapsto U_1 Z_j U_1 = -Z_j, \quad j \in \text{SL}_1, \quad U_1 := \prod_{k \in \text{SL}_1} X_k = U_1^\dagger, \quad (\text{D5b})$$

that acts trivially on the sites of the ring. Hence,  $H^{1D}$  has the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry that is generated by the two independent involutive unitary transformations (D5a) and (D5b).

#### APPENDIX E: A LOCAL UNITARY TRANSFORMATION IN 1D

We verify the transformation law

$$Q_{a,j}^{1D} \mapsto \tilde{Q}_{a,j}^{1D} = W Q_{a,j}^{1D} W^\dagger, \quad (\text{E1})$$

with  $\tilde{Q}_{a,j}^{1D}$  and  $W$  defined in Eqs. (4.4) and (4.3), respectively. To this end, it suffices to prove the identity

$$W X_i W^\dagger = Z_{i-1} X_i Z_{i+1}, \quad \forall i. \quad (\text{E2})$$

The terms in the exponent of  $W$  that do not contain  $X_i$  do not contribute to the transformation, i.e.,

$$\begin{aligned} W X_i W^\dagger &= e^{\pm i \frac{\pi}{4} Z_{i-1} Z_i \mp i \frac{\pi}{4} Z_i Z_{i+1}} X_i e^{\mp i \frac{\pi}{4} Z_{i-1} Z_i \pm i \frac{\pi}{4} Z_i Z_{i+1}} \\ &= X_i e^{\mp i \frac{\pi}{2} Z_{i-1} Z_i \pm i \frac{\pi}{2} Z_i Z_{i+1}}, \end{aligned} \quad (\text{E3})$$

where  $\pm = +$ ,  $\mp = -$  for  $i \in \text{SL}_1$ , and vice versa for  $i \in \text{SL}_2$ . Using additional relations

$$e^{\mp i \frac{\pi}{2} Z_{i-1} Z_i} = \mp i Z_{i-1} Z_i, \quad e^{\pm i \frac{\pi}{2} Z_i Z_{i+1}} = \pm i Z_i Z_{i+1}, \quad (\text{E4})$$

one acquires the identity in Eq. (E2).

#### APPENDIX F: OPEN BOUNDARY CONDITIONS IN 1D

Using the notation introduced in Eq. (4.1), we define the Hamiltonian

$$\begin{aligned} H_{\text{OBC}}^{1D} &:= H_{1,\text{OBC}}^{1D} + H_{2,\text{OBC}}^{1D}, \\ H_{1,\text{OBC}}^{1D} &:= H_1^{1D}(\beta_1 = 0) - Q_{1,1}^{1D}(\beta_1 = 0), \\ H_{2,\text{OBC}}^{1D} &:= H_2^{1D}(\beta_2 = 0) - Q_{2,2L}^{1D}(\beta_2 = 0). \end{aligned} \quad (\text{F1})$$

By inspection of the explicit representations

$$\begin{aligned} H_{1,\text{OBC}}^{1D} &= \sum_{j=1}^{L-1} (\mathbb{1} - Z_{2j} X_{2j+1} Z_{2j+2}), \\ H_{2,\text{OBC}}^{1D} &= \sum_{j=1}^{L-1} (\mathbb{1} - Z_{2j-1} X_{2j} Z_{2j+1}), \end{aligned} \quad (\text{F2})$$

we observe that  $\Lambda_1^{\text{OBC}} := X_1 Z_2$  and  $\Lambda_{2L}^{\text{OBC}} := Z_{2L-1} X_{2L}$  obey the vanishing commutation relations

$$\begin{aligned} [\Lambda_1^{\text{OBC}}, H_{1,\text{OBC}}^{1D}] &= [\Lambda_1^{\text{OBC}}, Z_{i-1} X_i Z_{i+1}] = 0 \\ &\quad \text{for } i = 3, \dots, 2L - 1, \\ [\Lambda_{2L}^{\text{OBC}}, H_{2,\text{OBC}}^{1D}] &= [\Lambda_{2L}^{\text{OBC}}, Z_{i-1} X_i Z_{i+1}] = 0 \\ &\quad \text{for } i = 2, \dots, 2L - 2. \end{aligned} \quad (\text{F3})$$

The two vanishing anticommutators

$$\{\Lambda_1^{\text{OBC}}, U_2\} = \{\Lambda_{2L}^{\text{OBC}}, U_1\} = 0, \quad (\text{F4})$$

along with the fact that  $\Lambda_1^{\text{OBC}}$ ,  $\Lambda_{2L}^{\text{OBC}}$  and the Hermitian operator  $U_a \equiv \prod_{j \in \text{SL}_a} X_j$  defined in Eq. (D5) commute with  $H_{\text{OBC}}^{1D}$ , imply that every eigenspace of  $H_{\text{OBC}}^{1D}$ , including the one of the scar state, is at least fourfold degenerate, and the quadruplet of states can be labeled by the eigenvalues of  $\Lambda_1^{\text{OBC}}$  and  $\Lambda_{2L}^{\text{OBC}}$ . The degeneracy is protected by the symmetries  $U_1$  and  $U_2$ . Since  $\Lambda_1^{\text{OBC}}$  and  $\Lambda_{2L}^{\text{OBC}}$  are local operators at the end of the chain, the Hamiltonian is in an SPT phase.

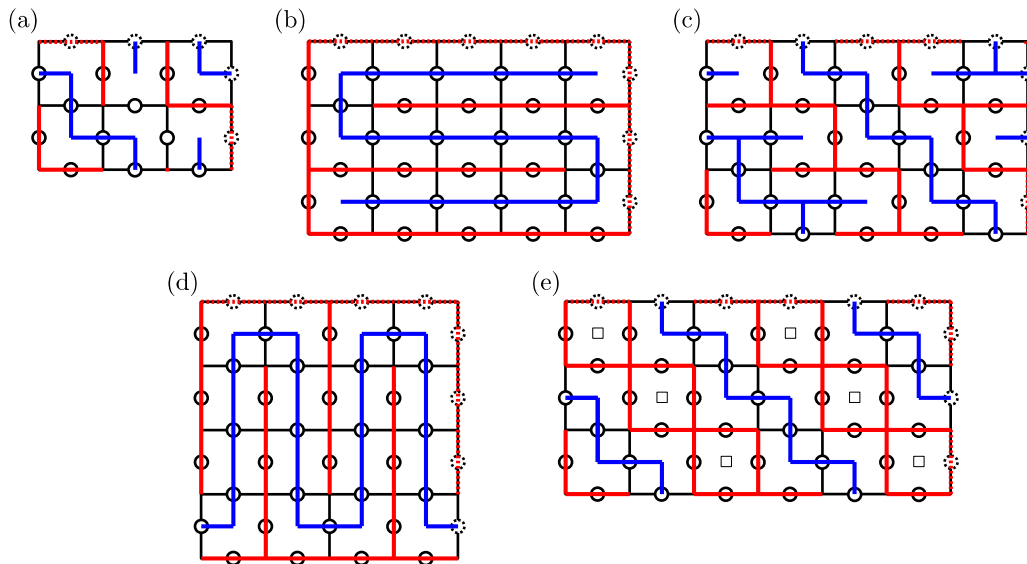


FIG. 6. Examples of lattice structures for the 2D model. Dashed sites and lines are used to represent periodic boundary conditions. Any path  $\mathcal{P}_1$  that is colored in red starts and ends by definition on the sites of the lattice  $\Lambda_*$ . Any path  $\mathcal{P}_2$  colored in blue starts and ends by definition on the sites of the dual lattice  $\Lambda_\square$ . The spin degrees of freedom are located on the sites of the median lattice  $\Lambda_\circ$  denoted by open circles. (a)–(d) Example of the path  $\mathcal{P}_1$  colored in red and the path  $\mathcal{P}_2$  colored in blue for a square lattice of given aspect ratio. Only the sites  $i$  of  $\Lambda_\circ$  represented by open circles are shown. With this choice for the paths  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , the condition  $\beta_1, \beta_2 > 0$  is sufficient to guarantee that the sum over  $s$  in  $H_1^{2D}$  (the sum over  $p$  in  $H_2^{2D}$ ) can never be arranged into the sum of two nonvanishing Hermitian operators that commute pairwise and commute with  $H_2^{2D}$  ( $H_1^{2D}$ ). (e) The choice made for the path  $\mathcal{P}_1$  colored in red and the path  $\mathcal{P}_2$  colored in blue fails to guarantee that the sum in  $H_a^{2D}$  can be arranged into the sum of two nonvanishing Hermitian operators that commute pairwise and with  $H_a^{2D}$  when  $\beta_a, \beta_{\bar{a}} > 0$ . Indeed, of all Hermitian operators  $B_p$  entering  $H_2^{2D}$ , those sites from the dual lattice  $\Lambda_\square$  that are identified by the symbol  $\square$  are not traversed by  $\mathcal{P}_2$ . They give a set of operators  $\{B_\square\}$ , whereby  $B_\square$  commutes with both  $H_1^{2D}$  and  $H_2^{2D}$ .

### APPENDIX G: EXAMPLES OF PATHS $\mathcal{P}_1$ AND $\mathcal{P}_2$ IN 2D

For convenience, we recall that we introduced the pair of Hamiltonians

$$H_1^{2D} := \sum_s \left[ \exp \left( -\beta_1 \sum_{i \in s \cap \mathcal{P}_1} Z_i \right) - A_s \right], \quad (\text{G1})$$

$$H_2^{2D} := \sum_p \left[ \exp \left( -\beta_2 \sum_{i \in p \cap \mathcal{P}_2} X_i \right) - B_p \right], \quad (\text{G2})$$

$$A_s := \prod_{i \in s} X_i, \quad B_p := \prod_{i \in p} Z_i, \quad (\text{G3})$$

in Eq. (5.1). The definition of the paths  $\mathcal{P}_1$  and  $\mathcal{P}_2$  was given below Eq. (5.1). An example for the choice of paths  $\mathcal{P}_1$  and  $\mathcal{P}_2$  was given in Fig. 3. Four more examples and one counterexample are given in Fig. 6.

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- [1] M. V. Berry and M. Tabor, *Proc. R. Soc. London, Ser. A* **356**, 375 (1977).  
 [2] J. M. Deutsch, *Phys. Rev. A* **43**, 2046 (1991).  
 [3] M. Srednicki, *Phys. Rev. E* **50**, 888 (1994).  
 [4] M. Rigol, V. Dunjko, and M. Olshanii, *Nature (London)* **452**, 854 (2008).  
 [5] H. Bernien, S. Schwartz, A. Keesling, H. Levine, A. Omran, H. Pichler, S. Choi, A. S. Zibrov, M. Endres, M. Greiner *et al.*, *Nature (London)* **551**, 579 (2017).  
 [6] T. Kinoshita, T. Wenger, and D. S. Weiss, *Nature (London)* **440**, 900 (2006).  
 [7] S. Trotzky, Y.-A. Chen, A. Flesch, I. P. McCulloch, U. Schollwöck, J. Eisert, and I. Bloch, *Nat. Phys.* **8**, 325 (2012).  
 [8] M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, *Science* **349**, 842 (2015).  
 [9] S. Moudgalya, S. Rachel, B. A. Bernevig, and N. Regnault, *Phys. Rev. B* **98**, 235155 (2018).

- [10] S. D. Geraedts, R. N. Bhatt, and R. Nandkishore, *Phys. Rev. B* **95**, 064204 (2017).
- [11] J. Z. Imbrie, V. Ros, and A. Scardicchio, *Ann. Phys.* **529**, 1600278 (2017).
- [12] C. Turner, A. Michailidis, D. Abanin, M. Serbyn, and Z. Papić, *Nat. Phys.* **14**, 745 (2018).
- [13] L. Fleishman and P. W. Anderson, *Phys. Rev. B* **21**, 2366 (1980).
- [14] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, *Phys. Rev. Lett.* **95**, 206603 (2005).
- [15] D. Basko, I. Aleiner, and B. Altshuler, *Ann. Phys.* **321**, 1126 (2006).
- [16] V. Oganesyan and D. A. Huse, *Phys. Rev. B* **75**, 155111 (2007).
- [17] A. Pal and D. A. Huse, *Phys. Rev. B* **82**, 174411 (2010).
- [18] M. Serbyn, Z. Papić, and D. A. Abanin, *Phys. Rev. Lett.* **111**, 127201 (2013).
- [19] D. A. Huse, R. Nandkishore, and V. Oganesyan, *Phys. Rev. B* **90**, 174202 (2014).
- [20] J. Z. Imbrie, *J. Stat. Phys.* **163**, 998 (2016).
- [21] O. Vafek, N. Regnault, and B. A. Bernevig, *SciPost Phys.* **3**, 043 (2017).
- [22] N. Shiraishi and T. Mori, *Phys. Rev. Lett.* **119**, 030601 (2017).
- [23] W. W. Ho, S. Choi, H. Pichler, and M. D. Lukin, *Phys. Rev. Lett.* **122**, 040603 (2019).
- [24] S. Moudgalya, N. Regnault, and B. A. Bernevig, *Phys. Rev. B* **98**, 235156 (2018).
- [25] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn, and Z. Papić, *Phys. Rev. B* **98**, 155134 (2018).
- [26] C.-J. Lin and O. I. Motrunich, *Phys. Rev. Lett.* **122**, 173401 (2019).
- [27] D. J. Luitz, N. Laflorencie, and F. Alet, *Phys. Rev. B* **91**, 081103(R) (2015).
- [28] V. Khemani, F. Pollmann, and S. L. Sondhi, *Phys. Rev. Lett.* **116**, 247204 (2016).
- [29] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, *Phys. Rev. B* **87**, 155114 (2013).
- [30] C. Castelnovo, C. Chamon, and D. Sherrington, *Phys. Rev. B* **81**, 184303 (2010).
- [31] S. Vijay, J. Haah, and L. Fu, *Phys. Rev. B* **94**, 235157 (2016).
- [32] C. Chamon, *Phys. Rev. Lett.* **94**, 040402 (2005).
- [33] S. Bravyi, B. Leemhuis, and B. M. Terhal, *Ann. Phys.* **326**, 839 (2011).
- [34] J. Haah, *Phys. Rev. A* **83**, 042330 (2011).
- [35] S. Vijay, J. Haah, and L. Fu, *Phys. Rev. B* **92**, 235136 (2015).
- [36] D. S. Rokhsar and S. A. Kivelson, *Phys. Rev. Lett.* **61**, 2376 (1988).
- [37] S. Sachdev, *Phys. Rev. B* **40**, 5204 (1989).
- [38] L. B. Ioffe and A. I. Larkin, *Phys. Rev. B* **40**, 6941 (1989).
- [39] C. L. Henley, *J. Stat. Phys.* **89**, 483 (1997).
- [40] R. Moessner and S. L. Sondhi, *Phys. Rev. Lett.* **86**, 1881 (2001).
- [41] C. Castelnovo, C. Chamon, C. Mudry, and P. Pujol, *Ann. Phys.* **318**, 316 (2005).
- [42] C. Castelnovo and C. Chamon, *Phys. Rev. B* **77**, 054433 (2008).
- [43] Y. Y. Atas, E. Bogomolny, O. Giraud, and G. Roux, *Phys. Rev. Lett.* **110**, 084101 (2013).
- [44] M. B. Hastings, *J. Stat. Mech.: Theory Exp.* (2007) P08024.
- [45] J. Eisert, M. Cramer, and M. B. Plenio, *Rev. Mod. Phys.* **82**, 277 (2010).
- [46] M. Hermanns, [arXiv:1702.01525](https://arxiv.org/abs/1702.01525).
- [47] Z.-C. Gu and X.-G. Wen, *Phys. Rev. B* **80**, 155131 (2009).
- [48] F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa, *Phys. Rev. B* **85**, 075125 (2012).
- [49] L. H. Santos, *Phys. Rev. B* **91**, 155150 (2015).
- [50] A. Yu. Kitaev, *Ann. Phys.* **303**, 2 (2003).
- [51] A. Hamma, P. Zanardi, and X.-G. Wen, *Phys. Rev. B* **72**, 035307 (2005).